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Problems on the Frontiers between General Relativity and Differential Geometry

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I. INTRODUCTION

The Contributions of Mathematics and of Physics

IN the work of Eugene Wigner one sees the basic harmony between the conceptual framework of physics and the structure of the mathematics associated with that physics. For example, the mathematical parameter that distinguishes one group representation from another measures a physical property of the system under study, such as its linear or angular momentum or its parity. From a first mention of these and many similar associations between mathematics and physics, it might seem that the main work of the physicist is over, once he has discovered the basic principles and equations in the given area of investigation. All that then remains, it might be argued, is to systematically develop the mathematics from this basic starting point (the work of the mathematician), and then to survey the derived quantities turned up in the course of the investigation, find with what physical magnitudes they are to be identified, and give them names—the work of the physicist! How far from the truth this perspective lies is shown throughout Wigner's work. There, physical motivation and mathematical analysis are inextricably entwined. A small example is his study of the transition from the Lorentz group to the Galilean group in the limiting case where the speed of light becomes very great. Uncovered here was a significant type of connection between two distinct groups into which purely mathematical motivations would seem likely to lead one only by the rarest chance. Even more widely known, to turn to an example where more minds have contributed, is hydrodynamics. Here the basic equations can be written in two lines and have been known since the eighteenth century. A group of investigators locked up with those equations with the promise of release a year later would come out predicting sound waves and possibly even shock waves. But to discover (and to analyze on the basis of the equations of hydrodynamics) such effects as turbulence, cavitation, vortices, and boundary-layer phenomena has been a task demanding not *mathematics alone*, but decades of the closest collaboration between mathematics and physics at a high level of sophistication.[1] Physical reasoning guided the selection of idealized models for analysis, motivated the choice of approximation method when approximations were necessary, and gave meaning to the coefficients in power-series expansions (drag coefficient, heat-transfer coefficient, moments of the correlation in turbulent velocity between different points, etc.) when series were employed. Never otherwise would volumes of results have been tied to two lines of equations!

The Unexpected Richness of General Relativity

Only one line is required to write Einstein's equation

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi T^{*}_{\mu\nu}\,, \qquad (1)$$

telling how much curvature is produced in space by a given concentration of stress and energy, and telling, in addition, how a curvature—whether initiated in this way or already existing in its own right—will evolve in time.[2] However, in this "geometrodynamics" it has taken even deeper puzzling, and even closer liaison between mathematics and physics, than in the case of hydrodynamics to read out of the basic law some portion of the richness that it contains: effect of a center of attraction on the physics in the surrounding space; equation of motion of a concentration of mass-energy; precession of the perihelion of Mercury, bending of light by the sun, and the gravitational red shift; gravitational radiation; expanding

[1] For an analysis of the relation between mathematics and physics, see John von Neumann, *Collected Papers*, edited by A. Taub (Pergamon Press, New York, 1962), Vol. 1, paper 1.

[2] *Notation:* In a local Lorentz frame $T_{00} = -T_0^0 = T^{00}$ is the density of energy of matter plus radiation; $-T_{10} = -T_{01}$ is the density of flow of energy in the x direction divided by the speed of light; and $T_{12} = T_{21} = T^{12}$ is the force in the x direction (exerted, for example, by the electromagnetic field) per unit of area normal to the y direction. Asterisks, or in

and recontracting universe; electromagnetism as an aspect of the curvature—and the derivative of the curvature—of an empty *Riemannian* 4-manifold; electricity as lines of force trapped in the topology of a multiply connected space; geons—long lived concentrations of mass-energy built out of gravitational or electromagnetic radiation or both, exemplifying mass constructed from curved empty space; degrees of freedom in the selection of initial value data—which then determine the future dynamical evolution of the geometry ("Cauchy problem"); canonically conjugate "field coordinate" (the intrinsic geometry on a space-like hypersurface) and "field momentum" (the extrinsic geometry, measuring how this 3-space is curved—or is to be curved—with respect to the enveloping 4-manifold); and other concepts, entities, effects, and consequences[3] derived from general relativity. Moreover, the prospects of much immediate guidance out of experiment are far less in the dynamics of geometry than they were and are in the dynamics of fluids. It is interesting and important to study with increased detail such effects as the correlation of clocks, the bending of light, the gravitational red shift, and the precession of gyroscopes and planetary orbits. However, the physics of dramatically new character already predicted by general relativity (and the new phenomenology undoubtedly still to be predicted) comes at distances $\sim 10^{28}$ cm (radius of universe) and $\sim 10^{-33}$ cm [Planck's characteristic length for fluctuations in geometry, $(\hbar G/c^3)^{1/2}$], both as far removed as can be from the domain of everyday experimentation. Unexplained measurements and newly observed effects are infrequent sources of the insights one has needed to unravel the implications of Einstein's equations to the present incomplete stage of exploration. There is no sign that this situation will change. Therefore, there is every reason to believe that geometrodynamics will be forced to evolve by a pattern extraordinarily hard to find among any of the other branches of physics. It will require a long and arduous effort almost unassisted by immediately relevant experiments. It will demand the closest sharing and intermingling of the most modern developments of mathematics, together with physical analysis of concepts and of model situations, all at an unprecedented level of sophistication. Fortunately, efforts in this direction, from both professions, have increased in the past decade by a factor which one can estimate roughly to be in excess of 2. Happily also, the American Mathematical Society sponsored, in the summer of 1962, an institute on Relativity and Differential Geometry at which progress and problems were reported from both sides.

Survey of Contents

The present account, adapted from a report presented at that institute, consists of three parts. Section II recalls the basic principles of general relativity in brief outline. Section III reviews some of the exact solutions of Einstein's equations, cites new insights won from these solutions; also new issues that have come to light in this way. Section IV recapitulates these issues in the context of differential geometry. At the Santa Barbara institute, Professor S. Chern, in a wide ranging account of the developments in this branch of mathematics, noted that modern differential geometry is *global* differential geometry. It is a mark of the central position of this subject in general relativity that several of the major problems outlined in IV have to do with topology, boundary value data, and the behavior of solutions in the large.

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some cases lower-case letters, denote familiar physical quantities transcribed to geometrical units by the appropriate number of powers of Newton's constant of gravitation G and the velocity of light c:

$T^*_{\mu\nu}$ (length^{-2}) $= (G/c^4)\, T_{\mu\nu}$ (energy/length^3);
m^* (length) $= (G/c^2) m$ (mass);
q^* (length) $= (G^{1/2}/c^2) q$ (electric charge);
$f_{\mu\nu}$ (length^{-1}) $= (G^{1/2}/c^2) F_{\mu\nu}$ (electromagnetic field).

The coordinates of a point in space-time are x^α (Greek; $\alpha = 0,1,2,3$). The coordinates on a space-like hypersurface $x^0 =$ constant are x^k (Latin; $k = 1,2,3$). The distance between two nearby points on the hypersurface is

$$ds^2 = {}^{(3)}g_{ik} dx^i dx^k .$$

Here, and elsewhere, Einstein's convention is employed, that repetition of an index implies summation over that index. The distance between two nearby points in space-time, whether space-like ($d\sigma$) or time-like ($d\tau$), is given by the formula

$$d\sigma^2 = -d\tau^2 = g_{\alpha\beta} dx^\alpha dx^\beta .$$

The reciprocal of the metric tensor has upper labels: ${}^{(3)}g^{ik}$ or $g^{\alpha\beta}$. Note ${}^{(3)}g_{23} = g_{23}$ but ${}^{(3)}g^{23} \neq g^{23}$. In a local Lorentz frame of reference the Riemann curvature tensor is

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\left(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\beta\delta,\alpha\gamma} - g_{\alpha\gamma,\beta\delta}\right),$$

where labels after a comma signify variables with respect to which differentiation is performed:

$$g_{\alpha\beta,\gamma\delta} = \partial^2 g_{\alpha\beta}/\partial x^\gamma \partial x^\delta .$$

The Ricci curvature tensor is

$$R_{\beta\delta} = g^{\alpha\gamma} R_{\alpha\beta\gamma\delta}$$

and the curvature scalar is

$$R = g^{\beta\delta} R_{\beta\delta} .$$

For other notation from differential geometry, see, for example, L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1949), revised second printing.

[3] For a collection of chapters dealing with recent developments in relativity, see L. Witten, editor, *The Theory of Gravitation* (John Wiley & Sons, Inc., New York, to be published, 1962). See also J. A. Wheeler, *Geometrodynamics* (Academic Press Inc. New York, 1962), cited hereafter as GMD.

Is Geometrodynamics Valid at Small Distances?

Before entering on this account, it is hardly out of place to at least touch upon a larger question. What, of value to physics, can possibly come out of geometrodynamics extrapolated, as it will necessarily have to be when quantized, far below the range of distances directly accessible to experiment? (1) Conceivably nothing at all! Then the whole effort will have turned out to be only an academic exercise. Or (2), perhaps only a few considerations fundamental for thinking about the elementary particle problem. Or finally (3), an insight into the structure of space at small distances so nearly correct, and so penetrating, as to explain, or as to come a large part of the way towards explaining, how elementary particles are built out of pure geometry and nothing more. For general relativity to be of any use in discussing such a question, Einstein's equations must make sense not merely down to 10^{-13}cm, but to distances 20 orders of magnitude smaller. Such an outcome would not be completely surprising in view of what Wigner would call the "unreasonable" extrapolatory power discovered for another simple physical law, associated with the names of Coulomb, Faraday, and Maxwell:

1833: valid from 10^{3}cm to 10^{-1}cm;
1913: to 10^{-8}cm (stopping power and atomic structure);
1933: to 10^{-13}cm (nuclear structure).

Unprecedented Analysis of Consequences of Theory Required before any Correlation with Elementary Particle Physics can Take Place

What are the implications for physics if geometrodynamics *is* good down to 10^{-33}cm, when there is no way to test this point by direct measurement? By way of partial answer, ask what it would have meant if Hamilton had taken the step, perhaps within his reach in view of the physical optical motivations for his work, from Hamiltonian mechanics to the Schrödinger equation, and had, in addition, adopted Coulomb's law of force. Energy levels and spectra would come out; though with what a struggle over the probability interpretation and many another builtin concept! However, the technology of the time and for centuries to come, let it be imagined, is unable to do experiments at the atomic level. Only possible are measurements of density, elasticity, refractivity, viscosity, and others of those "properties of matter" which were so much the center of attention in Hamilton's time. Then, still another round of effort is imposed on theory, to work up from the atomic level to the level of the solid-state and macroscopic phenomena, before a direct check against experiment is possible. What a fantastic nonstop flight of analysis! But at the end there are checks—enormous in number and satisfying in accuracy!

Elementary Particles as Almost Negligible Perturbations in Vacuum Physics

This comparison is apt in an important sense. Every order of magnitude estimate[4] of what it means to put together the quantum principle and general relativity, if correct, leads to five conclusions: (1) The important features in the structure of space have a dimension $\sim L^* = (\hbar G/c^3)^{1/2} \sim 10^{-33}$cm. (2) This structure is a property of all space, not only of those parts of space where there are elementary particles. (3) The important part of the fluctuations in the energy density of the vacuum go on at a level of the order of $\hbar c/L^{*4}$ (energy) or $\hbar/cL^{*4} \sim 10^{94}$g/cm^3 (mass). (4) The supplementary energy density associated with an elementary particle, $\sim 10^{15}$g/cm^3, is negligible by comparison. (5) Thus, relative to the Planck distance L^*, elementary particles are large scale and very diffuse objects with an enormous amount of standard internal structure. According to such estimates, particles are almost to be compared with the clouds in the sky for the little that they mean to the balance of energy and pressure in the substrate. Towards puzzling out what goes on at distances of the order L^*, it would therefore seem of little help, at this stage, to ask questions about the origin of nuclear forces, and about the specificity of the interaction between one kind of particle and another. It would be as well to ask, in the early 1800's, how work hardening of metals was to be explained when one knew nothing about atomic structure, let alone about dislocations! There is a limit to the extent to which the natural order of development of a subject can be upset!

The phenomenology of the elementary particles is obviously a subject of the greatest importance in its own right. In addition, it will someday be decisive in testing a theory of matter. But for purposes of arriving at that theory of matter, this absorbing area of investigation may not supply the most natural point of entry. How can it, if twenty powers of ten supervene between the objects studied and the decisive structure?

No Substitute Known for Einstein's Geometrodynamics

General relativity, in brief, is a theory with an un-

[4] See, for example, GMD, pp 67–83.

precedented and so far unbounded scope. Some consequences of substantial interest to physics have already been read out of it. From it much more is yet to be learned, especially at Planck's scale of distances. Here no other source of guidance is available.

II. RELATIVITY IN RESUMÉ

From Riemann and Mach to Einstein and Equivalence

Einstein's principle of the equivalence of gravitational and accelerational forces, and the general relativity that came out of it, represented the union of two currents of thought. Of these, one went back to Ernst Mach's principle that inertia has to do with the acceleration of one object relative to all other objects and must therefore be tied to an interaction between mass and mass—an interaction that Einstein identified as the radiational component of the gravitational force itself. The other stream of thought traces back to Riemann's arguments. Space controls the motion of physical entities. Therefore, space must be a part of physics. Consequently, space cannot be assumed to have an ideal Euclidean character. Riemann supplied, in addition, the mathematical machinery to analyze the curvature not only of 3-space but also, fortunately, of a manifold of higher dimensionality.

The Geodesic Postulate in the Beginning and Today

Einstein's description of gravitation is based upon the idea of a field defined everywhere, following the model of electromagnetism. This idea has two parts: the action of the object upon the field, and the action of the field upon the object. The field relevant for gravitation had to be identified, according to the equivalence principle, with the curvature of space-time itself; in other words, with pure geometry. The effect of the geometry on the particle is described by saying that it follows a geodesic.

Two developments since the early days of relativity have given increased insight into Einstein's original postulate, that a particle follows through space-time a history of extremal length (measured in units of proper time). (1) The quantum principle has made it clear why laws of extremal action occur throughout physics. Some explanation has at last been supplied for the penetrating observation Newton made at the very beginning of his *Principia*: "... the description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn."[5] But how? Feynman's formulation of quantum mechanics[6] attributes a probability amplitude of standard magnitude to every history that leads from an initial configuration at an initial time to a final configuration at a final time. Only the phase differs from one history to another, being proportional to the classical "action integral" or "dynamic path length" I_H for the history in question:

$$\begin{pmatrix}\text{partial probability amplitude or propagation} \\ \text{amplitude associated with history } H\end{pmatrix} = \text{const} \times \exp\,(iI_H/\hbar)\,. \qquad (2)$$

When these partial amplitudes are added, destructive interference occurs. In this sense most of the histories are not relevant. Only those histories contribute effectively to the propagation in time, whose amplitudes combine constructively. They have phases $I_H/\hbar$ which differ from the extremal phase by amounts of the order of a radian and less. Thus, the classical law of motion originates in a mechanism which "feels out"—and effectively rejects—nonclassical motions.

The reasoning about extremal dynamical path length, that is illustrated here by the motion of a particle, equally applies to the dynamics of a field. In consequence, fields, like particles, come close, under many circumstances, to following the classical laws of motion. Therefore, most of the present account is concerned with the dynamics of geometry at the classical level.

(2) Einstein, Infeld, Hoffmann, and others[7] showed that the geodesic law of motion need not be introduced as a separate postulate, but can be derived from the equation of time development of the geometry itself. It is as if hydrodynamics had once contained laws of motion for vortices in addition to the standard laws of fluid motion, until one found that vortex movement is a direct consequence of these standard equations.

Gravitational and Electromagnetic Forces Compared and Contrasted

So much for the geodesic equation of motion; now for its use in studying the gravitational field, or the geometry of space. One is first tempted to think of gravitational forces as differing only in a little detail

[5] Isaac Newton, *Mathematical Principles of Natural Philosophy*, 1729 translation by Andrew Motte, revised by F. Cajori (University of California Press, Berkeley, California, 1947).

[6] R. P. Feynman, Princeton Ph.D. Thesis, 1942; Revs. Modern Phys. **20**, 367 (1948); Phys. Rev. **76**, 769 (1949).

[7] For an account of these researches see L. Infeld and J. Plebanski, *Motion and Relativity* (Pergamon Press, New York, 1960). Problem areas in the derivation of the equations of motion from the field equation are discussed by J. A. Wheeler, Revs. Modern Phys. **33**, 63 (1961).

from electromagnetic forces. Consider, for example, a center of attraction, massive and endowed with spherical symmetry. Consider a particle approaching with its line of approach offset by the amount b from the parallel line that goes straight to the center of attraction. Let this distance be great enough, so that the change in direction is small. Then, for small velocities one is accustomed to write

$$\theta \doteq \frac{\begin{pmatrix}\text{transverse impulse given}\\ \text{by center of attraction}\end{pmatrix}}{(\text{forward momentum of particle})}$$
$$\doteq \frac{\int(\text{transverse force})\, d\,(\text{forward distance})}{(\text{momentum})\cdot(\text{velocity})}$$
$$\doteq \begin{cases}(q_{\text{test}}q_{\text{center}}/bm_{\text{test}}v^2) & \text{for electromagnetism}\\ (Gm_{\text{test}}m_{\text{center}}/bm_{\text{test}}v^2) & \text{for gravitation}\end{cases} \quad (3)$$

The field acting on the test particle in the frame of reference of the particle is vectorial in the case of electromagnetism, and tensorial in the case of gravitation. Consequently, one is prepared for different laws of transformation in the two cases, and quite different angles of deflection at high velocities,

$$\theta \doteq \begin{cases}2(q^*_{\text{test}}q^*_{\text{center}}/bm^*_{\text{test}})\beta^{-2}(1-\beta^2)^{1/2}\\ \quad\text{for electromagnetism}\\ 2(m^*_{\text{center}}/b)(1+1/\beta^2)\\ \quad\text{for gravitation}\end{cases} \quad . \quad (4)$$

This comparison makes the difference between the two forces appear as only a matter of detail. However, the over-all deflection is quite a complex concept in the case of general relativity. It requires sophisticated analysis[8] of a network of light rays and world lines of distant geodesics for its sharp definition. Therefore, turn from the integrated force and the total deflection to the local force.

The electromagnetic force is measured by the deviation from a straight line path or, to use geometrical terms, from a fiducial geodesic. Sometimes it is useful to spell out this idea in the context of a particular geometry and, even more specifically, a particular coordinate system. Then, the fiducial geodesic is described by the equation[2]

$$(D/D\tau)(dx^\alpha/d\tau) \equiv d^2x^\alpha/d\tau^2 + \Gamma^\alpha_{\beta\gamma}(dx^\beta/d\tau)(dx^\gamma/d\tau) = 0\,. \quad (5)$$

Here D stands for the absolute derivative. It corrects for any apparent change in direction which is caused by mere curvilinearity in the coordinates. In the same notation the equation of Lorentz for the motion of the charged test particle is

$$m^*(D/D\tau)(dx^\alpha/d\tau) = q^*F^\alpha{}_\beta(dx^\beta/d\tau)\,. \quad (6)$$

A knowledge of the force per unit charge on three test particles passing through the same point, as defined by the difference between (6) and (5), is enough to allow one to determine all six independent components $F^\alpha{}_\beta$ or $F_{\alpha\beta} = -F_{\beta\alpha}$ of the field at that point.

In the case of the gravitational field the test particles are neutral from the start. Then nothing is to be learned from looking at only *one* test particle. Of all the lessons learned from general relativity, one of the harder ones to assimilate was this, that no feature of the motion of one particle, but the difference in position η^α between *two* nearby test particles at corresponding proper times, is the proper measure of the local gravitational field. This difference satisfies the equation

$$D^2\eta^\alpha/D\tau^2 + R^\alpha_{\beta\gamma\delta}(dx^\beta/d\tau)\eta^\gamma(dx^\delta/d\tau) = 0\,. \quad (7)$$

For example, when two particles are initially at rest in the chosen frame of reference, then their acceleration, relative to each other, is given by the equation

$$D^2\eta^k/D\tau^2 + R^k_{0l0}\eta^l = 0\,. \quad (8)$$

Often considered is the unexciting appearance of a single test particle in a freely falling elevator. To investigate the local gravitational field, consider instead a freely falling auditorium. Inside, well separated, and initially at rest with respect to each other and this container, are four test particles. The first, O, serves as standard of reference. A lies at an easterly separation Δx; B, at a northerly separation Δy; and C is located at a distance Δz above O. Though the particles are falling freely, their state of relative rest will not continue. The accelerations of O and A have the same magnitude, m^*/r^3, but differ in direction by the angle $\Delta x/r$. Therefore, the separation Δx changes with time—already in Newtonian theory—in accordance with the equation

$$D^2\Delta x/D\tau^2 + (m^*/r^3)\Delta x = 0\,. \quad (9)$$

A similar equation holds for the separation Δy between O and B. The separation between O and C is governed by the derivative of the Newtonian acceleration with respect to elevation:

$$D^2\Delta z/D\tau^2 - (2m^*/r^3)\Delta z = 0\,. \quad (10)$$

Thus, one arrives at a Newtonian estimate—and an actually accurate figure—for those components of the Riemann curvature tensor which are analogous to

[8] E. Newman and J. Goldberg, Phys. Rev. **114**, 1391 (1959); J. Plebanski, *ibid.* **118**, 1396 (1960).

the electrostatic field of Faraday and Maxwell:

$$\| R^k{}_{0l0} \| = \begin{Vmatrix} (m^*/r^3) & 0 & 0 \\ 0 & (m^*/r^3) & 0 \\ 0 & & -(2m^*/r^3) \end{Vmatrix}$$

$$\begin{pmatrix} k = 1,2,3 = \text{row} \\ l = 1,2,3 = \text{column} \end{pmatrix} . \qquad (11)$$

So much for one example of curvature and its measurement.

Curvature Tensor as the Physically Meaningful Measure of the Local Gravitational Field

When the curvature tensor differs from zero in one coordinate system, no change to a new coordinate system can annul it:

$$\overline{R}_{\alpha\beta\gamma\delta} = R_{\kappa\lambda\mu\nu}(\partial x^\kappa/\partial \overline{x}^\alpha)(\partial x^\lambda/\partial \overline{x}^\beta)(\partial x^\mu/\partial \overline{x}^\gamma)(\partial x^\nu/\partial \overline{x}^\delta) . \qquad (12)$$

It measures, not the curvature of coordinate surfaces, but the curvature of space-time itself. There is no way to make an immediate connection between the physics of gravitation and the mathematics of geometry except to recognize the physically relevant components of the gravitation field—defined by the relative motion of two test particles—as but names for geometric quantities, the components of the Riemann curvature tensor.

Effects of Mass on Geometry Local and Long Distance

Turn now from the effect of geometry on masses to the effect of masses on geometry, where the concept of curvature is again the key to the analysis. A distribution of mass in one region has two effects on the curvature; one localized in that region, the other felt at a distance. The local effects are measured by ten combinations

$$R_{\mu\nu} = R^\sigma_{\mu\sigma\nu} \qquad (13)$$

of the twenty independent quantities $R_{\alpha\beta\gamma\delta}$. The remote effects are measured by the remaining components of the curvature tensor.

One is familiar with a similar situation in electromagnetism. The electric field at point A depends upon the distribution of electric charge, not only at A, but elsewhere. To sort out the effects of the nearby charges from those of the faraway charges, one takes the surface integral of the normal component of the electric field over the boundary of a localized volume element, finding

$$\text{div}\,\mathbf{E} = 4\pi\rho . \qquad (14)$$

The part of the gravitational field of local origin is found by a similar analysis. However, here a straightforward application of Newtonian ideas leads to misleading results. Consider the volume defined by a cluster of particles originally at rest relative to one another—like the freely falling particles O,A,B,C in an earlier example. In the local Lorentz frame centered on O the position of the typical particle in the cluster is given to the second order in the time by the formula

$$\eta^i = \eta^i_{\text{initial}} - (\tau^2/2)R^i_{0k0}\eta^k_{\text{initial}} . \qquad (15)$$

The volume of the cluster, relative to its initial volume, and also calculated to terms of the second order in the time, is

$$\begin{aligned} \frac{\partial\,(\text{new volume})}{\partial\,(\text{initial volume})} &= \frac{\partial(\eta^1,\eta^2,\eta^3)}{\partial(\eta^1_{\text{in}},\eta^2_{\text{in}},\eta^3_{\text{in}})} \\ &= \begin{vmatrix} 1-(\tau^2/2)R^1_{010} & -(\tau^2/2)R^1_{020} & -(\tau^2/2)R^1_{030} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & 1-(\tau^2/2)R^3_{030} \end{vmatrix} \\ &= 1-(\tau^2/2)R^k_{0k0} = (1-\tau^2/2)R^\alpha_{0\alpha0} \\ &= 1-(\tau^2/2)R_{00} . \end{aligned} \qquad (16)$$

Newtonian physics looks apart from components of the gravitational field of the character of magnetic forces. The equation of motion of one particle in Cartesian coordinates is

$$d^2x^i/d\tau^2 = -\partial\varphi^*/\partial x^i , \qquad (17)$$

where φ^* is the gravitational potential in dimensionless units (usual Newtonian value divided by $c)^2$. The separation $\eta^i (i = 1,2,3)$ between two nearby particles therefore follows the law of change with time

$$d^2\eta^i/d\tau^2 = -(\partial^2\varphi^*/\partial x^i\partial x^k)\eta^k . \qquad (18)$$

Without consideration for the non-Newtonian components in the force, one would therefore make the identification

$$R^i_{0k0} = \partial^2\varphi^*/\partial x^i\partial x^k$$

and

$$\begin{aligned} R_{00} &= (d^2/d\tau^2)\begin{pmatrix}\text{fractional change in volume of} \\ \text{cluster of particles}\end{pmatrix} \\ &= \nabla^2\varphi^* = 4\pi\begin{pmatrix}\text{density of mass-energy in geo-} \\ \text{metrized units of length/length}^3\end{pmatrix} \\ &= 4\pi T^*_{00} . \end{aligned}$$

If this result were correct in one frame of reference, it would have to be true in every frame of reference, from which it would at once follow that the equality

$R_{\mu\nu} = 4\pi T^*_{\mu\nu}$ holds for all components of both tensors.

Cluster Volume Constant to Second Order in Source-Free Case

This incorrect analysis leads to a result which is to this extent correct: When the stress-energy tensor vanishes in a given region, then the Ricci curvature tensor in that region also vanishes. Then every cluster of comoving test particles preserves its volume to the second order in time regardless of the changes in the shape of that volume, a way of stating the content of Einstein's source-free field equation due to Roger Penrose (unpublished).

To arrive at the correct form of the field equation when matter is present,

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}R = 8\pi T^*_{\mu\nu} \tag{19}$$

requires a deeper analysis which will not be given here.

Now all of the foundations of relativity are at hand. What do they say about the relation between geometry and physics?

Geometry Uniquely Specified by the Distribution of Mass?

A first look at (19) makes it appear that geometry is the slave of mass-energy. This impression seems to be confirmed when one looks at the simplest and best known of the solutions of this field equation of Einstein's—that for the curvature produced by a static spherically symmetrical center of attraction. The geometry is described by Karl Schwarzschild's expression for the distance between any two nearby points in space-time,[9]

$$d\sigma^2 = -d\tau^2 = -(1 - 2m^*/r)dt^2 + (1 - 2m^*/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) . \tag{20}$$

A single parameter m^* determines the effect of the central mass on every ray of light and on every planetary orbit—thence all three well-known tests of general relativity.

Degrees of Freedom Belonging to the Geometry Itself

Geometry has a dynamics of its own. However, this dynamics has been asleep so far because the source was at rest. Let two masses oscillate or rotate with respect to each other. Then gravitational radiation comes off. It is impossible to escape such effects when, as relativity teaches, there is a limiting velocity for the propagation of disturbances. A sudden change in the position of two masses can and must make itself felt at a distance, but only after the lapse of a finite time.

A Source-Free Gravitational Wave

Consider an array of sources of gravitational radiation located in all directions but at great distances, and so timed, relative to one another, that the pulses or waves they produce arrive in the region under study with the right phase relationships to produce an imploding spherical wave. This wave can be discussed either as a consequence of the motion of the distant sources or, in the case of infinitely distant sources, as a phenomenon in its own right—a manifestation of the independent degrees of freedom of geometry itself.

By way of analogy it is only necessary to recall how the electromagnetic field, at first the quasi-static but responsive slave of the charged particles, and then the oscillatory carrier of energy from one group of charges to another, was in the end most usefully regarded as a dynamic entity in its own right. No one feels uncomfortable considering a source-free plane electromagnetic wave, nor even an imploding spherical wave!

The Lorentz-Poincaré Theory of the Electromagnetic Constitution of Matter

The inspiration of seeing this new dynamic object come to life led, in the early 1900's, to various proposals by Lorentz, Poincaré, and others, for a purely electromagnetic account of the internal structure of elementary particles. The world, on this view, would be constructed of pure electromagnetism and nothing more. These attempts foundered on the problem of imparting stability to a collection of identical charges.[10] It was possible to give up this theory and return without loss to the dualistic picture of field plus particles!

General Relativity With and Without the Introduction of "Real" Mass and "Real" Charge

Similarly, relativity splits naturally into two subjects. One comprises geometry plus "real particles" plus electromagnetism plus other "real fields." This

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[9] For a sketch of a catalog of the proper distances between every point and every other point, and an account of how these distances—like a table of airline distances between the principal cities of the globe—determine the curvature of the manifold to which they belong, quite without reference to any coordinates at all, see for example GMD, p 12.

[10] For an analysis of this problem, together with a survey of the earlier literature, see A. Pais, *Developments in the Theory of the Electron*, a printed article not formally published, Princeton University and Institute for Advanced Study, 1948.

is the dualistic world of standard general relativity. There is little problem about tracing out the connection between this description of physics and that currently in use in theories that disregard the curvature of space-time.

The other version of general relativity may be called geometrodynamics because it considers only those objects, structures, phenomena, and events which arise out of pure geometry and its evolution with time in accordance with Einstein's equations. It is the analog of the Lorentz-Poincaré world of pure electromagnetism. It includes source-free electromagnetism as well as gravitation because through the work of Rainich and Misner one has learned how to regard electromagnetism as an aspect of geometry within the framework of Einstein's standard 1916 general relativity.[11]

The world of geometrodynamics is more interesting to consider than the world of Lorentz and Poincaré for two reasons: (1) Electricity does not have to be brought in from outside as something of a "physical" or nongeometric character. Instead, it shows up naturally as a feature of electric lines of force—themselves defined in purely geometrical terms—trapped in the topology of a multiply connected space. (2) Model objects can be constructed out of pure geometry which are endowed with mass-energy and which have any desired degree of stability. The simplest such "geon" is composed of electromagnetic radiation or gravitational radiation held in closed orbit by its own gravitational attraction.[3]

The charge and mass that belong to certain geometrodynamical structures are purely classical charge and mass. They have not the slightest direct connection with the quantized charges and masses of the real physical world. Therefore, the world of geometrodynamics has to be regarded as a purely model world, to be studied in its own right for whatever modest insights come out of it of use for understanding the real world. The unexpected scope of this model world, as revealed in the past decade in purely geometrical descriptions for charge and for mass, has naturally caused renewed interest in the daring dream of Clifford and Einstein that the universe is built of pure geometry and nothing more—still only a vision, though an inspiring one!

Nature of the Problems Uncovered by the Answers!

So much for the basic principles and outlooks of general relativity and geometrodynamics. Now for the main problem: to discover what *is* the problem! What does one want to do with Einstein's equations? What does one want to find out? Where does one want to go?

Dazzled by relativity theory and unprepared for the perspectives which it opened up, physics made headway mainly by investigating a number of special solutions of Einstein's equations, each associated with a special problem. It is appropriate to review some of the more interesting of these solutions, the new concepts to which they led, and the problems they posed (Sec. III) before looking for some of the features common to some of these problems (Sec. IV).

III. SPECIAL SOLUTIONS AND THEIR FEATURES

Table I summarizes the special solutions considered in this section.

Application of General Relativity to Closed Universe

How one learned about the nature of general relativity by working with it is nowhere shown so spectacularly as in the history of the cosmological problem.

It was reasonable in a first analysis to idealize to a homogeneous isotropic universe and, according to arguments of Einstein[12] based in part upon Mach's principle, to treat this universe as closed. Today another consideration can be adduced which favors a manifold closed in space. The action integral I_H is no longer a mere formal device to yield, by extremization, the dynamic equation for the classical history of the geometry $H = H_{\text{classical}}$. It has acquired a position of its own, as a measure of the phase $I_H/\hbar$ of the contribution to the quantum propagator from the history in question. It is difficult to see how any well-defined value can be attached to this quantity except in a closed space of finite content.

Were one dealing with a 2-dimensional space whose radius $a(t)$ varies with time, it would have been reasonable to combine distances measured on the sphere, and lapses of time, in the familiar way to form the Lorentz interval $d\sigma$ (space-like) or $d\tau$ (time-like) between one event and another

$$d\sigma^2 = -d\tau^2 = -dt^2 + a^2(t)(d\theta^2 + \sin^2\theta d\varphi^2) . \quad (21)$$

The corresponding expression for a 3-dimensional sphere is easily written down. It can be simplified by representing the time coordinate in terms of a parameter η, defined up to a constant by the equation

$$d\eta = dt/a(t) . \quad (22)$$

The quantity $d\eta$ measures the angle of travel about

[11] See, for example, GMD, reference 3.

[12] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1950), 3rd ed., p. 107.

TABLE I. Special solutions and the insights or issues to which they lead.

Solution	Insights or Issues
Friedmann and Tolman (expanding and recontracting universe).	Doubt versus confidence in the applicability of general relativity. Singularity in geometry after a finite proper time. Motive to look at gravitational radiation as a source of the effective mass-energy needed to curve space up into closure.
Taub (closed universe where all the curvature is supplied by gravitational radiation of maximal wavelength).	Simplest known example of a universe curved up into the topology S^3 of the 3-sphere by the equivalent mass-energy contained in pure source-free gravitational radiation. Singularity in geometry after a finite proper time.
Karl Schwarzschild (geometry associated with a spherically symmetric and neutral center of attraction).	Asymptotic behavior furnishes guide to what one means by mass-energy of any system restricted to a finite region of an asymptotically flat space. Non-Euclidean topology (bridge or wormhole) brought to attention. Singularity in geometry after a finite proper time ("pinch-off of throat"). Causality preserved in a modified sense by this pinch-off. Generalization of this causality analysis to arbitrary topology?
Reissner-Nordstrøm (geometry associated with a spherically symmetric and charged center of attraction).	Throat oscillates in circumference. Is causality maintained? Singularity in geometry after a finite proper time. Simplest starting point for wormhole picture of electric charge.
Misner (3-geometry, at moment of time symmetry, descriptive of wormhole with arbitrary ratio between size of throat and spacing between the two mouths).	Initial conditions for the problem of two neutral masses within the framework of pure geometrodynamics. When the two mouths are very close together, they almost amalgamate, and from the outside appear like a simple Schwarzschild throat. Expectation that geometry will become singular after a finite proper time.
Many special solutions (Gödel, Harrison, Jordan, Ehlers, Kundt, and others), also general considerations of Lifshitz.	To what kind of singularities in the intrinsic geometry does the source-free Einstein field equation lead? No singularity-free solution can be periodic (Papapetrou, Avez).
Small deviations from flatness, analyzed to the first order.	Degrees of freedom Conserved quantities Energy Asymptotic flatness
Gravitational waves (Einstein and Rosen, Bonnor, Robinson, Bondi, Pirani, Araki, Brill, Sachs, Arnowitt, Deser, Misner).	Positive-definite character of energy of pure gravitational waves. Asymptotic character of radiation field. Situations where no space-like singularity-free initial value surface can be found for defining Cauchy problem.

the sphere which can be accomplished by a light ray in the time dt. In this terminology the interval between two events is

$$d\sigma^2 = -d\tau^2 = a^2(\eta) \times [-d\eta^2 + dx^2 + \sin^2 x(d\theta^2 + \sin^2\theta d\varphi^2)] . \tag{23}$$

So far the reasoning has been based upon symmetry considerations. Now comes the equation for the radius of curvature a as a function of the time—a relation long known to be completely determined by the formula connecting the pressure and the density of mass-energy of the homogeneous medium that fills the space (in this view of mass-energy as something separate from and added to geometry). Any actual case is bracketed between the two extremes in Table II. It is surprising to hear this said when one hears of many alternative cosmological models: de Sitter's open universe, the Lemaitre model, the speculations about continuous creation put forward by Bondi, Gold and Hoyle, and other concepts. The confusion arises from the odd history of the subject.

Relativity Questioned because It Predicted Nonstatic Universe

Einstein showed in his first investigation that a homogeneous isotropic universe could not be static.

TABLE II. Dynamics of a homogeneous isotropic closed universe in the two opposite limiting cases when the medium that fills the space supplies zero pressure and maximum pressure. Times are expressed in the table in the units of length. The constant $(3c^2/8\pi G)$ in the last formula has the value $(1796 \times 10^{-30}\text{g/cm}^3) \times (10^9 \text{ light year})^2$. The formulas listed here are depicted graphically in GMD, p. 116.

Name	Friedmann	Tolman
Description	Dust filled	Radiation filled
Pressure/energy density	0	1/3
Maximum radius	a_0	a_0
Relation between time coordinate (measured in cm or m) and the time parameter η	$t = (a_0/2)(\eta - \sin\eta)$	$t = a_0(1 - \cos\eta)$
Radius of curvature as a function of the time parameter	$a(\eta) = (a_0/2)(1 - \cos\eta)$	$a(\eta) = a_0 \sin\eta$
Curve for radius as a function of time	Cycloid	Semicircle
Qualitative description of dynamics	Universe in both cases starts in a singular condition, radius increases with time to a maximum, then system recontracts to a singular condition.	
How much progress does a photon make (η measured in radians) in getting around the universe during the whole time of expansion and recontraction?	2π	π
Time back to start of expansion as estimated from present radius and present rate of expansion (inverse Hubble "constant")	$H^{-1} = \dfrac{a}{da/dt} = a_0 \dfrac{\sin^3 \eta/2}{\cos \eta/2}$	$H^{-1} = \dfrac{a}{da/dt} = a_0 \dfrac{\sin^2 \eta}{\cos \eta}$
Ratio between this extrapolated time and actual time back to start of expansion	$\dfrac{H^{-1}}{t} = \dfrac{2 \sin^3 \eta/2}{(\eta - \sin\eta)\cos \eta/2}$	$\dfrac{\sin^2 \eta}{\cos\eta(1 - \cos\eta)}$
This ratio is never less than	$\left(\dfrac{H^{-1}}{t}\right)_{\min} = 1.5$	$\left(\dfrac{H^{-1}}{t}\right)_{\min} = 2.0$
Density of mass-energy at any specified phase of the expansion	$\left(\dfrac{3c^2}{8\pi G a_0^2}\right) \sin^{-4} \eta$	$\left(\dfrac{3c^2}{8\pi G a_0^2}\right) \sin^{-6} (\eta/2)$

This result could have been accepted and announced as a fourth conclusion from general relativity in addition to the three well-known predictions. Then, in a few years, the work of Hubble would have come as an exciting confirmation of this forecast that the universe is necessarily dynamic.

The prediction appeared too radical, however. From what one knew at the time, it seemed necessary to think of the universe as static. Yet general relativity (Table II) offers no way for a homogeneous isotropic closed universe to be static. Therefore, Einstein reluctantly concluded that he had made a mistake in the dynamic equation of general relativity. He changed it in the most minor way he could find, by adding a "cosmological term" $\Lambda g_{\mu\nu}$ to the left-hand side of Eq. (1); a term otherwise unreasonable because of its *ad hoc* origin and lack of correspondence with any other existing knowledge. A short time later the evidence became unmistakable that the universe is actually dynamic. The motivation for the cosmological term disappeared. Einstein thereafter advocated the removal of this term from relativity.

Relativity Questioned because It Predicted $H^{-1} > t$

Soon attention fell on another argument for changing relativity, an argument equally fated for collapse. The red shift and the velocity of recession were already known for many galaxies. Now supposedly good figures were obtained for the distances to these galaxies. These distances could be multiplied by an empirical constant of proportionality H, so chosen by Hubble as to give a good representation of the observed velocities. The reciprocal H^{-1} of this constant gives the time back to the start of the expansion as estimated on the assumption that the rate of expansion did not change with time. According to relativity, the expansion is slower now than it was in the past. The extrapolated time, H^{-1}, should be at least 1.5 times as great as the actual time, t (Table II). In contrast, the empirical value for this extrapolated time, $H^{-1} \sim 2 \times 10^9$ yr, was far less than the geophysical value for the age, $t \sim 5 \times 10^9$ yr, of that part of the universe which has been the most thoroughly studied, the earth itself. Therefore, some who took seriously the astrophysical evidence as it stood concluded that relativity was inadequate to account for the situation. This was the era of theories of the continuous creation of matter.

This objection to relativity collapsed when it was discovered that the scale of galactic distances had been drastically in error. Revised figures for the distances raised the linearly extrapolated time back to the start of the expansion to 13×10^9 yr (plus or minus perhaps 50%).[13]

A less clear change also occurred in the time t which had to be made available for astrophysical evolution. Some stars with high content of heavy elements appear to require many times 10^9 yr for their development. However, this subject is in such an active state of development that it appears dangerous to draw conclusions from the apparent age, as presently estimated, for a few exceptional stars. The decisive time at present is not that estimated from the internal evolution of stars, nor from the evolution of the earth or of the solar system, but that required for the evolution of globular clusters ($t \sim 7 \times 10^9$ to 10×10^9 yr). There is no longer any well-established discrepancy between the observational value for the ratio H^{-1}/t and the minimum value of 1.5 predicted for this ratio by general relativity.

Relativity Questioned because It Predicts a Higher Density of Mass-Energy Than What Is Observed So Far

Regarding a cosmological possibility to test and disprove relativity, attention has turned from the question whether the universe is dynamic, and whether the time scale checks, to the issue whether the actual density of mass-energy—between (1/3) and 3×10^{-30} g/cm^3, according to a careful study by Jan Oort[14]—is seriously in disagreement with that required by relativity to curve the universe up into closure—between 7×10^{-30} g/cm^3 and 100×10^{-30} g/cm^3, depending upon the values ascribed to H^{-1} and t.[15] The difference between past and present figures for all the relevant numbers warns one not to overestimate the present apparent discrepancy. It is impossible to overthrow relativity, or even question it seriously, on the basis of this inadequate evidence.

So much for the past ups and downs of faith in general relativity as a reliable guide for the study of cosmology. They symbolize the uncertainties expressed from time to time in other branches of physics about Einstein's central idea, that space-time is curved! Now for a few current applications of general relativity—more confident than ever—to cosmology.

The call to look for new sources of mass-energy is less urgent than the need to obtain increasingly reliable figures for presently known sources—and firmer values for H^{-1} and t. Nevertheless, it must be noted that the present figure for the density includes no allowance for neutral hydrogen or neutrinos nor for the equivalent mass-energy of gravitational radiation. The reason is simple: The upper limits allowed by today's inadequate means of observation are too fantastically high to contemplate seriously. Among these potential sources of density missed in present bookkeeping, the one of greatest interest from the standpoint of general relativity is gravitational radiation.

Gravitational Radiation as Contributor to the Effective Density of Mass-Energy

Pure gravitational radiation, or ripples in the geometry, of reduced wavelength $\lambda\!\!\!^- = \lambda/2\pi$ and of magnitude $\delta g_{\mu\nu} \sim \delta g$, make zero contribution to the real energy density—the $T_{\mu\nu}$ on the right-hand side of Einstein's Eq. (1)—but averaged over several wavelengths, they make an effective contribution to the Ricci curvature of order $(\delta g/\lambda\!\!\!^-)^2$, and therefore a contribution to the *effective* density of mass-energy of the order

$$\rho \text{ effective (g/cm}^3) \backsim (c^2/8\pi G)(\delta g/\lambda\!\!\!^-)^2 \tag{24}$$

or in geometrized units (length/length^3),

$$\rho^* \text{ effective} \backsim (1/8\pi)(\delta g/\lambda\!\!\!^-)^2 . \tag{25}$$

All by themselves, without any stars or dust to help, they can curve a universe up into closure with a radius of the order a_0 provided only (see Table II) that the condition is satisfied

$$(\delta g/\lambda\!\!\!^-)^2 \backsim (1/a_0)^2 . \tag{26}$$

(Example: $\delta g \sim 10^{-4}$; $\lambda\!\!\!^- \sim 10^6$ light years; $a_0 \sim 10^{10}$

[13] A. Sandage, in Onzième conseil de physique Solvay, *La structure et l'évolution de l'univers* (Editions Stoops, Brussels, 1958) (referred to hereafter as SÉU).

[14] J. Oort in SÉU (reference 13).

[15] See, for example, M. Wakano *et al.* in SÉU (reference 13), also an updating of this work in GMD, p. 116.

light years). If the wavelength is small compared to the scale of the universe, one deals here with the equivalent of the Tolman universe, filled however not with electromagnetic radiation, but with gravitational radiation.

Nothing will be said here about the pioneering experiments that have already been made seeking, so far unsuccessfully, to detect gravitational radiation coming from outer space at one or another selected wavelength.[16] It is difficult to imagine the exploration of this area ever ending.

In connection with the mathematical analysis of such radiation it is appropriate to cite the work of Lifshitz.[17] He considers an ideal spherical universe which derives its curvature from a uniform filling of dust. He analyzes the small perturbations of this geometry into harmonics on the 3-sphere. He studies how the amplitude of each must vary with time as a consequence of the expansion and reconstruction of the universe. The results of this analysis have been spelled out by Adams[18] *et al.* and compared and contrasted with the growth of small perturbations in shape of an oscillating underwater bubble. There, small ripples in the sphere S^2 in late phases of recontraction grow up into horns and spikes—the more rapid the growth the smaller their scale. In contrast, small ripples δg in the metric of the sphere S^3 vary with time approximately as (1/radius) $= 1/a$ independently of wavelength. The wavelength itself varies in proportion to a. Therefore, the derivative of the metric goes as $1/a^2$ and the energy density as $1/a^4$, just as in Tolman's photon-filled universe.

The Taub Universe

Restrict oneself to the case where the wavelength of the gravitational radiation has the maximum possible value compatible with the size of the universe. The hyperspherical harmonic of Lifshitz then has the minimum possible index number. Let the perturbation in the metric of this order be so magnified in amplitude that it supplies all of the needed density of effective mass-energy. Let the dust be removed. What does the resulting universe look like, and what is its evolution with time? This question deals with a disturbance of large amplitude in a nonlinear system and looks difficult to answer.

Fortunately, the principle of pairing off of questions can be followed here as in other parts of physics. Taub[19] discovered the following geometry, an exact solution of Einstein's equation for source free geometrodynamics:

$$d\sigma^2 = -d\tau^2 = \gamma_1 dx^2 + (\gamma_1 \sin^2 x + \gamma_3 \cos^2 x) dy^2 + 2\gamma_3 \cos x dy dz + \gamma_3 dz^2 - \gamma_1^2 \gamma_3 dt^2 , \quad (27)$$

where

$$\gamma_1 = (1/4) \cosh t / \cosh^2 (t/2) ,$$
$$\gamma_3 = 1/\cosh t , \quad (28)$$

and the coordinates are angles with the ranges

$$0 \leqslant x < \pi , \quad 0 \leqslant y,z < 2\pi . \quad (29)$$

The motivations and reasoning which led to the construction of this remarkable solution did not supply an account of the physics involved. How can it be interpreted?

One has no alternative but to interpret Taub's solution as the low-order harmonic of Lifshitz, endowed with the critical amplitude required by the condition of closure! One has an exact solution for a universe containing nothing but gravitational radiation!

Some properties of the solution of Taub have already been discussed by Brill[20] who notes that "it is invariant under a three-parameter group which is isomorphic to the group of four-dimensional rotations without fixed points; . . . this universe is homogeneous. . . . but not isotropic." Brill also gives an argument to show that the geometry becomes singular after a finite proper time.

Brill's Closed Universe

Elsewhere, Brill had already analyzed a class of universes curved up to closure by their content of gravitational radiation. Each is characterized by a 3-hypersurface of "time symmetry" on which the distance between nearby points is

$$ds^2 = \psi^4 [e^{2\lambda q_1(\rho,z)} (d\rho^2 + dz^2) + \rho^2 d\varphi^2] . \quad (30)$$

Here $q_1(\rho,z)$ is an arbitrarily specified function, measuring what might be called the "distribution of gravitational wave energy". It is regular everywhere and is assumed to go to zero very fast outside of a limited region in the ρ,z plane. The quantity λ measures the "strength" of this distribution. It has to have a certain characteristic value to give proper closure to the space. The function $\psi(\rho,z)$ measures what might be called the "static gravitational effect" of this distribution of wave energy. The Einstein

[16] J. Weber, Phys. Rev. **117**, 306 (1960).
[17] E. Lifshitz, J. Phys. (U.S.S.R.) **10**, 116 (1946).
[18] J. B. Adams *et al.* in SÉU (reference 13).
[19] A. H. Taub, Ann Math. **53**, 472 (1951).

[20] Dieter Brill, *General Relativity: Selected Topics of Current Interest*, photolithoprinted report SUI 61–4, State University of Iowa, Iowa City, May 1961.

field equation not only determines the time evolution of this initial geometry, but also imposes upon it a solitary condition, that ψ satisfy the equation

$$\nabla_3^2\psi + 4\lambda\psi\nabla_2^2 q_1 = 0 , \qquad (31)$$

where

$$\nabla_2^2 = \partial^2/\partial\rho^2 + \partial^2/\partial z^2$$

and

$$\nabla_3^2 = \rho^{-1}(\partial/\partial\rho)\rho(\partial/\partial\rho) + (\partial^2/\partial z^2) + \rho^{-2}(\partial^2/\partial\varphi^2) .$$

The solution of interest at large $r = (\rho^2 + z^2)^{1/2}$ has the asymptotic form

$$\psi \backsim A + B/r .$$

The ratio of the constants, B/A, approaches infinity as the eigenvalue parameter λ, starting from zero, approaches the critical value required for closure. In the limit $\lambda = \lambda_{\text{crit}}$, the metric coefficient has the asymptotic value

$$\psi \backsim B/r .$$

That the distances are finite at $r \to \infty$ may be shown by introducing the new variable

$$R = B/r$$

in terms of which the asymptotic value of the metric ($R \to 0$!) is

$$ds^2 = dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) . \qquad (32)$$

Brill shows that the Taub universe—like these universes of his—is time symmetric, in the sense that there exists a space-like hypersurface on which the "extrinsic curvature" or "second fundamental form" $K_{ij}(x,y,z)$ (defined below in more detail) is zero.

A singularity develops in the Taub universe in a finite proper time. Is there not one of Brill's universes which will *not* become singular? Can a universe not oscillate periodically in time? The answer to the second question is definitely no, according to Papapetrou and Avez.[21] Consistent with their general analysis is another finding of Brill. He studies, by power-series expansion, the first stages in the development of the general one of his time-symmetric universes. He finds that the volume just before and just after the moment of time symmetry is smaller than the volume at that moment itself. There is no place for a time-symmetric stage of maximum contraction! Therefore, the "singularity conjecture" has been advanced[22]: that "in every closed universe which has the topology of the 3-sphere, and which obeys the equations of geometrodynamics, there always exist test-particle geodesics which cannot be continued indefinitely, because they run into a region where the metric geometry is singular."

[21] A. Papapetrou, Ann. Phys. (Leipzig) **3**, 360 (1959); A. Avez, Compt. rend. **250**, 3585 (1960).
[22] GMD, reference 3, p. 61.

A Geon as Special Example of a Gravitational Wave

A gravitational geon is a special instance of a gravitational wave held into a closed orbit by its own gravitational attraction. In a space that is asymptotically Euclidean, the gravitational energy leaks away at a rate which can be made as small as one pleases—but not zero—by making the number of wavelengths in the orbit sufficiently great. With this slow decrease in mass-energy, the size of the geon goes down. In the simplest type of geon the reduction is homogeneous and is represented by a scale factor. Then simple arguments[23] show that the mass and radius decrease linearly with time up to a "zero instant" when the geon vanishes:

$$m^*(t) = -m_1 t ;$$
$$R(t) = -R_1 t .$$

The derivative of the metric goes to infinity simply because of this decrease in scale. Shall this be called a singularity? Whatever be the answer in this case of an open universe, a gravitational geon contained in a closed universe with the topology S^3 is destined to be caught up eventually in a singularity of a more serious kind, if the "singularity conjecture" is correct.

Schwarzschild Solution

General relativity is evidently far from wanting in consequences important for cosmology; but the three best-known predictions of Einstein's theory are associated with another type of solution, Eq. (20), of the field equation, due to Karl Schwarzschild. As this solution has been studied more closely it has presented surprises, several of them recently.

In the beginning the treatment of the geometry strictly followed the model of the Newtonian gravitational potential. This potential varies as $1/r$ outside a spherically symmetric center of attraction, is continuous, has continuous first derivative at the surface, and inside a sphere of matter of uniform density, for example, varies as a harmonic oscillator potential. Similarly, one distinguished the "exterior" model-independent solution (20) from an "interior" solution governed by the equation of state of the material responsible for the attraction. In the Newtonian case the interior domain could not be made indefinitely

[23] GMD, reference 3, p. 144.

small without leading to a singularity in the potential; not so in the Schwarzschild case. There it makes sense to go to the limit in which there is no "real" mass at all, only mass due to the curvature of space itself. There is a moment of time symmetry. At this time the 3-geometry,

$$ds^2 = (1 + m^*/2R)^4 (dR^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2) ,$$

conformally related to Euclidean geometry, is quite free of singularity.

A Non-Euclidean Topology

The geometry at this moment of time symmetry has an unexpected character: a throat connecting two nearly Euclidean spaces. The circumference of a circle of coordinate radius R is

$$(1 + m^*/2R)^2 R 2\pi .$$

This expression reaches its minimum value $4\pi m^*$ at the center of the throat, $R = m^*/2$. The geometry is symmetric on the two sides of this throat, as seen by representing the metric in terms of a new coordinate R', defined by

$$RR' = (m^*/2)^2 .$$

Instead of speaking of a bridge in the sense of Einstein and Rosen between two spaces, one can think of the Schwarzschild geometry as a wormhole or handle[24] connecting two regions ("mouths") far away from each other in one Euclidean space. Moreover, that surrounding space need not even be Euclidean; a modification will impart to it a slight curvature of such a character that the space is actually closed in the sense of a 3-sphere at large distances.

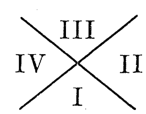
Singularity in Schwarzschild Geometry after a Finite Proper Time

So far the geometry has been examined as a function of position at one time. At earlier and later times the Schwarzschild throat has a circumference which is less than $4\pi m^*$. After a finite proper time (and also before a certain moment) the throat has pinched-off and the curvature is infinite.[25] Here is one more example of a singularity, and one more occasion to ask what position to take towards singularities!

Catastrophic and Noncatastrophic Points and Regions

Turn now from the curvature of the space to the fate of a test particle moving in this space. Does the world line begin or end in a region of infinite curvature? This question leads to a classification of space-time into catastrophic and noncatastrophic regions[26]

A point (four coordinates!) is said to be catastrophic if every time-like world line through it ends up, or starts, in a region of infinite curvature. Through a noncatastrophic point there are at least some time-like world lines which have no terminus. To make this classification of points in the Schwarzschild geometry, look apart from the angular coordinates θ and φ. Consider a 2-dimensional diagram in which the coordinates have to do with distances orthogonal to $\mathbf{d}\theta$ and $\mathbf{d}\varphi$—space-like distances horizontal, time-like ones vertical, and light-like displacements at $\pm$ 45°. The origin is to be the throat at the moment of maximum expansion. Light-like lines through it define a cross



in which regions I and III are catastrophic and II and IV are not. This classification is useful in dealing with what looks like a violation of causality in a space endowed with a wormhole.

Violation of Causality in a Doubly Connected Space

Consider a handle 3 m long providing a supplementary connection between two regions A and B which are separated by 3×10^{10} m in the surrounding nearly Euclidean manifold. Then a signal can get from A to B in 10^{-8} sec rather than in the 100 sec that one would expect. This possibility may be seen to violate customary ideas of causality, as follows: View events from a Lorentz frame moving in the direction $A \rightarrow B$ at a speed exceeding 10^{-10} c. Then one will see the signal arrive at B before A has sent—or even decided to send—a signal! Is there any escape from this difficulty?[26]

It is possible to signal through the throat and violate causality only if the source, or the receptor, or both are in the catastrophic regions of space-time. In contrast, no light signal will pass from a point in the noncatastrophic region IV through the throat to a point in the noncatastrophic region II or the converse. The throat is not open long enough! One still can and must uphold the principle of causality. One can do so only because one has been forced to limit the application of this principle to noncatastrophic regions of

[24] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).

[25] This pinch-off is best seen in the coordinates introduced by M. Kruskal, Phys. Rev. **119**, 1743 (1960), or in those introduced earlier by C. Fronsdal, *ibid.* **116**, 778 (1959).

[26] R. W. Fuller and J. A. Wheeler, Phys. Rev. **128**, 919 (1962).

space-time (in which one might reasonably think of making experiments!). This is a restriction that one previously never had occasion to consider. The resulting distinction between catastrophic and noncatastrophic regions is most interesting. It links two distinctive features of a particular 4-dimensional manifold: (1) the topology of the light cone, and (2) the regions where the curvature become infinite.

General Causality Analysis based on Topology Plus the Light Cone

What about the general 4-manifold of signature $-+++$? Is there any way to classify the regions here as catastrophic and noncatastrophic? The answer is not known. Traditional topology classifies manifolds with the aid of Betti numbers and other connectivity indices. It foregoes assignment of any particular metric to the space. Smith[27] has noted that one can stop short of a complete assignment of the metric and still give supplemental information of a topological character by specifying the light cone at each point in the manifold. He has examined some of the consequences of combining this information with the usual topological information. Either by use of his results or otherwise, it would seem essential to establish a theory valid for the general case which would connect the issues of topology, causality, and singularities.

Reissner-Nordstrøm Solution

For testing such a general theory there is available not only the Schwarzschild solution for a neutral center of attraction, but also the Reissner-Nordstrøm solution for a region endowed with both mass and charge,

$$d\sigma_2^2 = -d\tau^2 = -(1 - 2m^*r^{-1} + q^{*2}r^{-2})dt^2 + (1 - 2m^*r^{-1} + q^{*2}r^{-2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{33}$$

$$\begin{pmatrix}\text{Flux of electric lines of force through any sphere}\\ \text{of radius } r\end{pmatrix} = 4\pi q^* \text{ (independent of } r)\,. \tag{34}$$

This solution deserves discussion in its own right as well as for its connection with the subject of causality.

Oscillations

The traditional way (33) of writing the Reissner-Nordstrøm solution conceals from view the wormhole character of the solution at an initial moment and also the evolution of this doubly connected geometry with time. The flux of lines of force through the throat acts as a kind of elastic cushion against pinch-off of the throat. As the circumference of the throat shrinks to half-value, the density of lines of force quadruples, and the density of electromagnetic stress and energy rises sixteen-fold. Ultimately the back pressure builds up to the point where the throat re-expands. Oscillations in the geometry are therefore manifest to an observer stationed at the throat, as has been shown by Graves and Brill.[28] With the throat always open as a short cut for transmitting light signals, it might be supposed that wholesale violations of the principle of causality will occur. However, Graves and Brill show that the intrinsic curvature of space-time near the throat becomes infinite. Yet to be explored are the consequences of these singularities for defining catastrophic regions of space-time, and for preserving causality as between noncatastrophic regions of space-time.

Singularities All Characterized by the Same Type of Approach to Infinite Curvature?

The singularities that develop after the lapse of a finite proper time—the fifth example of infinites so far encountered in this account—also have an interest in and by themselves as guides to analyzing an important general issue: What kinds of singularities are possible, in principle, in the intrinsic geometry (as distinguished from the coordinates!). Infinite curvature is the characteristic feature of the singularities here as in the Schwarschild solution.

Wormhole Topology

In the Reissner-Nordstrøm solution the regions connected by the wormhole can be viewed either as portions of two distinct Euclidean spaces or as two remote parts of one Euclidean space. This second interpretation is especially interesting in connection with the net electrical neutrality observed in the physical world. It leads in the most direct way to the view of electricity as lines of force trapped in the topology of a multiply connected space.[29]

Electric Charge as Lines of Force Trapped in the Topology of a Multiply Connected Space

It is not necessary to think of charge as a region of space where the electric field becomes singular, or where Maxwell's equations break down, or where

[27] J. W. Smith, Proc. Natl. Acad. Sci. **46**, 111 (1960).

[28] J. C. Graves and D. R. Brill, Phys. Rev. **120** 1507 (1960).

[29] J. A. Wheeler, Phys. Rev. **97**, 511 (1955); C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957): also GMD, reference 3.

there resides some mysterious jelly called electricity. It is not necessary to arbitrarily restrict attention to those solutions of Einstein's equations which have a singly connected topology. The equations of general relativity and electrodynamics are purely local in character. They say nothing about the connectivity of the topology. Only the assumption that the topology of physical space should be Euclidean, E^3, or at most equivalent to that of the closed 3-sphere, S^3, prevented consideration of multiply connected topology in earlier times. On a 3-dimensional space-like (metric positive definite) hypersurface "sliced through" space-time let there be R_2 independent types of homologous closed 2-surfaces—or in other terms, let there be R_2 inequivalent wormhole mouths. Then one is led, in the following way, to the picture of electricity as trapped lines of force. Specify enough information about the electromagnetic field to make it possible to forecast its future evolution in time. As in the case of other dynamical systems, give here the "coordinate" of the field—the magnetic field $\mathbf{B}$ as a vector function of position over the space-like hypersurface—and also the "velocity" of the field—the time rate of change $\partial\mathbf{B}/\partial t$ as a function of position. These two vector fields are not specifiable quite arbitrarily, for it is required that $\mathbf{B}$ and its time derivative be divergence free:

$$\left.\begin{aligned}\operatorname{div}\mathbf{B}&=0\\ \operatorname{div}(\partial\mathbf{B}/\partial t)&=0\end{aligned}\right\}\ \text{vector notation}\qquad(35)$$

$$\left.\begin{aligned}\mathbf{d}*\mathbf{B}&=0\\ \mathbf{d}(\partial*\mathbf{B}/\partial t)&=0\end{aligned}\right\}\ \text{language of exterior differential forms}\,.\qquad(36)$$

From the field coordinate and its time rate of change it is possible in a singly connected 3-manifold to find a unique value for the "field momentum"—the electric field $\mathbf{E}$ as a vector function of position. With electrodynamical coordinate and momentum thus specified, the dynamical evolution of the sysetm in time is then uniquely specified. This momentum is obtained by solving the equations

$$\left.\begin{aligned}\operatorname{div}\mathbf{E}&=0\\ \operatorname{curl}\mathbf{E}&=-\partial\mathbf{B}/\partial t\end{aligned}\right\}\ \text{vector notation}\qquad(37)$$

$$\left.\begin{aligned}*\,\mathbf{d}*\mathbf{E}=\boldsymbol{\delta}\mathbf{E}&=0\\ \mathbf{d}\mathbf{E}&=-\partial*\mathbf{B}/\partial t\end{aligned}\right\}\ \text{differential forms}\qquad(38)$$

to obtain the solution

$$\mathbf{E}=\mathbf{E}_{\text{basic}}(x,y,z,)\,,\qquad(39)$$

(fixed by $\mathbf{B}$ and $(\partial\mathbf{B}/\partial t)$). However, in a space with second Betti number R_2 (number of wormholes) different from zero, the solution of (37) or (38) is not unique. Instead, it has the form

$$\mathbf{E}=\mathbf{E}_{\text{basic}}(x,y,z)+\sum_{k=1}^{R_2}q_k\mathbf{H}_k(x,y,z)\,.\qquad(40)$$

Here the constant coefficient q_k measures the *classical electric charge* associated with the kth wormhole; and the quantity $\mathbf{H}_k$ is a harmonic vector field with these properties:

1. Zero divergence $\left\{\begin{aligned}\operatorname{div}\mathbf{H}_k&=0\quad\text{(vector)}\\ \boldsymbol{\delta}\mathbf{H}_k&=0\quad\text{(form)}\end{aligned}\right.$ (41)

2. Zero curl $\left\{\begin{aligned}\operatorname{curl}\mathbf{H}_k&=0\quad\text{(vector)}\\ \mathbf{d}\mathbf{H}_k&=0\quad\text{(form)}\end{aligned}\right.$ (42)

3. Unit contribution to the apparent charge of the kth wormhole $\left\{\begin{array}{l}\int\mathbf{H}_k\cdot d\mathbf{S}=4\pi\quad\text{(vector)}\\ \text{surface around mouth of }k\text{th wormhole}\\ \int_{C_k}{}^{*}{}^{2}\,\mathbf{H}_k=4\pi\text{(form)}\end{array}\right.$ (43)

4. Zero contribution to the apparent charge of every other wormhole ($l\neq k$) $\left\{\begin{array}{l}\int_{C_l}^{2}\mathbf{H}_k\cdot d\mathbf{S}=0\quad\text{(vector)}\\ \int_{C_l}^{2}*\,\mathbf{H}_k=0\quad\text{(form)}\end{array}\right.$ (44)

When the kth pair of wormhole mouths are well isolated, and when the dimensions of the mouths are very small compared to the separation between them, then the kth harmonic vector field almost everywhere looks like the Coulomb field associated with a unit positive charge at the one location and a unit negative charge at the other. Only close up to the mouths and through the throat is the identity with a Coulomb field lost. The Coulomb field refers to a Euclidean space that has no throat in which to carry out the comparison! Moreover, the Coulomb field is singular. In contrast, the harmonic vector field remains divergence free, curl free, and singularity free right through the throat—as everywhere else.

Count of Equations Versus Unknowns in a Skeleton Space

The following elementary reasoning may illustrate how it comes about that the equations

$$\boldsymbol{\delta}\mathbf{E}=0\,,\quad \mathbf{d}\mathbf{E}=-\partial*\mathbf{B}/\partial t\qquad(45)$$

should have the R_2 independent solutions, the existence of which has been proven by Hodge.[30] Following

[30] W. V. Hodge, *The Theory and Application of Harmonic Integrals* (Cambridge University Press, New York, 1952), 2nd ed.

a type of reasoning long familiar in topology[31] replace the continuous 3-dimensional manifold by a lattice space of tetrahedrons fitted together, triangular face to triangular face. Also replace the continuous electric field $\mathbf{E}$ by the voltage differences V_e along the edges e which connect one vertex and another. If this simplicial decomposition of the space is fine enough, one will expect to be able to reproduce all details of the field with arbitrary accuracy.

Abbreviate the surface integral of the normal component of the magnetic field across the tetrahedral face f as

$$\int_{\text{face } f} {}^*\mathbf{B} = \Phi_f \,. \tag{46}$$

Then the condition on the curl of the electric field reduces to a condition on the sum of the voltages around the edges of each triangular face f, with due regard to sign:

$$\pm V_{e_p} \pm V_{e_q} \pm V_{e_r} = -\dot{\Phi}_f \,. \tag{47}$$

Let F denote the number of faces; then there are F equations of this type. These equations are not all independent. A knowledge of the circuital voltage around the edges of *three* faces of a tetrahedron suffices for predicting the circuital voltage around the boundary of the fourth face. Let T denote the number of the tetrahedrons that fill the space. Then there are T relations of this kind by which to deduce the curl conditions at one face from the curl conditions at other faces. However, not all of these T relations are independent. That for the last tetrahedron can be deduced as a consequence of these relations associated with all the other tetrahedrons that surround it. Thus, $F - (T - 1)$ is the number of curl relationships left over.

The number of divergence relations is equal to the number of vertices V. However, the divergence relation for the last vertex can be deduced from that for the remaining $V - 1$ vertices. Thus, $V - 1$ divergence conditions are left over.

Counting curl and divergence relations together, one has $F - T + 1 + V - 1$ linear equations for E unknown voltages V_e. *If* these equations are linearly independent, then the solution contains free parameters to the number

$$\begin{aligned} N &= (\text{unknowns}) - (\text{independent equations}) \\ &= T - F + E - V \,. \end{aligned} \tag{48}$$

This quantity is independent of the fineness of the simplicial decomposition. For example, introduce a new point at the center of one of the existing tetrahedrons. The resulting changes in the number of simplexes of the various dimensionalities are

$$\begin{aligned} \text{tetrahedrons: } \Delta T &= 4 - 1 = 3 \,; \\ \text{triangular faces: } \Delta F &= 6 \,; \\ \text{edges: } \Delta E &= 4 \,; \\ \text{vertices: } \Delta V &= 1 \,. \end{aligned} \tag{49}$$

The change in the number of free parameters is

$$\Delta N = \Delta T - \Delta F + \Delta E - \Delta V = 3 - 6 + 4 - 1 = 0 \tag{50}$$

The quantity $T - F + E - V$, the Euler-Poincaré characteristic, *itself* vanishes if the 3-space, in addition to being closed, is orientable in this sense: two gloves, originally compared and found to have the same chirality, and taken on the most different journeye, continue to be identical whenever they meet. The vanishing of $T - F + E - V$, or its higher dimensional equivalent, is known to be automatic in any closed orientable space of odd dimensionality.

The independence of the equations is a correct assumption in a 3-space with the topology of the 3-sphere. Then the field "momentum" $\mathbf{E}$, as represented in its skeletonized version by the E voltages V_e, is uniquely specified by the time rate of change of the field "coordinate", the magnetic field $\mathbf{B}$.

Final Value for Number of Free Parameters

In spaces of other topologies there are ordinarily relations between the linear equations to be solved for the V_e. The number of these relations is equal to the second Betti number, R_2, of the space, that is, to the number of independent classes of homologous closed 2-surfaces:

$$N = R_2 \neq T - F + E - V \,. \tag{51}$$

Consequently, there are R_2 free parameters in the solution. They represent electric charge in the purely classical, geometrodynamical, sense of electric charge.

[31] See, for example, P. Alexandroff, *Elementary Concepts of Topology,* translated by A. E. Farley (Dover Publications, New York, 1961); S. Lefschetz, *Introduction to Topology* (Princeton University Press, Princeton, New Jersey, 1949); H. Seifert and W. Threlfall, *Lehrbuch der Topologie* (photoreproduction by Chelsea Publishing Company, New York, 1947); and especially in connection with the application of algebraic topology to numerical analysis of bridge structures, electrical networks, and dynamic continua, see the paper of J. P. Roth, Quart. Appl. Math. **17**, 1 (1959) with its extensive bibliography. For a similar way to skeletonize space-time and formulate Einstein's field equations in purely algebraic form, see Tullio Regge, Nuovo cimento **19**, 558 (1960).

Parameters Enter the Quantum State Functional

These charge parameters have a consequence for the quantum-mechanical description of the wave function of the electromagnetic field in a multiply connected space. The amplitude, a complex number, cannot be represented as a functional of the magnetic field alone. It depends, in addition, upon R_2 parameters q_k, which are constants of the motion. Thus, a proper way of writing ψ is

$$\psi = \psi(\mathbf{B}(x,y,z);q_1,q_2,\cdots,q_{R_2};t) .$$

In mathematical terms, it is a mapping *from* the set of all real nonsingular divergence-free vector functions $\mathbf{B}(x,y,z)$ of position in the given 3-space, *onto* the space of complex numbers. All this applies to a given time t. At a new time the mapping changes according to the laws of quantum mechanics, but the parameters q_k remain unchanged.

The Initial Value Problem for Geometrodynamics

How to get field momentum from field coordinate and its time rate of change has been analyzed for electromagnetism; now for the same problem for geometrodynamics! The field "coordinate" is the geometry intrinsic to a 3-dimensional space-like hypersurface, defined, for example, by giving the six metric coefficients ${}^{(3)}g_{ik}$ as a function of three coordinates x,y,z *on* the hypersurface. A continuous parameter t may be used to single out a particular member of a family of such hypersurfaces. Then the "time rate of change of the field coordinate" is one terminology for the quantity $\partial\,{}^{(3)}g_{ik}(t,x,y,z)/\partial t$. None of this information allows one to calculate distances between two nearby points with different time coordinates. For this purpose it is necessary to know how the 3-manifold is curved with respect to the enveloping 4-manifold; to know the "extrinsic curvature," or in mathematical terms the "second fundamental form," or in physical terms, the "field momentum." This field momentum can be expressed in terms of four "potentials" plus other information derived from the field coördinate (and its time rate of change). The analogy with electromagnetism is close, where the momentum, the electric field $\mathbf{E}$, can be expressed as the gradiant of a *single* scalar potential plus a part (curl $\dot{\mathbf{A}}$) derived from the time rate of change of the field coordinate. A convenient choice of these potentials has been given by Arnowitt, Deser, and Misner.[32] These potentials have recently been employed to formulate the initial value problem in the form stated here.[33]

The Two-Surface Formulation of the Initial Value Problem

Given in this formulation are the metric coefficients ${}^{(3)}g'(x,y,z)$ and ${}^{(3)}g''(x,y,z)$ for the geometries intrinsic to two 3-spaces, not otherwise defined, which are "nearby" in a sense discussed in the reference. *To be found* are four functions of x, y, and z: η_0 (which in the end will represent the orthogonal time-like proper distance between the two surfaces) and $\eta^i (i = 1,2,3)$ (which in the end will represent the coordinate difference between points on the two surfaces connected by the same time-like normal). In terms of the given ${}^{(3)}g$ and ${}^{(3)}g$ and the yet to be found four potentials, everything else of relevance to the geometry of the "thin sandwich"—and hence the entire 4-manifold—can be derived. (1) The distance between a point on one surface and a nearby point on the other is

$$d\sigma^2 = -d\tau^2 = -(\eta_0^2 - \eta_i\eta^i) + 2\eta_i dx^i + {}^{(3)}g_{ik}dx^i dx^k , \tag{52}$$

where ${}^{(3)}g_{ik}$ is a suitable average of ${}^{(3)}g'_{ik}$ and ${}^{(3)}g''_{ik}$. (2) The extrinsic curvature is

$$K_{ik} = (g'_{ik} - g''_{ik} + \eta_{i|j} + \eta_{j|i})/2\eta_0 . \tag{53}$$

The four potentials, and hence the extrinsic curvature, are to be *found* from the four initial value equations[34]

$$(K_i^j - \delta_i^j \operatorname{Tr}\mathbf{K})_{|j} = \begin{pmatrix}\text{energy flux in suit-}\\ \text{able units}\end{pmatrix} \tag{54}$$

$$(\operatorname{Tr}\mathbf{K})^2 - \operatorname{Tr}(\mathbf{K}^2) + {}^{(3)}R = \begin{pmatrix}\text{energy density in}\\ \text{suitable units}\end{pmatrix} \tag{55}$$

These equations have been known for a long time, but their application to finding the time-like separation η_0 and the space-like "shift" η_i, and hence the field momentum, is new.

Uniqueness of Solution as Guide to a New Formulation of Mach's Principle

More decisive than any other questions about new implications of relativity would seem to be these: (1)

[32] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **122**, 997 (1961) and earlier papers cited therein.

[33] R. F. Baierlein, D. H. Sharp, and J. A. Wheeler, Phys. Rev. **126**, 1864 (1962).

[34] K. Stellmacher, Math. Ann. **115**, 136 (1937); A. Lichnerowicz, Helv. Phys. Acta. Suppl. **4**, 176 (1956); Y. Fourès-Bruhat, J. Rational Mech. Anal. **5**, 951 (1956).

In a 3-space with the topology of the 3-space is it sufficient to give ${}^{(3)}g_{ik}$ and $\partial{}^{(3)}g_{ik}/\partial t$ (or ${}^{(3)}g''_{ik} - {}^{(3)}g'_{ik}$)—plus the density and flow of energy, if any, as a function of position—in order to arrive at unique values for the four potentials η_α? If so, the extrinsic curvature will be determined, and thence the whole dynamic evolution of the geometry, past and future. In this event one will have a concrete expression for Mach's principle within the context of standard general relativity, in the sense that the geometry—the inertia, in physical terms—will be seen to be defined by the distribution of energy and momentum. It is presumably essential for the suggested uniqueness that the universe shall have the topology S^3, or one of a limited class of topologies. In this event *Mach's principle can be reformulated as a boundary condition on Einstein's field equation, demanding that the 3-geometry shall be "properly closed".*[35]

The Case of Multiple Connectivity: A New Kind of Charge?

(2) One class of topologies, let it be assumed, gives uniqueness. Can multiply connected topologies permit a number of independent solutions specified by certain parameters $Q_1, Q_2, \cdots$ up to a number fixed by the topological indices of the space? In this event one will be confronted by a *new kind of charge*, presumably very different in character, because of the nonlinearity of the equations, from the kind of electric charge already discussed.

Count of Degrees of Freedom

(3) These references to Mach's principle and a new kind of charge do nothing to satisfy the demands of those who want a listing of the degrees of freedom of the geometry analogous to that which one secures for the electromagnetic field by Fourier analysis. However, in geometrodynamics as in hydrodynamics, there is no evidence that such a count will ever be possible. One has a zoology of vortices, shocks, etc., in the one subject; one is on the way to acquiring a similar zoology in the other. Note, for example, that the wormhole is one example of "mass without mass"; the geon is another; and less organized forms of gravitational radiation provide a third. There are ways, unlimited in number, to combine these kinds of disturbance to get most complex geometrodynamical objects. This is the *nature* of general relativity!

[35] *Note added in proof:* A detailed description of this formulation of Mach's principle is given in a paper by the author to appear in the proceedings of the July 1962 Warsaw conference on relativistic theories of gravitation.

The Volume Element in the Space of Spaces

(4) Special solutions have application to cosmology and large-scale gravitational phenomena; but at the quantum level there is no more interest in these solutions than in the solutions of the last century for multitudes of special problems in mechanics! Such classical solutions may illuminate a quantum problem in the semiclassical or JWKB approximation. However, one has not progressed far enough to-day in quantizing general relativity to know how to profit from this circumstance. Questions are still at the level of principle. Let one central question be stated to illustrate the nature of the subject. It has to do with the space over which one integrates the quantum propagator.

The propagator $\langle x''t''|x't'\rangle$ in the quantum mechanics of a single particle is well known. Folded into the wave function at one time, it gives the wave function at a later time:

$$\psi(x'',t'') = \int \langle x'',t''|x',t'\rangle \psi(x',t')dx' \,. \qquad (56)$$

Out of it can be read all one wants to know about the quantum mechanics of the system. In electrodynamics, in a closed multiply connected space the corresponding formula is

$$\psi(\mathbf{B}''(x,y,z);q_1,q_2,\cdots;t'') = \int \langle \mathbf{B}'';q;t''|\mathbf{B}';q;t'\rangle \psi(\mathbf{B}';q_1,q_2,\cdots;t')\mathfrak{D}\mathbf{B}' \,. \qquad (57)$$

Here the functional integration goes over the space of all vector fields $\mathbf{B}(x,y,z)$ which (1) are defined over the specified three-manifold, (2) are divergence free, magnetic charge!). *Question:* What is the formula and (3) have zero flux through each wormhole (no which takes the place of (57) in geometrodynamics? This question does not have the ambitious object to find the propagator! Rather, if and when one succeeds in evaluating this quantity, how will one use it?

In relativity the specification of the 3-geometry ${}^{(3)}\mathcal{G}'$—by way of six metric coefficients ${}^{(3)}g'_{ik}(x,y,z)$ or otherwise—accomplishes in one stroke what is accomplished in electrodynamics by giving the *field* $\mathbf{B}'$ and the *time* t'. This unification of information—about *which* hypersurface and *what* gravitational field *on* this hypersurface—is one more consequence of the covariance of general relativity. How does one take apart this unity and integrate, not over the space of all 3-geometries, but over a subspace of (2/3) the dimensionality? What is the physical meaning of this subspace of spaces?

TABLE III. Issues at the frontiers between general relativity and differential geometry.

The problem	Implications and subsidiary questions
1. Initial value problem in the form in which the "coordinate"—the intrinsic geometry—and its time rate of change are given and one asks for the "momentum"—the extrinsic curvature.	1A. Mach's principle as a boundary condition ("properly closed universe") on Einstein's field equation. 1B. Charge-like parameters required in addition to foregoing for unique specification of solution in a multiply connected space. 1C. Means of bringing out what one can, and cannot hope, to have in way of a count of the degrees of freedom of the geometry. 1D. Volume element in, or measure for, the space of all geometrical configurations. 1E. Problem of classification of all relevant 3-dimensional topologies.
2. Singularities in Friedmann, Tolman, Taub, Schwarzschild, Reissner-Nordstrøm, and many other geometries	2A. Conjecture that every "properly closed space" ultimately develops a singularity. 2B. Problem of classifying all the types of singularities that can develop out of Einstein's field equation.[a] 2C. Question whether classification of singularities will, as in the case of analytic functions, result in a classification of the solutions themselves. 2D. Inevitability of singularity in classical solution implies conditions always develop where quantum character of geometry cannot be escaped.
3. Causality	3A. Existence of a correlation between (1) the topology of the space and of the light cone and (2) the location of singularities.
4. Action integral as a real valued function ("functional") over the space of all histories H of the 4-geometry which are compatible[b] with specified bounding 3-geometries ${}^{(3)}G'$ and ${}^{(3)}G''$.	4A. Morse theory of minima, maxima, and saddle points of this functional.

[a] Investigations of the subject of singularities have been made by E. M. Lifshitz and M. Khalatnikov, Zhur. Eksp. i Teoret. Fiz. **39**, 149 (1960); **39**, 800 (1960); Akad. Nauk USSR **40**, 1847 (1961); I. M. Khalatnikov, E. M. Lifshitz, and V. V. Sudakov, Phys. Rev. Letters **6**, 311 (1961). For a catalog of many solutions which display singularities, see B. K. Harrison, Phys. Rev. **116**, 1285 (1959) and P. Jordan, J. Ehlers and W. Kundt, Akad. Wiss. Mainz, No. 2 (1960).

[b] This topic is closely connected with the theory of cobordism: René Thom, Comentarii Math. Helvetici **28**, 17 (1954).

IV. PROBLEMS AND PROSPECTS IN GENERAL RELATIVITY HAVING TO DO WITH DIFFERENTIAL GEOMETRY AND TOPOLOGY

New insights have come out of the special solutions of Einstein's field equation just passed in review, and also new issues. A recapitulation of some of the problems which have shown up most strikingly is given in Table III.

This account omits many important topics and much interesting work. However, the issues discussed here are enough to display the central point: The analysis of the hidden aspects of Einstein's theory makes new demands, and offers unexpected insights, both to mathematics and to physics.

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