ously analytic in θ and that its modulus exp $(-\bar{z}\cdot C'(\theta)z)$ is uniformly bounded by the integrable function exp $(-\frac{1}{2}\,\bar{z}\cdot B'z)$. For imaginary θ, C is Hermitian and positive definite [see (A4a)], and in this case the equation $\Lambda(C(\theta)) = [\det C(\theta)]^{-1}$ has already been established. By analyticity it remains valid throughout the strip $|\theta_2| < \kappa$, in particular for $C(1) = B$.

Corollary.

$$I(A) = \int \exp\,(\bar{z}\cdot Az)d\mu_n(z) = [\det\,(1 - A)]^{-1} \quad \text{(A5)}$$

if $1 - A$ has a positive definite Hermitian part, in particular if A has sufficiently small matrix elements. In fact, by the definition of $d\mu_n(z)$ [see (1.5)], $I(A) = \Lambda(1 - A)$.

(b) Let

$$M(B,a,b) = \pi^{-6}\int \exp g(B,a,b;\xi,\eta)d^3\xi d^3\eta \quad \text{(A6)}$$

$$g = -\bar{\xi}\cdot B\xi - \bar{\eta}\cdot B\eta + D(\bar{a},\bar{\xi},\bar{\eta}) + D(b,\xi,\eta)\,. \quad \text{(A6a)}$$

Here, ξ and η are points inC_3, B is a 3×3 matrix, a,b are constant vectors in C_3, and D is a determinant as in Sec. 3f. As before, we proceed in three steps. (1) If $B = 1$, this is the integral in (3.23a), and for sufficiently small a,b, $M(1,a,b) = (1 - \bar{a}\cdot b)^{-2}$, by (3.23b). (2) If B is positive definite Hermitian, M is absolutely convergent for sufficiently small a,b (for example, $\bar{a}\cdot Ba < \det B$, and $\bar{b}\cdot Bb < \det B$). As before, set $B = S^*S$, let $\sigma = \det S$, and introduce new variables $\xi' = S\xi$, $\eta' = S\eta$. Set also $a' = Sa$ and $b' = Sb$. Then

$$\bar{\xi}\cdot B\xi = \bar{\xi}'\cdot\xi'\,, \quad \bar{\eta}\cdot B\eta = \bar{\eta}'\cdot\eta'$$

$$D(\bar{a},\bar{\xi},\bar{\eta}) = \bar{\sigma}^{-1}D(\bar{a}',\bar{\xi}',\bar{\eta}') = D(\bar{a}'',\bar{\xi}',\bar{\eta}')$$

$$D(b,\xi,\eta) = \sigma^{-1}D(b',\xi',\eta') = D(b'',\xi',\eta')\,,$$

where $a'' = \sigma^{-1}a'$, $b'' = \sigma^{-1}b'$. Thus,

$$g(B,a,b;\xi,\eta) = g(1,a'',b'';\xi',\eta')\,.$$

The Jacobian corresponding to (A3a) is now $(\sigma\bar{\sigma})^{-2}$. Hence $M(B,a,b) = (\sigma\bar{\sigma})^{-2}\ M(1,a'',b'') = [\sigma\bar{\sigma}(1 - \bar{a}''\cdot b'')^{-2}] = (\sigma\bar{\sigma} - \bar{a}'\cdot b')^{-2}$. Now $\sigma\bar{\sigma} = \det B$, and $\bar{a}'\cdot b' = \bar{a}\cdot Bb$. Therefore

$$M(B,a,b) = (\det B - \bar{a}\cdot Bb)^{-2}\,. \quad \text{(A7)}$$

(3) If B is no longer Hermitian, but has a positive definite Hermitian part, we may again show by analytic continuation that (A7) remains valid.

The integral to be evaluated in 4c is

$$N(H,\bar{u},v) = \int \exp g(H,\bar{u},v;\xi,\eta)d\mu_3(\xi)d\mu_3(\eta)\,.$$

Since $d\mu_3(\xi)\ d\mu_3(\eta)$ introduces the factor exp $(-\bar{\xi}\cdot\xi - \bar{\eta}\cdot\eta)$ it follows that $N(H,\bar{u},v) = M(1 + H,\bar{u},v)$, and hence

$$N(H,\bar{u},v) = [\det\,(1 + H) - u\cdot v - u\cdot Hv]^{-2}\,. \quad \text{(A8)}$$

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On the Localizability of Quantum Mechanical Systems*

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1. INTRODUCTION

FROM the very beginning of quantum mechanics, the notion of the position of a particle has been much discussed. In the nonrelativistic case, the proof of the equivalence of matrix and wave mechanics, the discovery of the uncertainty relations, and the development of the statistical interpretation of the theory led to an understanding which, within the inevitable limitations of the nonrelativistic theory, may be regarded as completely satisfactory.

Historically, confusion reigned in the relativistic case, because situations requiring a description in terms of many particles were squeezed into a formalism built to describe a single particle. I have in mind the difficulties with wave functions for a single particle which seem to yield nonzero probability for finding it in a state of negative energy. Soon attention shifted to the problems of the quantum theory of fields and the question of the status of position

* Dedicated to Eugene Wigner on his sixtieth birthday.

operators for relativistic particles was left without a clear resolution. That does not mean that papers were not written on the subject, but that those papers had completely different objectives in mind: They permitted the particles in question to be in nonphysical (negative energy) states or they studied operators which could not serve as position *observables* since their three components did not commute.

In the opinion of the present author, the decisive clarification of the relativistic case occurs in a paper of Newton and Wigner.[1] These authors show that, if the notion of localized state satisfies certain nearly inevitable requirements, for a single free particle it is uniquely determined by the transformation law of the wave function under inhomogeneous Lorentz transformations. The resulting position observables turn out, in the case of spin-$\frac{1}{2}$, to be identical with the Foldy-Wouthuysen "mean position" operators.[2] An analogous investigation for the case of Galilean relativity was carried out by Inönü and Wigner.[3]

The essential result of Newton and Wigner is that for single particles a notion of localizability and a corresponding position observable are uniquely determined by relativistic kinematics when they exist at all. Whether, in fact, the position of such a particle is observable in the sense of the quantum theory of measurement is, of course, a much deeper problem; *that* probably can only be decided within the context of a specific consequent dynamical theory of particles. All investigations of localizability for relativistic particles up to now, including the present one, must be regarded as preliminary from this point of view: They construct position observables consistent with a given transformation law. It remains to construct complete dynamical theories consistent with a given transformation law and then to investigate whether the position observables are indeed observable with the apparatus that the dynamical theories themselves predict.

In Newton and Wigner's formulation, the set $S_{\mathbf{a}}$ of states localized at a point $\mathbf{a}$ of the three-dimensional space at a given time, must satisfy the following axioms:

(a) $S_{\mathbf{a}}$ is a linear manifold;

(b) $S_{\mathbf{a}}$ is invariant under rotations about $\mathbf{a}$, reflections in $\mathbf{a}$, and time inversions;

(c) $S_{\mathbf{a}}$ is orthogonal to all its space translates;

(d) certain regularity conditions.

The solutions of (a) . . . (d) for elementary systems, i.e., for systems whose states transform according to an irreducible representation of the inhomogeneous Lorentz group, turn out to be continuum wave functions when they exist at all, i.e., according to the usual definitions of Hilbert space, there is no manifold $S_{\mathbf{a}}$. However, it is physically and mathematically clear that Newton and Wigner's formulation ought to be regarded as the limiting case of a notion of localizability in a region.

In the present paper, I propose a reformulation of the physical ideas of (a) . . . (d) in terms of a notion of localizability in a region. When the ideas are so formulated, one sees that the existence and uniqueness of a notion of localizability for a physical system are properties which depend only on the transformation law of the system under the Euclidean group, i.e., the group of all space translations and rotations. The analysis of localizability in the Lorentz and Galilei invariant cases is then just a matter of discussing what representations of the Euclidean group can arise there. To obtain uniqueness, one must add invariance under time inversion and an analogy of Newton and Wigner's regularity assumption. As would be expected, all the results obtained earlier in the old formulation come out. One can ask what is the point of the present extended footnote to Newton and Wigner's paper. First, it seems worthwhile to me to have a mathematically rigorous proof of the fundamental result of Newton and Wigner that a single photon is *not* localizable. Second, the work of Newton and Wigner can be regarded as a contribution to the general problem of determining what physical characteristics of a quantum mechanical system are consistent with a given relativistic transformation law. In this connection, it is interesting to regard the axioms I . . . V below for localizability in a region as a very special case of the notion of particle observables for a quantum theory. Elsewhere[4] I gave a set of axioms for the notion of a particle interpretation which yield I . . . V when specialized to the case of a single particle. One of the main reasons for giving full mathematical detail in

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[1] T. D. Newton and E. P. Wigner, Revs. Modern Phys. **21**, 400 (1949).

[2] L. Foldy and S. Wouthuysen, Phys. Rev. **78**, 29 (1950). This paper was widely read because of its exceptional clarity. The mean position operators themselves were discussed before by A. Papapetrou, Acad. Athens **14**, 540 (1939); R. Becker, Gött. Nach. p. 39 (1945); and M. H. L. Pryce, Proc. Roy. Soc. (London) **A150**, 166 (1935); **A195**, 62 (1948). For further references and discussion see A. S. Wightman and S. Schweber, Phys. Rev. **98**, 812 (1955).

[3] E. Inönü and E. Wigner, Nuovo cimento **9**, 705 (1952). The main point of this paper is that laws of transformation of the states of a particle under the inhomogeneous Galilei group other than those in the ordinary Schrödinger mechanics are inconsistent with localizability.

[4] See, *Les problèmes mathématiques de la théorie quantique des champs*, (CNRS, Paris, 1959), especially pages 36–38.

the present simple case is in preparation for the problem of determining particle interpretations.

It turns out that the natural mathematical tool for the analysis of localizability as understood here is the theory of imprimitive representations of the Euclidean group. The notion of imprimitivity was introduced for finite groups early in the history of group theory. It was generalized to the case of a large class of topological groups by Mackey.[5] From a mathematical point of view, the present paper merely writes out Mackey's theory in detail for the case of the Euclidean group. However, I decided to make the exposition as self-contained as possible, and to incorporate certain elegant ideas of Loomis in the proofs.[6] The purpose of this expository account is to make it possible for the reader to understand how the mathematical arguments go for the Euclidean group without having to work through the general case, however character building that experience might be.

2. MATHEMATICAL FORMULATION OF THE AXIOMS AND PRELIMINARY HEURISTIC DISCUSSION

The axioms are formulated in terms of projection operators $E(S)$, where S is some subset of Euclidean space at a given time. The $E(S)$ are supposed to be observables. They must be projection operators because they are supposed to describe a *property* of the system, the property of being localized in S. That is, if Φ is a vector in a separable Hilbert space, $\mathcal{H}$, describing a state in which the system lies in S, then $E(S)\Phi = \Phi$. If the system does not lie in S then $E(S)\Phi = 0$. $E(S)$ can therefore only have proper values one or zero and, as an observable, must be self-adjoint. Thus, it is a projection operator.[7]

The axioms are:

I. For every Borel set, S, of three-dimensional Euclidean space, $\mathbf{R}^3$, there is a projection operator $E(S)$ whose expectation value is the probability of finding the system in S.

II. $E(S_1 \cap S_2) = E(S_1)E(S_2)$.

III. $E(S_1 \cup S_2) = E(S_1) + E(S_2) - E(S_1 \cap S_2)$.

If $S_i, i = 1, 2, \cdots$ are disjoint Borel sets then

$E(\bigcup S_i) = \sum_{i=1} E(S_i)$.

IV. $E(\mathbf{R}^3) = 1$.

V′. $E(RS + \mathbf{a}) = U(\mathbf{a},R)E(S)U(\mathbf{a},R)^{-1}$,

where $RS + \mathbf{a}$ is the set obtained from S by carrying out the rotation R followed by the translation $\mathbf{a}$, and $U(\mathbf{a},R)$ is the unitary operator whose application yields the wave function rotated by R and translated by $\mathbf{a}$.

The notation $S_1 \cap S_2$ and $S_1 \cup S_2$ is used to indicate the common part and union, respectively, of the sets S_1 and S_2. $\bigcup S_i$ is the union of the sets S_i.

The physical significance of these axioms is as follows.

The Borel sets form the smallest family of sets which includes cubes and is closed under the operations of forming complements and denumerable unions. One might try to replace the Borel sets by all sets obtained by forming complements and *finite* unions starting from cubes and require III only for *finite* sums. However, it can be shown that any such $E(S)$ could be extended to one defined on the Borel sets and satisfying III as it stands. (See Appendix I for further discussion of this point.) In fact, $E(S)$ can be extended even further to all Lebesgue measurable sets, but this extension will not be needed here.[8]

II states that a system which is in *both* S_1 and S_2 is in $S_1 \cap S_2$. It is immediately clear from II that $E(S_1)E(S_2) = E(S_2)E(S_1)$.

III states that the set of states of the system for which it is localized in $S_1 \cup S_2$ is the closed linear manifold spanned by the states localized in S_1 and those localized in S_2.

IV says that the system has probability one of being somewhere.

V′ says that if Φ is a state in which the system is localized in S, then $U(\mathbf{a},R)\Phi$ is a state in which the system is localized in $RS + \mathbf{a}$.

I venture to say that any notion of localizability in three-dimensional space which does *not* satisfy I . . . V′ will represent a radical departure from present physical ideas.

The $E(S)$ define a set of commuting coordinate

[5] G. W. Mackey, Proc. Natl. Acad. Sci. U.S. **35**, 537 (1949); Ann. Math. **55**, 101 (1952); **58**, 193 (1953); Acta Math. **99**, 265 (1958). That Mackey's theory applies to localizability in quantum mechanics was independently realized by Mackey himself. I thank Professor Mackey for correspondence on the subject. Mackey's treatment is summarized in his Colloquium Lectures to the American Mathematical Society, Stillwater, Oklahoma Aug. 29–Sept. 1, 1961. It is a part of a coherent axiomatic treatment of quantum mechanics given in his unpublished Harvard lectures 1960–61.

[6] L. H. Loomis, Duke Math. J. **27**, 569 (1960).

[7] For a general discussion of observables describing a property see J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955), pp. 247–254.

[8] An argument that the Lebesgue measurable sets form a physically natural class is contained in J. von Neumann, Ann. Math. **33**, 595 (1932).

operators q_1,q_2,q_3 which form a vector in 3-space. In fact,

$$q_i = \int_{-\infty}^{\infty} \lambda dE(\{x_i \leqslant \lambda\}) , \qquad (2.1)$$

where $E(\{x_i \leqslant \lambda\})$ is the projection operator for the set $\{x_i \leqslant \lambda\}$ of all points of three-space whose ith coordinates satisfy $x_i \leqslant \lambda$. Of course, (2.1) has to be interpreted as meaning that the Stieltjes integral

$$(\Phi,q_i\Psi) = \int_{-\infty}^{\infty} \lambda d(\Phi,E(\{x_i \leqslant \lambda\})\Psi)$$

holds for all Φ and all Ψ on which q_i can be defined. Thus, any set of $E(S)$ uniquely determines a position operator $\mathbf{q}$. Conversely, one can regard the requirement that the $E(S)$ exist as a precise way of stating that $\mathbf{q}$ exists and its components are simultaneously observable. A notion of localizability for which $[q_i,q_j] \neq 0$ does not fall under the above scheme if, indeed, such a notion makes sense at all.

Axiom V′ has been stated in terms of the unitary operators $U(\mathbf{a},R)$. It is well known that without loss of physical generality these can be assumed to form a representation up to a $\pm$ sign,[9] i.e.,

$$U(\mathbf{a}_1,R_1)U(\mathbf{a}_2,R_2) = \omega(\mathbf{a}_1,R_1;\mathbf{a}_2R_2)U(\mathbf{a}_1 + R_1\mathbf{a}_2,R_1R_2) ,$$

where $\omega = \pm 1$. It is more convenient, from a mathematical point of view, to deal with a true representation for which $\omega = +1$. It is also well known that this can be arranged by passing to the two-sheeted covering group of the Euclidean group $\mathcal{E}_3$.[10] It may be defined as the set of pairs $\mathbf{a}$, A, where $\mathbf{a}$ is again a three-dimensional translation vector and A is a 2×2 unitary matrix of determinant one. The matrices $\pm A$ determine the same rotation given by

$$A\mathbf{x}\cdot\boldsymbol{\tau}A^* = (R(\pm A)\mathbf{x})\cdot\boldsymbol{\tau}$$

Here $\boldsymbol{\tau}$ stands for the Pauli matrices

$$\tau^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The multiplication law of $\mathcal{E}_3$ is

$$\{\mathbf{a},A_1\}\{\mathbf{a}_2,A_2\} = \{\mathbf{a}_1 + A_1\mathbf{a}_2,A_1A_2\} .$$

Here, for brevity, instead of writing $R(A_1)\mathbf{a}_2$ we write $A_1\mathbf{a}_2$. This will be done throughout the following. Thus, V′ is replaced by

V. $U(\mathbf{a},A)E(S)U(\mathbf{a},A)^{-1} = E(AS + \mathbf{a})$ for all Borel sets S of $\mathbf{R}^3$, and all $\{\mathbf{a},A\} \in \mathcal{E}_3$. Here $AS + \mathbf{a}$ is the set obtained from S by the transformation $\{\mathbf{a},A\}$ and $\{\mathbf{a},A\} \rightarrow U(\mathbf{a},A)$ is the representation of $\mathcal{E}_3$ belonging to the physical system in question.

In the terminology of Mackey, I . . . V state that the set of operators $\{E(S)\}$ are a *system of imprimitivity* for the representation $U(\mathbf{a},A)$ of $\mathcal{E}_3$ with *base* $\mathbf{R}^3$. In order to see the present problem in the context of Mackey's general theory, recall that he considers a topological group G and two continuous realizations of G, one a representation by homeomorphisms of a topological space $M : x \rightarrow h(g)x$ (homeomorphism means one-to-one mapping continuous both ways) and the other a unitary representation of G in a Hilbert space $\mathcal{H} : g \rightarrow U(g)$. Then a system of imprimitivity with base M is a family of projection operators in $\mathcal{H}$ which satisfy I, II, III, IV with the sets S interpreted as Borel sets of M, and, in addition, the appropriate modification of V:

$$U(g)E(S)U(g)^{-1} = E(h(g)S) .$$

A representation $U(g)$ which has at least one system of imprimitivity (with respect to M) is said to be imprimitive (with respect to M). A system of imprimitivity is *transitive* if the group of homeomorphisms $g \rightarrow h(g)$ is, i.e., if each point x is carried into every other by a suitable h.

In the case of a transitive system of imprimitivity, the space M can be replaced by a coset space as follows. Let G_x be the subgroup of all $g \in G$ such that $h(g)x = x$. Notice that if $h(g_1)x = y = h(g_2)x$ then $h(g_2^{-1}g_1)x = x$, so $g_2g = g_1$ where $g \in G_x$. The set of all elements of the form g_2g, $g \in G_x$ is denoted g_2G_x and called the left coset of G_x belonging to g_2. Thus, each left coset corresponds to a point of M, distinct cosets corresponding to distinct points, and by a mere change of names M can be replaced by the space of left cosets, usually denoted G/G_x. In the more general case of a nontransitive system the space M will split into orbits and the points of an orbit can be labeled by the points of G/G_x where x is any point of the orbit.

In the problem of localizability considered here, the system of imprimitivity is transitive but for momentum observables and particle observables, in general, the system of imprimitivity is not transitive.

Mackey's theory shows that the transitive system of imprimitivity and its associated representation

[9] The argument (originally due to E. P. Wigner) is outlined in *Dispersion Relations and Elementary Particles* (John Wiley & Sons, Inc., 1961), pp. 176–181.

[10] The argument (originally due to E. P. Wigner for the rotation group and Poincaré group) is given for the Euclidean group in V. Bargmann, Ann. Math. **59**, 1 (1954).

can be brought into a standard form by a suitably chosen unitary transformation, V:

$$\{E(S),U(g)\} \to \{VE(S)V^{-1},VU(g)V^{-1}\} .$$

In this standard form the $VU(g)V^{-1}$ becomes a so-called induced representation associated with a unitary representation of G_x where x is some fixed point of M. Two pairs $\{E_1(S),U_1(g)\}$ and $\{E_2(S), U_2(g)\}$ are unitary equivalent:

$$E_1(S) = VE_2(S)V^{-1},U_1(g) = VU_2(g)V^{-1} ,$$

if and only if the unitary representations of G_x are equivalent.

Detailed proofs of these assertions of Mackey's theory for the special case of $\mathcal{E}_3$ will be offered in the following sections. For the moment, the results will be taken for granted and used to discuss the uniqueness of $E(S)$ for given $U(g)$. Clearly, for $U(g)$ given the only unitary transformations, V, which can give new $VE(S)V^{-1} \neq E(S)$ are ones which commute with the given $U(g)$ but not with the $E(S)$.

That this possibility is actually realized in simple physical examples can be seen by considering a compound system of two free spinless Schrödinger particles with wave function $\psi(\mathbf{x}_1,\mathbf{x}_2)$. Let the corresponding representation of the Euclidean group be $U(\mathbf{a},R)$:

$$\begin{aligned}\psi(\mathbf{x}_1,\mathbf{x}_2) &\to (U(\mathbf{a},R)\psi)(\mathbf{x}_1,\mathbf{x}_2)\\ &= \psi(R^{-1}(\mathbf{x}_1 - \mathbf{a}),R^{-1}(\mathbf{x}_2 - \mathbf{a})) .\end{aligned}$$

Define the operators $\mathbf{X}^{(\alpha)}$ by

$$\mathbf{X}^{(\alpha)} = \alpha\mathbf{x}_1^{\mathrm{op}} + (1 - \alpha)\mathbf{x}_2^{\mathrm{op}}$$

where α is any real number, and by definition

$$\begin{aligned}(\mathbf{x}_1^{\mathrm{op}}\Phi)(\mathbf{y}_1,\mathbf{y}_2) &= \mathbf{y}_1\Phi(\mathbf{y}_1,\mathbf{y}_2)\\ (\mathbf{x}_2^{\mathrm{op}}\Phi)(\mathbf{y}_1,\mathbf{y}_2) &= \mathbf{y}_2\Phi(\mathbf{y}_1,\mathbf{y}_2) .\end{aligned}$$

Then, for each α, $\mathbf{X}^{(\alpha)}$ defines a possible position operator (the spectral representation of $\mathbf{X}_j^{(\alpha)}$, $j = 1, 2, 3$ yields the projections appearing in (2.1), and the general $E(S)$ can be found from these). In particular, $\mathbf{X}^{(0)} = \mathbf{x}_2^{\mathrm{op}}$ and $\mathbf{X}^{(1)} = \mathbf{x}_1^{\mathrm{op}}$, are possible position operators.

Now there exists a unitary operator, V, which commutes with the representation of the Euclidean group

$$[V,U(\mathbf{a},R)]_- = 0$$

and carries $\mathbf{X}^{(\alpha)}$ into $\mathbf{X}^{(\beta)}$

$$V\mathbf{X}^{(\alpha)}V^{-1} = \mathbf{X}^{(\beta)} .$$

To obtain V, one may note first that the operator $T^{(\alpha)}$ defined by

$$\begin{aligned}(T^{(\alpha)}\Phi)(\mathbf{x}_1,\mathbf{x}_2) = \Phi(\alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2,(\alpha - 1)\mathbf{x}_1\\ + (2 - \alpha)\mathbf{x}_2)\end{aligned}$$

is unitary and satisfies

$$T^{(\alpha)}\mathbf{x}_1^{\mathrm{op}}T^{(\alpha)-1} = \mathbf{X}^{(\alpha)} .$$

Then V is given by

$$V = T^{(\beta)}T^{(\alpha)-1} .$$

Clearly, the kind of nonuniqueness appearing in this example may be expected to be absent only when one is dealing with a single particle. Theorem 4 obtained below gives a precise criterion for uniqueness and a parametrization of the possible answers when more than one exists.

The uniqueness of the notion of localizability for given representation of the Euclidean group has been discussed assuming Mackey's theory. Now I attempt to give an intuitive idea of the circumstances in which a notion of localizability exists.

Since all the $E(S)$ commute, diagonalize them. Then the state vectors are represented by quantities $\Phi(\mathbf{x})$ defined on space and with a number of components which may vary with $\mathbf{x}$. [In fact, these $\Phi(\mathbf{x})$ for $\mathbf{x} = \mathbf{a}$ are just Newton and Wigner's linear manifold $S_{\mathbf{a}}$.] In this realization the scalar product of two vectors Φ and Ψ is

$$(\Phi,\Psi) = \int d\mathbf{x}(\Phi(\mathbf{x}),\Psi(\mathbf{x})) ,$$

where the scalar product appearing under the integral sign is in the components of $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$ for fixed $\mathbf{x}$. The operators $E(S)$ take the form

$$(E(S)\Phi)(\mathbf{x}) = \chi_S(\mathbf{x})\Phi(\mathbf{x}) ,$$

where $\chi_S(\mathbf{x}) = 1$ if $\mathbf{x} \in S$, 0 if $\mathbf{x} \notin S$. From the transformation law of $E(S)$ it is plausible that by a suitable choice of basis it can be arranged that

$$(U(\mathbf{a},1)\Phi)(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{a}) .$$

From this equation, it follows that the number of components of $\Phi(\mathbf{x})$ is the same for all $\mathbf{x}$. It is also plausible that by a suitable choice of basis the transformation law under rotation can be made to look the same for each $\mathbf{x}$:

$$(U(0,A)\Phi)(\mathbf{x}) = \mathfrak{D}(A)\Phi(A^{-1}\mathbf{x}) ,$$

where $\mathfrak{D}(A)$ acts on the components of $\Phi(A^{-1}\mathbf{x})$ at each point. Once these results are accepted, one can pass by Fourier transform to momentum space amplitudes. There one has

$$(U(\mathbf{a},A)\Phi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}\mathfrak{D}(A)\Phi(A^{-1}\mathbf{p}) \quad (2.2)$$

with the scalar product

$$(\Phi,\Psi) = \int d\mathbf{p}(\Phi(\mathbf{p}),\Psi(\mathbf{p})) \; . \quad (2.3)$$

The canonical form (2.2) is to be compared with

$$(U(\mathbf{a},A)\Phi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}Q(\mathbf{p},A)\Phi(A^{-1}\mathbf{p}) \; , \quad (2.4)$$

where

$$Q(\mathbf{p},A)Q(A^{-1}\mathbf{p},B) = Q(\mathbf{p},AB) \; ,$$

and the scalar product is

$$(\Phi,\Psi) = \int d\mu(\mathbf{p})(\Phi(\mathbf{p}),\Psi(\mathbf{p})) \; ,$$

a form which will be derived in Sec. 3.

The comparison shows:

(i) When the representation is in the canonical form (2.4) the measure $d\mu(\mathbf{p})$ on momentum space is just Lebesgue measure $d\mathbf{p}$.

(ii) The dimension of the vectors $\Phi(\mathbf{p})$ is the same for all $\mathbf{p}$.

(iii) The operators $Q(\mathbf{p},A)$ are of the form $\mathfrak{D}(A)$, where $A \rightarrow \mathfrak{D}(A)$ is a representation of the unitary unimodular group.

Intuitively (i) and (ii) are accounted for because, if one makes any state whose $\mathbf{x}$ dependence is a δ function one gets all momenta. Thus, one would expect to have the same number of linearly independent states for each $\mathbf{p}$. (iii) is essentially a consequence of the rotational invariance of the states localized at a point.

All three restrictions are nontrivial if applied to an arbitrary representation of $\mathcal{E}_3$. However, as will be seen in Secs. 5 and 6, (i) and (ii) are always satisfied in any relativistic theory (provided one leaves out the vacuum state). (iii) excludes a very important physical system, the single photon. One can see this immediately by looking at the $Q(\mathbf{p},A)$ for those A which leave $\mathbf{p}$ invariant. Such Q's have two eigenvectors corresponding to right-circularly and left-circularly polarized photons having angular momentum along $\mathbf{p}$, $\pm \hbar$, respectively. On the other hand, in $\mathfrak{D}(A)$ one cannot have states with angular momentum $\pm \hbar$ along $\mathbf{p}$ without also having states with zero component of angular momentum along $\mathbf{p}$. The nonlocalizability of the photon (and all other particles of spin $\geqslant \frac{1}{2}$ and mass zero) is a consequence of this simple kinematical fact.[11] For spin-0, (iii) is satisfied and so the phonon is localizable.[12] It is an oddity that the same is not true for Wigner's particles of infinite spin,[13] as will be seen in Sec. 5, even though in that case each angular momentum along $\mathbf{p}$ appears just once.

There is one paradox to which the preceding discussion might appear to give rise. Suppose one describes a photon by a real-valued three-component field $\mathbf{B}(\mathbf{x})$ satisfying

$$\operatorname{div} \mathbf{B} = 0 \; , \quad (2.5)$$

defines a scalar product (this is a *real* Hilbert space) by

$$(\mathbf{B}_1,\mathbf{B}_2) = \int \mathbf{B}_1(\mathbf{x})\cdot\mathbf{B}_2(\mathbf{x})d\mathbf{x} \; ,$$

and a representation of the Euclidean group

$$(U(\mathbf{a},R)\mathbf{B})(\mathbf{x}) = R\mathbf{B}(R^{-1}(\mathbf{x} - \mathbf{a})) \; .$$

Attempt to define projection operators by the equation

$$(E(S)\mathbf{B})(\mathbf{x}) = \chi_S(\mathbf{x})\mathbf{B}(\mathbf{x}) \; .$$

Why does not this describe the photon as a localizable system? The answer is that the $E(S)$ carry vectors satisfying the condition (2.5) into vectors which do not satisfy it, so $E(S)$ is not a well-defined operator in the manifold of states and the $\mathbf{x}$ in $\mathbf{B}(\mathbf{x})$ has nothing to do with localizability.

The notion of localizability discussed here is concerned with states localized in space at a given time. It is natural to inquire whether there exists a corresponding property in space-time. Then the $E(S)$ would satisfy

$$U(a,\Lambda)E(S)U(a,\Lambda)^{-1} = E(\Lambda S + a) \; ,$$

where S is a Borel set of space-time and $\{a,\Lambda\}$ is an inhomogeneous Lorentz transformation of space-time translation, a, and homogeneous Lorentz transformation, Λ. However, a requirement analogous to (i) follows from Mackey's theory: All four-momenta must occur in the theory. This is in flat violation of the physical requirement that there be a lowest

[11] That the photon was nonlocalizable was stated and believed long before reference 1 was written. See, for example, L. Landau and R. Peierls Z. Physik **62**, 188 (1930); **69**, 56 (1931); especially p. 67 of the latter. While the arguments given could possibly be regarded as plausible, they do not make clear what is the heart of the problem.

[12] If the neutrino had turned out to possess states of both helicities, i.e., states with components $\pm\frac{1}{2}\hbar$ of the component of angular momentum along $\mathbf{p}$, then it too would be localizable. A neutrino of definite helicity is nòt localizable.

[13] E. P. Wigner, Ann. Math. **40**, 149 (1939); Z. Physik **124**, 665 (1947–8).

energy state. Thus, a sensible notion of localizability in space-time does not exist.

3. RECAPITULATION OF THE UNITARY REPRESENTATIONS OF $\mathcal{E}_3$ THE UNIVERSAL COVERING GROUP OF THE EUCLIDEAN GROUP

In this section a canonical form of the representations of $\mathcal{E}_3$ will be derived in which the translation subgroup is diagonalized.

Any continuous unitary representation of $\mathcal{E}_3$: $\{\mathbf{a},A\} \to U(\mathbf{a},A)$ gives rise to a continuous unitary representation of the translation group $\mathfrak{T}_3$: $\mathbf{a} \to U(\mathbf{a},1)$. The first step in the analysis is to describe all such representations. By a unitary transformation the $U(\mathbf{a},1)$ are to be diagonalized, i.e., brought into the form

$$(U(\mathbf{a},1)\Phi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}\Phi(\mathbf{p}) . \tag{3.1}$$

(The minus sign in the exponent is a matter of convention; it is adopted to conform with custom in quantum mechanics.)

For this purpose, the notion of direct integral of Hilbert spaces and representations is needed. It will be described briefly in the present special context.[14]

Let μ be a positive measure on three-dimensional (momentum) space $\mathfrak{T}_3^*$. For each point $\mathbf{p}$ of $\mathfrak{T}_3^*$, let there be given a Hilbert space $\mathcal{H}_\mathbf{p}$ whose dimension $\nu(\mathbf{p})$ is a μ-measurable function of $\mathbf{p}$. Then the direct integral of the $\mathcal{H}_\mathbf{p}$ with respect to μ is a Hilbert space denoted $\int_{\mathfrak{T}_3^*}^{\oplus} d\mu(_\mathbf{p})\mathcal{H}_\mathbf{p}$ whose elements are functions defined on $\mathfrak{T}_3^*$, with values satisfying $\Phi(\mathbf{p}) \in \mathcal{H}_\mathbf{p}$. Furthermore, the elements must satisfy $(\Phi_1(\mathbf{p}),\Phi_2(\mathbf{p}))$ is a μ-measurable function of $\mathbf{p}$ for any two

$$\Phi_1,\Phi_2 \in \int_{\mathfrak{T}_3^*}^{\oplus} d\mu(\mathbf{p})\mathcal{H}_\mathbf{p} \tag{3.2}$$

[here $(\Phi_1(\mathbf{p}),\Phi_2(\mathbf{p}))$ is the scalar product in $\mathcal{H}_\mathbf{p}$], and

$$\int (\Phi(\mathbf{p}),\Phi(\mathbf{p}))d\mu(\mathbf{p}) < \infty . \tag{3.3}$$

The scalar product in $\int_{\mathfrak{T}_3^*}^{\oplus} d\mu(\mathbf{p})\mathcal{H}_\mathbf{p}$ is defined by

$$(\Phi_1,\Phi_2) = \int d\mu(\mathbf{p})(\Phi_1(\mathbf{p}),\Phi_2(\mathbf{p})) . \tag{3.4}$$

With this notation, the following theorem holds.

Theorem 1. Every continuous unitary representation of the three-dimensional translation group $\mathfrak{T}_3$ is unitary equivalent to one of the following form:

$$(U(\mathbf{a})\Phi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}\Phi(\mathbf{p}) ,$$

where Φ is an element of a direct integral, $\int_{\mathfrak{T}_3^*}^{\oplus} d\mu(\mathbf{p})\mathcal{H}_\mathbf{p}$, over $\mathfrak{T}_3^*$ with measure μ and multiplicity function $\nu(\mathbf{p}) = \dim \mathcal{H}_\mathbf{p}$.

A bounded operator, B, which commutes with the operators of the representation can be written in the form

$$(B\Phi)(\mathbf{p}) = B(\mathbf{p})\Phi(\mathbf{p}) ,$$

where $B(\mathbf{p})$ is a bounded operator in $\mathcal{H}_\mathbf{p}$ and such that for all Φ_1 and $\Phi_2 \in \int_{\mathfrak{T}_3^*}^{\oplus} d\mu(\mathbf{p})\mathcal{H}_\mathbf{p}$,

$$(\Phi_1(\mathbf{p}),\ B(\mathbf{p})\Phi_2(\mathbf{p})) \text{ is measurable in } \mathbf{p}.$$

Two such representations $\mathbf{a} \to U_1(\mathbf{a})$ and $\mathbf{a} \to U_2(\mathbf{a})$, with measures μ_1 and μ_2 and multiplicity functions $\nu_1(\mathbf{p})$ and $\nu_2(\mathbf{p})$, respectively, are unitary equivalent if and only if

(1) $\mu_1 \equiv \mu_2$, i.e., μ_1 and μ_2 give zero measure for the same sets of $\mathfrak{T}_3^*$.

(2) $\nu_1(\mathbf{p}) = \nu_2(\mathbf{p})$ except, perhaps, in a set of μ_1 measure zero.

For a sketch of a proof of Theorem 1, the reader is referred to Appendix II, and the references quoted there.

The measures μ and multiplicity functions ν appearing in a general representation of the translation group are completely arbitrary. Those which can appear in a representation of $\mathfrak{T}_3$ obtained by restriction from a representation of $\mathcal{E}_3$ are quite special. This comes about because $\mathbf{a} \to U(A\mathbf{a},1)$ defines a representation of $\mathfrak{T}_3$ which is unitary equivalent to $\mathbf{a} \to U(\mathbf{a},1)$ as a consequence of

$$U(0,A)U(\mathbf{a},1)U(0,A)^{-1} = U(A\mathbf{a},1) .$$

Now when $U(\mathbf{a},1)$ is brought into the diagonal form (3.1) by an appropriate unitary transformation, the representation $\mathbf{a} \to U(A\mathbf{a},1)$ takes the form

$$(U(A\mathbf{a},1)\Phi)(\mathbf{p}) = e^{-i(A^{-1}\mathbf{p})\cdot\mathbf{a}}\Phi(\mathbf{p})$$

and this in turn can be brought into the standard form by the unitary transformation

$$(W\Phi)(\mathbf{p}) = \Phi(A\mathbf{p})$$

which yields

$$(WU(A\mathbf{a},1)W^{-1})(W\Phi))(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}(W\Phi)(\mathbf{p})$$

and carries the direct integral $\int_{\mathfrak{T}_3^*}^{\oplus} d\mu(\mathbf{p})\mathcal{H}_\mathbf{p}$ into

[14] For a full account of the notion of direct integral see J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien* (Gauthier-Villars, Paris, 1957). The theory gives a precise mathematical meaning to the Dirac formalism of "representations" in quantum mechanics. See P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, New York, 1947), 3rd ed., Chap. III.

$\int_{\mathfrak{X}_3^*}^{\oplus} d\mu(\mathbf{p})\mathcal{H}_{A\mathbf{p}}$ where $d\mu_A(\mathbf{p}) = d\mu(A\mathbf{p})$. The unitary equivalence criterion given in Theorem 1 then implies

$$\mu \equiv \mu_A \tag{3.5}$$

$\nu(\mathbf{p}) = \nu(A\mathbf{p})$ for all $\mathbf{p}$ except possibly on a set of μ measure zero. (3.6)

Now in Appendix 2, it is shown that the only measures on $\mathfrak{X}_3^*$ satisfying (3.5) are equivalent to ones of the form

$$\mu_0\delta(\mathbf{p}) + d\rho(|\mathbf{p}|)d\omega(\mathbf{p})\,, \tag{3.7}$$

where $\mu_0 \geqslant 0$, $d\omega(\mathbf{p})$ is the area on the sphere of radius $|\mathbf{p}|$ and $d\rho$ is a measure on the positive real axis. Since, if $\mu \equiv \mu_1$, the unitary mapping

$$(W\Phi)(\mathbf{p}) = \Phi(\mathbf{p})\left[\frac{d\mu(\mathbf{p})}{d\mu_1(\mathbf{p})}\right]^{1/2}$$

carries the direct integral $\int_{\mathfrak{X}_3^*}^{\oplus} d\mu(\mathbf{p})\mathcal{H}_{\mathbf{p}}$ into $\int_{\mathfrak{X}_3^*}^{\oplus} d\mu_1(\mathbf{p})\mathcal{H}_{\mathbf{p}}$, one may for convenience choose μ in the form (3.7).[15] Later on μ will be taken in this form but for the moment a general μ satisfying (3.5) will be carried along. Furthermore since any two Hilbert space of the same dimension can be mapped on one another by unitary transformation, there is no loss in generality in taking $\mathcal{H}_{\mathbf{p}} = \mathcal{H}_{A\mathbf{p}}$ for all A.

The next task is to put the operators $U(0,A)$ in standard form. They will be written as a product $U(0,A) = Q(A)T(A)$ where $T(A)$ is defined by

$$(T(A)\Phi)(\mathbf{p}) = \Phi(A^{-1}\mathbf{p})\left[\frac{d\mu(A^{-1}\mathbf{p})}{d\mu(\mathbf{p})}\right]^{1/2}.$$

(Here the convention $\mathcal{H}_{\mathbf{p}} = \mathcal{H}_{A\mathbf{p}}$ has made it possible to equate vectors from two different Hilbert spaces.) It is easy to verify that $T(A)$ is unitary, with an adjoint given by

$$(T(A)^*\Phi)(\mathbf{p}) = \Phi(A\mathbf{p})\left[\frac{d\mu(A\mathbf{p})}{d\mu(\mathbf{p})}\right]^{1/2}. \tag{3.8}$$

An elementary computation shows that

$$T(A)U(\mathbf{a},1)T(A)^{-1} = U(A\mathbf{a},1)\,,$$

so $U(0,A)$ and $T(A)$ satisfy the same commutation relation with $U(\mathbf{a},1)$. Therefore $Q(A)$ commutes with $U(\mathbf{a},1)$. Thus, by Theorem 1, $Q(A)$ can be written in the form

$$(Q(A)\Phi)(\mathbf{p}) = Q(\mathbf{p},A)\Phi(\mathbf{p}) \tag{3.9}$$

[15] We here use the Radon-Nikodym theorem which asserts that if two measures μ_1 and μ_2 are equivalent, i.e., take the value zero for the same sets, then there exists a positive measurable function $\rho(\mathbf{p})$ such that $d\mu_1(\mathbf{p}) = \rho(\mathbf{p})d\mu_2(\mathbf{p})$. $\rho(\mathbf{p})$ is customarily denoted $(d\mu_1/d\mu_2)(\mathbf{p})$. See, for example, P. R. Halmos, *Measure Theory* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1950), p. 128.

Since $Q(A)$ is unitary $Q(\mathbf{p},A)$ must be unitary for almost all $\mathbf{p}$. Furthermore, the group multiplication law implies

$$Q(A)T(A)Q(B)T(B) = Q(AB)T(AB)\,,$$

which yields

$$Q(\mathbf{p},A)Q(A^{-1}\mathbf{p},B) = Q(p,AB) \tag{3.10}$$

for each A and B and almost all $\mathbf{p}$.

At this point a measure-theoretic technicality arises. It is possible *a priori*, that the set of measure zero on which (3.10) does not hold could depend on A and B in such a way that when one took the union over all such sets one would get a set of measure greater than zero. Actually, one can show that one can alter $Q(\mathbf{p},A)$ on a set of measure zero in $\mathbf{p}$ so that $Q(A)$ is unaffected, but (3.10) holds for all $\mathbf{p},A,B$ and $Q(\mathbf{p},A)$ is measurable in both variables. This argument is deferred to Appendix IV, because of its technical character. The result will be assumed in what follows.

The representation has now been reduced to the standard form

$$(U(a,A)\Phi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}Q(\mathbf{p},A) \times \Phi(A^{-1}\mathbf{p})\left[\frac{d\mu(A^{-1}\mathbf{p})}{d\mu(\mathbf{p})}\right]^{1/2}. \tag{3.11}$$

To understand the physical meaning of the $Q(\mathbf{p},A)$ it is helpful to consider some elementary examples. For a single free particle in Schrödinger theory, the wave function may be taken as a complex-valued function of $\mathbf{p}$, the scalar product is

$$(\Phi,\Psi) = \int d\mathbf{p}\ \Phi(\mathbf{p})^*\Psi(\mathbf{p}) \tag{3.12}$$

and the representation of the Euclidean group is

$$(U(\mathbf{a},A)\Phi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}\Phi(A^{-1}\mathbf{p})\,. \tag{3.13}$$

Thus, for a single free particle of spin zero $Q(\mathbf{p},A) = 1$. On the other hand, for a single free particle of spin-$\frac{1}{2}$ (Pauli theory), Φ has two components, the integrand in the formula (3.4) for the scalar product is

$$\sum_{i=1}^{2}\Phi_i(\mathbf{p})^*\Psi_i(\mathbf{p})\,,$$

and the transformation law (3.11) becomes

$$(U(\mathbf{a},A)\Phi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}A\Phi(A^{-1}\mathbf{p})\,.$$

Here, evidently $Q(\mathbf{p},A) = A$ and describes the transformation properties of the spin degree of freedom. In this case $Q(\mathbf{p},A)$ is independent of $\mathbf{p}$. To get an example in which $Q(\mathbf{p},A)$ cannot be brought by unitary transformation of $U(\mathbf{a},A)$ to a form independent of $\mathbf{p}$ one can consider the case of a single photon described in Sec. 5. [One of the results of section 4 is that for a localizable system $Q(\mathbf{p},A)$ can always be chosen independent of $\mathbf{p}$.] Clearly, in all these examples the $Q(\mathbf{p},A)$ gives the transformation law of the internal degrees of freedom of the system under rotations.

A detailed analysis of the consequences of the multiplication law of the Q's, Eq. (3.10), will be undertaken shortly. For the moment, only the fact that for those A which satisfy $A\mathbf{p} = \mathbf{p}$, (3.10) implies

$$Q(\mathbf{p},A)Q(\mathbf{p},B) = Q(\mathbf{p},AB) \qquad (3.14)$$

is needed. Such A form a group called the *little group of* $\mathbf{p}$, and (3.14) means that $A \rightarrow Q(\mathbf{p},A)$ defines a continuous unitary representation of the little group of $\mathbf{p}$. (Again see Appendix IV for a proof that every measurable unitary representation is continuous.) Evidently, when $\mathbf{p} = 0$ the little group of $\mathbf{p}$ is the group of all A, i.e., the unitary unimodular group itself. On the other hand, when $\mathbf{p} \neq 0$, the little group is the two sheeted covering group of the group of rotations around a fixed axis. It is therefore isomorphic to the multiplicative group of the complex numbers $e^{i\theta/2}$, $0 \leqslant \theta < 4\pi$.

The problem of determining when two representations of $\mathcal{E}_3$ are unitary equivalent can now be reduced to a related problem for their $Q(\mathbf{p},A)$. For, if $\{\mathbf{a},A\} \rightarrow U_1(\mathbf{a},A)$ and $\{\mathbf{a},A\} \rightarrow U_2(\mathbf{a},A)$ are equivalent representations, Theorem 1 implies $\mu_1 \equiv \mu_2$ and $\nu_1 = \nu_2$ almost everywhere. Thus, by a unitary transformation one can bring $U_1(\mathbf{a},A)$ into a form where $U_1(\mathbf{a},1) = U_2(\mathbf{a},1)$. Then U_1 and U_2 differ only in their $Q(\mathbf{p},A)$. If

$$U_1(\mathbf{a},A) = VU_2(\mathbf{a},A)V^{-1}, \qquad (3.15)$$

where V is a unitary operator, then, applying Theorem 1, one finds that V is of the form

$$(V\Phi)(\mathbf{p}) = V(\mathbf{p})\Phi(\mathbf{p}) \qquad (3.16)$$

and (3.15) reduces to

$$Q_1(\mathbf{p},A) = V(\mathbf{p})Q_2(\mathbf{p},A)V(A^{-1}\mathbf{p})^{-1}. \qquad (3.17)$$

If $\mathbf{p}$ rather than $A^{-1}\mathbf{p}$ occurred in the last factor, this would describe unitary equivalence of $Q_1(\mathbf{p},A)$ and $Q_2(\mathbf{p},A)$. When A belongs to the little group of $\mathbf{p}$, $A^{-1}\mathbf{p} = \mathbf{p}$ and that is indeed the case.

Again at this point a measure-theoretic technicality arises. Equation (3.17) holds for almost all $\mathbf{p}$, for each A. Again the reader is referred to Appendix IV for a proof that there is a fixed set of measure zero in $|\mathbf{p}|$ such that for all other $\mathbf{p}$ and all A, (3.17) holds.

Next, it will be shown that, if there exists a $V(\mathbf{p})$ for a single $\mathbf{p}$ which satisfies

$$Q_1(\mathbf{p},A) = V(\mathbf{p})Q_2(\mathbf{p},A)V(\mathbf{p})^* \qquad (3.18)$$

for all A in the little group of $\mathbf{p}$, then $V(\mathbf{q})$ can be extended to all $\mathbf{q}$ with $|\mathbf{q}| = |\mathbf{p}|$ so that (3.17) holds. (The statement holds trivially for $\mathbf{p} = 0$ so $\mathbf{p} \neq 0$ is assumed.) Solved for $V(A^{-1}\mathbf{p})$, (3.17) reads

$$V(A^{-1}\mathbf{p}) = Q_1(\mathbf{p},A)^{-1}V(\mathbf{p})Q_2(\mathbf{p},A). \qquad (3.19)$$

This will be consistent as a definition of V at $A^{-1}\mathbf{p}$ only if the right-hand side is constant on right cosets of the little group of $\mathbf{p}$, i.e., only if $A_1^{-1} = A_2^{-1}A_3^{-1}$ with A_3^{-1} in the little group of $\mathbf{p}$ implies that the right-hand side of (3.19) takes the same value for $A = A_1$ and A_2:

$$\begin{aligned} &Q_1(\mathbf{p},A_1)^{-1}V(\mathbf{p})Q_2(\mathbf{p},A_1)\\ &\quad= [Q_1(\mathbf{p},A_3)Q_1(A_3^{-1}\mathbf{p},A_2)]^{-1}V(\mathbf{p})\\ &\quad\quad\times [Q_2(\mathbf{p},A_3)Q_2(A_3^{-1}\mathbf{p},A_2)]\\ &\quad= Q_1(\mathbf{p},A_2)^{-1}[Q_1(\mathbf{p},A_3)^{-1}V(\mathbf{p})Q_2(\mathbf{p},A_3)]Q_2(\mathbf{p},A_2)\\ &\quad= Q_1(\mathbf{p},A_2)^{-1}V(\mathbf{p})Q_2(\mathbf{p},A_2). \end{aligned}$$

This defines $V(\mathbf{q})$ for all $\mathbf{q}$ with $|\mathbf{q}| = |\mathbf{p}|$. Next, it has to be verified that V so defined satisfies

$$Q_1(\mathbf{q}, A) = V(\mathbf{q})Q_2(\mathbf{q},A)V(A^{-1}\mathbf{q})^{-1}. \qquad (3.20)$$

Suppose that $\mathbf{q} = B^{-1}\mathbf{p}$. Then, the right-hand side of (3.20) is

$$\begin{aligned} &[Q_1(\mathbf{p},B)^{-1}V(\mathbf{p})Q_2(\mathbf{p},B)]Q_2(B^{-1}\mathbf{p},A)\\ &\quad\times [Q_1(\mathbf{p},BA)^{-1}V(\mathbf{p})Q_2(\mathbf{p},BA)]^{-1}\\ &= Q_1(\mathbf{p},B)^{-1}Q_1(\mathbf{p},BA) = Q_1(\mathbf{q},A), \end{aligned}$$

where, in the last step, the identity $Q_1(\mathbf{p},B)^{-1} = Q_1(B^{-1}\mathbf{p},B^{-1})$ which follows from (3.10), has been used.

Therefore, a necessary and sufficient condition that U_1 be unitary equivalent to U_2 is $\mu_1 = \mu_2$, $\nu_1 = \nu_2$ almost everywhere and the representations of the little groups $A \rightarrow Q_1(\mathbf{p},A)$, $A \rightarrow Q_2(\mathbf{p},A)$ be unitary equivalent for almost all $|\mathbf{p}|$ and at least one $\mathbf{p}$ for each $|\mathbf{p}|$.

Incidentally, in the course of the argument, it has been established that the little groups for $\mathbf{p}$ and $\mathbf{q}$ have unitary equivalent representations if $|\mathbf{p}| = |\mathbf{q}|$. Explicitly, if $\mathbf{q} = B\mathbf{p}$ and $A\mathbf{q} = \mathbf{q}$, then $B^{-1}AB\mathbf{p} = \mathbf{p}$ and

$$Q(\mathbf{q},A) = Q(\mathbf{p},B^{-1})^{-1}Q(\mathbf{p},B^{-1}AB)Q(\mathbf{p},B^{-1}). \qquad (3.21)$$

The mapping $A \rightarrow B^{-1}AB$ is an isomorphism between the little groups of $\mathbf{q}$ and $\mathbf{p}$ and (3.21) displays the unitary equivalence of the corresponding representations.

The classification of the unitary inequivalent representations of the little groups is well known. For $\mathbf{p}=0$, they are labeled by giving an integer-valued multiplicity function n_{0j} for $j=0, +\frac{1}{2}, 1, \frac{3}{2}, \cdots$. n_{0j} is the number of times the irreducible representation of angular momentum j appears. For $\mathbf{p} \neq 0$ the unitary inequivalent representations are labeled by an integer or $+$ infinity valued function, $n_{\mathbf{p}m}, m=0, \pm\frac{1}{2}, \pm 1, \cdots$ where $n_{\mathbf{p}m}$ is the number of times the one-dimensional irreducible representation $\phi \rightarrow e^{im\varphi}$ occurs.

All these results are collected in Theorem 2.

Theorem 2. Every continuous unitary representation of $\mathcal{E}_3$, the universal covering group of the Euclidean group, is unitary equivalent to one of the following form.

Let $\mathcal{H}_{\mathbf{p}}$ be a family of Hilbert spaces, one for each $\mathbf{p} \in \mathfrak{T}_3^*$ identical for all $\mathbf{p}$ with the same $|\mathbf{p}|$. Let

$$\mathcal{H} = \int^{\oplus} d\rho(|\mathbf{p}|) d\omega(\mathbf{p}) \mathcal{H}_{\mathbf{p}} \oplus \int \mathcal{H}_0$$

where $d\rho$ is a non-negative measure on the positive real axis, such that $\nu(\mathbf{p}) = \dim \mathcal{H}_{\mathbf{p}}$ is measurable, and $d\omega(\mathbf{p})$ is the measure on the unit sphere of the vectors $\mathbf{p}/|\mathbf{p}|$, invariant under rotations. $\mathcal{H}_0$, the contribution from $\mathbf{p}=0$, may or may not occur.

The representation is defined by

$$(U(\mathbf{a},A)\Phi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}} Q(\mathbf{p},A)\Phi(A^{-1}\mathbf{p}) \quad (3.22)$$

where $Q(\mathbf{p},A)$ is a unitary operator in $\mathcal{H}_{\mathbf{p}}$ satisfying $Q(\mathbf{p},1)=1$ and

$$Q(\mathbf{p},A)Q(A^{-1}\mathbf{p},B) = Q(\mathbf{p},AB)$$

for all A, B.

Two representations U_1 and U_2 are unitary equivalent if and only if

(1) $\rho_1 = \rho_2$, i.e., the measures ρ_1 and ρ_2 have the same null sets as measures on the positive real axis.

(2) $\mathcal{H}_0$ either occurs or not in both representations

(3) $\nu_1(\mathbf{p}) = \nu_2(\mathbf{p})$ for almost all $\mathbf{p}$.

(4) the representations of the little groups whose elements are all A such that $A\mathbf{p}=\mathbf{p}$ given by

$$A \rightarrow Q_1(\mathbf{p},A) \quad A \rightarrow Q_2(\mathbf{p},A)$$

are unitary equivalent for almost all $|\mathbf{p}|$.

The conditions (2), (3), and (4) are satisfied if the multiplicity functions of the representations of the little groups satisfy

$$n_{|\mathbf{p}|m}{}^{(1)} = n_{|\mathbf{p}|m}{}^{(2)} \quad \text{for all } m = 0, \pm\tfrac{1}{2}, \pm 1, \cdots$$

and almost all $|\mathbf{p}|$

$$n_{0j}{}^{(1)} = n_{0j}{}^{(2)} \quad \text{for all} \quad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots .$$

Any Euclidean invariant theory has a manifold of states whose transformation law is unitary equivalent to one of this form. It is to be expected (and may be seen in detail from the discussion of Secs. 6 and 7) that the imposition of requirements of relativistic invariance will eliminate some of these representations.

Up to this point, the only assumption that has been made about the quantum mechanical system under consideration is its invariance under the Euclidean group. Now the operation of time inversion I_t will be adjoined. It is well known that I_t has to be represented by an antiunitary operator, $U(I_t)$, whose square is $\omega(I_t) = \pm 1$, and that by suitable choice of phase it can be arranged that[16]

$$U(I_t)U(\mathbf{a},A)U(I_t)^{-1} = U(\mathbf{a},A)$$
$$U(\mathbf{a},A)U(I_t) = U(\{\mathbf{a},A\}I_t)$$
$$U(\{\mathbf{a}_1,A_1\}I_t)U(\{\mathbf{a}_2,A_2\}I_t) = \omega(I_t)U(\{\mathbf{a}_1,A\}I_t\{\mathbf{a}_2,A\}I_t)$$
$$U(\{\mathbf{a}_1,A_1\}I_t)U(\{\mathbf{a}_2,A_2\}) = U(\{\mathbf{a}_1,A_1\}I_t\{\mathbf{a}_2,A_2\}) .$$

Notice that if $\omega(I_t) = -1$, this is a representation only up to a sign.

To get a standard form for $U(I_t)$ when $U(\mathbf{a},A)$ is in the form (3.12), an extension of Theorem 1 to the case of antiunitary operators is needed. It will be assumed here. The result is

$$(U(I_t)\phi)(\mathbf{p}) = Q(\mathbf{p},I_t)\phi(-\mathbf{p})^* , \quad (3.23)$$

with the Q unitary operators satisfying

$$Q(\mathbf{p},I_t)Q(-\mathbf{p},I_t)^* = \omega(I_t)$$

and

$$Q(\mathbf{p},I_t)Q(-\mathbf{p},A)^* = Q(\mathbf{p},A)Q(A^{-1}\mathbf{p},I_t) .$$

The full analysis of these equations yields a proof that two representations including time inversion are unitary equivalent if and only if their measures μ are equivalent, they have the same multiplicity functions, and their representations of the little group extended by time inversion:

$$A \rightarrow Q(\mathbf{p},A) \quad A\mathbf{p} = \mathbf{p}$$
$$I_t \rightarrow Q(\mathbf{p},I_t)K$$

are equivalent for at least one $\mathbf{p}$ and almost all $|\mathbf{p}|$.

[16] See E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., 1959), Chap. 26.

Here K stands for complex conjugation. Only a special case will be considered here, namely, that in which $Q(\mathbf{p},A) = \mathfrak{D}(A)$ where $A \rightarrow \mathfrak{D}(A)$ is a continuous unitary representation of the unimodular group. As will be shown in the next section, for localizable systems this can always be arranged. A second specialization will be made. Only time inversion transformation laws for which

$$Q(\mathbf{p},I_t) = \mathfrak{D}(\tau^2) \qquad (3.24)$$

will be considered. This amounts to considering the case of ordinary type.[17] Time inversion invariance will be used only to get Theorem 4 on the uniqueness of the position observables.

4. REPRESENTATIONS OF $\mathcal{E}_3$ WHICH POSSESS A TRANSITIVE SYSTEM OF IMPRIMITIVITY[5]

The discussion of this section is in three parts. First, Mackey's standard form of an imprimitive representation is given and shown to be equivalent, in the special case at hand, to a simpler form which will be more convenient for present purposes. Second, for a given imprimitive representation a unitary transformation is found which brings it into Mackey's form. Third, the unitary transformations which commute with an imprimitive representation $U(\mathbf{a},A)$ but not with its system of imprimitivity $E(S)$ are parametrized. This yields a parametrization of the nonuniqueness in the definition of a position operator.

Suppose there is given a continuous unitary representation $A \rightarrow \mathfrak{D}(A)$ of the 2×2 unitary unimodular group in a Hilbert space $\mathcal{H}(\mathfrak{D})$. Then the representation of $\mathcal{E}_3$ *induced by* $\mathfrak{D}(A)$ is denoted $U^{\mathfrak{D}}$ and constructed as follows. Consider functions $\Phi(\mathbf{a},A)$ which are defined on $\mathcal{E}_3$, whose values lie in $\mathcal{H}(\mathfrak{D})$, and which satisfy

(a) $(\Phi(\mathbf{a},A),\chi)$ is a measurable function of $\{\mathbf{a},A\}$, for every $\chi \in \mathcal{H}(\mathfrak{D})$. [The indicated scalar product is in $\mathcal{H}(\mathfrak{D})$.]

(b) $\Phi(A\mathbf{b},AB) = \mathfrak{D}(A)\Phi(\mathbf{b},B)$ (4.1)

(c) $\int ||\Phi(\mathbf{a},A)||^2 d\mathbf{a} < \infty$

Notice that (b) implies

$$(\Phi(\mathbf{a},A),\Psi(\mathbf{a},A)) = (\Phi(B\mathbf{a},BA),\Psi(B\mathbf{a},BA))$$

so the integral in (c) is independent of A. Clearly, any linear combination of functions satisfying (a), (b), (c) also satisfies (a), (b), (c) so these functions form a vector space. If a scalar product of Φ and Ψ is defined

$$(\Phi,\Psi) = \int d\mathbf{a}(\Phi(\mathbf{a},A)\ \Psi(\mathbf{a},A))$$

the vector space becomes a Hilbert space $\mathcal{H}$.[18] The representation $U^{\mathfrak{D}}$ is defined in $\mathcal{H}$ by

$$(U(\mathbf{a},A)\Phi)(\mathbf{b},B) = \Phi(\mathbf{b} + B\mathbf{a},BA)\ . \qquad (4.2)$$

This representation possesses a transitive system of imprimitivity defined by

$$(E(S)\Phi)(\mathbf{a},A) = \chi_S(\mathbf{a})\Phi(\mathbf{a},A)$$

defined for Borel sets S of $\mathfrak{T}_3$ where, as usual, χ_S is the characteristic function of S: $\chi_S(\mathbf{a}) = 1$ if $\mathbf{a} \in S$, 0 if $\mathbf{a} \notin S$. It is easy to verify using (4.2) that the $E(S)$ transform correctly under $U(\mathbf{a},A)$, i.e., satisfy V.

Because of the smooth fashion in which A acts on $\mathbf{a}$ this representation can be put in a simpler form. If, for the moment, attention is restricted to continuous functions $\Phi(\mathbf{a},A)$, Eq. (4.1) can be used to write

$$\Phi(\mathbf{a},A) = \mathfrak{D}(A)\Phi(A^{-1}\mathbf{a},1) \qquad (4.3)$$

which expresses $\Phi(\mathbf{a},A)$ for general values of A in terms of its value for $A = 1$. Conversely, given any continuous function $\Phi(\mathbf{a})$ with values in $\mathcal{H}(\mathfrak{D})$, one can define a continuous $\Phi(\mathbf{a},A)$ by (4.3) and it will then satisfy (4.1). The scalar product of two such $\Phi(\mathbf{a})$ and $\Psi(\mathbf{a})$, $\int d\mathbf{a}(\Phi(\mathbf{a}),\Psi(\mathbf{a}))$ is equal to that of the corresponding $\Phi(\mathbf{a},A),\Psi(\mathbf{a},A)$ so the one to one correspondence can be extended by continuity to a unitary mapping between the Hilbert space $\mathcal{H}$ and the Hilbert space of the measurable square integrable $\Phi(\mathbf{a})$.

The representation (4.2) determines a corresponding representation on the $\Phi(\mathbf{a})$ given by

$$\mathfrak{D}(B)(U(\mathbf{a},A)\Phi)(B^{-1}\mathbf{b}) = \mathfrak{D}(BA)\Phi((BA)^{-1}(\mathbf{b} + B\mathbf{a}))$$

or

$$(U(\mathbf{a},A)\Phi)(\mathbf{b}) = \mathfrak{D}(A)\Phi(A^{-1}(\mathbf{b} + \mathbf{a}))\ .$$

Now this looks just like the standard form of Euclidean transformation appearing in Schrödinger

[17] See reference 16, especially pp. 343–344.

[18] The details of the proof involve identifying functions which differ only on a set of measure zero and establishing that the space is closed. For a proof which is easily adapted to the present circumstances see M. H. Stone, *Linear Transformations in Hilbert Space* (American Mathematical Society, Providence, Rhode Island, 1932), pp. 23–32.

theory except that there one has $-\mathbf{a}$ instead of $\mathbf{a}$ on the right-hand side. That just means that one uses as representative of the function $\Phi(-\mathbf{b})$ instead of $\Phi(\mathbf{b})$. This will be done from this point on. Thus, in the present context, Mackey's form of the imprimitive representation induced by $\mathfrak{D}$ may be taken as

$$(U(\mathbf{a},A)\Phi)(\mathbf{b}) = \mathfrak{D}(A)\Phi(A^{-1}(\mathbf{b} - \mathbf{a})) \quad (4.4)$$

$$(E(S)\Phi)(\mathbf{b}) = \chi_S(\mathbf{b})\Phi(\mathbf{b}) \quad (4.5)$$

with the scalar product

$$(\Phi,\Psi) = \int d\mathbf{b}(\Phi(\mathbf{b}),\Psi(\mathbf{b})) . \quad (4.6)$$

Now, the second step of the argument is undertaken; it is to be shown that for each pair consisting of a continuous unitary representation $\{\mathbf{a},A\} \to U(\mathbf{a},A)$ and a system of imprimitivity $E(S)$, there exists a unitary operator V such that $VU(\mathbf{a},A)V^{-1}$ and $VE(S)V^{-1}$ are of the form (4.4) and (4.5), respectively. Available to show this are several lines of argument, not one of them trivial. Here the elegant proof of Loomis[19] will be written out for the present simple case.

The first step in the argument is to express the problem in terms of certain complex-valued functions defined on the group. This is quite analogous to the study of general unitary representations in terms of positive definite functions on the group. To motivate Loomis' method, a brief sketch will first be given of the relation of positive definite functions and representations.

A function φ defined on a group G is *positive definite* if for each $n = 1,2,\cdots$ and all complex numbers $\alpha_1\cdots\alpha_n$ and $g_1\cdots g_n \in G$

$$\sum_{i,j=1}^{n} \alpha_i^*\varphi(g_i^{-1}g_j)\alpha_j \geqslant 0$$

Clearly, taking $n = 1$, one gets

$$\varphi(e) \geqslant 0 . \quad (4.7)$$

For $n = 2$,

$$(|\alpha_1|^2 + |\alpha_2|^2)\varphi(e) + \alpha_1^*\alpha_2\varphi(g_1^{-1}g_2) + \alpha_2^*\alpha_1\varphi(g_2^{-1}g_1) \geqslant 0 .$$

From the reality of the left-hand side one concludes $\varphi(g_1^{-1}g_2) = \varphi(g_2^{-1}g_1)^*$ which is equivalent to

$$\varphi(g^{-1}) = \varphi(g)^* \quad \text{all} \quad g \in G , \quad (4.8)$$

since G is a group. The positivity of the quadratic form then implies that the determinant of its matrix is positive, i.e.,

$$|\varphi(g)| \leqslant \varphi(e) .$$

Any unitary representation of G, $g \to U(g)$, yields examples of positive definite functions[20]

$$\varphi(g) = (\Phi,U(g)\Phi)$$

because, in this case,

$$\sum \alpha_i^*\alpha_k\varphi(g_i^{-1}g_k) = ||\sum \alpha_i U(g_i)\Phi||^2 \geqslant 0 .$$

If the representation is continuous then $\varphi(g)$ is continuous.

Conversely, given a continuous positive definite function one can construct a continuous representation of G. Let r, s be complex-valued functions on G which are different from zero only at a finite number of points. (Such functions form a vector space.) Introduce the form

$$(r,s) = \sum_{g,h\in G} \overline{r(g)}\, \varphi(g^{-1}h)s(h) \quad (4.9)$$

(r,s) is sesqui-linear, i.e.,

$$(r,s_1 + s_2) = (r,s_1) + (r,s_2) , \quad (r,\alpha s) = \alpha(r,s) \quad (4.10)$$

$$(r,s) = \overline{(s,r)} \quad (4.11)$$

by virtue of (4.8), and

$$(r,r) \geqslant 0 . \quad (4.12)$$

Now it may happen that there are some r for which

$$(r,r) = 0 .$$

If so, it is easy to see that they form a linear subspace and the components orthogonal to this linear subspace form a vector space on which (r,s) again satisfies (4.10), (4.11), and (4.12) but, in addition, $(r,r) = 0$ implies $r = 0$. This space may or may not be complete. If not, complete it and get a Hilbert space H_φ. To get a continuous representation of G in H_φ, define, first on functions with only a finite number of values different from zero,

$$(U(g)r)(h) = r(g^{-1}h) \quad (4.13)$$

with the inverse $U(g^{-1})$.

So defined $U(g)$ leaves the scalar product invariant

$$(U(g)r,U(g)s) = \sum_{h,k} r(g^{-1}h)^*\varphi(h^{-1}k)s(g^{-1}k) = \sum r(h')^*\varphi((gh')^{-1}(gk'))s(k') = (r,s)$$

[19] See reference 6. One of the main virtues of Loomis' treatment is that it applies to nonseparable Hilbert spaces. Since separability is assumed here this advantage will not be apparent.

[20] Positive definite functions were used in the proofs of the celebrated theorems of Bochner and Gelfand-Raikov for Abelian and locally-compact groups, respectively. A systematic account of their properties is found in R. Godement, Trans. Am. Math. Soc. 63, 1 (1948).

so the subspace of those r for which $(r,r) = 0$ is left invariant by $U(g)$. Therefore, so is its orthogonal complement. Because $U(g)$ is therefore defined and continuous on a dense subset of H_φ it can be extended by continuity to be a unitary operator in H_φ. Clearly, on the original functions

$$U(g_1)U(g_2) = U(g_1g_2) ,$$

so by continuity, $g \to U(g)$ defines a representation. To prove $U(g)$ is continuous in g, consider

$$\|(Ug) - U(g'))r\|^2 = \|(U(g^{-1}g') - 1)r\|^2 = 2[(r,r) - \mathrm{Re}\,(U(g^{-1}g')r,r)] .$$

Clearly, this equation implies that it suffices to verify $(U(g)r,r)$ is continuous in g at $g = e$ for all $r \in \mathsf{H}_\varphi$. For r of the special kind appearing in (4.9), which only take values different from zero at a finite number of points the continuity is easy to verify:

$$(U(g)r,r) = \textstyle\sum_{h,k} r(g^{-1}h)^*\varphi(h^{-1}k)r(k) = \textstyle\sum_{h,k} r(h)^*\varphi(gh)^{-1}k)r(k) ,$$

which clearly converges to (r,r) as $g \to e$ because φ is continuous and there is only a finite number of terms in the sum. For a general r, there always exists an s of the above form so that $||r - s|| < \epsilon/3$. By the above argument a neighborhood of e can be found so that $||U(g)s - s|| < \epsilon/3$. Then

$$||U(g)r - r|| < ||U(g)r - U(g)s|| + ||U(g)s - s|| + ||s - r|| ,$$

which completes the proof that $g\ U(g)$ is a continuous unitary representation of G.

Actually, if the continuous positive definite function from which one starts is of the form $(\Phi,V(g)\Phi)$, the representation constructed by the above process will be closely related to V itself. For, if the subspace (of the Hilbert space $\mathfrak{H}$ in which φ lies) spanned by vectors of the form $V(g)\Phi$ is denoted $\tilde{\mathfrak{H}}$, the constructed representation as unitary equivalent to the restriction of V to $\tilde{\mathfrak{H}}$. The required unitary equivalence is obtained by making $\sum r(g)V(g)$ correspond to r, for r differing from zero only at a finite number of points. Equation (4.9) is just arranged to make scalar products correspond. Clearly $(U(g)r)$ corresponds to $V(g) \sum r(h)V(h)\Phi$. The correspondence can be extended by continuity to yield the required unitary equivalence.

A representation V for which there is a vector Φ such that the $V(g)\Phi$ span the representation space is called *cyclic* and Φ is then a *cyclic vector*. Note that the function which is one at $g = e$ and zero elsewhere is a cyclic vector for the representation defined above. Thus, what has been established in the preceding paragraphs is that all cyclic representations are unitary equivalent to those of the form (4.9) and (4.13). Since any representation can be written as a direct sum of cyclic representations, it suffices for many purposes to study cyclic representations.

In the present case, there is a system of imprimitivity $E(S)$ in addition to the group representation $U(g)$ so one has to consider cyclic vectors and representations of $E(S)$ and $U(g)$ together. This suggests studying the function $(E(S)\Phi,U(g)\Phi) = \varphi_S(g)$ and using it to construct a pair unitary equivalent to $\{E(S),U(g)\}$ and in Mackey's form.

Now return to the special case of $\mathcal{E}_3$. When the representation and system of imprimitivity is in Mackey's form (4.4) and (4.5), the function $\varphi_S(\mathbf{a},A)$ is

$$\varphi_S(\mathbf{a},A) = \int d\mathbf{b}((E(S)\Phi)(\mathbf{b}),(U(\mathbf{a},A)\Phi)(\mathbf{b})) = \int_S d\mathbf{b}(\Phi(\mathbf{b}),\mathfrak{D}(A)\Phi(A^{-1}(\mathbf{b} - \mathbf{a}))) \quad (4.14)$$

The next task is to show that $\varphi_S(\mathbf{a},A)$ has a form closely related to this for any representation and transitive system of imprimitivity.

Before the discussion can begin a preliminary remark is necessary. Extensive use is going to be made of the part of the Radon-Nikodym theorem which says that if, for two measures μ_1 and μ_2, $\mu_1(S) = 0$ implies $\mu_2(S) = 0$, then there exists a measurable function $\rho(x)$ such that $d\mu_2(x) = \rho(x)d\mu_1(x)$. To make these applications it is essential to know that $E(S) = 0$ for all Borel sets S of Lebesgue measure zero. To obtain this result, it is convenient to use the fact that the $E(S)$ possess a *separating vector*, i.e., a vector Φ such that $E(S)\Phi = 0$ implies $E(S) = 0$. Although this is a standard result[21] a proof will be outlined. Choose an arbitrary unit vector Φ_1, and let $\mathfrak{H}_1$ be the subspace spanned by the $E(S)\Phi_1$. Choose a unit vector Φ_2 orthogonal to $\mathfrak{H}_1$ and let $\mathfrak{H}_2$ be the subspace spanned by the $E(S)\Phi_2$. Continuing in this way one gets a family of orthogonal subspaces such that $\mathfrak{H}$ is the direct sum of the $\mathfrak{H}_i$ and Φ_i is a cyclic unit vector for $\mathfrak{H}_i$. Take as separating vector $\Phi = \sum_n 2^{-n}\Phi_n$. Clearly, if $E(T)\Phi = 0$ then $E(T)\Phi_i = 0$ for all i. Consequently, $E(T)$ yields zero when applied to a dense set of vectors, the linear combinations of the $E(S)\Phi_i$. It is therefore zero and Φ is a separating vector. Note first that if $E(S) = 0$, then $E(AS + \mathbf{a}) = U(\mathbf{a},A)E(S)U(\mathbf{a},A)^{-1} = 0$. Thus if Φ

[21] See reference 14, p. 20.

is a separating vector, $(\Phi,E(S)\Phi) = ||E(S)\Phi||^2$ is quasi-invariant under Euclidean transformation, i.e., for all $\{\mathbf{a},A\}$, $(\Phi,E(S)\Phi) = 0$ if and only if $(\Phi,E(AS + \mathbf{a})\Phi) = 0$. Furthermore, $(\Phi,E(S)\Phi)$ defines a σ-additive positive measure on the Borel sets S of $\mathbf{R}^3$. Now in Appendix II it is shown that any measure defined on the Borel sets of $\mathbf{R}^3$ and quasi-invariant under translations is equivalent to Lebesgue measure. That implies in particular that $(\Phi,E(S)\Phi)$ and therefore $E(S) = 0$ whenever S is a Borel set of measure zero. Thus, the Radon-Nikodymn theorem implies that if Φ is any vector there exists a non-negative measurable function ρ such that

$$(\Phi,E(S)\Phi) = \int_S \rho(\mathbf{b})d\mathbf{b} \qquad (4.15)$$

$\rho(\mathbf{b})$ is clearly integrable over all space.

This equation can be used to get an expression for $\varphi_S(\mathbf{a},A)$ which is the first step in proving that it can always be arranged to have the form (4.14). Note that

$$|\varphi_S(\mathbf{a},A)| = |(E(S)\Phi,U(\mathbf{a},A)\Phi)| \leqslant ||E(S)\Phi||$$

where for convenience it has been assumed that $||\Phi|| = 1$. It therefore follows that if T is any Borel set of $\mathcal{E}_3$

$$\int_T \varphi_S(\mathbf{a},A)d\mathbf{a}dA \leqslant \int_T d\mathbf{a}dA \int_S \rho(\mathbf{b})d\mathbf{b}\,. \quad (4.16)$$

Now the left-hand side as a function of S and T is initially defined for products of rectangles $S \times T$, and is bounded by the product $\mu(S)\nu(T)$ where $\mu(S) = \int_S\rho(\mathbf{b})d\mathbf{b}$ and $\nu(T)$ is the Lebesgue measure of T. It is finitely additive on such products in the sense that if $S \times T = \bigcup S_i \times T_i$ where $(S_i \times T_i) \cap (S_j \times T_j) = 0$ for $i \neq j$, then its value on $S \times T$ is the sum of its values on the $S_i \times T_i$. Consequently, it has a unique extension to a Borel measure on $\mathbf{R}^3 \times \mathcal{E}_3$. [The necessary argument first shows that it can be extended to be finitely additive on sets which are arbitrary finite unions of disjoint products $S_i \times T_i$. Second, it shows that the boundedness described by (4.16) implies that this extension is actually σ additive. Finally, it uses a standard extension theorem[22] to assert that the resulting set function has a unique extension to be a Borel measure.] Clearly, this measure is bounded by the product measure $\mu \times \nu$. Thus again by the Radon Nikodym theorem this time for complex valued measures there exists a measurable function q of absolute value less than or equal to 1 such that

$$\int d\mathbf{a}dA\varphi_S(\mathbf{a},A) = \int_T\int_S d\mathbf{a}dAd\mathbf{b}q(\mathbf{a},A;b)\rho(\mathbf{b})\,. \quad (4.17)$$

From (4.17), it follows that

$$\varphi_S(\mathbf{a},A) = \int_S d\mathbf{b}\mathbf{q}(\mathbf{a},A;\mathbf{b})\rho(\mathbf{b})$$

for almost all $\{\mathbf{a},A\}$ which begins to look like (4.14). This completes the first stage of the proof.

The next stage is the construction of the Hilbert space of the $\Phi(\mathbf{a})$ which appears in (4.4) . . . (4.6). This is done in close analogy with the construction carried out in connection with (4.9) but for technical reasons which will appear in the proof it is convenient to consider continuous functions of compact support on $\mathcal{E}_3$ rather than the functions differing from zero only at a finite number of points, which were used there. Therefore let f and g be continuous complex-valued functions of compact support on $\mathcal{E}_3$ and define

$$U(f) = \int d\mathbf{b}dBf(\mathbf{b},B)U(\mathbf{b},B)$$

$$U(g) = \int d\mathbf{c}dCg(\mathbf{c},C)U(\mathbf{c},C)\,. \qquad (4.18)$$

Then

$$\begin{aligned}(E(S)U(f)\Phi,U(\mathbf{a},A)U(g)\Phi) &= \int d\mathbf{b}dB\int d\mathbf{c}dCf(\mathbf{b},B)^*\\ &\times g(\mathbf{c},C)(E(S)U(\mathbf{b},B)\Phi,U(\mathbf{a},A)U(\mathbf{c},C)\Phi)\\ &= \int d\mathbf{b}dB\int d\mathbf{c}dCf(\mathbf{b},B)^*g(\mathbf{c},C)(E(B^{-1}S - B^{-1}\mathbf{b})\Phi\\ &\times U(\{\mathbf{b},B\}^{-1}\{\mathbf{a},A\}\{\mathbf{c},C\})\Phi) = \int d\mathbf{r}\int d\mathbf{b}dB\\ &\times \int_{B^{-1}S-B^{-1}\mathbf{b}} d\mathbf{c}dCf(\mathbf{b},B)^*g(\mathbf{c},C)\\ &\times q(\{\mathbf{b},B\}^{-1}\{\mathbf{a},A\}\{\mathbf{c},C\};\mathbf{r})\rho(\mathbf{r})\,,\\ &= \int_S d\mathbf{r}\int d\mathbf{b}dB\int d\mathbf{c}dCf(\mathbf{b},B)^*g(\mathbf{c},C)\\ &\times q(\{\mathbf{b},B\}^{-1}\{\mathbf{a},A\}\{\mathbf{c},C\};B^{-1}(\mathbf{r}-\mathbf{b}))\\ &\times \rho(B^{-1}(\mathbf{r}-\mathbf{b}))\,. \end{aligned} \qquad (4.19)$$

For $\{\mathbf{a},A\} = \{0,1\}$ this reduces to

$$\begin{aligned}(E(S)U(f)\Phi,U(g)\Phi) &= \int_S d\mathbf{r}\int d\mathbf{b}dB\int d\mathbf{c}dC\\ &\times f(\mathbf{b}+\mathbf{r},B)^*g(\mathbf{c}+\mathbf{r},C)q(\{\mathbf{b},B\}^{-1}\{\mathbf{c},C\};-B^{-1}\mathbf{b})\\ &\times \rho(-B^{-1}\mathbf{b})\,,\end{aligned} \qquad (4.20)$$

which suggests introducing $(U(f)\Phi)(\mathbf{r})$ as the function $f(\mathbf{b}+\mathbf{r},B)$ of $\mathbf{b}$ and B.

[22] See reference 15, p. 54, Theorem A.

Then the form

$$((U(f)\Phi)(\mathbf{r}),(U(g)\Phi)(\mathbf{r})) = \int d\mathbf{b}dB \int d\mathbf{c}dC$$
$$\times f(\mathbf{b}+\mathbf{r},B)^* g(\mathbf{c}+\mathbf{r},C) q(\{\mathbf{b},B\}^{-1}\{\mathbf{c},C\}, -B^{-1}\mathbf{b})$$
$$\times \rho(-B^{-1}\mathbf{b}) \tag{4.21}$$

is suggested as the scalar product appearing in the integrand of (4.14).

With these definitions, one has

$$(E(S)U(f)\Phi,U(\mathbf{a},A)U(g)\Phi) = \int_S d\mathbf{r}$$
$$\times ((U(f)\Phi)(\mathbf{r}),W(\mathbf{a},A)(U(f)\Phi))(\mathbf{r})) , \tag{4.22}$$

where $W(\mathbf{a},A)$ is the operator defined by

$$f(\mathbf{b}+\mathbf{r},B) \to f(A^{-1}\mathbf{b}+A^{-1}(\mathbf{r}-\mathbf{a}),A^{-1}B) .$$

Notice that if linear transformation $\mathfrak{D}(A)$ is defined by the correspondence

$$f(\mathbf{b}+\mathbf{r},B) \to f(A^{-1}\mathbf{b}+\mathbf{r},A^{-1}B) . \tag{4.23}$$

Then W may be written

$$(W(\mathbf{a},A)U(f)\Phi)(\mathbf{r}) = \mathfrak{D}(A)(U(f)\Phi)(A^{-1}(\mathbf{r}-\mathbf{a}))$$

so that W is precisely of the form (4.4). It is obvious (by a simple change of variable) that $\mathfrak{D}(A)$ leaves the scalar product (4.21) invariant.

Now it has to be verified that (4.21) does indeed define a scalar product. First note that it is linear in g and conjugate linear in f. Furthermore, because (4.20) holds for every Borel set S,

$$(E(S)U(f)\Phi,U(g)\Phi) = [(E(S)U(g)\Phi,U(f)\Phi)]$$

and

$$(E(S)U(f)\Phi,U(f)\Phi) \geqslant 0$$

imply

$$((U(f)\Phi)(\mathbf{r}),(U(g)\Phi)(\mathbf{r}))$$
$$= [((U(g)\Phi)(\mathbf{r}),(U(f)\Phi)(\mathbf{r}))]^* \tag{4.24}$$

and

$$((U(f)\Phi)(\mathbf{r}),(U(f)\Phi)(\mathbf{r})) \geqslant 0 \tag{4.25}$$

for almost all $\mathbf{r}$. However, since f and g are continuous and of compact support the integral appearing in $\mathbf{r}$ is continuous in $\mathbf{r}$. Therefore (4.24) and (4.25) hold for all $\mathbf{r}$. Now, just as in the case of (4.12), one can introduce components of vectors orthogonal to the subspace for which (4.25) is an equality, and complete the resulting space to get a Hilbert space $\mathfrak{K}$ the same for each $\mathbf{r}$. $A \to \mathfrak{D}(A)$ is then a continuous unitary representation in $\mathfrak{K}$. Since the correspondence $U(f)\Phi \to (U(f)\Phi)(\mathbf{r})$ carries a dense set of vectors in the subspace spanned by the $U(f)\Phi$ into a dense set of vectors in the Hilbert space spanned by the functions of $\mathbf{r}$: $(U(f)\Phi)(\mathbf{r})$ and preserves scalar products it can be extended by continuity to become a unitary transformation V.

All this discussion is collected in Theorem 3.

Theorem 3. Let $\{\mathbf{a},A\} \to U(\mathbf{a},A)$ be a continuous unitary representation of $\mathcal{E}_3$ with a transitive system of imprimitivity, $E(S)$, based on $\mathbf{R}^3$. Then there exists a unitary transformation V, such that $VU(\mathbf{a},A)V^{-1} = W(\mathbf{a},A)$ and $VE(S)V^{-1} = F(S)$, respectively, given by

$$(W(\mathbf{a},A)\Phi)(\mathbf{b}) = \mathfrak{D}(A)\Phi(A^{-1}(\mathbf{b}-\mathbf{a})) \tag{4.26}$$

$$(F(S)\Phi)(\mathbf{b}) = \chi_S(\mathbf{b})\Phi(\mathbf{b}) . \tag{4.27}$$

Here $A \to \mathfrak{D}(A)$ is a continuous unitary representation of the 2×2 unitary unimodular group in a separable Hilbert space $\mathfrak{K}$ and the $\Phi(\mathbf{b})$ are functions on $\mathbf{R}^3$ with values in $\mathfrak{K}$ which are measurable in the sense that for all pairs of such functions $(\Phi(\mathbf{b}),\Psi(\mathbf{b}))$ is a measurable function of $\mathbf{b}$. In symbols

$$\mathfrak{H} = \int_{\mathbf{R}^3}^{\oplus} d\mathbf{b}\mathfrak{H}_\mathbf{p} \quad \text{with} \quad \mathfrak{H}_\mathbf{p} = \mathfrak{K} .$$

The remaining task of this section is to examine the arbitrariness in the definition of the position observable. For this purpose, one can bring the pair $\{E(S),U(\mathbf{a},A)\}$ into the form (4.26) and (4.27), and then determine all unitary operators which commute with $U(\mathbf{a},A)$ but not with $E(S)$. It is convenient for this purpose to rewrite (4.26) in momentum space

$$(U(\mathbf{a},A)\Phi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}\mathfrak{D}(A)\Phi(A^{-1}\mathbf{p}) .$$

If B is a unitary operator such that $[B,U(\mathbf{a},1)] = 0$, Theorem 1 shows that B can be written in the form

$$(B\Phi)(\mathbf{p}) = B(\mathbf{p})\Phi(\mathbf{p})$$

where $B(\mathbf{p})$ is a unitary operator in $\mathfrak{H}_\mathbf{p} = \mathfrak{K}$. The commutativity with $U(0,A)$ then implies

$$B(\mathbf{p})\mathfrak{D}(A) = \mathfrak{D}(A)B(A^{-1}\mathbf{p}) \tag{4.28}$$

for almost all $\mathbf{p}$.

This equation can be discussed along lines familiar from Sec. 3 and Appendix IV. For those A which satisfy $A\mathbf{p} = \mathbf{p}$, i.e., for A in the little group of $\mathbf{p}$, (4.28) reduces to

$$B(\mathbf{p})\mathfrak{D}(A) = \mathfrak{D}(A)B(\mathbf{p}) . \tag{4.29}$$

The set of all $B(\mathbf{p})$ satisfying this equation is easy to compute. Supposing them known one gets the general solution of (4.28) by using it as a definition:

$$B(\mathbf{p}) = \mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{p}})^{-1}B(\mathbf{q})\mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{p}}) . \quad (4.30)$$

Here the $A_{\mathbf{q}\leftarrow\mathbf{p}}$ satisfy $A_{\mathbf{q}\leftarrow\mathbf{p}}\mathbf{p} = \mathbf{q}$ and parametrize the cosets of the little group of $\mathbf{q}$. By virtue of (4.29) at $\mathbf{q}$ any parametrization yields the same $B(\mathbf{p})$. An argument just like that following equation (3.20) shows that the $B(\mathbf{p})$ defined for a fixed $\mathbf{q}/|\mathbf{q}|$ and all $\mathbf{p}$ by (4.30) satisfies (4.28).

To obtain all solutions of (4.29), one can decompose $A \rightarrow \mathfrak{D}(A)$ into irreducible representations of the 2×2 unitary unimodular group, and these in turn into irreducible representations of the little group

$$\mathfrak{D} = \sum_{j=0,1/2,1,\cdots} n_j\mathfrak{D}^{(j)} ,$$

$$\mathfrak{D}^{(j)} = \sum_{m=-j,-j-1,\cdots j} e^{im\phi} .$$

Thus $A \rightarrow \mathfrak{D}(A)$ restricted to the little group of $\mathbf{p}$ is unitary equivalent to

$$\sum n_m e^{im\phi} \quad (4.31)$$

with

$$n_m = \sum_{j\geqq|m|} n_j .$$

Here the summation over j is over integers if m is integral and half-odd integers if m is half an odd integer.

The $B(\mathbf{p})$ corresponding to a given set $\{n_m\}$, $m = 0, \pm\frac{1}{2}, \pm 1, \cdots$ is a direct sum of unitary operators acting in the subspaces of vectors with a definite value of m, and any such defines a possible $B(\mathbf{p})$. The number of real parameters free in an arbitrary $n_m \times n_m$ unitary matrix is n_m^2 so that $B(\mathbf{p})$ contains $\sum_m n_m^2$ arbitrary real parameters, each of which could be a function of $|\mathbf{p}|$.

Collecting the information acquired in the preceding discussion one has Theorem 4.

Theorem 4. If $E(S)$ is a system of imprimitivity for the unitary representation $\{\mathbf{a},A\} \rightarrow U(\mathbf{a},A)$ of $\mathcal{E}_3$ in the standard form (4.26), (4.27), then all other systems of imprimitivity consistent with U are given by

$$F(S) = BE(S)B^{-1} ,$$

where B is a unitary operator given by

$$(B\Phi)(\mathbf{p}) = \mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{p}})^{-1}B(\mathbf{q})\mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{p}})\Phi(\mathbf{p}) \quad (4.32)$$

so that

$$(F(S)\Phi)(\mathbf{p}) = \mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{p}})^{-1}B(\mathbf{q})^{-1}\mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{p}}) \times (2\pi^{-3/2}\int \tilde{\chi}_S(\mathbf{p}-\mathbf{r})d\mathbf{r}\mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{r}})^{-1}B(\mathbf{q})^{-1} \times \mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{r}})\Phi(\mathbf{r}) . \quad (4.33)$$

Here

$$\tilde{\chi}(\mathbf{p}) = [2\pi]^{-3/2}\int e^{i\mathbf{p}\cdot\mathbf{b}}\chi_S(\mathbf{b})d\mathbf{b}$$

and $B(\mathbf{q})$ is a solution of

$$[B(\mathbf{q}),\mathfrak{D}(A)] = 0$$

for all A satisfying $A\mathbf{q} = \mathbf{q}$.

In the discussion up to this point, symmetry under time inversion and any analog of the regularity assumption of Newton and Wigner have been ignored. This is natural in the case of Theorem 3 because the canonical form of a transitive system of imprimitivity can be obtained without the use of these additional assumptions. However, for Theorem 4, they are of decisive importance. Even in this case $\mathfrak{D}(A)$ one dimensional, (4.33) would give a wide variety of distinct position observable ($B(\mathbf{q})$) is then a complex-valued function of the form $B(\mathbf{q}) = e^{i\eta(|\mathbf{q}|)}, \eta$ real. In this case, the effect of the assumption of time inversion invariance is to force B to be real, and therefore to be equal to $+1$. However, it could be $+1$ for some $|\mathbf{p}|$ and -1 for others without violating either Euclidean or time inversion invariance. It is here that Newton and Wigner's assumption of regularity has the effect of making B a constant and $F(S) = E(S)$. They require (in a Lorentz invariant theory) that the infinitesimal Lorentz transformation operators be applicable to localized states in the sense that if $\Phi_n$is a sequence of vectors which converge to a state localized at a point $\mathbf{a}$, as $n \rightarrow \infty$, then $\lim_{\infty\rightarrow n} ||M_{\mu\nu}\Phi_n||/||\Phi_n|| < \infty$. Since $M_{0i}, i = 1,2,3$ are essentially differentiation operators this forces continuity on the momentum space representation of Newton and Wigner's localized (continuum) state. An analogous requirement in the present formulation has an analogous consequence. The details are as follows.

According to (3.14), the transformation law of states under time inversion is of the form

$$(U(I_t)\Phi)(\mathbf{p}) = \mathfrak{D}(\tau^2)\Phi(-\mathbf{p})^*$$

The requirement that B commute with $U(I_t)$ then forces

$$\mathfrak{D}(\tau^2)\overline{B(-\mathbf{p})} = B(\mathbf{p})\mathfrak{D}(\tau^2) \quad (4.34)$$

which is

$$\mathfrak{D}(\tau^2)\overline{\mathfrak{D}(A_{\mathbf{q}\leftarrow-\mathbf{p}})^{-1}B(\mathbf{q})\mathfrak{D}(A_{\mathbf{q}\leftarrow-\mathbf{p}})} = \mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{p}})^{-1}B(\mathbf{q})\mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{p}})\mathfrak{D}(\tau^2)$$

or using

$$\overline{\mathfrak{D}(A)} = \mathfrak{D}(\tau^2)^{-1}\mathfrak{D}(A)\mathfrak{D}(\tau^2),$$

$$\mathfrak{D}(A_{\mathbf{q}\leftarrow -\mathbf{p}}A^{-1}_{\mathbf{q}\leftarrow -\mathbf{p}})\mathfrak{D}(\tau^2)\overline{B(\mathbf{q})} = B(\mathbf{q})\mathfrak{D}(A_{\mathbf{q}\leftarrow \mathbf{p}}A^{-1}_{\mathbf{q}\leftarrow \mathbf{p}})\mathfrak{D}(\tau^2)\,. \quad (4.35)$$

For suitably chosen $\mathbf{p}$ the factors in $\mathfrak{D}$ cancel and one gets

$$\overline{B(\mathbf{q})} = B(\mathbf{q})\,, \quad (4.36)$$

provided that $\mathbf{q}$ is not along the 2 axis as will be assumed. The remaining condition on $B(\mathbf{q})$ says that it commutes with all $\mathfrak{D}(A_{\mathbf{q}\leftarrow\mathbf{p}}A^{-1}_{\mathbf{q}\leftarrow -\mathbf{p}}\, i\tau^2)$. This will be no further restriction since we may for convenience choose $\mathbf{q}$ along the 3-axis and then every such transformation is an element of the little group of $\mathbf{q}$, and $B(\mathbf{q})$ already commutes with them. [To see this it is convenient to choose a particular form for the $A_{\mathbf{p}\leftarrow\mathbf{q}}$:

$$A_{\mathbf{q}\leftarrow\mathbf{p}} = 2\left(1 + \frac{\mathbf{q}\cdot\mathbf{p}}{|\mathbf{p}||\mathbf{q}|}\right)^{-1/2} \times\left[1 + \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|} - i\frac{\mathbf{p}\times\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}\cdot\boldsymbol{\tau}\right].$$

This is well defined for all $\mathbf{p} \neq -\mathbf{q}$.
Then

$$A_{\mathbf{q}\leftarrow\mathbf{p}}A^{-1}_{\mathbf{q}\leftarrow -\mathbf{p}} = -i\frac{(\mathbf{p}\times\mathbf{q})}{|\mathbf{p}||\mathbf{q}|}\cdot\boldsymbol{\tau}\,.$$

It is easy to choose $\mathbf{p}$ so that $(\mathbf{p}\times\mathbf{q}/|\mathbf{p}||\mathbf{q}|\cdot\boldsymbol{\tau})\tau^2 = 1$; then (4.36) follows. However $\mathbf{p}$ is chosen provided $\mathbf{q}$ is along the 3 axis $(\mathbf{q}\times\mathbf{p}/|\mathbf{q}||\mathbf{p}|)\cdot\boldsymbol{\tau}\tau^2$ leaves $\mathbf{q}$ invariant. This proves the second statement.]

A comparison of these statements with the discussion just before Theorem 4 shows that the effect of time inversion invariance on the arbitrariness of $B(\mathbf{q})$ is to reduce the number of arbitrary real parameters from $\sum n_m^2$ to $\sum n_m(n_m - 1)$ each of which could be on a function of $|\mathbf{q}|$. It is clear that the position observable will be nonunique as long as $\mathfrak{D}(A)$ is not irreducible. If $\mathfrak{D}(A)$ is irreducible and the elements of the little group have $\mathfrak{D}(A)$ reduced to diagonal form $B(\mathbf{q})$ is diagonal with diagonal elements which are real functions of $|\mathbf{q}|$ of square 1; the position observable is still not unique. However, unless $B(\mathbf{q})$ is the constant matrix ± 1, the formula (4.33) will yield discontinuous functions of $\mathbf{p}$. [Take a compact set S, then the integral in (4.33) is differentiable, so discontinuities in the function outside the integral are discontinuities of $(F(S)\Phi)(\mathbf{p})$.] Such discontinuities will appear at any value of $\mathbf{q}$ where $B(|\mathbf{q}|)$ jumps so $B(|\mathbf{q}|)$ must be constant in $|\mathbf{q}|$. It must be a constant multiple of the identity if it is to be differentiable at $\mathbf{p} = -\mathbf{q}$. In summary, we have Theorem 5.

Theorem 5. In a Euclidean invariant system with time inversion symmetry the possible observables $F(S)$ which describe localization are given by (4.33) with $B(\mathbf{q})$ a real unitary operator.

If localized states are differentiable in $\mathbf{p}$ and $A \to \mathfrak{D}(A)$ is an irreducible representation of the 2×2 unitary unimodular group then $B(\mathbf{p})$ is a constant multiple of the identity and $F(S) = E(S)$. Conversely, if the $F(S)$ are unique $A \to \mathfrak{D}(A)$ must be irreducible and localized states differentiable.

In the following in Theorems 6 and 7, the regularity and time inversion invariance requirements assumed in Theorem 5 will always be taken for granted.

5. REPRESENTATIONS OF $\mathcal{E}_3$ WHICH ARE RESTRICTIONS OF REPRESENTATIONS OF THE COVERING GROUP OF THE INHOMOGENEOUS LORENTZ GROUP

It is well known that every continuous unitary representation of the covering group of the Poincaré group is unitary equivalent to one of the form

$$(U(\mathbf{a},A)\Phi)(\mathbf{p}) = e^{i\mathbf{p}\cdot\mathbf{a}}Q(\mathbf{p},A)\Phi(A^{-1}\mathbf{p})$$

in $\mathfrak{H}$,

$$\mathfrak{H} = \int^{\oplus} d\mu(p)\mathfrak{H}_p$$

with

$$d\mu(p) = \mu_0\delta(p)dp + d\rho_+(m)d\Omega_{m+}(p) + d\rho_-(m)d\Omega_{m-}(p) + d\rho(im)d\Omega_{im}(p)\,,$$

$d\Omega_{m\pm}(p) = d\,\mathbf{p}/[m^2 + \mathbf{p}^2]^{1/2}$ being the invariant measure on the hyperboloids $p^2 = m^2$, $p^0 \gtrless 0$, respectively. $d\Omega_{im}(p)$ is the invariant measure on the hyperboloid $p^2 = -m^2$. $Q(p,A)$ is unitary and satisfies

$$Q(p,A)Q(A^{-1}p,B) = Q(p,AB)\,.$$

For the subrepresentations with $m^2 > 0$, $Q(p,A)$ can be chosen in the form

$$Q(p,A) = \mathfrak{Q}(A^{-1}_{p\leftarrow k}AA_{A^{-1}p\leftarrow k}) \quad (5.1)$$

where $k = (m,0,0,0)$ and $A_{q\leftarrow k}$ is given by

$$A_{q\leftarrow k} = [2(q^0 + m)m]^{-1/2}[m1 + \tilde{q}]\,,\ \tilde{q} = q^0 + \mathbf{q}\cdot\boldsymbol{\tau}$$

and $A \to \mathfrak{Q}(A)$ is a continuous unitary representation of the unitary unimodular group. For $m = 0$, the $Q(p,A)$ are a direct sum of two parts, the first of which contains all the finite spin constituents while the second contains all infinite spin constituents. For both of these (5.1) again holds but k is some standard light-like vector, say (1,0,0,1), and $A_{q\leftarrow k}$ is a parametrization of the cosets of the little group of k. That little group is isomorphic to the two-sheeted covering group of the Euclidean group of the plane and $A \rightarrow \mathcal{Q}(A)$ is a continuous unitary representation of it. For the finite spin part this representation is trivial for the "translations" while for the infinite spin part it is not. The subspace of the mass zero representations can be written as a direct integral over two-dimensional Ξ space

$$\mathcal{H}_p = \int^{\oplus} d\sigma(\Xi)\mathcal{H}_{p\Xi}\,, \quad p^2 = 0\,,$$

with the scalar product

$$(\Phi,\Psi)_{m=0} = \int d\Omega_0(p) \int d\sigma(\Xi)(\Phi(p,\Xi),\Psi(p,\Xi))$$

where

$$d\sigma(\Xi) = \sigma_0\delta(\Xi)d\Xi + d\sigma_1(|\Xi|)d\varphi$$

with $\Xi = \Xi_1 + i\Xi_2 = |\Xi|e^{i\varphi}$ and

$$(Q(k,A)\Phi)(k,\Xi) = \exp(i\Xi\cdot\mathbf{t})Q_1(k,\Xi,A)\Phi(k,e^{-i\theta}\Xi)$$

for

$$A = [1 + \tfrac{1}{2}\,t(\mathbf{e}_1 + i\mathbf{e}_2)\cdot\boldsymbol{\tau}] \times [\cos\theta/2 - i\sin(\theta/2)(\mathbf{k}/k^0)\cdot\boldsymbol{\tau}]$$

with $\mathbf{e}_1^2 = 1 = \mathbf{e}_2^2$, $\mathbf{e}_1\cdot\mathbf{e}_2 = 0 = \mathbf{e}_1\cdot\mathbf{k} = \mathbf{e}_2\cdot\mathbf{k}$, $t = t_1 + it_2$. Here $Q_1(k,0,A)$ may be expressed in terms of a representation $\mathcal{Q}_1$ of the above A leaving k fixed.

$$Q_1(k,0,A) = \mathcal{Q}_1[\cos\theta/2 - i\sin(\theta/2)(\mathbf{k}/k^0)\cdot\boldsymbol{\tau}]$$

$Q_1(k,\Xi,A)$, $\Xi \neq 0$ may be expressed in terms of a representation $\mathcal{Q}$ of the two element groups $A = \pm 1$, which is the subgroup of those unitary unimodular A which leave k and some Ξ, say Ξ_1 fixed:

$$Q_1(k,\Xi,A) = \mathcal{Q}(A_{\Xi\leftarrow\Xi_1}AA_{A^{-1}\Xi\leftarrow\Xi_1})$$

where A is a transformation of the form $\cos\theta/2 - i\sin(\theta/2)(\mathbf{k}/k^0)\cdot\boldsymbol{\tau}$ carrying Ξ_1 into Ξ.

The representations of imaginary mass and null four-momentum (apart from the identity representation) will be ignored here as being irrelevant to the transformation properties of physical systems.

Clearly, when $\{a,A\}$ is restricted to lie in $\mathcal{E}_3$, the subrepresentation which comes from mass 0 is in precisely the form (2.2) and Theorems 4 and 5 apply directly. For the case of mass zero and $\Xi = 0$, the system is localizable if the representation of the little group $A \rightarrow Q_1(k,0,A)$ is the restriction of a representation of the unitary unimodular group. This happens for the spin-zero case but for no other irreducible representation. For the case of mass zero and $\Xi \neq 0$, the representation of the little group is a direct integral over irreducible representations which are determined by the value of $|\Xi|$ and the representation of the little group of the little group, $A = \pm 1$. The representatives of the state vectors, $\Phi(k,\Xi)$ can be expanded in Fourier series on the circle $|\Xi| = \text{const}$. This corresponds to a decomposition into irreducible representations of the subgroup of the unitary unimodular group that leaves k fixed. In case the little group of the little group is trivially represented, each integer angular momentum along $\mathbf{k}$ appears exactly once. In case it is nontrivially represented, each half odd integer angular momentum along $\mathbf{k}$ appears twice. Such representations can never be the restriction of a representation of the full unimodular group. Thus elementary systems with $\Xi \neq 0$ are never localizable. Reducible systems are localizable only if each representation $|\Xi|$ appears with infinite multiplicity or not at all.

Theorem 6. Lorentz invariant systems of $m^2 > 0$ are always localizable. Their position observables are unique if the systems are elementary, i.e., their representations are irreducible.

For $m = 0$, the only localizable elementary system has spin zero. For a reducible system to be localizable it is necessary and sufficient that each irreducible representation of infinite spin appear with zero or infinite multiplicity, and the finite spin parts contribute states of angular momentum along a fixed direction whose multiplicities coincide with those of the restriction of a representation of the unimodular group.

The identity representation for which $p = 0$ can not appear in the transformation law of any localizable system.

6. REPRESENTATIONS OF $\mathcal{E}_3$ ARISING IN GALILEI-INVARIANT SYSTEMS

Unlike the case of Lorentz invariance where all representations up to a factor are physically equivalent to representations of the covering group, Galilei invariance leads to factors which cannot be got rid of by passing to the covering group. However, as Bargmann showed,[23] one can regard them as true repre-

[23] V. Bargmann, Ann. Math. 59, 1 (1954), especially pp. 38–43.

sentations of a certain extension of the covering group of the Galilei group. The first task of this section is to express this statement in explicit formulas and summarize the classification of the representations.

The Galilei transformations will be denoted (a,Γ) or in more detail $(\tau,\mathbf{a},\mathbf{v},R)$ where $(0,\Gamma) = (0,0,\mathbf{v},R)$ $(a,1) = (\tau,\mathbf{a},0,1)$ and

$$(\tau,\mathbf{a}) \begin{Bmatrix} t \\ \mathbf{x} \end{Bmatrix} = \begin{Bmatrix} t + \tau \\ \mathbf{x} + \mathbf{a} \end{Bmatrix}, \quad (\mathbf{v},R) \begin{Bmatrix} t \\ \mathbf{x} \end{Bmatrix} = \begin{Bmatrix} t \\ R\mathbf{x} + \mathbf{v}t \end{Bmatrix}.$$

R is a rotation of the three space of the $\mathbf{x}$. The group multiplication law is

$$(\tau_1,\mathbf{a}_1,\mathbf{v}_1,R_1)(\tau_2,\mathbf{a}_2,\mathbf{v}_2,R_2) = (\tau_1 + \tau_2,\ \mathbf{a}_1 + \mathbf{v}_1\tau_2,\ \mathbf{v}_1 + R_1\mathbf{v}_2,\ R_1R_2) .$$

The covering group is obtained by replacing R by A, a 2×2 unitary matrix of determinant 1, just as in (2.1). For simplicity, $\{a,\Gamma\}$ will be written for the group elements in this case also.

Bargmann showed that by physically inessential changes of phase, one could bring all the factors into the following form:

$$\omega(a_1,\Gamma_1;a_2,\Gamma_2) = \exp i(M/\hbar) \times (\mathbf{v}_1 \cdot A_1 \cdot \mathbf{a}_2 + \tfrac{1}{2}\,\mathbf{v}_1^2\tau_2) .$$

Here $M/\hbar$ is a constant of the dimensions time/[length]^2, which has arbitrarily been written as a ratio in order that its interpretation shall come out automatically when applied to Schrödinger theory ($\hbar$ is Planck's constant divided by 2π).

Furthermore, Bargmann pointed out that every representation up to a factor with this factor arises from a representation of the group whose elements are $(\exp i\theta,a,\Gamma)$ $0 \leq \theta < 2\pi$ and whose multiplication law is

$$(\exp i\theta_1,a_1,\Gamma_1) \cdot (\exp i\theta_2,a_2,\Gamma_2) = (\exp i[\theta_1 + \theta_2 + (M/\hbar)(\mathbf{v}_1 \cdot R_1 \cdot \mathbf{a}_2 + \tfrac{1}{2}\,\mathbf{v}_1^2\tau_2)],\ a_1 + \Gamma_1 a_2,\ \Gamma_1\Gamma_2)$$

via the formula

$$U(\exp i\theta,a,\Gamma) = e^{i\theta}U(a,\Gamma) . \qquad (6.1)$$

For the case $M = 0$, this refinement is unnecessary. That case will be discussed later.

Now the elements of the group of the $(\exp i\theta,a,\Gamma)$ which are of the form $(\exp i\theta,a,1)$ form a normal subgroup and the group is the semi-direct product of this Abelian normal subgroup and the subgroup of the $(\phi,0,\Gamma)$. Just as in the case of the Euclidean group one diagonalizes the Abelian subgroup in terms of a direct integral over the character group whose elements are $\exp i[q\theta + \hbar^{-1}(E\tau - \mathbf{p}\cdot\mathbf{a})]$. The states are then functions $\Phi(q,p)$ labeled by integers q and a real four-component $p = (E/\hbar,\mathbf{p}/\hbar)$. The scalar product is

$$(\Phi,\Psi) = \int d\mu(q,p)(\Phi(q,p),\Psi(q,p))$$

and

$$(U(\exp i\theta,a,1)\Phi)(q,p) = \exp i[q\theta + \hbar^{-1}(E\tau - \mathbf{p}\cdot\mathbf{a})]\Phi(q,p) .$$

The action of Γ on the subgroup $(\exp i\theta,\tau,\mathbf{a},1)$ is

$$(1,0,\Gamma)(\exp i\theta,a,1)(1,0,\Gamma)^{-1} = (\exp i[\theta + (M/\hbar)(\mathbf{v}\cdot A\cdot\mathbf{a} + \tfrac{1}{2}\,\mathbf{v}^2\tau)],\Gamma a,1) .$$

It induces a corresponding transformation of the characters

$$q \to q \quad E \to E - \mathbf{p}\cdot\mathbf{v} + \tfrac{1}{2}\,qM\mathbf{v}^2$$
$$\mathbf{p} \to A^{-1}(\mathbf{p} - qM\mathbf{v}) .$$

From this, it follows that

$$d\mu(q,E,\mathbf{p}) \equiv d\mu(q,E + \mathbf{v}\cdot A\mathbf{p} + \frac{q}{2}\,M\mathbf{v}^2,A\mathbf{p} + qM\mathbf{v}).$$

To yield a representation of the form (6.1), $d\mu$ must be a product of δ_{q1} with a measure in (E,p) above, $d\nu(E,p)$, satisfying

$$d\nu(E,\mathbf{p}) \equiv d\nu(E + \mathbf{v}\cdot A\mathbf{p} + \frac{M}{2}\,\mathbf{v}^2,A\mathbf{p} + m\mathbf{v}) .$$

This in turn implies that $d\nu$ is equivalent to a measure constant on parabolas

$$d\nu(E,p) = d\rho(E_0)dN_{E_0}(\mathbf{p})$$

where $dN_{E_0}(\mathbf{p})$ is the measure $d\mathbf{p}$ on the parabola

$$E = E_0 + \frac{\mathbf{p}^2}{2M}$$

and $d\rho(E_0)$ is a measure on the real axis describing the spectrum of rest energy of the system. Again just in the case of the Euclidean group, there is a canonical form

$$U(\exp i\theta,a,\Gamma)\Phi)(E_0,p) = \exp(i\theta) \times \exp[(i/\hbar)(E\tau - \mathbf{p}\cdot\mathbf{a})]Q(E_0,\mathbf{p},\Gamma) \times \Phi(E_0,A^{-1}(\mathbf{p} - M\mathbf{v})) \qquad (6.2)$$

where

$$Q(E_0,\mathbf{p},\mathbf{v}_1,A_1)Q(E_0,A_1^{-1}(\mathbf{p} - M\mathbf{v}_1),\mathbf{v}_2,A_2) = Q(E_0,\mathbf{p},\mathbf{v}_1 + A_1\mathbf{v}_2,A_1A_2) \qquad (6.3)$$

and the scalar product is

$$(\Phi,\Psi) = \int d\rho(E_0) \int dN_{E_0}(\mathbf{p})(\Phi(E_0,\mathbf{p}),\Psi(E_0,\mathbf{p})) .$$

The little group of a vector $\mathbf{q}$ consists of all $(\mathbf{v},A)$ of the form $(M^{-1}(\mathbf{q} - A\mathbf{q}),A)$ and is isomorphic to the unitary unimodular group.

The Q's can be brought into the canonical form

$$Q(E_0,\mathbf{p},\mathbf{v},A) = Q(E_0,0, -\mathbf{p}/M,1)^{-1}Q(E_0,0,0,A) \times Q(E_0,0,A^{-1}(\mathbf{v} - \mathbf{p}/M,1) \quad (6.4)$$

where

$$A \to \mathfrak{Q}(A) = Q(E_0,0,0,A)$$

is a continuous unitary representation of the group of unitary unimodular matrices. Evidently, (6.4) just describes a superposition of Schrödinger particles of mass M, and various rest energies (described by E_0), and spins (described by the irreducible constituents of $\mathfrak{Q}$).

It is clear that the representation of $\mathcal{E}_3$ that is obtained from (6.4)

$$(U(\mathbf{a},A)\Phi(p)) = e^{-i\mathbf{p}\cdot\mathbf{a}}\mathfrak{Q}(A)\Phi(A^{-1}\mathbf{p})$$

with the scalar product

$$(\Phi,\Psi) = \int d\mathbf{p}(\Phi(\mathbf{p}),\Psi(\mathbf{p}))$$

and

$$(\Phi(\mathbf{p}),\Psi(\mathbf{p})) = \int d\rho(E_0)(\Phi(E_0,\mathbf{p}),\Psi(E_0,\mathbf{p})) .$$

Thus for $M > 0$, the situation is essentially identical with that in the Lorentz invariant case. There is always a position operator and the arbitrariness in it is that associated with the representation of the unimodular group which describes the transformation properties of the system under rotations in the rest system.

For $M < 0$, localizability still makes sense but such representations are rejected on the physical ground that the kinetic energy of a particle is negative.

For representations with $M = 0$, the preceding argument has to be reexamined. There is no need to introduce θ as in (6.1). The diagonalization of $U(a,1)$ leads to

$$(U(a,1)\Phi)(E,\mathbf{p}) = \exp[i(E\tau - \mathbf{p}\cdot\mathbf{a})/\hbar]\Phi(E,\mathbf{p})$$

with a scalar product

$$(\Phi,\Psi) = \int d\mu(E,\mathbf{p})(\Phi(E,\mathbf{p}),\Psi(E,\mathbf{p}))$$

but the measure μ now satisfies

$$d\mu(E,\mathbf{p}) \equiv d\mu(E + \mathbf{v}\cdot A\cdot\mathbf{p},A\mathbf{p})$$

and this implies that $d\mu$ is equivalent to

$$d\omega(\mathbf{p})d\rho(|\mathbf{p}|)dE + \mu_0\delta(\mathbf{p})dEd\mathbf{p} ,$$

where $d\omega(\mathbf{p})$ is the area on the sphere of radius $|\mathbf{p}|$. (The fact that the energy spectrum of the system runs to $-\infty$, makes these representations of dubious physical interest, but does not exclude their being localizable.) The full transformation law is of the form

$$(U(a,\Gamma)\Phi)(E,\mathbf{p}) = \exp[i(E\tau - \mathbf{p}\cdot\mathbf{a})/\hbar]Q(E,\mathbf{p},\Gamma) \times \Phi(E - \mathbf{v}\cdot\mathbf{p},A^{-1}\mathbf{p})$$

where $Q(E,p,\Gamma)$ satisfies

$$Q(E,\mathbf{p},\mathbf{v}_1,A_1)Q(E - \mathbf{v}_1\cdot\mathbf{p},A_1^{-1}\mathbf{p},\mathbf{v}_2,A_2) = Q(E,\mathbf{p},\mathbf{v}_1 + A_1\mathbf{v}_2,A_1A_2) .$$

The little group of p, $p \neq 0$ consists of all $(\boldsymbol{\omega},A)$ such that $A\mathbf{p} = \mathbf{p}$ and $\mathbf{v}\cdot\mathbf{p} = 0$. This is a group isomorphic to the two-sheeted covering group of the Euclidean group of the plane. For $\mathbf{p} = 0$, the little group is the full unitary unimodular group. There can be no contribution of this latter kind in any localizable system because the criterion (i) of Sec. 2 is not satisfied so only the former case will be considered. There, the criterion (i) forces $d\rho(|\mathbf{p}|)$ to be equivalent to $|\mathbf{p}|^2d|\mathbf{p}|$. From this, it is clear that no irreducible representation is localizable because an irreducible representation has $d\mu$ concentrated on an orbit $|\mathbf{p}| = \text{const.}$[24]

The general $Q(E,\mathbf{p},\mathbf{v},A)$ is expressed in terms of the representation of the little group of the vector $(0,\mathbf{q})$, where $|\mathbf{p}| = |\mathbf{q}|$, in the following way:

$$Q(E,\mathbf{p},\Gamma) = Q(0,\mathbf{q},\Gamma((E,\mathbf{p}) \leftarrow (0,\mathbf{q}))^{-1}$$
$$\times Q(0,\mathbf{q},\Gamma((E,\mathbf{p}) \leftarrow (0,\mathbf{q}))^{-1}\Gamma\Gamma((\Gamma^{-1}(E,\mathbf{p})) \leftarrow (0,\mathbf{q}))$$
$$\times Q(0,\mathbf{q},\Gamma((\Gamma^{-1}(E,\mathbf{p}) \leftarrow (0,\mathbf{q}))) .$$

Here $\Gamma((E,\mathbf{p}) \leftarrow (0,\mathbf{q}))$ is a Galilei transformation which carries $(0,\mathbf{q})$ into $(E,\mathbf{p})$, so

$$\Gamma((E,\mathbf{p}) \leftarrow (0,\mathbf{q}))^{-1}\Gamma\Gamma((\Gamma^{-1}(E,\mathbf{p})) \leftarrow (0,\mathbf{q}))$$

belongs to the little group of $(0,\mathbf{q})$.

The same procedure can be applied to analyze the representation $(\mathbf{v},A) \to Q(0,\mathbf{q},\mathbf{v},A)$, $\mathbf{v}\cdot\mathbf{q} = 0$, $A\mathbf{q} = \mathbf{q}$, of the little group of $(0,\mathbf{q})$ as was applied to $\mathcal{E}_3$ itself. One diagonalizes the "translations" $\mathbf{v}$. Then the

[24] This result agrees with that of Inönü and Wigner, reference 3.

representation takes the form

$$Q(0,\mathbf{q},\mathbf{v},A)\Phi)(\mathbf{q},\mathbf{n}) = e^{i\mathbf{v}\cdot\mathbf{n}}Q_1(\mathbf{q},\mathbf{n},A)\Phi(\mathbf{q},A^{-1}\mathbf{n}) .$$

Here $\mathbf{n}$ is a two-component vector in the plane perpendicular to $\mathbf{q}$ which labels the characters of the "translation" subgroup. The scalar product is

$$(\Phi(\mathbf{q}),\Psi(\mathbf{q})) = \int d\sigma(\mathbf{n})(\Phi(\mathbf{q},\mathbf{n}),\Psi(\mathbf{q},\mathbf{n})) ,$$

where the measure σ is equivalent to one of the form

$$d\sigma(\mathbf{n}) = \sigma_0\delta(\mathbf{n})d\mathbf{n} + d\sigma_1(|\mathbf{n}|)d\varphi,\ \mathbf{n}_1 + i\mathbf{n}_2 = |\mathbf{n}|e^{i\varphi} .$$

The little group of the little group is the little group itself if $\mathbf{n} = 0$, while it is the two-element group: $A = \pm 1$ if $\mathbf{n} \neq 0$. In the former case $A \to Q_1(\mathbf{q},0,A)$ is any continuous unitary representation of the little group of $\mathbf{q}$. In the latter case, $\pm 1 \to Q_1(\mathbf{q},\mathbf{n}, \pm 1)$ is any unitary representation of the 2-element group and the Q_1 of general argument is expressed in terms of the elements of the little group by

$$Q_1(q,\mathbf{n},A) = [Q_1(\mathbf{q},\mathbf{n}_0,A_{\mathbf{n}\leftarrow\mathbf{n}_0}{}^{-1})]^{-1}$$
$$\times\ Q_1(\mathbf{q},\mathbf{n}_0,A_{\mathbf{n}\leftarrow\mathbf{n}_0}AA_{A^{-1}\mathbf{n}\leftarrow\mathbf{n}_0})Q_1(\mathbf{q},\mathbf{n}_0,A_{A^{-1}\mathbf{n}\leftarrow\mathbf{n}_0})$$

The irreducible representations of the little group of the little group have either $\sigma_0 > 0$, $d\sigma_1 = 0$ or $\sigma_0 = 0$, $d\sigma_1(|\mathbf{n}|) = \delta(|\mathbf{n}| - \alpha)d|\mathbf{n}|$, for some $\alpha > 0$. The corresponding Q_1 are one dimensional.

In the case $\sigma_0 > 0$, $Q_1(\mathbf{q},0,A)$ is just $Q(0,\mathbf{q},\mathbf{v},A)$ for $\mathbf{v}\cdot\mathbf{q} = 0$ $A\mathbf{q} = \mathbf{q}$, so the system will be localizable if $A \to Q(0,\mathbf{q},0,A)$ for $A\mathbf{q} = \mathbf{q}$ defines a representation which is a restriction of a representation of the full unimodular group. In the case $|\mathbf{n}| \neq 0$, $Q_1(q,\mathbf{n}, \pm 1) = +1$ yields a $Q(0,\mathbf{q},0,A)$, $A\mathbf{q} = \mathbf{q}$ which contains each integer angular momentum along $\mathbf{q}$ just once, so it is not localizable. A necessary condition for localizability is that the representation $Q_1(q,\mathbf{n}, \pm 1) = +1$ have zero or infinite multiplicity. The irreducible representation $Q_1(q,\mathbf{n}, \pm 1) = \pm 1$ yields a $Q(0,\mathbf{q},0,A)$, $A\mathbf{q} = \mathbf{q}$ which contains each half-odd integer angular momentum along $\mathbf{q}$ just once so it is not localizable. A necessary condition for localizability is that $Q_1(q,\mathbf{n}, \pm 1) = \pm 1$ appear with zero or infinite multiplicity.

All this is summarized in Theorem 7.

Theorem 7. Every Galilei invariant system with $M > 0$ is localizable.

For $M = 0$, no elementary system is localizable because such a system has momentum satisfying $|\mathbf{p}| = \text{const.}$ Systems with $M = 0$ and a reducible representation of the Galilei group are localizable if and only if:

(a). The measure on momentum space is equivalent to Lebesgue measure;

(b). The subrepresentation of the little group of $(0,\mathbf{q})$ for which the pure Galilei transformations $\boldsymbol{\Gamma} = (\mathbf{v},1)$ are trivially represented is, for almost all $|\mathbf{q}|$, the restriction to the group of A such that $A\mathbf{q} = \mathbf{q}$ of a fixed representation of the 2×2 unitary unimodular group;

(c). The subrepresentation of the little group of $(0,q)$ for which the pure Galilei transformations are non-trivially represented contains each irreducible with multiplicity zero or infinity, the same for almost all $|\mathbf{q}|$.

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APPENDIX I. FINITE ADDITIVITY ON FINITE UNIONS OF CUBES

In connection with axioms I, II, III, it was remarked that it might appear more natural from a physical point of view to weaken the axioms so that the existence of the observable $E(S)$ is required only for S a finite union of cubes, and finite additivity is required:

$$E(S_1 \cup S_2) = E(S_1) + E(S_2) \quad \text{if} \quad S_1 \cap S_2 = 0$$

instead of the complete or σ-additivity described in III.

In this Appendix, it is shown that such a weakening of the axioms is only apparent because any $E(S)$ satisfying the weakened axioms can be extended uniquely so as to satisfy I, II, III, as they stand.

Consider the family $\mathfrak{A}$ of all sets of $\mathbf{R}^3$ which are finite unions of half-open intervals. By a half-open interval is meant a set $[\mathbf{a},\mathbf{b})$ of the form

$$\{\mathbf{y};\ \ a_1 \leqslant y_1 < b_1,\ \ a_2 \leqslant y_2 < b_2,\ \ a_3 \leqslant y_3 < b_3\} ,$$

that is, the set of all y satisfying the listed inequalities. By assumption, the cases $a_j = -\infty$ and $b_j = +\infty$ are also included; in the former case, the equality sign in $a_j \leqslant y_j$ should be ignored. $\mathfrak{A}$ is referred to as an algebra of sets because it is closed under the operations of taking the complement of a set and taking the union of a finite number of sets. A *σ algebra* of sets is one closed under complementation and denumerable unions. A *projection-valued finitely-additive measure* on $\mathbf{A}$ is a function, E, with values which are projections in a Hilbert space $\mathcal{H}$, defined for all sets of $\mathbf{A}$ and satisfying II, and

$$\text{III}' \quad E(S_1 \cup S_2) = E(S_1) + E(S_2) - E(S_1 \cap S_2)$$

for any $S_1, S_2 \in \mathbf{A}$.

A projection-valued finitely additive measure that satisfies in addition

$$E(\bigcup S_i) = \sum_{i=1}^{\infty} E(S_i)$$

for any sequence of $S_i \in \mathbf{A}, i = 1,2,\cdots$ such that $S_i \cap S_j = 0$, $i \neq j$ and $\bigcup S_i \in \mathbf{A}$ is called *completely additive or σ additive*. The precise statement of the result of this Appendix is

Theorem A5. Any finitely-additive projection-valued measure on $\mathbf{A}$ which satisfies

$$E(S + \mathbf{a}) = U(\mathbf{a})E(S)U(\mathbf{a})^{-1}$$

for some continuous unitary representation of the translation group $\mathbf{a} \to U(\mathbf{a})$ is necessarily completely additive on $\mathbf{A}$. It then possesses a unique completely additive extension to the σ algebra of all Borel sets on $\mathbf{R}^3$.

Variants of the last statement of the theorem are quite standard in various contexts in measure theory, so it will not be proved here. (In Halmos' book, reference 15, p. 54, the theorem is stated: "If μ is a σ finite measure on a ring $\mathbf{R}$, then there is a unique measure $\bar{\mu}$ on the σ ring, $S(R)$, generated by $\mathbf{R}$ such that for E in $\mathbf{R}$, $\bar{\mu}\ (E) = \mu(E)$; the measure $\bar{\mu}$ is σ finite." The assumptions of the present Appendix are more general in that one has a projection-valued measure rather than a real-valued measure, but otherwise everything is more special: The ring of sets, $\mathbf{R}$, is an algebra because the whole space is in $\mathbf{R}$, the measure is finite rather than only σ finite.) The first part of the theorem is a consequence of the following chain of four theorems. The argument is a straightforward generalization of one due to Hewitt.[25]

If F is any function on $\mathbf{R}^3$ whose values can be added and subtracted and $[\mathbf{a},\mathbf{b})$ is an interval, define

$$\begin{aligned}\Delta_F[\mathbf{a},\mathbf{b}) = F(b_1,b_2,b_3) - F(a_1,b_2,b_3) - F(b_1,a_2,b_3) \\ - F(b_1,b_2,a_3) + F(a_1,a_2,b_3) + F(a_1,b_2,a_3) \\ + F(b_1,a_2,a_3) - F(a_1,a_2,a_3)\ .\end{aligned}$$

[25] E. Hewitt, Mat. Tidsskrift (1951B), pp. 81–94.

If F is real valued it is said to be *positively monotonic* if $\Delta_F[\mathbf{a},\mathbf{b}) \geqslant 0$ for all $[\mathbf{a},\mathbf{b})$.[26] If the values of F are commuting projections the analogous requirement is that $\Delta_F[\mathbf{a},\mathbf{b})$ be a projection for all intervals $[\mathbf{a},\mathbf{b})$. Notice that if $E(S)$ is any finitely additive projection valued measure defined on $\mathbf{A}$, it yields such an F from the definition

$$F(x_1,x_2,x_3) = E(\{\mathbf{y};y_1 < x_1,y_2 < x_2,y_3 < x_3\})\ . \quad \text{(A1)}$$

Conversely, the following theorem holds.

Theorem A1. Let F be a positively monotonic function defined on $\mathbf{R}^3$ with values which are commutative projections. Suppose

$$\begin{aligned}F(-\infty,x_2,x_3) = F(x_1,-\infty,x_3) \\ = F(x_1,x_2,-\infty) = 0\ . \quad \text{(A2)}\end{aligned}$$

Then there exists a finitely additive projection valued measure E on $\mathbf{A}$ satisfying (A1).

The proof is completely elementary and will be omitted.

Now consider the increasing sequence of projections

$$F(x_1 - 1/k,\cdots x_3 - 1/k) \quad k = 1,2,\cdots\ .$$

It converges to a projection $F_-(x_1,\cdots x_3)$ which may or may not be $F(x_1,\cdots x_3)$.

Example. Consider the function $E_{\mathbf{t}}$ defined on $\mathbf{A}$ which is the projection $E \neq 0$ for a set S if there is an interval of the form $\{\mathbf{y};t_1 - \epsilon \leqslant y_1 < t_1,t_2 - \epsilon \leqslant y_2 < t_2,t_3 - \epsilon \leqslant y_3 < t_3\}$ which lies in S and zero otherwise. It is easy to see that $E_{\mathbf{t}}$ is a finitely additive projection valued measure on $\mathbf{A}$. It is not completely additive because the interval $\{\mathbf{y};t_1 - 1 \leqslant y_1 < t_1,t_2 - 1 \leqslant y_2 < t_2,t_3 - 1 \leqslant y_3 < t_3\}$ can be written as a denumerable union of intervals for which the coordinates y_j lie in intervals where right-hand end points are less than t_j. For each such interval $E_{\mathbf{t}}(S) = 0$ but for the union $E_{\mathbf{t}}(S) = E$. Clearly, the F corresponding to $E_{\mathbf{t}}$ does not satisfy $F(t_1\cdots t_3) = F_-(t_1\cdots t_3)$.

If for each $\mathbf{x} \in \mathbf{R}^3$, $F(\mathbf{x}) = F_-(\mathbf{x})$, then the phenomenon occurring in the example cannot happen and the projection valued measure defined by F is σ additive on $\mathbf{A}$.

Theorem A2. Every projection valued positively monotonic function F on $\mathbf{R}^3$ which satisfies (A2) and

$$\lim_{k\to\infty} F(x_1 - 1/k,\cdots x_3 - 1/k) = F(x_1,\cdots x_3)\ . \quad \text{(A3)}$$

[26] A detailed discussion of positively monotonic functions is given in, E. J. McShane, *Integration* (Princeton University Press, Princeton, New Jersey, 1947), pp. 242–274.

defines a projection valued measure E on $\mathbf{\mathfrak{A}}$ which is σ additive.

Proof. Since each element of $\mathbf{\mathfrak{A}}$ is a finite union of disjoint intervals and E is finitely additive according to Theorem A2, it suffices to consider the case of a denumerable union of sets in $\mathbf{\mathfrak{A}}$ whose union is an interval. But such a union defines a monotonically increasing sequence of projections which converges to the projection belonging to the interval by virtue of (A3). Therefore E is completely additive.

A finitely-additive projection-valued measure E is called *purely finitely additive* if there is no nontrivial σ additive projection-valued measure which is zero on every set S for which $E(S) = 0$. (It is not difficult to see that the example $E_{\mathbf{t}}$ is purely finitely additive.)

Theorem A3. Every finitely additive projection valued measure on $\mathbf{\mathfrak{A}}$ is the sum of a purely finitely additive part and a σ additive part. This decomposition is unique.

Proof. The difference $F(\mathbf{x}) - F_{-}(\mathbf{x})$ is a projection, and two such, corresponding to distinct points $\mathbf{x}$ are orthogonal. Because the Hilbert space is separable, there can be at most a denumerable set of points $\mathbf{x}$ where $F(\mathbf{x}) - F_{-}(\mathbf{x}) \neq 0$; call them $\mathbf{t}^{(k)}$. Let $E_{\mathbf{t}^{(k)}}(S)$ be the projection-valued measure given in the example above with $E = F(\mathbf{t}^{(k)}) - F_{-}(\mathbf{t}^{(k)})$. Then

$$E(S) - \textstyle\sum_k E_{\mathbf{t}^{(k)}}(S)$$

defines a finitely-additive projection-valued measure whose F satisfies (A3) for all points $\mathbf{x}$ and so by Theorem A2 is σ additive. Thus

$$E(S) = \textstyle\sum_k E_{\mathbf{t}^{(k)}}(S) + E^{(2)}(S)$$

defines a decomposition into a purely finitely additive part and a σ additive part. For the case in which $E(S)$ is purely finitely additive, $E^{(2)}(S) = 0$ because otherwise $E^{(2)}(S)$ would be a σ additive projection-valued measure vanishing whenever $E(S)$ does in contradiction with the definition of a purely finitely-additive measure. This shows that the purely finitely-additive part of any E is uniquely determined by the discontinuities of the corresponding F.

Now note that if $E(S)$ is quasi-invariant under translations in the sense that $E(S + \mathbf{a}) = 0$ if and only if $E(S) = 0$, then the same applies to the purely finitely-additive part, $E^{(1)}(S)$, of $E(S)$ and the σ additive part of S. [$E(S)$ is surely quasi-invariant if there exists a representation $\mathbf{a} \to U(\mathbf{a})$ of the translation group such that $E(S + \mathbf{a}) = U(\mathbf{a})E(S)U(\mathbf{a})^{-1}$] Furthermore, if $F^{(1)}$ has a nonzero discontinuity $F^{(1)}(\mathbf{x}) - F_{-}^{(1)}(\mathbf{x})$ at $\mathbf{x} = \mathbf{t}$, it must also have a nonzero discontinuity at $\mathbf{x} = \mathbf{t} + \mathbf{a}$. This statement is in conflict with the denumerability of the points of discontinuity unless $F^{(1)} = 0$. Thus, there are no nontrivial purely finitely-additive projection valued measures quasi-invariant under translations.

Theorem A4. Every finitely additive projection valued measure on $\mathbf{\mathfrak{A}}$ which is quasi-invariant under translations is σ additive.

From Theorem A4 and the result already cited that σ additive projection-valued measures on $\mathbf{\mathfrak{A}}$ have unique extensions to the Borel sets of $\mathbf{R}^3$, Theorem A5 follows.

While the results of this Appendix make it clear that the assumptions of I to V can be weakened without impairing the results of the paper, it should be noted that the particular weakened assumptions used have been chosen primarily for reasons of mathematical elegance. A deeper physical analysis would ask whether the existence of some kind of approximate position measurement implied the existence of precise position measurements in the sense of I to V.

APPENDIX II. SKETCH OF THE DERIVATION OF THE CONTINUOUS UNITARY REPRESENTATIONS OF THE TRANSLATION GROUP

The result of Theorem 1 which describes all unitary representations of the translation groups has been used in physics since the beginning of quantum mechanics, but explicit mathematical statements and proofs of it are relatively recent. The purpose of this Appendix is to outline some of the ideas involved in the proofs.

The translation group of n-dimensional real Euclidean space $\mathbf{R}^n$ will here be denoted $\mathfrak{T}$ with elements $\mathbf{a}$. (The whole machinery works in the same way for any dimension n so the assumption $n = 3$ is dropped.) The derivation of Theorem 1 can be divided into three parts:

(1) Determination of the character group $\mathfrak{T}^*$ of $\mathfrak{T}$,

(2) Derivation of the spectral representation

$$U(\mathbf{a}) = \int_{\mathfrak{T}^*} e^{-i\mathbf{p}\cdot\mathbf{a}} dF(\mathbf{p}) \,,$$

(3) Spectral multiplicity theory for the projection valued measure F on $\mathfrak{T}^*$.

These stages actually reflect the historical development of the theorem and I will follow them here at least in part.

A *character* of $\mathfrak{T}$ is a one-dimensional continuous unitary representation of $\mathfrak{T}$, i.e., a complex-valued continuous function χ of modulus one, which satisfies

$$\chi(\mathbf{a} + \mathbf{b}) = \chi(\mathbf{a})\chi(\mathbf{b}) \,. \qquad \text{(A4)}$$

It is well known that any such χ is of the form $\chi_{\mathbf{p}}$, where

$$\chi_{\mathbf{p}}(\mathbf{a}) = \exp -i(\mathbf{p}\cdot\mathbf{a}) \quad \text{and} \quad \mathbf{p}\cdot\mathbf{a} = \sum_{j=1}^{n} p^j a^j .$$

[The argument goes as follows. From (A4), $\chi(0) = 1$ and $\chi(\mathbf{a})$ can be written

$$\chi(\mathbf{a}) = \chi(a^1,0,\cdots 0)\chi(0,a^2,0,\cdots 0)\cdots\chi(0,0,\cdots a^n) ,$$

where $\chi(0\cdots \mathbf{a}^j\cdots)$ is a character of the one-dimensional translation group of a^j. Thus the problem is reduced to finding all characters for the translation group of the real line. By introducing $i \ln \chi = f$ one reduces the problem to that of finding all real continuous $f(a)$ defined mod 2π such that

$$f(a) + f(b) = f(a + b) \bmod 2\pi \qquad \text{(A.5)}$$

To complete the proof it is convenient to specify $f(a)$ completely instead of mod 2π. Because χ is continuous, a unique specification is obtained in some neighborhood of $a = 0$ by requiring $f(0) = 0$ and $f(a)$ continuous in the neighborhood. From (A5), one then derives $qf(q^{-1}c) = f(c)$ for any c in the neighborhood and any integer q. Thus, again using (A5), $f((p/q)c) = (p/q)f(c)$ for any rational number $p/q < 1$. The continuity of f then implies $f(yc) = yf(c)$ for every real number <1, i.e., $f(y) = yf(c)/c$ for y in the neighborhood. Finally, using (A5) again, one gets $f(y) = yf(c)/c \bmod 2\pi$ for all y. Q.E.D.]

The characters clearly form a group under multiplication

$$\chi_{\mathbf{p}_1}(\mathbf{a})\chi_{\mathbf{p}_2}(\mathbf{a}) = \chi_{\mathbf{p}_1+\mathbf{p}_2}(\mathbf{a})$$

and, if the usual topology of Euclidean space is introduced for the **p**'s, the group operations are continuous. The set of all characters (or equivalently the set of all **p**'s) is denoted $\mathfrak{T}^*$ and called the character group of $\mathfrak{T}$[27].

The step (2) alone can be regarded as a decomposition of an arbitrary continuous unitary representation into irreducibles. This operation is familiar in quantum mechanics for the one dimensional translation group as Stone's theorem: Any one-parameter continuous unitary group is of the form

$$U(a) = \exp -iaH , \qquad \text{(A6)}$$

where H is self-adjoint. Then by the spectral theorem for self-adjoint operators $H = \int_{-\infty}^{\infty} p dF(p)$ so (A6) can be written

$$U(a) = \int e^{-ipa} dF(p) . \qquad \text{(A7)}$$

Here F defines a projection valued measure via $F(S) = \int_S dF(p)$. The extension of (A7) to arbitrary Abelian groups was carried out by a number of authors.[28] Since the step from the one-dimensional to n-dimensional translation group is easy, and excellent textbook accounts of Stone's theorem are available,[29] no more details of (2) will be given here.

The problem of determining when two representations are unitary equivalent is reduced by the SNAG theorem to the corresponding problem for their F's. A solution of this problem is provided by (3), the theory of spectral multiplicity. It shows that the unitary equivalence class of an F can be characterized by two objects, a measure class on $\mathfrak{T}^*$ and a multiplicity function on $\mathfrak{T}^*$, which described, respectively (and roughly), tell which irreducible representations of $\mathfrak{T}$ occur in $\mathbf{a} \to U(\mathbf{a})$ and how often. This theory is to the theory of (2) what the Hellinger-Hahn theory of a self-adjoint operator[30] is to the spectral resolution of a self-adjoint operator.

There are available nearly as many approaches to the theory of spectral multiplicity as there are authors who have written on the subject. One may make a direct analysis of the commutative algebra of projections.[31] This leads to a decomposition of the Hilbert space into orthogonal subspaces $\mathcal{H}_j$ on which the projections are uniformly j-dimensional. That means that $\mathcal{H}_j$ is a direct sum of j subspaces $\mathcal{H}_1^j \cdots \mathcal{H}_j^j$ such that the projections E take of the form

$$E(\Phi_1,\cdots \Phi_j) = (E_1\Phi_1,\cdots E_j\Phi_j)$$

and on $\mathcal{H}_k^j$ the E_k are uniformly one dimensional. Finally, a uniformly one-dimensional algebra of projections is one which is maximal Abelian, i.e., any projection which commutes with all the given projections is one of them. It is shown that a uni-

[27] This construction of the character group can be carried out for an arbitrary locally compact Abelian group. See, for example, L. Pontrjagin, *Topological Groups* (Princeton University Press, Princeton, New Jersey, 1939), Chap. V.

[28] Stone's original paper is Ann. Math. **33**, 643 (1932). The extension to any locally compact Abelian group is contained in M. Naĭmark, Izvest. Akad. Nauk U.S.S.R. **7**, 237 (1943); W. Ambrose, Duke Math. J. **11**, 589 (1944); R. Godement, Compt. rend. **218**, 901 (1944). It is sometimes referred to as the SNAG theorem.

[29] See for example F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionelle* (Budapest, 1953), p. 377.

[30] See M. H. Stone, *Linear Transformations in Hilbert Space* (American Mathematical Society, Providence, Rhode Island, 1932), Chap. VII.

[31] See, for example, H. Nakano, Ann. Math. **42**, 657 (1941); I. E. Segal, Memoirs Am. Math. Soc. **9** (1951), Secs. I and II; P. R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity* (Chelsea Publishing Company, New York, 1951).

formly one-dimensional algebra of projections is unitary equivalent to one in which the projections have the form

$$(E(S)\Phi)(x) = \chi_S(x)\Phi(x)\ ,$$

where $\Phi(x)$ are complex-valued functions square integrable with respect to a measure μ. Reassembling the Hilbert space, one just gets a form of the projection operators just like that indicated for the projection operators on momentum space given in Theorem 1.

Alternatively, one can imbed the projection operators in an appropriately chosen commutative algebra of bounded operators and then use the spectral theory of such commutative algebras to obtain the required canonical form.[32]

APPENDIX III. QUASI-INVARIANT MEASURES

In this Appendix, the structure of quasi-invariant measures defined on the Borel sets of $\mathbf{R}^3$ is determined for two different situations. In the first, the group acting on $\mathbf{R}^3$ is $\mathbf{R}^3$ itself. Then, every finite quasi-invariant measure is equivalent to Lebesgue measure. In the second, the group acting on $\mathbf{R}^3$ is the rotation group. Then the most general finite quasi-invariant measure is equivalent to a measure of the form

$$\mu(S) = \mu_0\chi_S(0) + \int_0^\infty d\rho(a) \int_{S\cap\{\mathbf{p};|\mathbf{p}|=a\}} d\omega_a(\mathbf{p})\ , \quad \text{(A8)}$$

where $\mu_0 \geqslant 0, \chi_S$ is the characteristic function of the set S, $d\omega_a(\mathbf{p})$ is the invariant surface element on the sphere $|\mathbf{p}| = a$, and $d\rho(a)$ is a measure on the positive real axis.

The result for the first situation is a special case of the general result that any Borel measure on a locally compact group quasi-invariant with respect to the action of the group on itself is equivalent to Haar measure.[33] The proof of Loomis given in[33] is so simple that it will be repeated here in the special context of $\mathbf{R}^3$.

Let S be any Borel set in $\mathbf{R}^3$. Denote the finite measure quasi-invariant with respect to Lebesgue measure by μ. Let S^* be the set in $\mathbf{R}^6$ defined by $\mathbf{x} - \mathbf{y} \in S$. (It is a Borel set because $\mathbf{x} - \mathbf{y}$ is a continuous function of $\mathbf{x}$ and $\mathbf{y}$. Then the characteristic functions of S^* and S are related by

$$\chi_{S^*}(\mathbf{x},\mathbf{y}) = \chi_S(\mathbf{x} - \mathbf{y})$$

and so because χ_S is positive

$$\iint \chi_{S^*}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mu(\mathbf{y}) = \int\left(\int d\mathbf{x}\chi_S(\mathbf{x}-\mathbf{y})\right)d\mu(\mathbf{y})$$

$$= \int\left(\int d\mu(\mathbf{y})\chi_S(\mathbf{x}-\mathbf{y})\right)d\mathbf{x}\ .$$

Because $d\mathbf{x}$ is invariant under translation the second expression is $(\int_S d\mathbf{x})\ \int d\mu(\mathbf{y})$. On the other hand, $\chi_S(\mathbf{x} - \mathbf{y}) = \chi_{-S+\mathbf{x}}(\mathbf{y})$ so the last equality becomes

$$\left(\int_S d\mathbf{x}\right)\int d\mu(\mathbf{y}) = \int \mu(-S+\mathbf{x})dx\ . \quad \text{(A9)}$$

From this equality, the required equivalence can be deduced as follows. Note that $\int_S d\mathbf{x} = 0$ if and only if $\int_{-S} d\mathbf{x} = 0$. Thus, from (A9), $\int_S d\mathbf{x} = 0$ implies $\mu(S + \mathbf{x}) = 0$ for almost all $\mathbf{x}$. By the quasi-invariance of μ, this, in turn, implies $\mu(S) = 0$. Conversely, if $\mu(S) = 0$ and therefore $\mu(S + \mathbf{x}) = 0$, (A9) implies $\int_S d\mathbf{x} = 0$.

This completes the proof of the equivalence of μ with Lebesgue measure. The Radon-Nikodym theorem guarantees that $d\mu(\mathbf{x}) = \rho(\mathbf{x})d\mathbf{x}$ where $\rho(\mathbf{x})$ is positive and measurable.

In the second situation, one has a finite measure μ on the Borel sets of $\mathbf{R}^3$ such that for every Borel set S and every rotation R, $\mu(S) = 0$ if and only if $\mu(RS) = 0$.

It is easy to see that any such quasi-invariant measure is equivalent to an invariant measure. In fact, consider the non-negative set function

$$\tilde{\mu}(S) = \int dR\mu(RS)\ ,$$

where the integration is over all the rotation group and dR is the invariant measure on the rotation group for which $\int dR = 1$. It is not difficult to verify that $\tilde{\mu}$ is σ additive. Furthermore, it is equivalent to μ, because $\tilde{\mu}(S) = 0$ implies $\mu(RS) = 0$ for almost all R, which, because of the quasi-invariance of μ, yields $\mu(S) = 0$. Conversely, $\mu(S) = 0$ implies $\mu(RS) = 0$, which implies $\tilde{\mu}(S) = 0$. Thus, it suffices to consider invariant μ.

It is convenient in completing the proof to use an alternative characterization of a finite measure on $\mathbf{R}^3$ as a non-negative bounded linear functional on the continuous functions of compact support, $\mathcal{C}(\mathbf{R}^3)$. That the functional, μ, is non-negative means $\mu(f)$

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[32] R. Godement, Ann. Math. **53**, 68 (1951); J. Dixmier, "Les Algebres d'Operateurs dans l'Espace Hilbertien," *Algebres de von Neumann* (Gauthier-Villars, Paris, 1957), Chap. II; see especially pp. 216–224.

[33] See, for example, G. W. Mackey, Duke Math. J. **16**, 313 (1949), Lemma **33**; J. von Neumann, Bull. Amer. Math. Soc. **42**, 343 (1936).

$\geqslant 0$ for $f \geqslant 0$, $f \in \mathcal{C}$. That μ is bounded means

$$\sup_{|f| \leqq 1} |\mu(f)| < \infty , \quad \text{where} \quad |f| = \sup_{\mathbf{x} \in \mathbf{R}^3} |f(\mathbf{x})| .$$

The relation between the functional μ and the corresponding measure μ is simply

$$\mu(f) = \int f(\mathbf{x}) d\mu(\mathbf{x}) .$$

Since the measure is uniquely determined by the functional, to verify the equality of two measures it suffices to verify the equality of the corresponding functionals.[34]

Now, for an invariant measure

$$\mu(f) = \int dR\, \mu(Rf) = \mu\left(\int (Rf) dR\right),$$

because the approximating sums to the integral $\int (Rf)(\mathbf{x})dR$ converge uniformly in $\mathbf{x}$, and $\mu(f)$ is continuous for uniform convergence of its argument. But $f(\mathbf{x}) \rightarrow \int (Rf)(\mathbf{x})dR = \int f(R\mathbf{x})dR$ maps the continuous functions of compact support on $\mathbf{R}^3$ onto the continuous functions of compact support on $0 \leqslant |\mathbf{x}| < \infty$ and convergence in $\mathcal{C}(\mathbf{R}^3)$ implies convergence in $\mathcal{C}([0,\infty))$. Thus, the functional μ regarded as defined on $\mathcal{C}([0,\infty))$ defines a finite measure on the non-negative real axis. Splitting it into a contribution with support at 0, and the rest, one has just the μ_0 and $d\rho$ of (A8). In fact, (A8) is just an explicit form in terms of measure of

$$\mu(f) = \mu\left(\int Rf dR\right).$$

APPENDIX IV. SOME MEASURE-THEORETIC NICETIES CONNECTED WITH EQS. (3.10) AND (3.17)

This Appendix is devoted to some fine points which arise in the otherwise elementary derivation of Sec. 3.

Recall that Theorem 1 states that if

$$[Q(A), U(\mathbf{a},1)] = 0 ,$$

then $Q(A)$ is of the form

$$(Q(A)\Phi)(\mathbf{p}) = Q(\mathbf{p},A)\Phi(\mathbf{p}) ,$$

where for each unitary unimodular A, $Q(\mathbf{p},A)$ is measurable in $\mathbf{p}$ in the sense that for each $\Psi_1,\Psi_2 \in \mathcal{H}$, $(\Psi_1(\mathbf{p}), Q(\mathbf{p},A)\Psi_2(\mathbf{p}))$ is μ measurable. The first step in the argument is to prove that $Q(\mathbf{p},A)$ can, if necessary, be altered on a set of μ-measure zero so that it becomes measurable in both variables relative to the measure $\mu \times \alpha$, where α is the invariant measure on the 2×2 unitary unimodular group.

Let $\Phi_j, j = 1,2,\cdots$ be a complete orthonormal set in $\mathcal{H}$. Then it suffices to treat the functions $(\Phi_j(\mathbf{p}), Q(\mathbf{p},A)\Phi_k(\mathbf{p}))$ separately because the general case then follows by the expansions

$$\Psi_1 = \sum a_j\Phi_j , \quad \Psi_2 = \sum b_j\Phi_j , \quad \text{and}$$
$$(\Psi_1(\mathbf{p}),Q(\mathbf{p},A)\Psi_2(\mathbf{p})) = \sum_{j,k=1} a_j^* b_k \times (\Phi_j(\mathbf{p}),Q(\mathbf{p},A)\Phi_k(\mathbf{p})) .$$

An ugly little lemma is necessary.

Lemma. Let $f(\mathbf{p},A)$ be a complex-valued function on $\mathbf{R}^3 \times G$ which is μ measurable and μ essentially bounded on $\mathbf{R}^3$ for each $A \in G$, the 2×2 unitary unimodular group. Suppose $\int f(\mathbf{p},A)\chi_E(\mathbf{p})d\mu(\mathbf{p})$ is α measurable on G for each μ measurable subset E of $\mathbf{R}^3$ of finite measure. Here, α-measurability on G is with respect to the invariant measure dA.

Then there exists a function, g, $\mu \times \alpha$ measurable on $\mathbf{R}^3 \times G$, and such that for a certain μ-measurable subset N of $\mathbf{R}^3$ of zero measure

$$f(\mathbf{p},A) = g(\mathbf{p},A) \quad \text{for all} \quad A \in G \quad \text{and} \quad \mathbf{p} \notin N .$$

This lemma is a special case of Lemma 3.1 of reference 33, and will not be proved here.

The lemma shows that by a suitable redefinition of $Q(\mathbf{p},A)$ which does not affect the corresponding operator $Q(A)$, one can have $Q(\mathbf{p},A),\mu \times A$ measurable.

The next step in the argument is to show that in the equation

$$\sum\nolimits_l (\Phi_j(\mathbf{p}),Q(\mathbf{p},A)\Phi_l(\mathbf{p}))(\Phi_l(\mathbf{p}),Q(A^{-1}\mathbf{p},B)\Phi_k(\mathbf{p})) = (\Phi_j(\mathbf{p}),Q(\mathbf{p},AB)\Phi_k(\mathbf{p})) \tag{A10}$$

which holds for each $A,B \in G$ and $\mathbf{p} \in \mathbf{R}^3$ such that $\mathbf{p} \notin N_1(A,B), A^{-1}\mathbf{p} \notin N_2(A,B)$ where N_1 and N_2 are μ-measurable sets of μ-measure zero, the right- and left-hand sides are $\mu \times \alpha \times \alpha$ measurable on $\mathbf{R}^3 \times G \times G$. Because a Borel-measurable function of a Borel-measurable function is Borel measurable, it suffices to prove that the mappings $T_1:\{\mathbf{p},A,B\} \rightarrow \{\mathbf{p},AB\}$ and $T_2:\{\mathbf{p},A,B\} \rightarrow \{A^{-1}\mathbf{p},B\}$ are Borel-measurable functions.

Now T_1 and T_2 are continuous, and a set F which is $\mu \times \alpha$ measurable in $\mathbf{R}^3 \times G$ differs from a Borel set by a subset of a Borel set of zero $\mu \times \alpha$ measure.[35]

Furthermore, a continuous function has the property that the antecedent of any Borel set of its range

[34] See, for example, P. R. Halmos, *Measure Theory* (D. van Nostrand Company, Inc., Princeton, New Jersey, 1950), pp. 243–9.

[35] See reference 15, pp. 55–56.

is a Borel set of its domain. Thus to prove the $\mu \times \alpha \times \alpha$ measurability of T_1 and T_2, it suffices to show that for any Borel set F of $\mathbf{R}^3 \times G$ of zero $\mu \times \alpha$ measure $T_1^{-1}(F)$ and $T_2^{-1}(F)$ have zero $\mu \times \alpha \times \alpha$ measure. Consider T_2, the proof for T_1 being similar.

Let $F_{\mathbf{p}}$ denote $\{A;\{\mathbf{p},A\} \in F\}$. Clearly, $\{\mathbf{p},AB\} \in F$ if and only if $AB \in F_{\mathbf{p}}$ (or $A \in F_{\mathbf{p}}B^{-1}$) for some $\mathbf{p}$. Now

$$\mu \times \alpha(F) = 0 = \int d\mu(\mathbf{p})dA\chi_{F\mathbf{p}}(A) = \int d\mu(\mathbf{p})\alpha(F_{\mathbf{p}})$$

This implies $\alpha(F_{\mathbf{p}}) = 0$ for μ almost all $\mathbf{p}$. By the invariance of α, $\alpha(F_{\mathbf{p}}B^{-1}) = 0$ for each B and μ almost all $\mathbf{p}$. Because $\{\mathbf{p},A,B\} \in T_2^{-1}(F)$ if and only if $A \in F_{\mathbf{p}}B^{-1}$ for some $\mathbf{p}$ and B,

$$(\mu \times \alpha \times \alpha)(T_2^{-1}F) = \int d(\mu \times \alpha \times \alpha)(\mathbf{p},A,B) \times \chi_{T_2^{-1}(F)}(\mathbf{p},A,B)$$
$$= \int d(\mu \times \alpha)(\mathbf{p},B)\alpha(F_{\mathbf{p}}B^{-1}) = 0 .$$

The preceding argument shows that (A10) is a relation between $\mu \times \alpha \times \alpha$ measurable functions, which, for fixed A,B can fail to hold only on a set of $\mathbf{p}$'s of μ-measure zero. However, the union of these null sets as A and B run over G could, *a priori*, be a set of measure greater than zero. That this is, in fact, not the case is seen as follows. Since the set of $\{\mathbf{p},A,B\}$ where (A10) fails is of $(\mu \times \alpha \times \alpha)$-measure zero its section for $\mathbf{p}$ and B fixed is of α measure zero. But, as A runs over a set of α measure zero, $A^{-1}\mathbf{p}$ runs over a set of μ measure zero. [Here the equivalence of μ to a measure of the form (A8) is being used.] Thus, the set of $A^{-1}\mathbf{p}$ where

$$Q(A^{-1}\mathbf{p},B) = Q(\mathbf{p},A)^{-1}Q(\mathbf{p},AB) \qquad \text{(A11)}$$

fails, $\mathbf{p}$ and B being fixed, is a set of μ measure zero. By redefining $Q(A^{-1}\mathbf{p},B)$ at those $A^{-1}\mathbf{p}$ by (A11), one obtains a new family of $Q(\mathbf{q},B)$ measurable in $\{\mathbf{q},B\}$, which still yield the old $Q(B)$ but for which (A11) [or equivalently (A10)] is always valid. This completes the justification of the statement just after Eq. (3.10).

A second measure theoretic point arises in connection with Eq. (3.14). Using the $Q(\mathbf{p},A)$ whose existence has just been established, one gets a measurable but, *a priori*, not necessarily continuous unitary representation of the little group. In fact, every measurable unitary representation of any locally compact group G is continuous, as will now be shown by a well-known argument which has not yet crept into the text books.

Because

$$||(U(x) - U(y))\Phi||^2 = ||(U(y^{-1}x) - 1)\Phi||^2$$
$$= 2(\Phi,\Phi) - 2 \operatorname{Re} (U(y^{-1}x)\Phi,\Phi)$$

the strong continuity of $U(x)$, i.e., the requirement that for each $\Phi \in \mathcal{H}$ and $y \in G$ the first of these expressions be small when x is close to y, is implied by the weak continuity of $U(x)$ at the identity, i.e., the requirement that for each $\Phi,\Psi \in \mathcal{H}$, $(\Phi,U(x)\Psi)$ is close to (Φ,Ψ) for x close to the identity. The continuity of $(\Phi,U(x)\Psi)$ at the identity for all Φ,Ψ is implied by the continuity of $(\chi,U(x)\chi)$ for all χ as one sees by considering $\chi = \Phi + \Psi, \Phi + i\Psi, \Phi, \Psi$ in turn, and taking appropriate linear combinations. Because $U(x)$ is unitary, it suffices to verify the weak continuity for the elements of any dense set of vectors in $\mathcal{H}$, say Φ_i. To see this one can look at the identity

$$(\Phi,(U(x) - 1)\Phi) = (\Phi - \Phi_i,(U(x) - 1)\Phi)$$
$$+ (\Phi_i,(U(x) - 1)(\Phi - \Phi_i))$$
$$+ (\Phi_i,(U(x) - 1)\Phi_i) ,$$

which yields the estimate

$$(\Phi,(U(x) - 1)\Phi) \leq 2||\Phi|| \, ||\Phi - \Phi_i||$$
$$+ 2||\Phi_i|| \, ||\Phi - \Phi_i|| + |(\Phi_i,(U(x) - 1)\Phi_i)| .$$

The first two terms on the right-hand side can be made small by appropriate choice of $\Phi_i \cdot \Phi_i$ having been chosen, the last term can be made small by an appropriate choice of x according to the assumed continuity of $(\Phi_i,U(x)\Phi_i)$.

Since $U(x)$ is measurable and unitary $(\Phi,U(x)\Phi)$ is a bounded measurable function for each Φ. Thus, for each continuous function of compact support, φ, it makes sense to talk about

$$(\Phi,U(\varphi)\Phi) = \int \varphi(x)dx(\Phi,U(x)\Phi)$$

and if φ_y is defined by $\varphi_y(x) = \varphi(y^{-1}x)$,

$$|(\Phi,(U(\varphi_y) - U(\varphi))\Phi)| \leqslant \int |\varphi_y(x) - \varphi(x)|dx||\Phi||^2. \qquad \text{(A12)}$$

Here dx is the left invariant integral on G. The right-hand side of this inequality is small for y sufficiently close to the identity. Now $\mathcal{H}$ is a direct sum of subspaces in which there is a vector Φ such that vectors of the form $U(\varphi)\Phi$ are dense, φ being continuous and of compact support so to prove $U(x)$ continuous it suffices to verify that $(U(\varphi)\Phi,U(x)U(\varphi)\Phi)$ is continuous for any such φ and Φ. But

$$(U(\varphi)\Phi,U(x)U(\varphi)\Phi) = (U(\varphi)\Phi,U(\varphi_x)\Phi)$$

so that the required continuity follows from (A12) and the proof is complete.

Finally, there is the matter of sets of measure zero in the criterion for unitary equivalence (3.17). Solved for $V(A^{-1}\mathbf{p})$ it reads

$$V(A^{-1}\mathbf{p}) = Q_1(\mathbf{p},A)^{-1}V(\mathbf{p})Q_2(\mathbf{p},A) .$$

By an argument just like that used in the first few paragraphs of this Appendix, one concludes that both sides of this equality are $(\mu \times \alpha)$ measurable functions of $\mathbf{p}$ and A and the set on which the equality fails is of $\mu \times \alpha$ measure zero. It then follows that, for fixed $\mathbf{p}$, the set of A on which it fails is of α measure zero. That in turn implies that the set of $A^{-1}\mathbf{p}$ for which it fails is of μ measure zero. Picking one $\mathbf{p}$ from each orbit and altering $V(\mathbf{p})$ on the corresponding set of measure zero one gets a new family $V(\mathbf{p})$ which is also measurable and yields the same V but for which (3.17) always holds.