On the Representations of the Rotation Group

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The present paper contains hardly any new result and can claim only a methodological interest. In a recent article[1] I studied a family of Hilbert spaces $\mathfrak{F}_n$, whose elements are entire analytic functions of n complex variables. The methods developed there appear appropriate for a fairly effortless treatment of the representation theory of the rotation group, and this paper is offered in the hope that it may suggest further applications of these methods. (Here, and in the following, the term "rotation group" actually refers to the group $\mathfrak{U}$ of unitary unimodular transformations of a two-dimensional vector space, the spin space of quantum mechanics. It is this group that is basic for the quantum mechanical applications.)

The application of the function spaces $\mathfrak{F}_n$ to the study of the rotation group is related to the long known fact that its irreducible representations may be obtained by considering homogeneous polynomials in two complex variables. (All these polynomials are elements of $\mathfrak{F}_2$, and may thus be treated simultaneously.) This fact has been used, in one form or another, in almost every treatment of the representation theory of the rotation group. It has been most systematically exploited by Kramers and his school,[2] who have applied the concepts and the methods of the theory of binary invariants. Van der Waerden also used it very effectively in his book[3]—for example, in the derivation of the vector coupling coefficients.

It was shown by Wigner—in his profound investigation of simply reducible groups[4]—that remarkably many properties of the 3-j symbols, 6-j symbols, etc. and of their interrelations are shared by all simply reducible groups, and are not confined to the rotation group. By contrast, the present paper is restricted to the rotation group. Naturally, this restriction permits simplifications and short cuts. In addition, we know from Regge's intriguing discovery of unsuspected symmetries of the 3-j and the 6-j symbols[5] that there are important relations which do no longer hold for all simply reducible groups. While the following analysis does not lead to a deeper understanding of the Regge symmetries it yields, at least, a fairly transparent formulation and derivation of the symmetries.

Ten years ago Schwinger published a highly ingenious treatment of the rotation group based on a certain operator method.[6] In a strict mathematical sense, the Hilbert space method of the present paper is isomorphic to Schwinger's operator method. (For a detailed comparison see Sec. 2e below.) The generating functions for the 3-j and the 6-j symbols, in particular, are due to Schwinger.

There are, however, characteristic differences in our approach. (1) Schwinger introduces certain operators $a_\mathfrak{k}$ (and their adjoints) for which the commutation rules of the annihilation and creation operators of boson fields are postulated. All other objects to be studied are defined in terms of the $a_\mathfrak{k}$, including the orthonormal vector basis of the Hilbert space on which the operators $a_\mathfrak{k}$ act. In the present paper, however, the Hilbert space is *a priori* given as a function space, and the standard methods of analysis are available at each step. (2) Schwinger is primarily concerned with angular momenta—in group theoretical terms: with infinitesimal rotations—and he constructs the representations from their infinitesimal generators, while in the present paper the representations are directly defined on the function space $\mathfrak{F}$.

The present paper may be read without any knowledge of the content of the paper (H) of reference 1. To the extent that they are needed the results of (H) are reproduced in Sec. 1. Sections 2 through 4 deal with the rotation group. The representation theory of the rotation group is developed from its beginning—for the convenience of the reader, for the

[1] V. Bargmann, Comm. Pure Appl. Math. **14**, 187 (1961). Hereafter quoted as (H).

[2] For a survey of these methods see H. C. Brinkmann, *Applications of Spinor Invariants in Atomic Physics* (Interscience Publishers, Inc., New York, 1956).

[3] B. L. van der Waerden, *Die gruppentheoretische Methode in der Quantenmechanik* (Verlag Julius Springer, Berlin, Germany, 1932).

[4] An excellent exposition of this investigation is given by W. T. Sharp, "Racah Algebra and the Contraction of Groups." CRT—935 (AECL—1098) Atomic Energy of Canada Ltd., Chalk River, Ontario, 1960 (unpublished).

[5] T. Regge, Nuovo cimento **10**, 544 (1958); **11**, 116 (1959).

[6] J. Schwinger, "On Angular Momentum," U.S. Atomic Energy Commission, NYO—3071, 1952 (unpublished).

sake of logical coherence, and also in order to show that those definitions and constructions which appear natural in the framework of the function space $\mathfrak{F}$ are, at the same time, useful and relevant from a group theoretical point of view. The decomposition of the direct product and the 3-j symbols are treated in Sec. 3, the 6-j symbols in Sec. 4.—No loss in generality is caused by the fact that the representations are constructed on $\mathfrak{F}$, because the main results—for example, the properties of the 3-j and the 6-j symbols—depend only on the representations and not on the vector space on which the representations are realized.

Remarks on the notation. I adopt the definitions and the notation of Wigner's book,[7] with a few exceptions. (1) Complex conjugation is indicated by a bar ($\bar{\alpha}$ is the conjugate of α). (2) The (Hermitian) adjoint of an operator or a matrix A is denoted by A^*. (3) The transpose of a matrix A is denoted by ${}^tA$, and A's determinant by det A. (4) The product of a vector f by a scalar λ will be written either λf or $f\lambda$, whichever appears more convenient.

1. THE HILBERT SPACE $\mathfrak{F}_n$

a. Introductory remarks. The elements of $\mathfrak{F}_n$ are entire analytic functions $f(z)$, where $z = (z_1,z_2,\cdots,z_n)$ is a point of the n-dimensional complex Euclidean space C_n. Every entire $f(z)$ may be expanded in an everywhere converging power series

$$f(z) = \sum_{h_1,\cdots,h_n} \alpha_{h_1 h_2 \cdots h_n} z_1^{h_1} z_2^{h_2} \cdots z_n^{h_n} . \quad (1.1)$$

It will be convenient to use the following shorthand notation. We set

$$h = (h_1,\cdots,h_n)$$

for an ordered set of non-negative integers h_i, and $h = 0$ if all $h_i = 0$. We write α_h for the coefficient $\alpha_{h_1}\cdots_{h_n}$ and denote the power products in (1.1) by

$$z^{[h]} = z_1^{h_1} z_2^{h_2} \cdots z_n^{h_n} ,$$

so that the power series (1.1) takes the form

$$f(z) = \sum_h \alpha_h z^{[h]} . \quad (1.2)$$

We shall also use the abbreviations

$$|h| = h_1 + h_2 + \cdots + h_n , \quad [h!] = h_1! h_2! \cdots h_n! . \quad (1.3)$$

The elements of the n-dimensional space C_n will be called points or vectors (synonymously); $a\cdot b = \sum_{k=1}^n a_k b_k$ is the scalar product of a and b. In particular, $\bar{a}\cdot a = \sum_k |a_k|^2$.

[7] E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959).

b. Definition of the Hilbert space $\mathfrak{F}_n$. The inner product of two elements f,f' of $\mathfrak{F}_n$ is

$$(f,f') = \int \overline{f(z)} f'(z) d\mu_n(z) , \quad (1.4)$$

where

$$d\mu_n(z) = \pi^{-n} \exp(-\bar{z}\cdot z) \prod_k dx_k dy_k , \quad (z_k = x_k + iy_k) . \quad (1.4a)$$

Here and in the following all integrals are extended over the whole space C_n.

The definition (1.4) is meant to imply that an entire function $f(z)$ belongs to $\mathfrak{F}_n$ if and only if

$$(f,f) = \int |f(z)|^2 d\mu_n(z) < \infty . \quad (1.4b)$$

[The norm of f is $||f|| = (f,f)^{1/2}$.] Separating the Gaussian in (1.4a) we shall occasionally write

$$d\mu_n(z) = \rho_n(z) d^n z , \quad \rho_n(z) = \pi^{-n} \exp(-\bar{z}\cdot z) , \quad (1.5)$$

$$d^n z = \prod_{k=1}^n dx_k dy_k . \quad (1.5a)$$

In order to express the inner product of f and f' in the expansion coefficients of their power series, we first compute $(z^{[h]},z^{[h']})$. Introducing polar coordinates, $z_k = r_k e^{i\phi_k}$, we have $(z^{[h]}, z^{[h']}) = \omega_1\omega_2\cdots\omega_n$,

$$\omega_k = \frac{1}{\pi}\int_0^{2\pi} \exp(i(h_k' - h_k)\phi_k) d\phi_k \times \int_0^\infty r_k^{h_k+h'_k+1} e^{-r_k^2} dr_k .$$

It follows that $\omega_k = \delta_{h_k,h'_k} h_k!$ Hence

$$(z^{[h]},z^{[h']}) = \begin{cases} 0 & , h \neq h' , \\ [h!], & h = h' . \end{cases} \quad (1.6)$$

For two functions of $\mathfrak{F}_n, f(z) = \sum \alpha_h z^{[h]}$ and $f'(z) = \sum \alpha'_h z^{[h]}$, one now readily obtains

$$(f,f') = \sum_h [h!] \bar{\alpha}_h \alpha_h' . \quad (1.7)$$

In particular,

$$(f,f) = \sum_h [h!] |\alpha_h|^2 . \quad (1.8)$$

This last equation may be interpreted as follows. For an entire function $f(z)$ either both sides are infinite—in which case f does not belong to $\mathfrak{F}_n$—or both have the same finite value.

The orthonormal set u_h. According to (1.6), the simplest orthonormal set in $\mathfrak{F}_n$ is given by

$$u_h = z^{[h]}/[h!]^{1/2} , \quad (1.9)$$

and Eq. (1.8) expresses its completeness.

The subspaces $\mathfrak{P}_s$. Let $\mathfrak{P}_s$ be the set of all homogeneous polynomials in $\mathfrak{F}_n$ of order s. It is spanned by those u_h for which $|h| = h_1 + \cdots + h_n = s$. $\mathfrak{P}_s$ and $\mathfrak{P}_{s'}$ are clearly orthogonal if $s \neq s'$, and

$$\mathfrak{F}_n = \mathfrak{P}_0 + \mathfrak{P}_1 + \mathfrak{P}_2 + \cdots . \tag{1.10}$$

is a decomposition into mutually orthogonal subspaces. It will be useful to introduce

$$\mathfrak{Q}_j = \mathfrak{P}_{2j}\,, \quad (j = 0,\tfrac{1}{2},1,\cdots)\,. \tag{1.10a}$$

An element f of $\mathfrak{F}_n$ belongs to $\mathfrak{P}_s$ if and only if

$$f(\lambda z) = \lambda^s f(z) \tag{1.10b}$$

for every constant λ, or alternatively if and only if Euler's equation

$$\sum\nolimits_k z_k(\partial f/\partial z_k) = s\cdot f \tag{1.10c}$$

is satisfied.

c. The principal vectors $\mathbf{e}_a$. Define for every a in C_n the function $\mathbf{e}_a$ by

$$\mathbf{e}_a(z) = \exp\,(\bar{a}\cdot z)\,. \tag{1.11}$$

It is clear that $\mathbf{e}_a$ belongs to $\mathfrak{F}_n$. Its power series is

$$\mathbf{e}_a(z) = \sum_h \frac{\bar{a}^{[h]}z^{[h]}}{[h!]}\,. \tag{1.11a}$$

It follows therefore from (1.7) that for any f in $\mathfrak{F}_n$

$$(\mathbf{e}_a,f) = \sum\nolimits_h a^{[h]}\alpha_h = f(a)\,, \tag{1.12}$$

or, in integral form

$$\int \exp\,(a\cdot\bar{z})f(z)d\mu_n(z) = f(a)\,. \tag{1.12a}$$

The existence of these "principal vectors" $\mathbf{e}_a$ is a characteristic feature of $\mathfrak{F}_n$. It is seen that they play here a role similar to that of the δ functions $\delta(q-a)$ in the standard Hilbert space of quantum mechanics, but unlike the δ functions they are elements of Hilbert space.

Applying (1.12) to $f = \mathbf{e}_b$ we have

$$(\mathbf{e}_a,\mathbf{e}_b) = \mathbf{e}_b(a) = \exp\,(\bar{b}\cdot a) \tag{1.13}$$

and hence $(\mathbf{e}_a,\mathbf{e}_a) = \exp\,(\bar{a}\cdot a)$.

By Schwarz's inequality we conclude from (1.12) that

$$|f(z)| \leq ||f||\cdot||\mathbf{e}_z|| \leq ||f||\exp\,(\tfrac{1}{2}\,\bar{z}\cdot z) \tag{1.13a}$$

Conversely, if an entire function $f(z)$ satisfies the inequality

$$|f(z)| \leq c\exp\,(\tfrac{1}{2}\,\gamma\bar{z}\cdot z) \tag{1.13b}$$

where c and γ are positive constants and $\gamma < 1$, then, by the integral definition (1.4b), f belongs to $\mathfrak{F}_n$. (The constant $\gamma < 1$ must not be omitted!)

d. Product decomposition of $\mathfrak{F}_n$. To every decomposition of n into the sum of two positive integers, $n = n' + n''$, corresponds a decomposition of $\mathfrak{F}_n$ into the direct product

$$\mathfrak{F}_n = \mathfrak{F}_{n'} \otimes \mathfrak{F}_{n''}\,. \tag{1.14}$$

Set $z' = (z_1,\ldots,z_{n'})$ and $z'' = (z_{n'+1},\ldots,z_n)$. If $f'(z')$ and $f''(z'')$ belong to $\mathfrak{F}_n'$ and $\mathfrak{F}_n''$, respectively, the product $f(z) = f'(z')\,f''(z'')$ belongs to $\mathfrak{F}_n$. Furthermore, $d\mu_n(z) = d\mu_{n'}(z')\,d\mu_{n''}(z'')$ by (1.4a), and for the inner product of f with $g(z) = g'(z')\,g''(z'')$ one obtains

$$(f,g) = (f',g')(f'',g'')\,,$$

the two factors (f',g') and (f'',g'') being taken on $\mathfrak{F}_{n'}$ and $\mathfrak{F}_{n''}$. The orthonormal functions u_h as well as the principal vectors $\mathbf{e}_a$ are decomposed accordingly.

Similarly one can form products of subspaces of $\mathfrak{F}_{n'}$ and $\mathfrak{F}_{n''}$, for example,

$$\mathfrak{P}_{s's''} = \mathfrak{P}'_{s'} \otimes \mathfrak{P}''_{s''}\,, \quad \mathfrak{Q}_{j'j''} = \mathfrak{Q}'_{j'} \otimes \mathfrak{Q}''_{j''} \tag{1.14a}$$

[see (1.10) and (1.10a)], which contains all polynomials homogeneous in z' of order s' and in z'' of order s''. The functions f in $\mathfrak{P}_{s's''}$ are characterized by

$$f(\lambda'z',\lambda''z'') = \lambda'^{s'}\lambda''^{s''}f(z',z'')$$

for any complex constants λ',λ''.

e. Operators on $\mathfrak{F}_n$. We turn now to a brief review of some operators which occur in the following.

(α) *The operators* z_k *and* d_k. Here d_k stands for the differential operator $\partial/\partial z_k$. Since the elements f of $\mathfrak{F}_n$ are analytic, z_kf and d_kf are always defined as analytic functions, but they do not necessarily belong to $\mathfrak{F}_n$. We shall apply, however, the operators z_k and d_k only to polynomials, so that no difficulties arise.

The d_k, z_l evidently satisfy the commutation rules

$$[z_k,z_l] = 0\,, \quad [d_k,d_l] = 0\,, \quad [d_k,z_l] = \delta_{kl}\,. \tag{1.15}$$

Furthermore, z_k and d_k are *adjoint* [with respect to the inner product (1.4)],

$$z_k = d_k^*\,, \tag{1.15a}$$

i.e., for any f,g in $\mathfrak{F}_n$,

$$(z_kf,g) = (f,d_kg) \tag{1.16}$$

whenever z_kf and d_kg are in $\mathfrak{F}_n$. For simplicity, set $k = 1$. Write, for any $h = (h_1,h_2,\cdots,h_n)$, $h' = (1$

$+ h_1, h_2, \cdots, h_n)$. If $f = \sum \alpha_h z^{[h]}$ and $g = \sum \beta_h z^{[h]}$, we have

$$z_1 f = \sum \alpha_h z^{[h']}, \quad d_1 g = \sum (1+h_1)\beta_{h'} z^{[h]},$$
$$(z_1 f, g) = \sum_h [h'!]\bar{\alpha}_h \beta_{h'},$$
$$(f, d_1 g) = \sum_h (1+h_1)[h!]\bar{\alpha}_h \beta_{h'},$$

which proves (1.16) because $(1+h_1)[h!] = [h'!]$.

It follows from (1.15) and (1.15a) that the operators d_k, z_k satisfy the defining relations for the annihilation and creation operators of boson fields.[8]

(β) *The unitary transformations* T_U. For every unitary transformation U on C_n we define an operator T_U on $\mathfrak{F}_n$ by[9]

$$(T_U f)(z) = f({}^tUz) \qquad (1.17)$$

where ${}^tU$ is the transpose of the matrix U. T_U is clearly a linear operator (i.e., linear in f), and for two unitary transformations U, U'

$$T_U T_{U'} = T_{UU'}. \qquad (1.17a)$$

If $U = 1$, then $T = 1$ (identity), so that $T_{U^{-1}} = T_U^{-1}$.

In addition T_U is *unitary*. Introducing the variables $z' = {}^tUz$ in the integral (1.4) one finds that

$$(T_U f, T_U g) = (f, g) \qquad (1.18)$$

because the measure $d\mu_n(z)$ is invariant under unitary transformations of the z.

It follows that the T_u form a *unitary representation* of the n-dimensional unitary group, and also of any of its subgroups.

The representation is decomposed because any subspace $\mathfrak{P}_s$ is clearly carried into itself [apply, for example, the criterion (1.10b)]. In the case $n = 2$ this will provide the basis for our discussion of the rotation group.

(γ) *The conjugation* K. The last operator to be considered is the conjugation K, which is defined as follows. Let $g = Kf$, then

$$g(z) = \overline{f(\bar{z})}, \qquad (1.19)$$

where the bar, as before, denotes complex conjugation. For $f = \sum \alpha_h z^{[h]}$ we find

$$g(z) = \sum \bar{\alpha}_h z^{[h]}, \qquad (1.19a)$$

i.e., the power series with complex conjugate coefficients.

We note the following properties of K:

(1) K is *antilinear*, i.e.,

$$K(f_1 + f_2) = Kf_1 + Kf_2, \quad K(\lambda f) = \bar{\lambda} Kf$$

for any complex constant λ.

(2) $K^2 = 1$.

(3) $(Kf, Kf') = (f', f) = \overline{(f, f')}$,

i.e., K is *antiunitary*. [(3) follows from either definition of the inner product, (1.4) or (1.6).]

A function f may be called *real* if $Kf = f$ (so that its power series has real coefficients). Thus $z^{[h]}$ and u_h are real.

With the help of K we may also define the complex conjugate of a linear operator A on $\mathfrak{F}_n$ by setting

$$\bar{A} = KAK. \qquad (1.20)$$

$\bar{A}$ itself is *linear* since K appears an even number of times in the definition (1.20). If $B = \bar{A}$, then $\bar{B} = A$. Let

$$Au_h = \sum_{h'} u_{h'} a_{h'h},$$

where $a_{h'h}$ are the matrix elements of A in the system u_h. Then, since $Ku_h = u_h$,

$$\bar{A}u_h = K(Au_h) = \sum_{h'} u_{h'} \overline{a_{h'h}}. \qquad (1.21)$$

Thus, $\bar{A}$'s matrix elements are complex conjugate to those of A.

Application to T_U. If $\bar{U}$ is the matrix complex conjugate to U,

$$\overline{T_U} = T_{\bar{U}}. \qquad (1.22)$$

Proof. Let $g = \overline{T_U} f$ and set, successively, $f_1 = Kf$, $f_2 = T_U f_1$, $g = Kf_2$. By definition, $g(z) = \overline{f_2(\bar{z})}$, $f_2(\bar{z}) = f_1({}^tU\bar{z}) = f_1(y)$, and, finally, $f_1(y) = \overline{f(\bar{y})} = \overline{f({}^t\bar{U}z)}$. Hence $h(z) = f({}^t\bar{U}z)$, Q.E.D.

2. THE REPRESENTATIONS $\mathfrak{D}_j$

a. The group $\mathfrak{U}$. We start with a brief review of the group $\mathfrak{U}$ of unimodular unitary transformations in two dimensions and its connection with the rotation group.

The vectors in C_2 will be denoted by ζ, with components ζ_1, ζ_2. (In dealing with several vectors ζ, we shall often denote their components by ξ, η instead of ζ_1, ζ_2 in order to avoid a profusion of indices.) The (Hermitian) inner product of two vectors ζ, ζ' is

$$\bar{\zeta} \cdot \zeta' = \bar{\zeta}_1 \zeta_1' + \bar{\zeta}_2 \zeta_2'.$$

Denoting the Hermitian Pauli spin matrices by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (2.1)$$

[8] I. E. Segal has used a generalization of $\mathfrak{F}_n$ to $\mathfrak{F}_\infty$ for a comprehensive study of the canonical operators of quantum field theory, where infinitely many d_k, z_k occur. (Lectures at the Summer Seminar on Applied Mathematics, 1960, Boulder, Colorado, unpublished.)

[9] This differs somewhat from the corresponding definition in (H) (reference 1), Eq. (3.4), p. 205.

we write

$$b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3 = \mathbf{b}\cdot\sigma\ ; \quad \mathbf{b} = (b_1,b_2,b_3)$$

for a three-vector $\mathbf{b}$ with real or complex components. Every 2×2 matrix B may be expressed in the form

$$B = b_0\cdot 1 + \mathbf{b}\cdot\sigma \tag{2.1a}$$

with uniquely determined $b_0,\mathbf{b}$.

The algebraic properties of the spin matrices are summarized in

$$\begin{aligned}(\mathbf{a}\cdot\sigma)(\mathbf{b}\cdot\sigma) + (\mathbf{b}\cdot\sigma)(\mathbf{a}\cdot\sigma) &= 2(\mathbf{a}\cdot\mathbf{b})1\\ (\mathbf{a}\cdot\sigma)(\mathbf{b}\cdot\sigma) - (\mathbf{b}\cdot\sigma)(\mathbf{a}\cdot\sigma) &= 2i(\mathbf{a}\times\mathbf{b})\cdot\sigma\end{aligned} \tag{2.2}$$

for any two vectors $\mathbf{a},\mathbf{b}$, where $\mathbf{a}\times\mathbf{b}$ denotes the vector product.

In the following, the matrix

$$\Gamma = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix} \tag{2.3}$$

will play an important role. (It is the basic matrix ϵ of the spinor calculus.) We note that

$$^t\Gamma = -\Gamma\,, \quad \Gamma^2 = -1\,, \quad ^t\Gamma\cdot\Gamma = 1\,, \quad \det\Gamma = 1\,, \tag{2.3a}$$

where "det" denotes the determinant.

For every 2×2 matrix B we define the *associate* matrix B_a by

$$B_a = \Gamma B \Gamma^{-1} \tag{2.4}$$

If $B = (\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix})$, then $B_a = (\begin{smallmatrix}\delta & -\gamma\\ -\beta & \alpha\end{smallmatrix})$. (The elements of B_a are the minors of B.) It follows that

$$(^tB)_a = {}^t(B_a)\,, \quad (B^{-1})_a = (B_a)^{-1}\,, \quad (BC)_a = B_a\cdot C_a\,, \tag{2.4a}$$

$$B\cdot{}^tB_a = {}^tB\cdot B_a = (\det B)\cdot 1\,. \tag{2.4b}$$

Since for the spin matrices $\sigma_k^2 = 1$, $\det\sigma_k = -1$, we obtain from (2.4b)

$$(^t\sigma_k)_a = -\sigma_k\,. \tag{2.4c}$$

Hence, for any B written in the form (2.1a),

$$\begin{aligned}B\cdot{}^tB_a &= (b_0 + \mathbf{b}\cdot\sigma)(b_0 - \mathbf{b}\cdot\sigma) = (b_0^2 - b^2)\cdot 1\\ \det B &= b_0^2 - b^2\,, \quad b^2 = \mathbf{b}\cdot\mathbf{b}\,.\end{aligned} \tag{2.4d}$$

The group $\mathfrak{U}$. A matrix U belongs to $\mathfrak{U}$ if and only if $^tU\cdot\bar{U} = 1$, and $\det U = 1$. In view of (2.4b) these conditions may be replaced by

$$U_a = \Gamma U \Gamma^{-1} = \bar{U}\ ; \quad \det U = 1\,. \tag{2.5}$$

Let $U = b_0 + \mathbf{b}\cdot\sigma\epsilon\mathfrak{U}$. Then $U^* = \bar{b}_0 + \bar{b}\cdot\sigma = U^{-1}$. By (2.4d), $U^{-1} = b_0 - \mathbf{b}\cdot\sigma$. Hence b_0 is real, and $\mathbf{b}$ imaginary. Setting $b_0 = a_0$, $\mathbf{b} = -i\mathbf{a}$, we find that U belongs to $\mathfrak{U}$ if and only if

$$U = a_0 - i\mathbf{a}\cdot\sigma\,, \quad \det U = a_0^2 + a^2 = 1\,, \tag{2.6}$$

a_0, $\mathbf{a}$ real. In matrix form

$$U = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix} a_0 - ia_3 & -ia_1 - a_2\\ -ia_1 + a_2 & a_0 + ia_3\end{pmatrix}$$

$$\delta = \bar{\alpha}\,, \quad \gamma = -\bar{\beta}\,, \quad \alpha\bar{\alpha} + \beta\bar{\beta} = 1\,. \tag{2.6a}$$

Connection with the rotation group. Every U in $\mathfrak{U}$ defines a rotation $\mathbf{r}' = R_U\mathbf{r}$ by

$$\mathbf{r}'\cdot\sigma = U(\mathbf{r}\cdot\sigma)U^{-1}\,, \tag{2.7}$$

so that $R_{U_1}R_{U_2} = R_{U_1U_2}$, and $R_{-U} = R_U$. Using (2.2) one obtains by straightforward computation

$$\mathbf{r}' = R_U\mathbf{r} = (a_0^2 - a^2)\mathbf{r} + 2(\mathbf{a}\cdot\mathbf{r})\mathbf{a} + 2a_0(\mathbf{a}\times\mathbf{r})\,, \tag{2.7a}$$

the well known expression of a rotation in terms of Euler's *homogeneous parameters*. Specifically,

$$a_0 = \cos(\tfrac{1}{2}\phi)\,, \quad \mathbf{a} = \sin(\tfrac{1}{2}\phi)\mathbf{n}\,, \quad (\mathbf{n}\cdot\mathbf{n} = 1)\,, \tag{2.7b}$$

where $\mathbf{n}$ is the axis and ϕ the angle of the rotation R_U.

To the one-parametric subgroup of rotations about the axis $\mathbf{n}$ corresponds the subgroup

$$\begin{aligned}U(\phi) &= \cos(\tfrac{1}{2}\phi) - i\sin(\tfrac{1}{2}\phi)\mathbf{n}\cdot\sigma\\ &= \exp[-\tfrac{1}{2}i\phi(\mathbf{n}\cdot\sigma)]\end{aligned} \tag{2.7c}$$

of $\mathfrak{U}$.

b. The representations $\mathfrak{D}^j$. It is now easy to obtain some of the basic results concerning the representations $\mathfrak{D}^j$ of $\mathfrak{U}$.

On the Hilbert space $\mathfrak{F}_2$ of analytic functions $f(\zeta)$ (we write now ζ instead of z) the operators T_U,

$$(T_Uf)(\zeta) = f(^tU\zeta)\,, \tag{2.8}$$

provide a unitary representation of the group $\mathfrak{U}$, as was shown in Sec. 1e.

The subspace $\mathfrak{Q}_j = \mathfrak{P}_{2j}$ of homogeneous polynomials of order $2j$—where $2j = 0,1,2,\cdots$—is *invariant* under the transformations T_U, and $\mathfrak{D}^j$ is the representation of $\mathfrak{U}$ defined by the *restriction* of T_U to $\mathfrak{Q}_j$. Since different $\mathfrak{Q}_j$ have different dimensions, the various representations $\mathfrak{D}^j$ are clearly *inequivalent.*

According to the first section—see Eq. (1.9)—$\mathfrak{Q}_j$ is spanned by the $2j + 1$ orthonormal functions

$$\zeta_1^\kappa\zeta_2^\lambda/(\kappa!\lambda!)^{1/2} = \xi^\kappa\eta^\lambda/(\kappa!\lambda!)^{1/2}\,, \quad (\kappa + \lambda = 2j) \tag{2.9}$$

or, with $m = j, j-1, \cdots, -j$,

$$v_m^j = \xi^{j+m}\eta^{j-m}/[(j+m)!(j-m)!]^{1/2}\,, \quad (\kappa - \lambda = 2m) \tag{2.9a}$$

If U is given by (2.6a), then

$$T_U v_m^j = \frac{(\alpha\xi + \gamma\eta)^{j+m}(\beta\xi + \delta\eta)^{j-m}}{[(j+m)!(j-m)!]^{1/2}} . \quad (2.10)$$

The matrix elements $\mathfrak{D}_m^{jm'}(U)$ are defined by

$$T_U v_m^j = \sum_{m'} v_{m'}^j \mathfrak{D}_m^{jm'}(U) , \quad \mathfrak{D}_m^{jm'}(U) = (v_{m'}^j, T_U v_m^j) , \quad (2.10a)$$

and their explicit form may be deduced from (2.10).

For rotations about the z-axis, $U(\phi) = \cos(\frac{1}{2}\phi) - i \sin(\frac{1}{2}\phi)\sigma_3$, so that, in (2.6a), $\alpha = e^{-i\phi/2}$, $\delta = e^{i\phi/2}$, $\beta = \gamma = 0$, and

$$T_U v_m^j = e^{-im\phi} v_m^j \quad (2.10b)$$

c. Infinitesimal transformations. Consider the one-parametric subgroup (2.7c), and the corresponding transformations $T_{U(\phi)}$. The infinitesimal generator of $T_{U(\phi)}$ may then be defined by

$$(\mathbf{n}\cdot\mathbf{M})f = i(d/d\phi)T_{U(\phi)}f|_{\phi=0} . \quad (2.11)$$

One obtains from (2.8) the expression

$$((\mathbf{n}\cdot\mathbf{M})f)(\zeta) = \tfrac{1}{2}\sum_{\alpha,\beta=1}^{2} \zeta_\alpha(\mathbf{n}\cdot\sigma)_{\alpha\beta}\frac{\partial f(\zeta)}{\partial \zeta_\beta} , \quad (2.11a)$$

where $(\mathbf{n}\cdot\sigma)_{\alpha\beta}$ are the matrix elements of $\mathbf{n}\cdot\sigma$. Hence

$$\mathbf{n}\cdot\mathbf{M} = n_1 M_1 + n_2 M_2 + n_3 M_3$$
$$M_k = \tfrac{1}{2}\sum_{\alpha,\beta} \zeta_\alpha(\sigma_k)_{\alpha\beta} d_\beta , \quad d_\beta = \partial/\partial\zeta_\beta \quad (2.12)$$

The operators M_k transform each $\mathfrak{Q}_j$ into itself. [If f is a homogeneous polynomial of order $2j$, so is $M_k f$, by (2.11a).] Furthermore, they are self-adjoint. This may be inferred from the fact that $-i(\mathbf{n}\cdot\mathbf{M})$ is an infinitesimal unitary operator, or from the explicit expression (2.12) because σ_k is a Hermitian matrix, and $(\zeta_\alpha d_\beta)^* = \zeta_\beta d_\alpha$, by (1.15a).

For the *commutator* of $\mathbf{n}\cdot\mathbf{M}$ and $\mathbf{n}'\cdot\mathbf{M}$ one readily obtains

$$\begin{aligned}[\mathbf{n}\cdot\mathbf{M}, \mathbf{n}'\cdot\mathbf{M}] &= \tfrac{1}{4}\sum_{\alpha,\beta} \zeta_\alpha[\mathbf{n}\cdot\sigma, \mathbf{n}'\cdot\sigma]_{\alpha\beta} d_\beta \\ &= (i/2)\sum \zeta_\alpha((\mathbf{n}\times\mathbf{n}')\cdot\sigma)_{\alpha\beta} d_\beta \\ &= i(\mathbf{n}\times\mathbf{n}')\cdot\mathbf{M}\end{aligned}$$

where (2.2) has been used. Thus

$$[M_1, M_2] = iM_3 , \quad [M_2, M_3] = iM_1 ,$$
$$[M_3, M_1] = iM_2 . \quad (2.12a)$$

From (2.12),

$$M_1 + iM_2 = \zeta_1 d_2 , \quad M_1 - iM_2 = \zeta_2 d_1 ,$$
$$M_3 = \tfrac{1}{2}(\zeta_1 d_1 - \zeta_2 d_2) , \quad (2.13)$$

so that, for example

$$M_3 v_m^j = m v_m^j$$

in accordance with (2.10b).

Lastly,

$$\begin{aligned}M^2 = \sum_{k=1}^{3} M_k^2 &= M_3^2 + M_3 \\ &+ (M_1 - iM_2)(M_1 + iM_2) \\ &= \tfrac{1}{4}(\zeta_1 d_1 + \zeta_2 d_2)^2 + \tfrac{1}{2}(\zeta_1 d_1 + \zeta_2 d_2) \\ &= N(N+1) ,\end{aligned}$$

where

$$N = \tfrac{1}{2}(\zeta_1 d_1 + \zeta_2 d_2) .$$

On $\mathfrak{P}_{2j}$, $Nf = jf$ [see (1.10c)], hence $M^2 f = j(j+1)f$.

Remark. Two questions have not yet been considered, (1) the *irreducibility*, (2) the *completeness* of the representations constructed so far. (1) To prove the irreducibility of $\mathfrak{D}^j$ it suffices to show that every linear operator A defined on $\mathfrak{Q}_j$ which commutes with all T_U is necessarily of the form $A = \alpha\cdot 1$. If A commutes with all T_U, it also commutes with all M_k [by (2.11)], and a standard computation, using (2.13), shows that this indeed implies $A = \alpha\cdot 1$. (2) The completeness is a much deeper problem, and it is doubtful whether the existing proofs by integral methods (Wigner, reference 7, p. 166) or by differential (Lie group) methods (Waerden, reférence 3, Sec. 17) can be essentially simplified. In any event, the particular method of this paper does not seem to contribute anything to this problem.

d. Complex conjugation. At the end of the first section we saw that $T_{\bar U} = \overline{T_U}$. Since the transition to $\overline{T_U}$ implies also the transition to the *complex conjugate* matrix elements in the system v_m^j, we have

$$\mathfrak{D}^j(\bar U) = \overline{\mathfrak{D}^j(U)} . \quad (2.14)$$

It follows from the unitarity of the matrices $\mathfrak{D}^j$ that

$$\mathfrak{D}^j(U^*) = \mathfrak{D}^j(U^{-1}) = (\mathfrak{D}^j(U))^{-1} = (\mathfrak{D}^j(U))^*$$

and hence

$$\mathfrak{D}^j({}^tU) = \mathfrak{D}^j(\overline{U^*}) = \overline{(\mathfrak{D}^j(U))^*} = {}^t\mathfrak{D}^j(U) . \quad (2.14a)$$

The matrix Γ introduced in (2.3) belongs to the group $\mathfrak{U}$. Therefore the relation $\Gamma U \Gamma^{-1} = \bar U$ implies that $\overline{T_U} = T_\Gamma T_U T_\Gamma^{-1}$, in particular[10]

$$\overline{\mathfrak{D}^j(U)} = C^j \mathfrak{D}^j(U)(C^j)^{-1} ; \quad C^j = \mathfrak{D}^j(\Gamma) . \quad (2.15)$$

[10] On $\mathfrak{Q}_{1/2}$, $D^{1/2}(U) = U$, and $C^{1/2} = \Gamma$.

The relations ${}^t\Gamma = -\Gamma = \Gamma^{-1}$, $\Gamma^2 = -1$ imply

$$ {}^tC^j = (-1)^{2j}C^j = (C^j)^{-1}\,, \quad (C^j)^2 = (-1)^{2j}\,, \quad (2.15a)$$

because $\mathfrak{D}^j(-1) = (-1)^{2j}$.

Setting

$$w_m^j = T_\Gamma^{-1} v_m^j\,, \quad (2.16)$$

we obtain a new orthonormal system for which

$$T_U w_m^j = \sum_{m'} w_{m'}^j \overline{\mathfrak{D}_m^{jm'}(U)}\,. \quad (2.16a)$$

In fact,

$$T_U w_m^j = T_U T_\Gamma^{-1} v_m^j = T_\Gamma^{-1} \overline{T}_U v_m^j$$
$$= T_\Gamma^{-1}\left(\sum_{m'} v_{m'}^j \overline{\mathfrak{D}_m^{jm'}(U)}\right) = \sum_{m'} w_{m'}^j \overline{\mathfrak{D}_m^{jm'}(U)}$$

For any function $f(\zeta)$, set $T_\Gamma{}^{-1}f = g$. Then $g(\zeta) = f({}^t\Gamma^{-1}\zeta) = f(\Gamma\zeta)$, i.e.,

$$g(\zeta_1,\zeta_2) = f(-\zeta_2,\zeta_1)\,. \quad (2.16b)$$

Thus,

$$w_m^j = (-1)^{j+m} v_{-m}^j\,. \quad (2.16c)$$

Now
$$w_m^j = \sum_{m'} v_{m'}^j (C^j)^{-1}_{m'm} = \sum_{m'} C^j_{mm'} v_{m'}^j\,,$$
$$v_m^j = \sum_{m'} w_{m'}^j C^j_{m'm} \quad (2.16d)$$

(where ${}^tC^j = (C^j)^{-1}$ has been used). Hence

$$C^j_{mm'} = (-1)^{j+m}\delta_{m,-m'} = (-1)^{j-m'}\delta_{m,-m'}\,. \quad (2.16e)$$

e. Comparison with Schwinger's method. Schwinger starts with the introduction of operators a which correspond to the d_α, ζ_α introduced above:

$$a_+ \to d_1\,, \quad a_- \to d_2\,, \quad a_+^{+} \to \zeta_1\,, \quad a_-^{+} \to \zeta_2\,. \quad (2.17)$$

For them he postulates the commutation rules (1.15) as well as the adjointness (1.15a). In terms of the operators a he next defines the operators J_k corresponding to the M_k of (2.12) above, as well as the orthonormal system of vectors which span the Hilbert space on which the a operate. The basic vector is ψ_0 which corresponds to $v_0^0 = 1$ used here, since $a_+\psi_0 = a_-\psi_0 = 0$ (or $\partial\psi_0/\partial\zeta_1 = \partial\psi_0/\partial\zeta_2 = 0$), and $\psi(jm)$ is defined by

$$\psi(jm) = \frac{(a_+^{+})^{j+m}(a_-^{+})^{j-m}}{[(j+m)!(j-m)!]^{1/2}}\psi_0$$

which, by (2.17), corresponds to

$$\frac{\zeta_1^{j+m}\zeta_2^{j-m}}{[(i+m)!(j-m)!]^{1/2}}\cdot 1\,,$$

i.e., to v_m^j. In addition, the action of the operators a on the ψ_{jm} is precisely the same as the action of the cooresponding d_α, ζ_α on the v_m^j, so that the *isomorphism* of the two methods is established. One may say that the function space $\mathfrak{F}$ with its operators d_α, ζ_α is a realization of Schwinger's more abstractly defined system.

3. THE DECOMPOSITION OF THE DIRECT PRODUCT AND THE 3-j SYMBOLS

In terms of the quantum-mechanical vector addition model the decomposition of the direct product $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2}$ answers the question how two angular momenta $\mathbf{j}_1,\mathbf{j}_2$ combine to a third one, $\mathbf{j}' = \mathbf{j}_1 + \mathbf{j}_2$. The details of the answer are contained in the vector coupling coefficients. Setting $\mathbf{j}_3 = -\mathbf{j}'$ one may, alternatively, ask under what conditions $\mathbf{j}_1 + \mathbf{j}_2 + \mathbf{j}_3 = 0$. This latter problem leads to Wigner's 3-j symbols, and its greater symmetry (in $\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3$) is the cause for the greater symmetry of the 3-j symbols.

a. Preliminary remarks on representation theory. We recall the following facts. Let V_α be a family of unitary operators defined on the unitary vector space $\mathfrak{B}$, and let $e_1,e_2,\cdots,e_m$ and $f_1,f_2,\cdots,f_n$ be two sets of vectors in $\mathfrak{B}$ which transform under V_α as follows:

$$V_\alpha e_i = \sum_{j=1}^{m} e_j\rho_{ji}(\alpha)\,; \quad V_\alpha f_r = \sum_{s=1}^{n} f_s\sigma_{sr}(\alpha)\,. \quad (3.1)$$

(The case $m = n$, $f_i = e_i$ is not excluded!) The matrices $\rho_{ji}(\alpha)$ and $\sigma_{sr}(\alpha)$ are assumed *unitary* and *irreducible*.

Consider the inner products

$$\beta_{ir} = (e_i,f_r)$$

By the unitarity of V_α we obtain from (3.1)

$$\beta_{ir} = (V_\alpha e_i, V_\alpha f_r) = \sum_{j,s} \overline{\rho_{ji}(\alpha)}\beta_{js}\sigma_{sr}(\alpha)$$

In matrix form $\beta = \rho^*(\alpha)\beta\sigma(\alpha)$, and since ρ is unitary,

$$\rho(\alpha)\beta = \beta\sigma(\alpha)\,.$$

Schur's lemma now implies the following:
(1) If ρ and σ are *inequivalent*, then $\beta = 0$, i.e.,

$$(e_i,f_r) = 0\,, \quad \text{for all } i,r\,. \quad (3.2)$$

(2) If $\rho = \sigma$ (hence $m = n$), $(e_i,f_r) = \beta_{ir} = \lambda\delta_{ir}$. This holds in particular for $f_i = e_i$, so that

$$(e_i,e_j) = \lambda\delta_{ij}\,; \quad \|e_1\|^2 = \|e_2\|^2 = \cdots = \|e_n\|^2 = \lambda\,. \quad (3.2a)$$

b. The product representation $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2}$. Our treatment of the direct products $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2}$ is based on the decomposition of $\mathfrak{F}_n$ discussed in Sec. 1d, specifically the decomposition of $\mathfrak{F}_4$. Set $\zeta' = (\xi_1,\eta_1)$, $\zeta'' = (\xi_2,\eta_2)$ and let $\mathfrak{F}_2'$ and $\mathfrak{F}_2''$ be the Hilbert spaces of analytic functions $f(\zeta')$ and $f(\zeta'')$ respectively. Then $\mathfrak{F}_4 = \mathfrak{F}_2' \otimes \mathfrak{F}_2''$ is a Hilbert space of analytic functions $f(\zeta',\zeta'')$ or $f(z)$ where $z = (z_1,z_2,z_3,z_4) = (\xi_1,\eta_1,\xi_2,\eta_2)$.

For any U in $\mathfrak{U}$ the operators T_U' and T_U'' are defined on $\mathfrak{F}_2'$ and $\mathfrak{F}_2''$, respectively, by Eq. (2.8). For a function $f(\zeta',\zeta'')$ in $\mathfrak{F}_4$ we set correspondingly

$$(T_U^{(2)}f)(\zeta',\zeta'') = f({}^tU\zeta',{}^tU\zeta'') . \qquad (3.3)$$

As shown in Sec. 1e the operators $T_U^{(2)}$ form a unitary representation of $\mathfrak{U}$, and furthermore

$$T_U^{(2)} = T_U' \otimes T_U'' , \qquad (3.3a)$$

for if $f(\zeta',\zeta'') = f_1(\zeta')f_2(\zeta'')$, then $T_U^{(2)}f = (T_U'f_1)(T_U''f_2)$.

It follows from (3.3) that the infinitesimal transformations corresponding to $T_U^{(2)}$ are

$$M_k^{(2)} = M_k' + M_k'' , \quad (k = 1,2,3) \qquad (3.3b)$$

where M'_k and M''_k are formed according to (2.12) for ζ' and ζ'', respectively. All M'_k commute with all M''_l.

The subspace $\mathfrak{Q}_{j_1j_2} = \mathfrak{Q}_{j_1}' \otimes \mathfrak{Q}_{j_2}''$ of $\mathfrak{F}_4$ (see (1.14)) is spanned by the $(2_{j_1}+1)(2_{j_2}+1)$ orthonormal functions

$$\begin{aligned} v_{m_1}^{j_1}(\zeta')v_{m_2}^{j_2}(\zeta'') &= \frac{\xi_1^{j_1+m_1}\eta_1^{j_1-m_1}\xi_2^{j_2+m_2}\eta_2^{j_2-m_2}}{[(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!]^{1/2}} \\ &= \frac{\xi_1^{\kappa_1}\xi_2^{\kappa_2}\eta_1^{\lambda_1}\eta_2^{\lambda_2}}{[\kappa_1!\kappa_2!\lambda_1!\lambda_2!]^{1/2}} , \end{aligned} \qquad (3.4)$$

$$\kappa_\alpha + \lambda_\alpha = 2j_\alpha , \quad \kappa_\alpha - \lambda_\alpha = 2m_\alpha , \quad (\alpha = 1,2) . \qquad (3.4a)$$

$\mathfrak{Q}_{j_1j_2}$ is invariant under $T_U^{(2)}$, and $T_U^{(2)}$ restricted to $\mathfrak{Q}_{j_1j_2}$ provides the product representation $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2}$.

It is clear how this is generalized to the product of more than two spaces, for example $\mathfrak{F}_6 = \mathfrak{F}_2 \otimes \mathfrak{F}_2'' \otimes \mathfrak{F}_2'''$, the Hilbert space of analytic functions $f(\zeta',\zeta'',\zeta''')$. The subspace $\mathfrak{Q}_{j_1j_2j_3} = \mathfrak{Q}_{j_1}' \otimes \mathfrak{Q}_{j_2}'' \otimes \mathfrak{Q}_{j_3}''' = \mathfrak{Q}_{j_1j_2} \otimes \mathfrak{Q}_{j_3}'''$ is spanned by

$$\begin{aligned} v_{m_1}^{j_1}(\zeta')v_{m_2}^{j_2}(\zeta'')v_{m_3}^{j_3}(\zeta''') &= \frac{\xi_1^{\kappa_1}\xi_2^{\kappa_2}\xi_3^{\kappa_3}\eta_1^{\lambda_1}\eta_2^{\lambda_2}\eta_3^{\lambda_3}}{[\kappa_1!\kappa_2!\kappa_3!\lambda_1!\lambda_2!\lambda_3!]^{1/2}} \\ &= \frac{\xi^{[\kappa]}\eta^{[\lambda]}}{([\kappa!][\lambda!])^{1/2}} , \end{aligned} \qquad (3.5)$$

where $\xi = (\xi_1,\xi_2,\xi_3)$, $\eta = (\eta_1,\eta_2,\eta_3)$, $\kappa = (\kappa_1,\kappa_2,\kappa_3)$, $\lambda = (\lambda_1,\lambda_2,\lambda_3)$, and

$$\kappa_\alpha + \lambda_\alpha = 2j_\alpha , \quad \kappa_\alpha - \lambda_\alpha = 2m_\alpha , \quad (\alpha = 1,2,3) . \qquad (3.5a)$$

Defining

$$(T_U^{(3)}f)(\zeta',\zeta'',\zeta''') = f({}^tU\zeta',{}^tU\zeta'',{}^tU\zeta''') , \qquad (3.6)$$

we have for the representation $T_U^{(3)}$ of $\mathfrak{U}$

$$T_U^{(3)} = T_U^{(2)} \otimes T_U''' = T_U' \otimes T_U'' \otimes T_U''' , \qquad (3.6a)$$

the infinitesimal transformations are $M_k^{(3)} = M_k' + M_k'' + M_k'''$, and $T_U^{(3)}$ restricted to the invariant subspace $\mathfrak{Q}_{j_1j_2j_3}$ yields the representation $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2} \otimes \mathfrak{D}^{j_3}$.

c. The decomposition of $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2}$. Suppose the representation $\mathfrak{D}^{j_3}$ is contained in $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2}$, i.e., there are $2j_3+1$ orthonormal functions $\psi_m^{j_3}$ in $\mathfrak{Q}_{j_1j_2}$ such that

$$T_U^{(2)}\psi_m^{j_3} = \sum_{\mu=-j_3}^{j_3} \psi_\mu^{j_3}\mathfrak{D}_m^{j_3\mu}(U) , \quad m = j_3,j_3-1,\cdots,-j_3 \qquad (3.7)$$

Consider the function

$$a = \textstyle\sum_m \psi_m^{j_3}w_m^{j_3}(\zeta''')$$

in $\mathfrak{Q}_{j_1j_2j_3}$, where $w_m^{j_3}(\zeta''') = \sum_{m'} C_{mm'}^{j_3}\, v_{m'}^{j_3}(\zeta''')$ (see (2.16d)). As a sum of orthonormal functions, $a \neq 0$. Since

$$\begin{aligned} T_U'''w_m^{j_3} &= \textstyle\sum_\nu w_\nu^{j_3}\overline{\mathfrak{D}_m^{j_3\nu}(U)} , \\ T_U^{(3)}a &= \textstyle\sum_m (T_U^{(2)}\psi_m^{j_3})(T_U'''w_m^{j_3}) \\ &= \textstyle\sum_{m,\mu,\nu} \psi_\mu^{j_3}w_\nu^{j_3}\mathfrak{D}_m^{j_3\mu}(U)\overline{\mathfrak{D}_m^{j_3\nu}(U)} \\ &= \textstyle\sum_{\mu,\nu} \psi_\mu^{j_3}w_\nu^{j_3}\delta_{\mu\nu} = a . \end{aligned} \qquad (3.7a)$$

Thus, a is invariant under $T_U^{(3)}$, and $M_k^{(3)}a = (M_k' + M_k'' + M_k''')a = 0$ (which is equivalent to saying that $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2} \otimes \mathfrak{D}^{j_3}$ contains the identical representation). This is the precise mathematical content of the remarks at the beginning of this section.

Conversely, let h be a function of unit norm in $\mathfrak{Q}_{j_1j_2j_3}$ such that

$$T_U^{(3)}h = h . \qquad (3.8)$$

As the $w_m^{j_3}(\zeta''')$ span $\mathfrak{Q}_{j_3}'''$, h has an expansion

$$h = \textstyle\sum_m \chi_m w_m^{j_3} \qquad (3.8a)$$

with *uniquely determined* χ_m in $\mathfrak{Q}_{j_1j_2}$. Now, by (3.7a),

$$\begin{aligned} T_U^{(3)}h &= \textstyle\sum_m (T_U^{(2)}\chi_m)(T_U'''w_m^{j_3}) \\ &= \textstyle\sum_m \{\sum_{m'} (T_U^{(2)}\chi_{m'})\overline{\mathfrak{D}_{m'}^{j_3m}(U)}\}w_m^{j_3} . \end{aligned}$$

Since $T_U^{(3)}h = h$,

$$\textstyle\sum_{m'} (T_U^{(2)}\chi_{m'})\overline{\mathfrak{D}_{m'}^{j_3m}(U)} = \chi_m ,$$

and hence

$$T_U^{(2)}\chi_m = \textstyle\sum_\mu \chi_\mu\mathfrak{D}_m^{j_3\mu}(U) .$$

By (3.2a), $(\chi_m,\chi_{m'}) = \lambda\delta_{mm'}$. Thus, h is a sum of orthogonal functions, and since it was assumed normalized, $||h||^2 = \sum_m ||\chi_m||^2 = (2j_3+1)\lambda = 1$. Thus

$$\psi_m^{j_3} = (2j_3+1)^{1/2}\chi_m \tag{3.8b}$$

are orthonormal functions in $\mathfrak{Q}_{j_1j_2}$ which transform under $\mathfrak{D}^{j_3}$.

d. The functions F_k, H_k, and the 3-j symbols. The invariant functions h in $\mathfrak{Q}_{j_1j_2j_3}$ [see (3.8)] may be constructed as follows.[11] Since the U are unimodular, the three determinants

$$\delta_1 = \xi_2\eta_3 - \xi_3\eta_2\,,\quad \delta_2 = \xi_3\eta_1 - \xi_1\eta_3\,,\quad \delta_3 = \xi_1\eta_2 - \xi_2\eta_1 \tag{3.9}$$

are invariant under $T_U^{(3)}$, and so is every monomial in δ_α,

$$F_k = \frac{\delta_1^{k_1}\delta_2^{k_2}\delta_3^{k_3}}{k_1!k_2!k_3!} = \frac{\delta^{[k]}}{[k!]} \quad k = (k_1,k_2,k_3) \tag{3.9a}$$

where k_α are any non-negative integers and the factorials in the denominator are included for convenience. [Depending on the circumstances we shall indicate the variables on which F_k depends by writing either $F_k(\xi,\eta)$ or $F_k(\zeta',\zeta'',\zeta''')$.]

F_k belongs to $\mathfrak{Q}_{j_1j_2j_3}$, i.e., it is homogeneous in ζ', ζ'', ζ''' of the orders $2j_1$, $2j_2$, $2j_3$ if and only if

$$k_2+k_3 = 2j_1\,,\quad k_3+k_1 = 2j_2\,,\quad k_1+k_2 = 2j_3 \tag{3.10}$$

or equivalently

$$k_\alpha = J - 2j_\alpha(\alpha = 1,2,3)\;;\; J = j_1+j_2+j_3 \tag{3.10a}$$

$$k_1 = j_2+j_3-j_1\,,\quad k_2 = j_3+j_1-j_2\,,$$
$$k_3 = j_1+j_2-j_3\,. \tag{3.10b}$$

Note that

$$k_1+k_2+k_3 = J\,. \tag{3.10c}$$

As will be shown below [see (3.24)], $||F_k||^2 = (J+1)!/[k!]$. The corresponding normalized h is therefore

$$H_k = \Delta(j_1,j_2,j_3)F_k\;;\quad \Delta(j_1,j_2,j_3) = ([k!]/(J+1)!)^{1/2} \tag{3.11}$$

where Δ is the so-called "quantum mechanical triangle coefficient."

Corresponding to every H_k there are $2j_3+1$ orthonormal functions $\psi_m^{j_3}$ in $\mathfrak{Q}_{j_1j_2}$ [see (3.8b)] which transform under $\mathfrak{D}^{j_3}$ provided that $j_3 = j_1+j_2-k_3$ for an integral $k_3 \geq 0$, and $j_3 \geq |j_1-j_2|$, as follows from (3.10b). Since the ψ belonging to different j_3 are *orthogonal* to each other [by (3.2)] we thus obtain altogether $n = (2j_1+1)(2j_2+1)$ orthonormal functions in $\mathfrak{Q}_{j_1j_2}$. As n is the dimension of $\mathfrak{Q}_{j_1j_2}$, the decomposition of $\mathfrak{D}^{j_1}\otimes\mathfrak{D}^{j_2}$ is thus completed.

The 3-j symbols. H_k may be expanded in the products (3.5):

$$H_k = \sum_{m_1,m_2,m_3}\begin{pmatrix} m_1 & m_2 & m_3\\ j_1 & j_2 & j_3\end{pmatrix} v_{m_1}^{j_1}(\zeta')v_{m_2}^{j_2}(\zeta'')v_{m_3}^{j_3}(\zeta''')\,, \tag{3.12}$$

and the expansion coefficients are the 3-j symbols.[12]

The invariance relation $T_U^{(3)}H_k = H_k$ is equivalent to the equations

$$\sum_{\mu_1\mu_2\mu_3}\mathfrak{D}_{\mu_1}^{j_1m_1}(U)\mathfrak{D}_{\mu_2}^{j_2m_2}(U)\mathfrak{D}_{\mu_3}^{j_3m_3}(U)\begin{pmatrix}\mu_1 & \mu_2 & \mu_3\\ j_1 & j_2 & j_3\end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m_3\\ j_1 & j_2 & j_3\end{pmatrix}.$$

Using the relations $v_\mu^{j_3} = \sum_{m_3} w_{m_3}^{j_3} C_{m_3\mu}^{j_3}$, we have

$$H_k = \sum_{m_1m_2m_3}\begin{pmatrix} m_1 & m_2 & j_3\\ j_1 & j_2 & m_3\end{pmatrix} v_{m_1}^{j_1}(\zeta')v_{m_2}^{j_2}(\zeta'')w_{m_3}^{j_3}(\zeta''') \tag{3.13}$$

$$\begin{pmatrix} m_1 & m_2 & j_3\\ j_1 & j_2 & m_3\end{pmatrix} = \sum_\mu C_{m_3\mu}^{j_3}\begin{pmatrix} m_1 & m_2 & \mu\\ j_1 & j_2 & j_3\end{pmatrix} = (-1)^{j_3+m_3}\begin{pmatrix} m_1 & m_2 & -m_3\\ j_1 & j_2 & j_3\end{pmatrix} \tag{3.13a}$$

Hence, by (3.8a) and (3.8b),

$$\psi_{m_3}^{j_3} = (2j_3+1)^{1/2}\sum_{m_1m_2}\begin{pmatrix} m_1 & m_2 & j_3\\ j_1 & j_2 & m_3\end{pmatrix} v_{m_1}^{j_1}(\zeta')v_{m_2}^{j_2}(\zeta'')\,. \tag{3.14}$$

This last equation relates the vector coupling (V-C) coefficients to the 3-j symbols. [In standard form the V-C coefficients differ from those of (3.14) by the factor $(-1)^{k_1}$, see Wigner,[12] Eq. (24.16), p. 294.]

For later use we add here a few remarks. (1) If in (3.12) or (3.13), F_k is substituted for H_k, the co-

[11] We follow B. L. van der Waerden's derivation (reference 3, p. 69).

[12] The position of the indices m in (3.12) corresponds to Wigner's general definition of co- and contravariant indices (reference 7, pp. 292–296). Since, however, the fully contravariant and the fully covariant 3-j symbols are numerically equal [reference 7, Eq. (24.18a), p. 295] the coefficients in (3.12) are the same as the more familiar ones with the position of j and m reversed. We have also written the matrix elements of D^j in accordance with Wigner's rules, but we follow Wigner in writing v_m^j, w_m^j, etc., irrespective of their transformation properties.

efficients will be divided by $\Delta(j_1,j_2,j_3)$, and we shall write

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix}_F = \left(\frac{(J+1)!}{[k!]}\right)^{1/2} \begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix}, \tag{3.15a}$$

and similarly for the 3-j symbol in (3.13a). (2) By (2.16) and (2.16b), $w^j_m(\zeta) = v^j_m(\Gamma\zeta)$. Consequently, if in (3.13), H_k is evaluated for ζ', ζ'', $\Gamma^{-1}\zeta'''$, there appears on the right-hand side $w^{j_3}_{m_3}(\Gamma^{-1}\zeta''') = v^{j_3}_{m_3}(\zeta''')$. If a similar transformation is carried out on ζ'', one obtains

$$F_k(\zeta',\Gamma^{-1}\zeta'',\Gamma^{-1}\zeta''') = \sum_{m_1,m_2,m_3} \begin{pmatrix} m_1 & j_2 & j_3 \\ j_1 & m_2 & m_3 \end{pmatrix}_F \times v^{j_1}_{m_1}(\zeta')v^{j_2}_{m_2}(\zeta'')v^{j_3}_{m_3}(\zeta''') \tag{3.15b}$$

$$\begin{pmatrix} m_1 & j_2 & j_3 \\ j_1 & m_2 & m_3 \end{pmatrix}_F = (-1)^{j_2+m_2+j_3+m_3} \begin{pmatrix} m_1 & -m_2 & -m_3 \\ j_1 & j_2 & j_3 \end{pmatrix}_F \tag{3.15c}$$

e. Computation of the 3-*j symbols.* We introduce two closely related sets of coefficients f, h by setting

$$F_k(\xi,\eta) = \textstyle\sum_{\kappa,\lambda} f_{k\kappa\lambda}\xi^{[\kappa]}\eta^{[\lambda]} \tag{3.16a}$$

$$H_k(\xi,\eta) = \sum_{\kappa,\lambda} h_{k\kappa\lambda} \frac{\xi^{[\kappa]}\eta^{[\lambda]}}{([\kappa!][\lambda!])^{1/2}} \tag{3.16b}$$

$$h_{k\kappa\lambda} = \left(\frac{\prod_{\alpha=1}^{3} k_\alpha!\kappa_\alpha!\lambda_\alpha!}{(J+1)!}\right)^{1/2} f_{k\kappa\lambda}\,. \tag{3.16c}$$

In view of (3.5), comparison of (3.12) and (3.16b) shows that

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = h_{k\kappa\lambda} \tag{3.17}$$

$$k_\alpha = J - 2j_\alpha\,,\quad \kappa_\alpha = j_\alpha + m_\alpha\,,$$
$$\lambda_\alpha = j_\alpha - m_\alpha\ (\alpha = 1,2,3)\,. \tag{3.17a}$$

Although the nine integers k, κ, λ may seem highly redundant they are better suited to expressing the full symmetry of the 3-j symbols than are the customary j and m. (A similar situation prevails in the case of the 6-j symbols as will be seen in the next section.)

Equations (3.15) define the coefficients f and h for all k,κ,λ, but since [by (3.9)] F_k is homogeneous of order $k_1+k_2+k_3$ in the ξ_α as well as the η_α, f and h vanish unless

$$\kappa_1+\kappa_2+\kappa_3 = \lambda_1+\lambda_2+\lambda_3 = k_1+k_2+k_3 = J\,. \tag{3.18}$$

This condition corresponds to $m_1+m_2+m_3 = 0$.

To compute $f_{k\kappa\lambda}$ we simply apply the binomial theorem to the powers of δ_α. Let

$$\frac{\delta_1^{k_1}}{k_1!} = \sum_{p_1+q_1=k_1} \frac{(\xi_2\eta_3)^{p_1}(-\xi_3\eta_2)^{q_1}}{p_1!q_1!}\,,$$
$$\frac{\delta_2^{k_2}}{k_2!} = \sum_{p_2+q_2=k_2} \frac{(\xi_3\eta_1)^{p_2}(-\xi_1\eta_3)^{q_2}}{p_2!q_2!}\,,$$
$$\frac{\delta_3^{k_3}}{k_3!} = \sum_{p_3+q_3=k_3} \frac{(\xi_1\eta_2)^{p_3}(-\xi_2\eta_1)^{q_3}}{p_3!q_3!}\,.$$

Then,

$$f_{k\kappa\lambda} = \sum \frac{(-1)^{q_1+q_2+q_3}}{p_1!p_2!p_3!q_1!q_2!q_3!}\,. \tag{3.19}$$

The summation extends over all non-negative integers p_α, q_α which satisfy the conditions summarized in the following matrix equation:

$$L \equiv \begin{pmatrix} k_1 & k_2 & k_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = \begin{pmatrix} q_1+p_1 & q_2+p_2 & q_3+p_3 \\ q_2+p_3 & q_3+p_1 & q_1+p_2 \\ q_3+p_2 & q_1+p_3 & q_2+q_1 \end{pmatrix} \equiv Q\,. \tag{3.19a}$$

Equation (3.19) reduces to a simple sum, because all p_α, q_α may be expressed by any one of them. Let $q_3 = z$. Then $p_1 = \kappa_2 - z$, $p_2 = \lambda_1 - z$, $p_3 = k_3 - z$; $q_1 = k_1 - \kappa_2 + z$, $q_2 = k_2 - \lambda_1 + z$, and the sum extends over those z for which all p and q are non-negative. (This is Racah's expression.[13]) If μ is the minimum of the entries of L the sum has $\mu+1$ terms.

In L the elements of each row as well as the elements of each column add up to J, [see (3.17a) and (3.18)]. In Q all row sums and column sums are equal by definition, the common value being $\sum_{\alpha=1}^{3}(p_\alpha+q_\alpha)$.

Finally, one may write f_L instead of $f_{k\kappa\lambda}$, and similarly h_L. Denoting L's matrix elements by $l_{i\alpha}$ (where i denotes the row and α the column), (3.16) and (3.17) may be summarized by

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = h_L = \left(\frac{\prod_{i,\alpha} l_{i\alpha}!}{(J+1)!}\right)^{1/2} f_L\,. \tag{3.19b}$$

f. The generating function Φ *and the symmetries of the* 3-*j symbol.* The generating function of the 3-j symbols is defined by[14]

$$\Phi(\tau,\xi,\eta) = \textstyle\sum_k \tau^{[k]}F_k(\xi,\eta) = \displaystyle\sum_{k,\kappa,\lambda} f_{k\kappa\lambda}\tau^{[k]}\xi^{[\kappa]}\eta^{[\lambda]}$$
$$= \textstyle\sum_L f_L\tau^{[k]}\xi^{[\kappa]}\eta^{[\lambda]}\,, \tag{3.20a}$$

[13] G. Racah, Phys. Rev. **62**, 438 (1942), Eq. (16). See also A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), Eq. (3.6.11).

[14] This corresponds to the function defined by Schwinger (reference 6) in Eq. (3.42).

where $\tau = (\tau_1, \tau_2, \tau_3)$ is a triple of complex variables. It will be useful to arrange the nine variables τ, ξ, η in a matrix

$$\Xi = \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix}$$

in analogy to L. It follows at once from (3.9) that

$$\Phi(\tau,\xi,\eta) \equiv \Phi(\Xi) = \exp\left(\sum_{\alpha=1} \tau_\alpha \delta_\alpha\right) = \exp\left(D(\tau,\xi,\eta)\right) = \exp(\det \Xi) \quad (3.21)$$

$$D(\tau,\xi,\eta) = \det \Xi = \begin{vmatrix} \tau_1 & \tau_2 & \tau_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} . \quad (3.21\text{a})$$

The elementary symmetries of the determinant D now yield corresponding symmetries of the coefficients f and h.[15] We combine the following facts.

(α) For any 3×3 matrix A, let $P(A)$ be the matrix obtained by some fixed permutation of A's elements (such as the transposition of two rows or two columns, etc.). Then[16]

$$\Phi(P(\Xi)) = \sum f_{P(L)} \tau^{[k]} \xi^{[\kappa]} \eta^{[\lambda]}$$

(β) Set $\exp[-D(\tau,\xi,\eta)] = \Phi'(\tau,\xi,\eta) \equiv \Phi'(\Xi)$.

Evidently

$$\Phi'(\Xi) = \Phi(-\tau,\xi,\eta) = \sum (-1)^J f_L \tau^{[k]} \xi^{[\kappa]} \eta^{[\lambda]}$$

because $k_1 + k_2 + k_3 = J$.

By comparing coefficients we conclude therefore

(a) If $\det[P(\Xi)] = \det \Xi$, then $\Phi(P(\Xi)) = \Phi(\Xi)$, and hence $f_{P(L)} = f_L$.

(b) If $\det[P(\Xi)] = -\det \Xi$, then $\Phi(P(\Xi)) = \Phi'(\Xi)$, hence $f_{P(L)} = (-1)^J f_L$.

This leads to the final result:

First case: If P is (1) an even permutation of rows, (2) an even permutation of columns, (3) the interchange of rows and columns then

$$f_{P(L)} = f_L \quad \text{aud} \quad h_{P(L)} = h_L . \quad (3.22)_{\text{I}}$$

Second case: If P is (1) an odd permutation of rows, (2) an odd permutation of columns then

$$f_{P(L)} = (-1)^J f_L \quad \text{and} \quad h_{P(L)} = (-1)^J h_L . \quad (3.22)_{\text{II}}$$

[The equations for the coefficients f, which follow immediately from the above analysis, imply those for the coefficients h because neither the numerator nor the denominator of the normalization constant in (3.19b) is affected by the operations P in (3.22).]

The operations listed under I and II generate the symmetry group of 72 elements discovered by Regge. Previously, only the following more evident symmetry operations had been noticed: (1) Permutation of the *columns of* L, i.e., simultaneous permutation of j_α and m_α. (2) Transposition of the second and third row in L, i.e., changing the sign of all m_α.

g. The norm of F_k. As follows from (1.13b), for fixed τ the generating function $\Phi_\tau \equiv \Phi(\tau,\xi,\eta)$ is an element of $\mathfrak{F}_6$ as long as the τ_α are small enough. (The precise condition, which is $\sum_{\alpha=1}^{3} |\tau_\alpha|^2 < 1$, need not concern us.) The inner product of two such functions Φ_τ and Φ_τ' (taken on $\mathfrak{F}_6$) is then, by (3.20a)

$$(\Phi_\tau, \Phi_{\tau'}) = \sum_{k,k'} \bar{\tau}^{[k]} \tau'^{[k']} (F_k, F_{k'}) \quad (3.23)$$

In the computation of the inner product according to (1.4) we may separate the ξ and the η integrations, so that

$$(\Phi_\tau, \Phi_{\tau'}) = \int \left[\int \overline{\exp D(\tau,\xi,\eta)} \exp D(\tau',\xi,\eta) d\mu_3(\eta) \right] \times d\mu_3(\xi) . \quad (3.23\text{a})$$

In ordinary vector notation $D(\tau,\xi,\eta) = (\tau \times \xi) \cdot \eta$. Thus the inner integral is of the form (1.13) (if η is identified with z), with

$$a = \overline{\tau \times \xi} , \quad b = \overline{\tau' \times \xi} ,$$

and it has the value $\exp(\bar{b} \cdot a)$, where

$$\bar{b} \cdot a = (\tau' \cdot \bar{\tau})(\bar{\xi} \cdot \xi) - (\tau' \cdot \bar{\xi})(\bar{\tau} \cdot \xi) = \bar{\xi} \cdot A \xi ,$$

A denoting the matrix with elements

$$a_{\alpha\beta} = (\tau' \cdot \bar{\tau}) \delta_{\alpha\beta} - \tau'_\alpha \bar{\tau}_\beta .$$

Hence, (Φ_τ, Φ_τ') is a Laplacian integral of the form

$$(\Phi_\tau, \Phi_\tau') = \int \exp(\bar{\xi} \cdot A \xi) d\mu_3(\xi)$$

and, by Eq. (A5) in the Appendix,

$$(\Phi_\tau, \Phi_{\tau'}) = [\det(1 - A)]^{-1} = (1 - \bar{\tau} \cdot \tau')^{-2} . \quad (3.23\text{b})$$

Expanding in a power series one obtains

$$(\Phi_\tau, \Phi_{\tau'}) = \sum_{\mu=0}^{\infty} (\mu + 1)(\bar{\tau} \cdot \tau')^\mu = \sum_k \frac{(|k| + 1)!}{[k!]} \bar{\tau}^{[k]} \tau'^{[k]} ,$$

---

[15] This proof is essentially the same as Regge's [reference 5(a)].

[16] Consider a power series in n variables $x_r, G(x_1, x_2, \ldots, x_n) = \Sigma_{i_1, i_2, \ldots} \gamma_{i_1 i_2 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$, and let $G'(x_1, x_2, \ldots, x_n) \equiv G(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$ for some permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of the integers $1, 2, \ldots, n$. Then $G'(x_1, x_2, \ldots, x_n) = \Sigma_{i_1, i_2 \ldots} \times \gamma_{i_{\pi_1} i_{\pi_2} \ldots i_{\pi_n}} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$. In our case the variables τ, ξ, η correspond to the x_r, and f_L to the coefficients $\gamma_{i_1 i_2 \ldots i_n}$.

where $|k| = k_1 + k_2 + k_3$. Comparison with (3.23) yields

$$(F_k, F_{k'}) = \begin{cases} 0 & \text{if } k' \neq k \\ (J+1)!/[k!] & \text{if } k' = k \end{cases} \quad (J = |k|)\,, \tag{3.24}$$

as announced in Sec. 3d.

h. Recursion relations. For the derivatives of Φ one finds

$$\partial\Phi/\partial\tau_1 = (\xi_2\eta_3 - \xi_3\eta_2)\Phi\,, \quad \partial\Phi/\partial\xi_1 = (\eta_2\tau_3 - \eta_3\tau_2)\Phi\,,$$
$$\partial\Phi/\partial\eta_1 = (\tau_2\xi_3 - \tau_3\xi_2)\Phi$$

and six more equations obtained by cyclic permutations. If the expansion (3.20b) is inserted numerous relations between the coefficients f result, most of which are of course known. We mention two examples.

(1) $\partial\Phi/\partial\tau_1 = (\xi_2\eta_3 - \xi_3\eta_2)\Phi$ leads to

$$k_1 f_{k\kappa\lambda} = f_{k_1-1\cdots\kappa_2-1\cdots\lambda_3-1} - f_{k_1-1\cdots\kappa_3-1\cdots\lambda_2-1}\,.$$

(On the right-hand side only those indices are marked which differ from the corresponding ones of the left-hand terms.)

(2) $\tau_3\,\partial\Phi/\partial\tau_2 + \xi_3\,\partial\Phi/\partial\xi_2 + \eta_3\,\partial\Phi/\partial\eta_2 = 0$ leads to

$$-k_2 f_{k\kappa\lambda} = (1+\kappa_2) f_{\cdot k_2-1,k_3+1,\kappa_2+1,\kappa_3-1\cdots} + (1+\lambda_2) f_{\cdot k_2-1,k_3+1,\cdots\lambda_2+1,\lambda_3-1}$$

Upon insertion of the normalization constant and "translation" into the (j,m)-notation one obtains the following two formulas for 3-j symbols.

$$(1)\quad [(J+1)(J-2j_1)]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = [(j_2+m_2)(j_3-m_3)]^{1/2}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ m_1 & m_2-\frac{1}{2} & m_3+\frac{1}{2} \end{pmatrix} - [(j_2-m_2)(j_3+m_3)]^{1/2}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ m_1 & m_2+\frac{1}{2} & m_3-\frac{1}{2} \end{pmatrix}$$

$$(2)\quad [(J-2j_2)(J+1-2j_3)]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + ([j_2+m_2+1)(j_3+m_3)]^{1/2}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3+\frac{1}{2} \\ m_1 & m_2-\frac{1}{2} & m_3+\frac{1}{2} \end{pmatrix} + [(j_2-m_2+1)(j_3-m_3)]^{1/2}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3+\frac{1}{2} \\ m_1 & m_2+\frac{1}{2} & m_3-\frac{1}{2} \end{pmatrix} = 0\,.$$

4. RACAH COEFFICIENTS

This section deals with the Racah coefficients (in the form of Wigner's 6-j symbols). The main objective is the construction and analysis of a generating function, and its application to a discussion of the symmetries of the Racah coefficients.

a. Formal preliminaries. In terms of 3-j symbols the 6-j symbol is defined as follows.

$$\begin{Bmatrix} j_{23} & j_{31} & j_{12} \\ j_{01} & j_{02} & j_{03} \end{Bmatrix} = \sum_{m_\alpha, m'_\alpha} \begin{pmatrix} m'_1 & m'_2 & m'_3 \\ j_{23} & j_{31} & j_{12} \end{pmatrix}\begin{pmatrix} j_{23} & m_2 & j_{03} \\ m'_1 & j_{02} & m_3 \end{pmatrix} \times \begin{pmatrix} j_{01} & j_{31} & m_3 \\ m_1 & m'_2 & j_{03} \end{pmatrix}\begin{pmatrix} m_1 & j_{02} & j_{12} \\ j_{01} & m_2 & m'_3 \end{pmatrix} \tag{4.1}$$

(The summation of the m_α and m'_α extends over all values compatible with the associated j.)

The notation introduced here is meant to emphasize the tetrahedral symmetry of the 6-j symbol. It alludes to a tetrahedron with vertices V_α ($\alpha = 0,1,2,3$) and edges $j_{\alpha\beta}$. The four 3-j symbols in (4.1) correspond, respectively, to the triangles opposite V_0, V_1, V_2, V_3. We define

$$j_{\alpha\beta} \equiv j_{\beta\alpha} \quad (\alpha \neq \beta; \alpha,\beta = 0,1,2,3)\,. \tag{4.2}$$

In the sequel *different* subscripts in an equation containing $j_{\alpha\beta}$ or $k_{\alpha\beta}$ denote *different* integers taken from the sequence 0,1,2,3. Thus the perimeter of the αth triangle (with vertices V_β, V_γ, V_δ) is

$$J_\alpha = j_{\beta\gamma} + j_{\gamma\delta} + j_{\delta\beta}\,. \tag{4.3}$$

In accordance with (3.10a) and (3.10b), we set

$$k_{\alpha\beta} = J_\alpha - 2j_{\gamma\delta} = j_{\gamma\beta} + j_{\delta\beta} - j_{\gamma\delta} \quad (\alpha \neq \beta)\,. \tag{4.4}$$

The twelve $k_{\alpha\beta}$ depend on the *ordered* pairs (α,β) while in the definition of $j_{\alpha\beta}$ the order is irrelevant; α refers to the triangle, β to the vertex opposite $j_{\gamma\delta}$. The inverse relation is

$$j_{\gamma\delta} = \tfrac{1}{2}(k_{\alpha\gamma} + k_{\alpha\delta})\,. \tag{4.5}$$

Since $j_{\gamma\delta}$ belongs to the two triangles opposite V_α and V_β, we have also $j_{\gamma\delta} = \frac{1}{2}(k_{\beta\gamma} + k_{\beta\delta})$, so that the k satisfy the *compatibility* conditions

$$k_{\alpha\gamma} + k_{\alpha\delta} = k_{\beta\gamma} + k_{\beta\delta}\,. \tag{4.6}$$

Further useful relations are

$$k_{\alpha\beta} + k_{\alpha\gamma} + k_{\alpha\delta} = J_\alpha\,, \tag{4.7}$$

$$k_{\alpha\beta} - k_{\beta\alpha} = k_{\gamma\beta} - k_{\delta\alpha} = J_\alpha - J_\beta\,. \tag{4.7a}$$

The (triangle) conditions on the $j_{\alpha\beta}$ to lead to non-vanishing 3-j symbols in (4.1) are simply: *$k_{\alpha\beta}$ are non-negative integers.*[17]

Any set of twelve numbers $k_{\alpha\beta}$ ($\alpha\neq\beta$) which satisfy the compatibility conditions (4.6) will be called *"tetrahedral."* Given such a tetrahedral set. If $j_{\gamma\delta}$ is defined by (4.5) then $j_{\gamma\delta}=j_{\delta\gamma}$, and the relations (4.4) hold.

The ordered pairs $\alpha\beta$ are conveniently arranged in four triads T_α defined as follows:

$$T_\alpha \textit{ contains the three pairs with first element } \alpha\,. \quad (4.8)$$

We shall also need the four "transposed" triads T^*_α:

$$T^*_\alpha \textit{ contains the three pairs with second element } \alpha\,. \quad (4.8a)$$

Lastly we introduce three tetrads W_i:

$$W_1\!:\!(01,10,23,32)\,,\quad W_2\!:\!(02,20,31,13)\,,$$
$$W_3\!:\!(03,30,12,21)\,. \quad (4.8b)$$

In the functions F_k connected with the four 3-j symbols in (4.1) the numbers $k_{\alpha\beta}$ appear in the order which corresponds to the order of the $j_{\gamma\delta}$ by Eq. (4.4). Thus we have in succession

$$(k_{01},k_{02},k_{03})(k_{10},k_{13},k_{12})(k_{23},k_{20},k_{21})(k_{32},k_{31},k_{30})\,. \quad (4.9)$$

It is seen that the four triples correspond to the four triads T_α. Moreover, the first, second, and third element in each triple corresponds, respectively, to W_1, W_2, W_3.

Let $\alpha\to\pi_\alpha$ be a permutation of the four integers 0,1,2,3 (which may be interpreted as a permutation of the four vertices V_α of the tetrahedron), and define

$$j'_{\alpha\beta}=j_{\pi_\alpha,\pi_\beta}\quad k'_{\alpha\beta}=k_{\pi_\alpha,\pi_\beta} \quad (4.10)$$

Then Eqs. (4.4) to (4.6) remain valid. The triads are permuted accordingly ($T\to T_{\pi\alpha}$, $T^*\to T^*{}_{\pi\alpha}$) while the tetrads are subject to a permutation π' of three integers, which depends on π ($W_i\to W_{\pi'_i}$).

b. The generating function $R(\tau)$. The 6-j symbol (4.1) is a function $r(k)$, where k represents the $k_{\alpha\beta}$. If we replace, on the right-hand side of (4.1), the 3-j symbols ( ) by the corresponding symbols ( )$_F$ [see (3.15a)], i.e., if we divide by all four triangle coefficients, we obtain a function $s(k)$ such that

$$r(k)=\left(\frac{\prod_{\alpha,\beta}k_{\alpha\beta}!}{\prod_\alpha (J_\alpha+1)!}\right)^{1/2} s(k)\,. \quad (4.11)$$

[17] These conditions are, however, insufficient to insure the existence of a tetrahedron with edges $j_{\alpha\beta}$. We deal here with the combinatorial rather than with the metric properties of a tetrahedron.

So far $r(k)$ and $s(k)$ are defined for "tetrahedral" sets $k_{\alpha\beta}$. In all other cases we set $r(k)=s(k)=0$.

In terms of 12 complex variables $\tau_{\alpha\beta}(\alpha\neq\beta)$ we now define the generating function of the 6-j symbols by

$$R(\tau)=\sum\nolimits_{k_{\alpha\beta}} s(k)\prod\nolimits_{\alpha,\beta}\tau_{\alpha\beta}^{k_{\alpha\beta}}\,. \quad (4.12)$$

The function $R(\tau)$ may be expressed as an integral over the product of four generating functions Φ [see (3.20a)], which are related to the 3-j symbols in (4.1). To this end we introduce six pairs of complex variables $\zeta^\alpha=(\xi_\alpha,\eta_\alpha)$ and $\theta^\alpha=(\xi'_\alpha,\eta'_\alpha)$ ($\alpha=1,2,3$) corresponding, respectively, to the summation indices m_α and m'_α in (4.1). Then

$$R(\tau)=\int\Phi_0\Phi_1\Phi_2\Phi_3 d\mu_6(\xi',\eta')d\mu_6(\xi,\eta)\,, \quad (4.13)$$

$$\Phi_0=\Phi(\tau_{01},\tau_{02},\tau_{03};\theta^1,\theta^2,\theta^3)\,,$$
$$\Phi_1=\Phi(\tau_{10},\tau_{13},\tau_{12};{}^t\Gamma\overline{\theta^1},\zeta^2,{}^t\Gamma\overline{\zeta^3})\,,$$
$$\Phi_2=\Phi(\tau_{23},\tau_{20},\tau_{21};{}^t\Gamma\overline{\zeta^1},{}^t\Gamma\overline{\theta^2},\zeta^3)\,,$$
$$\Phi_3=\Phi(\tau_{32},\tau_{31},\tau_{30};\zeta^1,{}^t\Gamma\overline{\zeta^2},{}^t\Gamma\overline{\theta^3})\,. \quad (4.13a)$$

[The variables $\tau_{\alpha\beta}$ in the four functions Φ correspond to the four triples (4.9), to an upper index m_α in (4.1) corresponds ζ^α, to a lower index m_α corresponds ${}^t\Gamma\,\overline{\zeta^\alpha}$ as an argument of Φ, similarly for the m'_α. Note that ${}^t\Gamma\,\overline{\zeta^\alpha}=(\bar\eta_\alpha,-\bar\xi_\alpha)$.]

The proof for the integral representation (4.13) is straightforward, but writing it out in full would lead to rather unmanageable equations. It will suffice to consider the contribution of the ζ^1 integration to (4.13). Φ_0 and Φ_1 are free of ζ^1, so that only Φ_2 and Φ_3 need be considered. Now

$$\Phi_2=\sum_{k_{23},k_{20},k_{21}}\tau_{23}^{k_{23}}\tau_{20}^{k_{20}}\tau_{21}^{k_{21}}F_{k_{23},k_{20},k_{21}}({}^t\Gamma\overline{\zeta^1},{}^t\Gamma\overline{\theta^2},\zeta^3)\,,$$
$$\Phi_3=\sum_{k_{32},k_{31},k_{30}}\tau_{32}^{k_{32}}\tau_{31}^{k_{31}}\tau_{30}^{k_{30}}F_{k_{32},k_{31},k_{30}}(\zeta^1,{}^t\Gamma\overline{\zeta^2},{}^t\Gamma\overline{\theta^3})\,.$$

The problem is further simplified by studying the contribution of just one F chosen from Φ_2 and one F from Φ_3. By (3.15b), since ${}^t\Gamma=\Gamma^{-1}$,

$$F_{k_{23},k_{20},k_{21}}({}^t\Gamma\overline{\zeta^1},{}^t\Gamma\overline{\theta^2},\zeta^3)=\sum_{\mu_1,\mu_2,\mu_3}\begin{pmatrix}j_1 & j_2 & \mu_3\\ \mu_1 & \mu_2 & j_3\end{pmatrix}_F \times\overline{v^{j_1}_{\mu_1}(\zeta^1)}\,\overline{v^{j_2}_{\mu_2}(\theta^2)}\,v^{j_3}_{\mu_3}(\zeta^3)\,,$$

where $2j_1=k_{20}+k_{21}$, $2j_2=k_{23}+k_{21}$, $2j_3=k_{23}+k_{20}$. Similarly,

$$F_{k_{32},k_{31},k_{30}}(\zeta^1,{}^t\Gamma\overline{\zeta^2},{}^t\Gamma\overline{\theta^3})=\sum_{\nu_1\nu_2\nu_3}\begin{pmatrix}\nu_1 & l_2 & l_3\\ l_1 & \nu_2 & \nu_3\end{pmatrix}_F \times v^{l_1}_{\nu_1}(\zeta^1)\,\overline{v^{l_2}_{\nu_2}(\zeta^2)}\,\overline{v^{l_3}_{\nu_3}(\theta^3)}\,,$$

with $2l_1 = k_{30} + k_{31}$, $2l_2 = k_{32} + k_{30}$, $2l_3 = k_{32} + k_{31}$.

If one multiplies the two functions F and integrates over ζ^1, one finds, due to the orthonormality of the v_μ^j: (1) The result is zero if $j_1 \neq l_1$. (2) If $j_1 = l_1$, i.e.,

$$k_{20} + k_{21} = k_{30} + k_{31} (= 2j_{01}) \qquad (4.14)$$

the result is

$$\sum_{\mu_2,\mu_3,\nu_2,\nu_3} \left\{ \sum_{\mu} \begin{pmatrix} j_{01} & j_2 & \mu_3 \\ \mu & \mu_2 & j_3 \end{pmatrix}_F \begin{pmatrix} \mu & l_2 & l_3 \\ j_{01} & \nu_2 & \nu_3 \end{pmatrix}_F \right\} \times \overline{v_{\mu_2}^{j_2}(\theta^2)}\, v_{\mu_3}^{j_3}(\zeta^3)\, \overline{v_{\nu_2}^{l_2}(\zeta^2)}\, \overline{v_{\nu_3}^{l_3}(\theta^3)}\,.$$

Continuing, step by step, with the remaining variables, one obtains, in analogy to (4.14), the remaining five compatibility relations (4.6), which shows that only the tetrahedral sets $k_{\alpha\beta}$ give a nonvanishing contribution, and in addition it is seen that the contribution of a tetrahedral set is precisely the 6-j symbol divided by the four triangle coefficients, i.e., $s(k)$, as it should be.

The computation of $R(\tau)$ carried out below gives a simple result,[18] viz.,

$$R(\tau) = [G(\tau)]^{-2}\,, \quad G(\tau) = 1 + \sum_{\alpha=0}^{3} a_\alpha + \sum_{i=1}^{3} b_i \qquad (4.15)$$

$$a_0 = \tau_{10}\tau_{20}\tau_{30}\,, \quad a_1 = \tau_{01}\tau_{31}\tau_{21}\,, \quad a_2 = \tau_{32}\tau_{02}\tau_{12}\,, \quad a_3 = \tau_{23}\tau_{13}\tau_{03} \qquad (4.15\text{a})$$

$$b_1 = \tau_{01}\tau_{10}\tau_{23}\tau_{32}\,, \quad b_2 = \tau_{02}\tau_{20}\tau_{13}\tau_{31}\,, \quad b_3 = \tau_{03}\tau_{30}\tau_{12}\tau_{21}\,. \qquad (4.15\text{b})$$

If the τ are sufficiently small (for example, $|\tau_{\alpha\beta}| \leq \frac{1}{2}$ for all a,β), the integral (4.13) converges absolutely, the operations carried out below are legitimate, and the power series (4.12) may be obtained from a term-by-term integration.

c. Computation of $R(\tau)$. By (3.21), the integrand in (4.13) has the form $\exp(D_0 + D_1 + D_2 + D_3)$, where

$$D_0 = \begin{vmatrix} \tau_{01} & \tau_{02} & \tau_{03} \\ \xi_1' & \xi_2' & \xi_3' \\ \eta_1' & \eta_2' & \eta_3' \end{vmatrix}, \quad D_1 = \begin{vmatrix} \tau_{10} & \tau_{13} & \tau_{12} \\ \bar\eta_1' & \xi_2 & \bar\eta_3 \\ -\bar\xi_1' & \eta_2 & -\bar\xi_3 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} \tau_{23} & \tau_{20} & \tau_{21} \\ \bar\eta_1 & \bar\eta_2' & \xi_3 \\ -\bar\xi_1 & -\bar\xi_2' & \eta_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} \tau_{32} & \tau_{31} & \tau_{30} \\ \xi_1 & \bar\eta_2 & \bar\eta_3' \\ \eta_1 & -\bar\xi_2 & -\bar\xi_3' \end{vmatrix}.$$

The cyclic symmetry of the exponent $\sum_{\alpha=0}^{3} D_\alpha$ in the indices 1,2,3 greatly reduces the work in this computation. In fact, only a few terms need actually be calculated. We have

$$D_1 + D_2 + D_3 = \sum_{\alpha=1}^{3} (c_\alpha \bar\xi_\alpha' + d_\alpha \bar\eta_\alpha') - E$$

$$c_1 = \tau_{12}\xi_2 - \tau_{13}\bar\eta_3\,, \quad c_2 = \tau_{23}\xi_3 - \tau_{21}\bar\eta_1\,,$$
$$c_3 = \tau_{31}\xi_1 - \tau_{32}\bar\eta_2$$
$$d_1 = \tau_{12}\eta_2 + \tau_{13}\bar\xi_3\,, \quad d_2 = \tau_{23}\eta_3 + \tau_{21}\bar\xi_1\,,$$
$$d_3 = \tau_{31}\eta_1 + \tau_{32}\bar\xi_2$$
$$E = \tau_{10}(\bar\xi_3\xi_2 + \bar\eta_3\eta_2) + \tau_{20}(\bar\xi_1\xi_3 + \bar\eta_1\eta_3) + \tau_{30}(\bar\xi_2\xi_1 + \bar\eta_2\eta_1)\,.$$

First step: Integration over ξ',η'. By (1.12a),

$$\int \exp(c\cdot\bar\xi' + d\cdot\bar\eta')\exp(D_0 - E)\,d\mu_6(\xi',\eta') = \exp f\,,$$

$$f = \begin{vmatrix} \tau_{01} & \tau_{02} & \tau_{03} \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} - E\,.$$

Inserting c_α and d_α one obtains

$$f = \sum_{\alpha=1}^{3} u_\alpha \bar\delta_\alpha + \sum_{\alpha=1}^{3} v_\alpha \delta_\alpha - \bar\xi\cdot H\xi - \bar\eta\cdot H\eta\,, \qquad (4.16)$$

where δ_α are the determinants in (3.9),

$$u_1 = \tau_{02}\tau_{13}\tau_{32}\,, \quad u_2 = \tau_{03}\tau_{21}\tau_{13}\,, \quad u_3 = \tau_{01}\tau_{32}\tau_{21}$$
$$v_1 = \tau_{03}\tau_{12}\tau_{23}\,, \quad v_2 = \tau_{01}\tau_{23}\tau_{31}\,, \quad v_3 = \tau_{02}\tau_{31}\tau_{12}$$

and H is the matrix

$$H = \begin{pmatrix} a_1 & -\tau_{03}\tau_{12}\tau_{21} & \tau_{20} \\ \tau_{30} & a_2 & -\tau_{01}\tau_{23}\tau_{32} \\ -\tau_{02}\tau_{13}\tau_{31} & \tau_{10} & a_3 \end{pmatrix}$$

the a_α being defined in (4.15a).

Second step: Integration over ξ,η. By (A8) of the Appendix

$$\int \exp f\,d\mu_6(\xi,\eta) = [\det(1 + H) - u\cdot v - u\cdot Hv]^{-2}\,, \qquad (4.16\text{a})$$

and a straightforward computation gives the expression (4.15).

d. Symmetries of the 6-j symbols. It is useful to arrange $k_{\alpha\beta}$ and $\tau_{\alpha\beta}$ in matrix form

$$\mathcal{K} = \begin{pmatrix} k_{10} & k_{20} & k_{30} \\ k_{01} & k_{31} & k_{21} \\ k_{32} & k_{02} & k_{12} \\ k_{23} & k_{13} & k_{03} \end{pmatrix} \qquad \mathfrak{J} = \begin{pmatrix} \tau_{10} & \tau_{20} & \tau_{30} \\ \tau_{01} & \tau_{31} & \tau_{21} \\ \tau_{32} & \tau_{02} & \tau_{12} \\ \tau_{23} & \tau_{13} & \tau_{03} \end{pmatrix} \qquad (4.17)$$

such that the rows correspond to the four transposed triads T_α^*, and the columns to the three tetrads W_1, W_2, W_3. In the generating function (4.15), a_α is the product of the elements in the αth row, and b_i the product of the elements in the ith column of $\mathfrak{J}$.

[18] Apart from the notation this coincides with Schwinger's Eq. (4.18) in reference 6.

By (4.7), J_α is the sum of all k belonging to the triad T_α. Similarly, we may introduce J^*_α as the sum of all k belonging to the transposed triad T^*_α, and w_i as the sum of all k belonging to the tetrad W_i. (Equivalently, J^*_α is the sum of the elements in the αth row, and w_i the sum of the elements in the ith column of $\mathcal{K}$.) Clearly,

$$\sum_{\alpha=0}^{3} J_\alpha = \sum_{\alpha=0}^{3} J^*_\alpha = \sum_{i=1}^{3} w_i = |\mathcal{K}| \equiv \sum_{\alpha,\beta} k_{\alpha\beta} \,. \quad (4.17\text{a})$$

For a tetrahedral $\mathcal{K}$, by (4.7a),

$$J^*_\alpha - J_\alpha = \textstyle\sum_\beta (k_{\beta\alpha} - k_{\alpha\beta}) = \sum_\beta (J_\beta - J_\alpha) \,,$$

or

$$J^*_\alpha = |\mathcal{K}| - 3J_\alpha \,. \quad (4.17\text{b})$$

In analogy to (3.20) we write

$$R(\mathfrak{J}) = \textstyle\sum_K s(\mathcal{K}) \prod_{\alpha,\beta} \tau_{\alpha\beta}^{k_{\alpha\beta}} \,.$$

If we denote, as in 3f, by $P(\mathfrak{J})$ and $P(\mathcal{K})$ the matrices obtained from $\mathfrak{J}$ and $\mathcal{K}$, respectively, by a fixed permutation of their elements, then

$$R(P(\mathfrak{J})) = \textstyle\sum_K s(P(\mathcal{K})) \prod_{\alpha,\beta} \tau_{\alpha\beta}^{k_{\alpha\beta}} \,.$$

Consequently, if $R(P(\mathfrak{J})) = R(\mathfrak{J})$, then $s(P(\mathcal{K})) = s(\mathcal{K})$ for all $\mathcal{K}$.

This remark yields the *Regge group of symmetry operations*. In fact, R is invariant (1) under any permutation of the rows (this permutes the a_α and leaves the b_i invariant), (2) under any permutation of the columns (this permutes the b_i, but leaves the a_α invariant).

Note that the J_α are permuted by the first type and left unchanged by the second type of operations, as follows from (4.17b). Hence the normalization factor in (4.11) is unaffected by all these operations, and what we proved for $s(\mathcal{K})$ holds also for $r(\mathcal{K})$, i.e., for the 6-j symbols. [For any permutation of the k, the corresponding transformation of the j may be derived from (4.4) and (4.5).]

The symmetry operations listed above generate the group $S_4 \times S_3$ (the direct product of the symmetric groups S_4 and S_3) of order $24 \cdot 6 = 144$. Its elements are the products of permutations π of the rows and σ, say, of the columns of $\mathcal{K}$, which may be chosen independently of each other. The previously known symmetry operations are the transformations (4.10), induced by a permutation of the vertices of the tetrahedron, where σ is no longer independent of π, but equals π'.

e. Explicit expression for the 6-j symbol. Expanding $R = G^{-2}$ in a power series one obtains

$$R = \sum_{z=0}^{\infty} (-1)^z (z+1) \left(\sum_{\alpha=0}^{3} a_\alpha + \sum_{i=1}^{3} b_i \right)^z$$
$$= \sum_{\nu_\alpha, \omega_i} (-1)^z (z+1)! \left(\prod_{\alpha=0}^{3} \frac{a_\alpha^{\nu_\alpha}}{\nu_\alpha!} \prod_{i=1}^{3} \frac{b_i^{\omega_i}}{\omega_i!} \right),$$

where ν_α, ω_i run independently over all non-negative integers, and $z = \sum_\alpha \nu_\alpha + \sum_i \omega_i$. This leads to

$$s(\mathcal{K}) = \sum \frac{(-1)^z (z+1)!}{\nu_0!\nu_1!\nu_2!\nu_3!\omega_1!\omega_2!\omega_3!} \,. \quad (4.18)$$

The summation extends over all non-negative integers which satisfy the matrix equation

$$\mathcal{K} \equiv \begin{pmatrix} k_{10} & k_{20} & k_{30} \\ k_{01} & k_{31} & k_{21} \\ k_{32} & k_{02} & k_{12} \\ k_{23} & k_{13} & k_{03} \end{pmatrix} = \begin{pmatrix} \nu_0+\omega_1 & \nu_0+\omega_2 & \nu_0+\omega_3 \\ \nu_1+\omega_1 & \nu_1+\omega_2 & \nu_1+\omega_3 \\ \nu_2+\omega_1 & \nu_2+\omega_2 & \nu_2+\omega_3 \\ \nu_3+\omega_1 & \nu_3+\omega_2 & \nu_3+\omega_3 \end{pmatrix} \equiv \mathfrak{N} \quad (4.19)$$

or

$$k_{\alpha\beta} = \nu_\beta + \omega_i \quad (\alpha,\beta) \in W_i \,. \quad (4.19\text{a})$$

The 6-j symbol is then given by

$$\begin{Bmatrix} j_{23} & j_{31} & j_{12} \\ j_{01} & j_{02} & j_{03} \end{Bmatrix} = \left(\frac{\prod_{\alpha,\beta} k_{\alpha\beta}!}{\prod_\alpha (J_\alpha + 1)!} \right)^{1/2} s(\mathcal{K}) \quad (4.19\text{b})$$

[see (4.11)], the j and k being related by Eqs. (4.4) and (4.5).

Apart from its immediate application to the expression (4.18) the equation $\mathcal{K} = \mathfrak{N}$ (where we assume k,ν,ω to be non-negative integers) is of interest as a parametrization of *tetrahedral* $\mathcal{K}$. It is not difficult to show that $\mathcal{K}$ is tetrahedral if and only if it satisfies (4.19) for some $\mathfrak{N}$. Furthermore, for a given $\mathcal{K}$ the equation $\mathcal{K} = \mathfrak{N}$ has $\mu + 1$ solutions, where μ is the value of $\mathcal{K}$'s smallest element, and hence (4.18) has $\mu + 1$ terms.

Action of the symmetry operations. Let P be an element of the Regge group characterized by the permutations π and σ. If $\mathcal{K} = \mathfrak{N}$, then $P(\mathcal{K}) = \mathfrak{N}'$, the parameters of $\mathfrak{N}'$ being given by $\nu'_\alpha = \nu_{\pi\alpha}$, $\omega'_i = \omega_{\sigma i}$.

Racah's formula.[19] To obtain Racah's famous expression for $s(\mathcal{K})$ we have merely to express ν_α and ω_i by z. From (4.19) and (4.7a), $k_{\alpha 0} - k_{0\alpha} = \nu_0 - \nu_\alpha = J_\alpha - J_0$. Furthermore,

$$\nu_0 + J_0 = \nu_0 + k_{01} + k_{02} + k_{03} = \sum_{\alpha=0}^{3} \nu_\alpha + \sum_{i=1}^{3} \omega_i = z$$

[19] Racah, reference 13, Eq. (36.) Edmonds, reference 13, Eq. (6.3.7).

[by (4.19)], hence,

$$\nu_\alpha = z - J_\alpha . \quad (4.20)$$

The first row in (4.19) yields $\omega_i = k_{i0} - \nu_0$, i.e.,

$$\omega_i = t_i - z , \quad t_i = k_{i0} + J_0 . \quad (4.20a)$$

In terms of the j,

$$t_1 = j_{02} + j_{03} + j_{12} + j_{13} ,$$
$$t_2 = j_{03} + j_{01} + j_{23} + j_{21} ,$$
$$t_3 = j_{01} + j_{02} + j_{31} + j_{32} . \quad (4.20b)$$

Inserting ν_α and ω_i in (4.18) one obtains Racah's formula

$$s(\mathcal{K}) = \sum_z \frac{(-1)^z (z+1)!}{\prod_\alpha (z - J_\alpha)! \prod_i (t_i - z)!}$$

the summation to be extended over those z for which all ν_α and $\omega_i \geq 0$.

It is readily shown that $4t_i = w_i + |\mathcal{K}|$. Hence the Regge operations are also described by $J'_\alpha = J_{\pi\alpha}$, $t'_i = t_{\sigma i}$ [see (4.17a)].

Remark. Schwinger has also computed the generating function for the 9-j symbol [reference 6, Eq. (4.37)]. This does not reveal any new symmetries—at least none to be obtained by a permutation of the relevant quantities $k_{\alpha\beta}$.

f. Recursion relations. Let $\Omega_{\alpha\beta}$ be the differential operator $\tau_{\alpha\beta}\, \partial/\partial\tau_{\alpha\beta}$. Then $\Omega_{\alpha\beta} G = g_{\alpha\beta}$, where $g_{\alpha\beta} = a_\beta + b_i$ if $(\alpha,\beta) \in W_i$. Hence

$$\Omega_{\alpha\beta} R = -2 g_{\alpha\beta} G^{-3}$$

and $g_{\gamma\delta}\Omega_{\alpha\beta} R = g_{\alpha\beta}\Omega_{\gamma\delta} R$, which leads to recursion relations for the $s(k)$. As an example consider $g_{32}\Omega_{01}R = g_{01}\Omega_{32}R$. Now $g_{32} = a_2 + b_1$, $g_{01} = a_1 + b_1$, so that

$$a_2\Omega_{01}R = a_1\Omega_{32}R + b_1(\Omega_{32} - \Omega_{01})R .$$

From the power series for R we obtain

$$\begin{aligned} k_{01}s(k) = {} & (k_{32}+1)s(\cdots k_{02}+1, k_{12}+1, k_{32} \\ & +1, \cdots k_{01}-1, k_{21}-1, k_{31}-1, \cdots) \\ & + (k_{32}+1-k_{01})s(\cdots k_{02}+1, k_{12} \\ & +1, \cdots k_{01}-1, k_{10}-1, k_{23}-1, \cdots) , \end{aligned}$$

where again on the right-hand side only those k are marked which differ from the corresponding ones on the left-hand side. For the 6-j symbols one finds

$$\begin{aligned} & [(J_2+1)k_{01}(k_{02}+1)(k_{12}+1)]^{1/2} \begin{Bmatrix} j_{23} & j_{31} & j_{12} \\ j_{01} & j_{02} & j_{03} \end{Bmatrix} \\ & = [(J_1+2)k_{21}k_{31}(k_{32}+1)]^{1/2} \begin{Bmatrix} j_{23}+\frac{1}{2} & j_{31}-\frac{1}{2} & j_{12} \\ j_{01}-\frac{1}{2} & j_{02}+\frac{1}{2} & j_{03} \end{Bmatrix} \\ & \quad + (k_{32}+1-k_{01})[k_{01}k_{23}]^{1/2} \\ & \quad \times \begin{Bmatrix} j_{23}+\frac{1}{2} & j_{31}-\frac{1}{2} & j_{12} \\ j_{01} & j_{02} & j_{03}-\frac{1}{2} \end{Bmatrix} \end{aligned}$$

APPENDIX. EVALUATION OF SOME LAPLACIAN INTEGRALS

(a) Let

$$\Lambda(B) = \pi^{-n} \int \exp(-\bar{z}\cdot Bz) d^n z \quad (A1)$$

B is an $n \times n$ (complex) matrix with elements b_{kl}, so that

$$\bar{z}\cdot Bz = \sum_{k,l=1}^{n} \bar{z}_k b_{kl} z_l$$

The integral extends over all of C_n, and $d^n z = \prod_{k=1}^n dx_k\, dy_k$ $(z_k = x_k + iy_k)$.

Every B has a unique decomposition $B = B' + iB''$, with Hermitian B' and B'', and we call B' the *Hermitian part* of B.

If B' is *positive definite* the integral in (A1) converges absolutely, and

$$\Lambda(B) = (\det B)^{-1} . \quad (A2)$$

Proof. We proceed in three steps. (1) If $B = 1$, then $\Lambda = 1$. [This is (1.6) for $h = h' = 0$] ,(2) If $B'' = 0$ and B is *positive definite* there exists a nonsingular matrix S such that

$$B = S^* S . \quad (A3)$$

Introducing new variables $z' = Sz$ (and $\bar{z}' = \bar{S}\bar{z}$) we obtain $\bar{z}\cdot Bz = \bar{z}'\cdot z'$, which proves the absolute convergence of the integral. Setting $z'_k = x'_k + iy'_k$ we find for the Jacobian of the transformation

$$\frac{\partial(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial(x'_1,\cdots,x'_n,y'_1,\cdots,y'_n)} = (\det S\cdot\det \bar{S})^{-1} = (\det B)^{-1}. \quad (A3a)$$

Hence, $\Lambda(B) = (\det B)^{-1}\,\Lambda(1)$, Q.E.D. (3) Consider now $B = B' + iB''$ with positive definite B' and arbitrary B''. The modulus of the integrand is $\exp(-\bar{z}\cdot B'z)$, which establishes absolute convergence.

Introduce a complex parameter $\theta = \theta_1 + i\theta_2$, and set $C(\theta) = B' + i\theta B''$, so that

$$C(0) = B' , \quad C(1) = B . \quad (A4)$$

$C(\theta)$ has the decomposition $C'(\theta) + iC''(\theta)$ with

$$C'(\theta) = B' - \theta_2 B'' , \quad C''(\theta) = \theta_1 B'' \quad (A4a)$$

For small θ_2, $C'(\theta)$ is close to B' so that, for a suitable constant κ,

$$\bar{z}\cdot C'(\theta)z > \tfrac{1}{2}\,\bar{z}\cdot B'z \quad \text{if} \quad |\theta_2| < \kappa .$$

We now restrict θ to the strip $|\theta_2| < \kappa$, and show that $\Lambda(C(\theta))$ is *analytic* in θ. For this it is sufficient to observe that the integrand $\exp(-\bar{z}\cdot C(\theta)z)$ is obviously analytic in θ and that its modulus exp $(-\bar{z}\cdot C'(\theta)z)$ is uniformly bounded by the integrable function exp $(-\frac{1}{2}\,\bar{z}\cdot B'z)$. For imaginary θ, C is Hermitian and positive definite [see (A4a)], and in this case the equation $\Lambda(C(\theta)) = [\det C(\theta)]^{-1}$ has already been established. By analyticity it remains valid throughout the strip $|\theta_2| < \kappa$, in particular for $C(1) = B$.

Corollary.

$$I(A) = \int \exp\,(\bar{z}\cdot Az)d\mu_n(z) = [\det\,(1 - A)]^{-1} \quad \text{(A5)}$$

if $1 - A$ has a positive definite Hermitian part, in particular if A has sufficiently small matrix elements. In fact, by the definition of $d\mu_n(z)$ [see (1.5)], $I(A) = \Lambda(1 - A)$.

(b) Let

$$M(B,a,b) = \pi^{-6}\int \exp g(B,a,b;\xi,\eta)d^3\xi d^3\eta \quad \text{(A6)}$$

$$g = -\bar{\xi}\cdot B\xi - \bar{\eta}\cdot B\eta + D(\bar{a},\bar{\xi},\bar{\eta}) + D(b,\xi,\eta)\,. \quad \text{(A6a)}$$

Here, ξ and η are points inC_3, B is a 3×3 matrix, a,b are constant vectors in C_3, and D is a determinant as in Sec. 3f. As before, we proceed in three steps. (1) If $B = 1$, this is the integral in (3.23a), and for sufficiently small a,b, $M(1,a,b) = (1 - \bar{a}\cdot b)^{-2}$, by (3.23b). (2) If B is positive definite Hermitian, M is absolutely convergent for sufficiently small a,b (for example, $\bar{a}\cdot Ba < \det B$, and $\bar{b}\cdot Bb < \det B$). As before, set $B = S^*S$, let $\sigma = \det S$, and introduce new variables $\xi' = S\xi$, $\eta' = S\eta$. Set also $a' = Sa$ and $b' = Sb$. Then

$$\bar{\xi}\cdot B\xi = \bar{\xi}'\cdot\xi'\,,\quad \bar{\eta}\cdot B\eta = \bar{\eta}'\cdot\eta'$$
$$D(\bar{a},\bar{\xi},\bar{\eta}) = \bar{\sigma}^{-1}D(\bar{a}',\bar{\xi}',\bar{\eta}') = D(\bar{a}'',\bar{\xi}',\bar{\eta}')$$
$$D(b,\xi,\eta) = \sigma^{-1}D(b',\xi',\eta') = D(b'',\xi',\eta')\,,$$

where $a'' = \sigma^{-1}a'$, $b'' = \sigma^{-1}b'$. Thus,

$$g(B,a,b;\xi,\eta) = g(1,a'',b'';\xi',\eta')\,.$$

The Jacobian corresponding to (A3a) is now $(\sigma\bar{\sigma})^{-2}$. Hence $M(B,a,b) = (\sigma\bar{\sigma})^{-2}\ M(1,a'',b'') = [\sigma\bar{\sigma}(1 - \bar{a}''\cdot b'')^{-2}] = (\sigma\bar{\sigma} - \bar{a}'\cdot b')^{-2}$. Now $\sigma\bar{\sigma} = \det B$, and $\bar{a}'\cdot b' = \bar{a}\cdot Bb$. Therefore

$$M(B,a,b) = (\det B - \bar{a}\cdot Bb)^{-2}\,. \quad \text{(A7)}$$

(3) If B is no longer Hermitian, but has a positive definite Hermitian part, we may again show by analytic continuation that (A7) remains valid.

The integral to be evaluated in 4c is

$$N(H,\bar{u},v) = \int \exp g(H,\bar{u},v;\xi,\eta)d\mu_3(\xi)d\mu_3(\eta)\,.$$

Since $d\mu_3(\xi)\ d\mu_3(\eta)$ introduces the factor exp $(-\bar{\xi}\cdot\xi - \bar{\eta}\cdot\eta)$ it follows that $N(H,\bar{u},v) = M(1 + H,\bar{u},v)$, and hence

$$N(H,\bar{u},v) = [\det\,(1 + H) - u\cdot v - u\cdot Hv]^{-2}\,. \quad \text{(A8)}$$

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On the Localizability of Quantum Mechanical Systems*

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1. INTRODUCTION

FROM the very beginning of quantum mechanics, the notion of the position of a particle has been much discussed. In the nonrelativistic case, the proof of the equivalence of matrix and wave mechanics, the discovery of the uncertainty relations, and the development of the statistical interpretation of the theory led to an understanding which, within the inevitable limitations of the nonrelativistic theory, may be regarded as completely satisfactory.

Historically, confusion reigned in the relativistic case, because situations requiring a description in terms of many particles were squeezed into a formalism built to describe a single particle. I have in mind the difficulties with wave functions for a single particle which seem to yield nonzero probability for finding it in a state of negative energy. Soon attention shifted to the problems of the quantum theory of fields and the question of the status of position

* Dedicated to Eugene Wigner on his sixtieth birthday.