Wigner Coefficients for the SU_3 Group and some Applications[*]

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I. INTRODUCTION AND SUMMARY

ONE of the contributions of Professor E. P. Wigner to the development of modern physics was the determination of the coefficients[1] associated with the Clebsch-Gordan series for the unitary unimodular group in two dimensions (SU_2). Because of the homomorphism of the SU_2 group onto the rotation group, these Wigner coefficients have been of particular importance in atomic and nuclear spectroscopy.

In this paper we determine a class of Wigner coefficients for the three-dimensional unitary unimodular group (SU_3). The representations of the unitary group in three dimensions U_3 are characterized[2] by a partition $[h_1h_2h_3]$. The representations of the SU_3 group are characterized by the differences $[h_1 - h_3, h_2 - h_3]$ of these numbers,[2] that is to say, by a Young diagram of two rows. The Wigner coefficients that we intend to determine are those associated with the Clebsch-Gordan development of the Kronecker product of two irreducible representations, expressed diagrammatically in Fig. 1. This is not the most general Kronecker product, as the representation $[h_1'']$ has only one row. The restriction allows, by the rules of decomposition of products of irreducible representations,[3] each representation $[h_1h_2h_3]$ to occur only once, i.e., the Kronecker product of $[h_1'h_2']$ and $[h_1'']$ is simply reducible.[4]

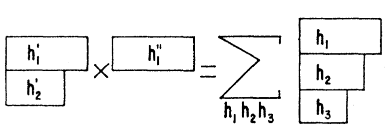

FIG. 1. Clebsch-Gordan development of a class of Kronecker products of irreducible representations of SU_3.

At this point the reader could well ask if the SU_3 group corresponds to symmetries of any problem of physical importance, as otherwise the present paper would be of purely mathematical significance.

The author would like to point out three problems of interest in physics, in which the SU_3 group plays a fundamental role.

I.1 Collective Motions in Nuclei

Since the revival of the nuclear shell model by Goeppert-Mayer and Jensen[5] and others, it was quite apparent that some nuclear properties, for example, the large quadrupole moments of some nuclei, could not be explained by this model, at least in its simple original form. This led Bohr and Mottelson[6] and others to propose the collective model in which the individual particles in the shell picture interact with the vibrations of the liquid drop. It was shown by Elliott[2] that the essential features of the collective model could be reproduced within the shell model framework if one considered the particles as moving in a harmonic oscillator potential and interacting through a quadrupole-quadrupole (Q^2) force. This last problem is invariant under the SU_3 group of transformations, and this invariance was used by Bargmann and Moshinsky[7,8] to obtain the operators of a complete set of integrals of motion for it, as well as the eigenvalues of these operators. There remained the problem of the eigenfunctions and it is precisely for their construction that the Wigner coefficients associated with Clebsch-Gordan development of Fig. 1 are of use.

If we are permitted an analogy, we could think of the nuclear shell model problem as an island we want to occupy. To do this we must establish a beachhead,

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1 E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959), pp. 191, 293.

2 J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128 (1958).

3 D. E. Littlewood, *The Theory of Group Characters* (Oxford University Press, New York, 1940), p. 94.

4 M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), pp. 151, 152.

5 M. Goeppert-Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, Inc., New York, 1955).

6 A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd **26**, No. 14 (1952); A. Bohr and B. R. Mottelson, *ibid.* **27**, No. 16 (1953).

7 V. Bargmann and M. Moshinsky, Nuclear Phys. **18**, 697 (1960).

8 V. Bargmann and M. Moshinsky, Nuclear Phys. **23**, 177 (1961).

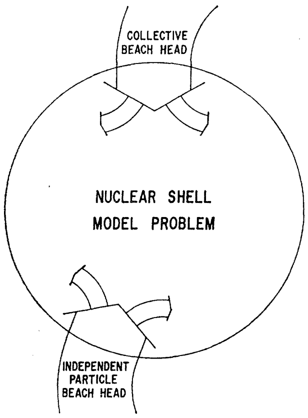

FIG. 2. Approaches to the nuclear shell-model problem.

and the usual one has been the independent-particle beachhead, i.e., an exact solution, with the necessary symmetry, for the independent particles from which we could proceed by perturbation or variational methods, to the interior of the island as illustrated in Fig. 2.

The development of this and previous[7,8] papers allows us to establish a collective beachhead in which we start with an exact solution of a problem where the Q^2 interaction is included. With the help of this collective wave function we could proceed to the interior of the island, again with the help of perturbation or variational techniques as illustrated in Fig. 2.

I.2 Elementary Particles

In the last few years there has been a great deal of interest in a possible symmetry of the strong interactions of elementary particles associated with the SU_3 group. The first efforts in this direction were due to Ikeda *et al.*[9] and others, who assumed the Sakata model in which the proton, neutron, and Λ particles are taken as fundamental, and the others are formed from them and their antiparticles. The symmetry with respect to SU_3 then comes as a direct extension of the symmetry with respect to SU_2 associated with the isospin concept as applied to proton and neutron. This symmetry would allow the classification of hyperons and mesons in terms of irreducible representations of the SU_3 group, from which selection rules and reaction rates could be obtained.

Other models with SU_3 symmetry have been proposed by Gell-Mann,[10] Ne'eman,[11] etc., and even general discussions of symmetries associated with simple groups[12] have been given.

In all models with SU_3 symmetry one needs to construct wave functions that are a basis for irreducible representations of the SU_3 group, in which the SU_2 subgroup, associated with the good quantum number of isospin, is explicitly reduced. The Wigner coefficients necessary for the construction of these wave functions are obtained in this paper and their application to elementary particles is briefly discussed in Sec. VI.

I.3 The Fractional Parentage Coefficients in the p Shell

The fractional parentage coefficients in a shell of angular momentum l are closely related to the Wigner coefficients of the unitary unimodular group in $2l+1$ dimensions, when we consider the Clebsch-Gordan development in which the representation $[h_1'' \cdots h_{2l+1}'']$ is reduced to a single block, i.e., to [1]. For $l = 1$, this would mean that the Wigner coefficient associated with the development of Fig. 1 when $h_1'' = 1$ will be connected with the fractional parentage coefficients of the p shell. These coefficients are well known,[13] but the procedure used in this paper to derive them could, in principle, be generalized to other shells for which fractional parentage coefficients are not yet available.

We shall start in Sec. II of this paper with the derivation of the polynomial expressions which are basis for the irreducible representation of the SU_3 group in which the SU_2 subgroup is explicitly reduced ($SU_3 \supset SU_2$ chain). As in previous papers,[7,8] these polynomials will be given in terms of creation operators of a three-dimensional harmonic oscillator. The polynomials that are a basis for an irreducible representation of SU_3 in which R_3 is explicitly reduced[8] ($SU_3 \supset R_3$ chain) are also discussed.

In Sec. III we define and determine the Wigner coefficient in the $SU_3 \supset SU_2$ chain and in Sec. IV we determine the coefficients that relate the basis for the representation of the $SU_3 \supset SU_2$ chain with the one for the $SU_3 \supset R_3$ chain.

In Secs. V, VI, and VII we discuss, respectively, the applications of the Wigner coefficients to collective motions, elementary particles, and fractional parentage coefficients.

The explicit expression for the Wigner coefficient in the $SU_3 \supset SU_2$ chain is given in (3.25) and the

[9] M. Ikeda, S. Ogawa, and Y. Ohnuki, Progr. Theoret. Phys. (Kyoto) **22**, 715 (1959); **23**, 1073 (1960); Suppl. Progr. Theoret. Phys. (Kyoto) No. 19, 44 (1961).
[10] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[11] Y. Ne'eman, Nuclear Phys. **26**, 222 (1961).
[12] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Revs. Modern Phys. **34**, 1 (1962).
[13] H. A. Jahn and H. van Wieringen, Proc. Roy. Soc. (London) **A209**, 502 (1951).

coefficient for going from the $SU_3 \supset SU_2$ chain to the $SU_3 \supset R_3$ chain is given in (4.13). The algebraic expressions for these coefficients are rather complicated and so a program is being developed by T. A. Brody for their eventual tabulation. Another approach to the tabulation of the Wigner coefficients (3.25) is being considered by H. V. McIntosh and I. Renero who are developing a symbolic program for an electronic computer that would evaluate directly the scalar product (3.23) from which (3.25) was derived.

II. POLYNOMIAL BASIS FOR THE IRREDUCIBLE REPRESENTATIONS OF U_3.

It is well known[14] that the basis for the irreducible representations of unitary groups of a given dimension can be expressed in terms of homogeneous polynomials in the components of vectors in a space of the same dimension. For the basis of the representations of U_3 we could then take homogeneous polynomials in the components of n three-dimensional vectors. It is very convenient to think of these vectors as creation operators[7,8] $\overset{+}{a}_{ms}, s = 1,\cdots,n$ associated with particles in a three-dimensional harmonic oscillator with the components $m = 1,0,\bar{1}$ of the vectors[15] being spherical rather than Cartesian, so that the metric[1] is

$$(-1)^m \delta_{m,-m'}\,, \tag{2.1}$$

and the raising and lowering of indices follows the rule

$$a^{+m}{}_s = (-1)^m a^+{}_{-ms}\,. \tag{2.2}$$

If we now consider the family of linearly independent homogeneous polynomials of a given degree in the $\overset{+}{a}_{ms}$,

$$P(\overset{+}{a}_{ms}) \equiv P(\overset{+}{a}_{11},\overset{+}{a}_{01},\overset{+}{a}_{\bar{1}1},\cdots,\overset{+}{a}_{\bar{1}n})\,, \tag{2.3}$$

it is clear that under the substitution

$$\overset{+}{a}_{ms} \rightarrow \textstyle\sum_{m'} U_m^{m'} \overset{+}{a}_{m's}\,, \tag{2.4}$$

where $U_m^{m'}$ is a three-dimensional unitary matrix, the polynomials (2.3) would be a basis for a generally reducible representation of U_3.

Before restricting the polynomials (2.3) further, so that they constitute a basis for irreducible representations of U_3, we shall introduce the following definitions and notations.[16] First, with the creation operators $\overset{+}{a}_{ms}$ we shall associate annihilation operators $a^m{}_s$ satisfying the commutation relation

$$[a^m{}_s, \overset{+}{a}_{m's'}] = \delta_{m'}^m \delta_{ss'}\,. \tag{2.5}$$

From this commutation relation we conclude that when applied to the polynomials (2.3) the annihilation operator could be interpreted as the differential operator

$$a^m{}_s = (\partial/\partial \overset{+}{a}_{ms})\,. \tag{2.6}$$

From (2.5) or (2.6) we see that under the substitution (2.4), $a^m{}_s$ transforms as

$$a^m{}_s \rightarrow \textstyle\sum_{m'} (U^\dagger)^m_{m'} a^{m'}{}_s\,, \tag{2.7}$$

where $U^\dagger$ is the transposed conjugate of the matrix U.

We now designate[7,8,16] by $|0\rangle$ the ground state for our system of particles in the harmonic oscillator, and with its help we can define the scalar product (P,P') of two polynomials of the type (2.3) as

$$(P,P') \equiv \langle 0|P^+P'|0\rangle\,, \tag{2.8}$$

where P^+ is obtained by replacing all $\overset{+}{a}_{ms}$ by $a^m{}_s$, and we use the commutation relation (2.5) to evaluate (2.8). This definition of the scalar product will be very convenient to discuss orthogonality relations between polynomials, as well as to normalize them.

Once we have both $\overset{+}{a}_{ms}$ and $a^m{}_s$, we can define the following operators[16]:

$$\mathcal{C}_m{}^{m'} = \sum_{s=1}^{n} \overset{+}{a}_{ms} a^{m'}{}_s\,, \tag{2.9}$$

$$C_{ss'} = \sum_{m=\bar{1}}^{1} \overset{+}{a}_{ms} a^m{}_{s'}\,. \tag{2.10}$$

From (2.5) we obtain the following commutation relations for these operators[16]:

$$[\mathcal{C}_m{}^{m'}, \mathcal{C}_{m''}{}^{m'''}] = \mathcal{C}_m{}^{m'''}\delta_{m''}^{m'} - \mathcal{C}_{m''}{}^{m'}\delta_m^{m'''}\,, \tag{2.11}$$

$$[C_{ss'}, C_{s''s'''}] = C_{ss'''}\delta_{s''s'} - C_{s''s'}\delta_{ss'''}\,, \tag{2.12}$$

$$[\mathcal{C}_m{}^{m'}, C_{ss'}] = 0\,. \tag{2.13}$$

Because of the commutation rules (2.11) we shall refer[8] to $\mathcal{C}_m{}^{m'}$ as the operators associated with the infinitesimal unitary transformations in three dimensions. The operators $C_{ss'}$ that annihilate one quantum in particle s' and create one in particle s, will be referred to as transfer operators. It is clear from (2.4) and (2.7) that the transfer operators remain invariant under unitary transformations affecting the creation and annihilation operators. When applying functions of the operators (2.9) and (2.10) to polynomials P of the type (2.3), we shall always think of

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[14] M. Hamermesh, reference 4, Chap. X.

[15] In what follows the negative of a number index will be indicated with a bar above, e.g., $-1 = \bar{1}$.

[16] M. Moshinsky, Nuclear Phys. **31**, 384 (1962).

$\mathcal{C}_m{}^{m'}, C_{ss'}$ as first-order partial differential operators in the variables a^+_{ms}, according to the rule (2.6).

We can now further characterize the homogeneous polynomials P of (2.3) by making them satisfy the equations

$$C_{ss'}P = 0,\ s < s'\ ;\ C_{ss}P = h_s P\ . \tag{2.14}$$

As the $C_{ss'}$ are invariant under U_3, the polynomial solutions of (2.14) would still be solutions after we carry out the substitution (2.4) and so the linearly independent polynomial solutions of (2.14) form a basis for a representation of U_3. It has been shown[7,8] that (2.14) implies that only the first three of the h_s are different from zero with $h_1 \geqslant h_2 \geqslant h_3 \geqslant 0$, and the sum of the h_s

$$h \equiv h_1 + h_2 + h_3\ , \tag{2.15}$$

is equal to the degree of the polynomial. The most general solution of (2.14) then depends only on the creation operators of three particles and to write it we first introduce the notation[16]

$$\begin{aligned} \Delta^s_m &\equiv a^+_{ms}, \Delta^{ss'}_{mm'} \equiv a^+_{ms}a^+_{m's'} - a^+_{m's}a^+_{ms'}\ , \\ \Delta^{ss's''}_{10\bar{1}} &\equiv \epsilon^{mm'm''} a^+_{ms}a^+_{m's'}a^+_{m''s''}\ , \end{aligned} \tag{2.16}$$

where repeated indices are summed over 1, 0, and $\bar{1}$ and $\epsilon^{mm'm''}$ is a completely antisymmetric tensor, i.e., $\epsilon^{10\bar{1}} = 1$, $\epsilon^{1\bar{1}0} = -1$, etc. The solution of (2.14) then has the form[16]

$$P = (\Delta^1_1)^{h_1-h_2}(\Delta^{12}_{10})^{h_2-h_3}(\Delta^{123}_{10\bar{1}})^{h_3} Z\left(\frac{\Delta^1_0}{\Delta^1_1}, \frac{\Delta^1_{\bar{1}}}{\Delta^1_1}, \frac{\Delta^{12}_{1\bar{1}}}{\Delta^{12}_{10}} \right), \tag{2.17}$$

where Z is an arbitrary polynomial in the variables indicated, subject to the condition that after multiplying by the first part of (2.17) no negative powers of the a^+_{ms} remain, i.e., P should be a polynomial in the a^+_{ms}.

The family of linearly independent polynomials (2.17) forms a basis for a representation of U_3 characterized by the partition $[h_1h_2h_3]$. If we restrict ourselves to the SU_3 group for which det $||U^{m'}_m|| = 1$, the factor $(\Delta^{123}_{10\bar{1}})^{h_3}$ in (2.17) will be an invariant under SU_3, so that by eliminating it from (2.17) we have a set of polynomials that form a basis for a representation of SU_3 characterized by the partition

$$[\kappa_1,\kappa_2],\ \kappa_1 \equiv h_1 - h_3,\ \kappa_2 \equiv h_2 - h_3\ . \tag{2.18}$$

The representations of U_3 and SU_3, whose basis we have obtained, are not unitary representations[17] as the linearly independent polynomials (2.17) have not, in general, orthonormal scalar products of the type (2.8). To obtain unitary representations, we further characterize the polynomials in terms of Hermitian operators built up from the $\mathcal{C}_m{}^{m'}$, which, because of (2.13), commute with the $C_{ss'}$ and so can be applied simultaneously with (2.14). These Hermitian operators can be constructed in several ways in accordance with the subgroups of U_3 in which we are interested. In the elementary-particle problem we are concerned with a two-dimensional unitary subgroup U_2 of U_3, i.e., in the $U_3 \supset U_2$, chain, while in the collective motion problem we are interested in the three-dimensional rotation subgroup R_3, i.e., in the $U_3 \supset R_3$ chain. We shall discuss the two cases separately.

II.1 Polynomial Basis for the $U_3 \supset U_2$ Chain

Let us consider the subgroup U_2 of U_3 associated with matrices of the form

$$\begin{pmatrix} U^1_1 & 0 & U^{\bar{1}}_1 \\ 0 & 1 & 0 \\ U^1_{\bar{1}} & 0 & U^{\bar{1}}_{\bar{1}} \end{pmatrix}. \tag{2.19}$$

The operators associated with the infinitesimal unitary transformations for this subgroup will be $\mathcal{C}_1{}^1$, $\mathcal{C}_1{}^{\bar{1}}$, $\mathcal{C}_{\bar{1}}{}^1$, $\mathcal{C}_{\bar{1}}{}^{\bar{1}}$, and from them we could form the Hermitian linear combinations

$$\begin{aligned} T_1 &\equiv \tfrac{1}{2}\,(\mathcal{C}_1{}^{\bar{1}} + \mathcal{C}_{\bar{1}}{}^1),\ T_2 \equiv -i\tfrac{1}{2}\,(\mathcal{C}_1{}^{\bar{1}} - \mathcal{C}_{\bar{1}}{}^1), \\ T_3 &\equiv \tfrac{1}{2}\,(\mathcal{C}_1{}^1 - \mathcal{C}_{\bar{1}}{}^{\bar{1}}), \end{aligned} \tag{2.20}$$

$$K \equiv \mathcal{C}_1{}^1 + \mathcal{C}_{\bar{1}}{}^{\bar{1}} \tag{2.21}$$

From the commutation rules (2.11) we obtain

$$[T_1,T_2] = iT_3 \text{ and cycl., } [T_i,K] = 0,\ i = 1,2,3\ . \tag{2.22}$$

Clearly we could then use the three commuting operators K,T_3, and

$$\mathbf{T}^2 = T_1^2 + T_2^2 + T_3^2\ , \tag{2.23}$$

to characterize further the polynomial solutions of (2.14). Instead of K we shall use an operator S defined as

$$S \equiv K - \tfrac{2}{3}\,(\mathcal{C}_1{}^1 + \mathcal{C}_0{}^0 + \mathcal{C}_{\bar{1}}{}^{\bar{1}}) = -\mathcal{C}_0{}^0 + \tfrac{1}{3}H\ , \tag{2.24}$$

where H is the operator that gives the degree h of the polynomial, i.e.,

$$H \equiv \mathcal{C}_1{}^1 + \mathcal{C}_0{}^0 + \mathcal{C}_{\bar{1}}{}^{\bar{1}} = \sum_{s=1}^{n} C_{ss}\ . \tag{2.25}$$

From (2.11), and (2.12), H clearly commutes with all $\mathcal{C}_m{}^{m'}$, $C_{ss'}$.

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[17] E. P. Wigner, reference 1, p. 74.

For reasons to be discussed in Sec. VI, the operators T_i will be referred to as the Cartesian components of isospin and the operator S as the reduced strangeness. As these operators can be constructed from linear combinations of the operators $\mathcal{C}_m{}^{m'} - \frac{1}{3}H\delta_m^{m'}$, they are associated[8] with the infinitesimal transformations of a subgroup of the unitary unimodular group SU_3 rather than with those of a subgroup of U_3.

The polynomial solutions of (2.14) can now be further specified by the equations

$$\mathbf{T}^2 P = t(t+1)P, \quad T_3 P = \tau P\,, \quad (2.26\text{ a,b})$$

$$SP = \sigma P\,. \quad (2.26\text{ c})$$

From the commutation rules (2.22) of the T_i, we see that $\tau = t, t-1, \cdots, -t$, and $2t$ is restricted to non-negative integer values. The operator $\mathcal{C}_0{}^0$, when applied to polynomials of the form (2.3), will give the degree of the a_{0s}^+ in this polynomial, and so from (2.24) 3σ will be an integer congruent modulo 3 with the degree h of the polynomial.

Equations (2.14) and (2.26) determine the polynomial P up to an arbitrary multiplicative constant. We shall show this first for the case $\tau = t$ for which (2.26) is equivalent to

$$(T_1 + iT_2)P \equiv \mathcal{C}_1{}^{\bar{1}} P = 0,\ T_3 P = tP,\ SP = \sigma P\,. \quad (2.27\text{ a,b,c})$$

Equation (2.27 a) applied to (2.17) implies that

$$\sum\nolimits_s a_{1s}^+(\partial P/\partial a_{\bar{1}s}^+) = (\Delta_1^1)^{h_1-h_2}(\Delta_{10}^{12})^{h_2-h_3}(\Delta_{10\bar{1}}^{123})^{h_3} \times [\partial Z/\partial(\Delta_{\bar{1}}^1/\Delta_1^1)] = 0\,, \quad (2.28)$$

which means that Z is independent of $(\Delta_{\bar{1}}^1/\Delta_1^1)$. The most general form of (2.17) becomes then

$$P = (\Delta_{10\bar{1}}^{123})^{h_3} \sum_{l,p} [b_{lp}(\Delta_1^1)^{h_1-h_2-l}(\Delta_0^1)^l(\Delta_{10}^{12})^{h_2-h_3-p}(\Delta_{1\bar{1}}^{12})^p], \quad (2.29)$$

where the b_{lp} are arbitrary constants. If we now impose the conditions (2.27 b,c) on (2.29), we see that l,p are restricted to the values

$$l = h_1 - \tfrac{1}{3}h - \tfrac{1}{2}\sigma - t \geqslant 0\,, \quad (2.30\text{ a})$$

$$p = -h_3 + \tfrac{1}{3}h + \tfrac{1}{2}\sigma - t \geqslant 0\,, \quad (2.30\text{ b})$$

where as l,p are non-negative integers, $2t$ must be congruent modulo 2 with $(\frac{2}{3}h + \sigma)$. Defining

$$k \equiv h_1 - h_2 - l = -h_2 + \tfrac{1}{3}h + \tfrac{1}{2}\sigma + t \geqslant 0\,, \quad (2.30\text{ c})$$

$$n \equiv h_2 - h_3 - p = h_2 - \tfrac{1}{3}h - \tfrac{1}{2}\sigma + t \geqslant 0\,, \quad (2.30\text{ d})$$

we see that the polynomial characterized by the partition $[h_1h_2h_3]$ and by $\sigma, t, \tau = t$, has the form

$$P_{h_1h_2h_3\sigma tt} = b(\Delta_1^1)^k(\Delta_0^1)^l(\Delta_{10}^{12})^n(\Delta_{1\bar{1}}^{12})^p(\Delta_{10\bar{1}}^{123})^{h_3}\,, \quad (2.31)$$

where b is an arbitrary constant that can be given the value (A18) of Appendix A, so as to normalize the polynomial in the senses of (2.8).

The polynomial P for an arbitrary τ can now be written

$$P_{h_1h_2h_3\sigma t\tau} = [(t+\tau)!]^{1/2}[(t-\tau)!(2t)!]^{-1/2} \times (\mathcal{C}_{\bar{1}}{}^1)^{t-\tau} P_{h_1h_2h_3\sigma tt}\,, \quad (2.32)$$

as from the commutation rules (2.11), $P_{h_1h_2h_3\sigma t\tau}$ behaves as a Racah tensor[18] of order t and projection τ with respect to the operators (2.20) of isospin and it reduces to (2.31) when $\tau = t$.

Two polynomials $P_{h_1h_2h_3\sigma t\tau}$ that differ in at least one of the values of $\sigma t\tau$ have a zero scalar product as the operators (2.26) are Hermitian. As the constant b in (2.31) is chosen so that the polynomials are normalized, it is clear then that these provide a basis for a unitary representation of U_3.

The polynomial (2.32) can also be written as

$$P_{h_1h_2h_3\sigma t\tau} \equiv (\Delta_{10\bar{1}}^{123})^{h_3} P_{\kappa_1\kappa_2 0\sigma t\tau}(\Delta_m^1, \Delta_m^2)\,, \quad (2.33)$$

where κ_1, κ_2 are the differences (2.18). As $\Delta_{10\bar{1}}^{123}$ is invariant under the SU_3 group, it follows that the polynomials $P_{\kappa_1\kappa_2 0\sigma t\tau}$ constitute a basis for a representation of SU_3, i.e., under the substitution (2.4) with det $||U_m^{m'}|| = 1$, the polynomials transform as

$$P_{\kappa_1\kappa_2 0\sigma t\tau} \to \sum_{\sigma't'\tau'} P_{\kappa_1\kappa_2 0\sigma't'\tau'} \mathfrak{D}^{\kappa_1\kappa_2}_{\sigma't'\tau',\sigma t\tau}(U_m^{m'})\,. \quad (2.34)$$

The $\mathfrak{D}$'s in (2.34) are then a unitary representation of SU_3 characterized by the partition $[\kappa_1\kappa_2]$. The representation is irreducible, as we can prove that the only matrix $M_{\sigma't'\tau',\sigma t\tau}$ commuting with all the $\mathfrak{D}$'s is a multiple of $\delta_{\sigma\sigma'}\delta_{tt'}\,\delta_{\tau\tau'}$, i.e., by making use of Schur's lemma. The proof will be omitted as it can be derived straightforwardly by first considering a matrix M that commutes with all $\mathfrak{D}$'s corresponding to a unitary unimodular transformation of the type (2.19), then by further restricting the M to commute with the $\mathfrak{D}$'s associated with a transformation

$$\begin{pmatrix} \exp(-i\alpha) & 0 & 0 \\ 0 & \exp(i2\alpha) & 0 \\ 0 & 0 & \exp(-i\alpha) \end{pmatrix}, \quad (2.35)$$

and finally by restricting the M to commute with the $\mathfrak{D}$'s associated with a three-dimensional rotation R_3.

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[18] G. Racah, Phys. Rev. **62**, 438 (1942).

The family of polynomials $P_{h_1-h_3h_2-h_30\sigma t\tau}$ (Δ_m^1,Δ_m^2) for all values of σ,t consistent with the inequalities (2.30), with the fact that

$$3\sigma \equiv h \bmod 3\,, \tag{2.36a}$$

$$2t \equiv (\tfrac{2}{3}\,h + \sigma) \bmod 2\,, \tag{2.36b}$$

and for $\tau = t,\cdots, -t,$ forms a basis for an irreducible representation of SU_3 characterized by $[h_1 - h_3,h_2 - h_3]$. From these polynomials we are going to derive the Wigner coefficients of the SU_3 group. We shall use the fact that any polynomial function $R(C_{ss'})$ of the $C_{ss'}$ in which the $a^m{}_s$ are interpreted as the differential operators (2.6), when applied to our family of polynomials, will transform them into another family of polynomials which will continue to be a basis for an irreducible representation of SU_3 of the type (2.34), since the $C_{ss'}$ are invariant under SU_3.

II.2 Polynomial Basis for the $U_3 \supset R_3$ Chain

The operators associated with the infinitesimal three-dimensional rotations will be the components of the angular momentum vector and in spherical notation $(m = 1,0,\bar{1})$ they are given by

$$\mathfrak{L}_m = -\sum_{m'm''}\sum_{s} \epsilon_{mm'm''} a^{+m'}{}_s a^{m''}{}_s\,, \tag{2.37}$$

where ϵ is the completely antisymmetric tensor. From (2.2) and (2.9) we then obtain

$$\mathfrak{L}_1 = -(\mathfrak{C}_0{}^{\bar{1}} + \mathfrak{C}_1{}^0), \mathfrak{L}_0 = (\mathfrak{C}_1{}^1 - \mathfrak{C}_{\bar{1}}{}^{\bar{1}}) = 2T_3,$$
$$\mathfrak{L}_{\bar{1}} = (\mathfrak{C}_0{}^1 + \mathfrak{C}_{\bar{1}}{}^0)\,. \tag{2.38}$$

The Casimir operator[19] of the R_3 group will be

$$\mathfrak{L}^2 = \sum\nolimits_m (-1)^m\, \mathfrak{L}_m\mathfrak{L}_{-m}\,, \tag{2.39}$$

and we could use the Hermitian operators $\mathfrak{L}^2,\mathfrak{L}_0$ to further characterize the polynomials (2.14), i.e., P would satisfy

$$\mathfrak{L}^2\mathrm{P} = \lambda(\lambda + 1)\mathrm{P},\ \mathfrak{L}_0\mathrm{P} = \mu\mathrm{P}\,. \tag{2.40}$$

As in the previous subsection we could restrict our discussion of the Eq. (2.40) to the case when $\mu = \lambda$, for which (2.40) is equivalent to

$$\mathfrak{L}_1\mathrm{P} = 0,\ \mathfrak{L}_0\mathrm{P} = \lambda\mathrm{P}\,. \tag{2.41}$$

The polynomials that are solutions of (2.14) and (2.41) were obtained in a previous paper.[8,16] Introducing the notation

$$s = \sum\nolimits_m (-1)^m\Delta_m^1\Delta_{-m}^1,\ w_+ = \sum\nolimits_m (-1)^m\Delta_{1m}^{12}\Delta_{-m}^1\,,$$
$$v = \tfrac{1}{2}\sum\nolimits_{m,m'} (-1)^{m+m'}\Delta_{mm'}^{12}\Delta_{-m-m'}^{12}\,, \tag{2.42}$$

[19] G. Racah, *Group Theory and Spectroscopy* [Notes of lectures at the Institute of Advanced Study (Princeton, New Jersey, 1951)], pp, 44, 45.

we can write the polynomial as

$$\mathrm{P}_{h_1h_2h_3q\lambda\lambda} = w_+^{\gamma}(\Delta_1^1)^{\xi+2q}(\Delta_{10}^{12})^{\eta-2q}(s)^{\zeta-q}v^q(\Delta_{10\bar{1}}^{123})^{h_3}\,, \tag{2.43}$$

where for $h_1 - h_3 - \lambda$ even

$$\gamma = 0,\ \xi = -h_2 + h_3 + \lambda,\ \eta = h_2 - h_3,$$
$$\zeta = \tfrac{1}{2}\,(h_1 - h_3 - \lambda)\,, \tag{2.44a}$$

and q is an integer restricted by the inequalities

$$0 \leqslant 2q \leqslant h_2 - h_3,\ h_2 - h_3 - \lambda \leqslant 2q \leqslant h_1 - h_3 - \lambda. \tag{2.44b}$$

For $h_1 - h_3 - \lambda$ odd,

$$\gamma = 1,\ \xi = -h_2 + h_3 + \lambda,\ \eta = h_2 - h_3 - 1\,,$$
$$\zeta = \tfrac{1}{2}\,(h_1 - h_3 - \lambda - 1)\,, \tag{2.44c}$$

and

$$0 \leqslant 2q \leqslant h_2 - h_3 - 1\,,$$
$$h_2 - h_3 - \lambda \leqslant 2q \leqslant h_1 - h_3 - \lambda - 1\,. \tag{2.44d}$$

The polynomial P for an arbitrary μ can now be written as

$$\mathrm{P}_{h_1h_2h_3q\lambda\mu} = [(\lambda + \mu)!]^{1/2}\,[(\lambda - \mu)!(2\lambda)!]^{-1/2}$$
$$\times\,(\mathfrak{L}_{\bar{1}})^{\lambda-\mu}\mathrm{P}_{h_1h_2h_3q\lambda\lambda}\,. \tag{2.45}$$

The family of polynomials $\mathrm{P}_{h_1-h_3h_2-h_30q\lambda\mu}$ (Δ_m^1,Δ_m^2) obtained from (2.45) after removing the factor $(\Delta_{10\bar{1}}^{123})^{h_3}$, will form a basis for an irreducible representation of SU_3 characterized by the partition $[h_1 - h_3, h_2 - h_3]$ but with the rows of the representation being labeled by $q\lambda\mu$ instead of the $\sigma t\tau$ of the previous subsection. In Sec. IV we shall obtain the coefficients relating the basis (2.45) to the basis (2.32), which we need for the explicit construction of the wave functions associated with the collective motions.

The polynomials (2.45) form a basis for a nonunitary irreducible representation as the index q is not associated with any Hermitian operator. In reference 8 we showed, however, that it was possible to use a Hermitian operator Ω, which was essentially the quadrupole moment of the particles, to determine up to an arbitrary constant the solution of (2.14) and (2.41). If the polynomials P are then subject to the additional condition

$$\Omega\mathrm{P} = \omega\mathrm{P}\,, \tag{2.46}$$

they would form a basis for a unitary representation of SU_3 whose rows would be $\omega\lambda\mu$. In reference 8 we gave the explicit procedure for determining the coefficients α_q^ω in the development

$$\mathrm{P}_{h_1h_2h_3\omega\lambda\mu} = \sum\nolimits_q \alpha_q^\omega\mathrm{P}_{h_1h_2h_3q\lambda\mu}\,, \tag{2.47}$$

and so we could also connect the unitary representations whose rows are $\omega\lambda\mu$ with those whose rows are $\sigma t \tau$.

III. DETERMINATION OF THE WIGNER COEFFICIENTS OF THE SU_3 GROUP FOR THE $SU_3 \supset SU_2$ CHAIN

In subsection II.1 we obtained the polynomials in the creation operators a^{+}_{ms} that are a basis for the irreducible representations of SU_3 in the $SU_3 \supset SU_2$ chain. We shall now consider the product of two polynomials of this type, one associated with the partition $[h_1'h_2']$ and the other with the partition $[h_1'']$ and develop this product in terms of polynomials associated with the partitions $[h_1h_2h_3]$ according to the rules of decomposition of products of irreducible representations.[3] The coefficients in this development are the class of Wigner coefficients for SU_3 mentioned in the introduction.

The polynomials for the partition $[h_1'h_2']$ will be denoted as

$$P_{h_1'h_2'0\sigma't'\tau'}(\Delta^1_m,\Delta^2_m)\,, \tag{3.1}$$

and they have the form (2.32) with $h_1'h_2'0\sigma't'\tau'$ replacing $h_1h_2h_3\sigma t\tau$. For the polynomial associated with the partition $[h_1'']$, we can again use (2.32) denoting the particle by index $s = 3$ to distinguish it from the indices 1,2 appearing in (3.1), and so the normalized polynomials becomes

$$\begin{aligned} P_{h_1''00\sigma''t''\tau''}(\Delta^3_m) = {} & [(t''+\tau'')!(h_1''-2t'')! \\ & \times (t''-\tau'')!]^{-1/2}\,(\Delta^3_1)^{t''+\tau''} \\ & \times (\Delta^3_0)^{h_1''-2t''}(\Delta^3_{\bar{1}})^{t''-\tau''}\,. \end{aligned} \tag{3.2}$$

For the partition $[h_1'']$ the eigenvalue σ'' is not independent of h_1'',t'', as applying the operator S of (2.24) to (3.2) we obtain

$$\sigma'' = 2(t'' - \tfrac{1}{3}h_1'')\,. \tag{3.3}$$

If we now consider the set of products of the polynomials (3.1), (3.2), we see that they provide a basis for a generally reducible representation of SU_3. We want to develop these products in terms of polynomials that would be basis for irreducible representations of SU_3. To do this we first notice that in the $SU_3 \supset SU_2$ chain the rows of the irreducible representations are characterized by the indices $\sigma t\tau$ associated with the Hermitian operators $S,\mathbf{T}^2,T_3$. It is convenient therefore to choose linear combinations of the products of (3.1) and (3.2) that are eigenpolynomials of these operators. These linear combinations will have the form

$$\begin{aligned} \Pi^{h_1'h_2'h_1''}_{\sigma't't'',\sigma t\tau}(\Delta^1_m,\Delta^2_m,\Delta^3_m) \equiv {} & \sum_{\tau'\tau''}\langle t't''\tau'\tau''|t\tau\rangle \\ & \times P_{h_1'h_2'0\sigma't'\tau'}(\Delta^1_m,\Delta^2_m)P_{h_1''00\sigma''t''\tau''}(\Delta^3_m)\,, \end{aligned} \tag{3.4}$$

where $\langle|\rangle$ is a Wigner coefficient for the SU_2 group. Clearly the Π's are eigenpolynomials of $\mathbf{T}^2,T_3$ with eigenvalues $t(t+1)$, τ, respectively, and of S with eigenvalue $\sigma = \sigma' + \sigma''$, as S is an additive operator. The index σ'' is suppressed in Π, as from (3.3) it is not independent of the others.

In what follows we shall deal only with the development of the Π's, since the products of the polynomials on the right-hand side of (3.4) could be expressed in terms of the Π's by making use of the orthogonality properties of the Wigner coefficients of the SU_2 group.

From the Eqs. (2.14), satisfied by the polynomial (3.1), and from (3.2) we have[20]

$$C_{11}\Pi = h_1'\Pi\,,C_{22}\Pi = h_2'\Pi\,,C_{12}\Pi = 0 \tag{3.5a,b,c}$$

$$C_{33}\Pi = h_1''\Pi\,. \tag{3.5d}$$

The operators $C_{ss}(s = 1,2,3)$ in (3.5) are Hermitian, but C_{12} is not, as from (2.10) $C^{+}_{12} = C_{21}$. We would like to rewrite the Eqs. (3.5) in a form in which only Hermitian operators appear, and for this purpose we define the pseudospin vector[7] $\mathbf{F}$ whose spherical components $F_m(m = 1,0,\bar{1})$ are

$$\begin{aligned} & F_1 = -(1/\sqrt{2})C_{12}, F_0 = \tfrac{1}{2}\,(C_{11} - C_{22}), \\ & F_{\bar{1}} = (1/\sqrt{2})C_{21}\,. \end{aligned} \tag{3.6}$$

From the commutation rules (2.12) we immediately see that the Cartesian components of $\mathbf{F}$ satisfy the commutation rules of angular momentum and so the operator

$$\begin{aligned} \mathbf{F}^2 = \textstyle\sum_m (-1)^m F_m F_{-m} = {} & C_{21}C_{12} + \tfrac{1}{2}\,(C_{11} - C_{22}) \\ & \times [\tfrac{1}{2}\,(C_{11} - C_{22}) + 1]\,, \end{aligned} \tag{3.7}$$

commutes with all F_m. Equations (3.5) are now equivalent to

$$\mathbf{F}^2\Pi = f'(f'+1)\Pi\,,$$

$$F_0\Pi = f'\Pi\,,\ \text{where}\ f' = \tfrac{1}{2}\,(h_1' - h_2')\,, \tag{3.8a,b}$$

$$(C_{11} + C_{22})\Pi = (h_1' + h_2')\Pi\,,\ C_{33}\Pi = h_1''\Pi\,, \tag{3.8c,d}$$

where all the operators are Hermitian.

To obtain the Wigner coefficients for the terms in the Clebsch-Gordon development, we must determine

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[20] We shall suppress the indices in the Π, P etc., in all equations where no misunderstanding can occur.

the set of polynomials that are a basis for an irreducible representation of U_3 characterized by the partition $[h_1h_2h_3]$ and at the same time eigenpolynomials of the operators (3.8) with the same eigenvalues as Π. This can be achieved by applying polynomial functions $R(C_{ss'})$ in the operators $C_{ss'}$ to the P's of (2.32), as we have shown in subsection II.1 that this would give a new set of polynomials that are a basis for the same representation $[h_1h_2h_3]$ of U_3. For the polynomials RP the equivalence between the degrees of the creation operators in particles 1,2,3 and the partition numbers h_1,h_2,h_3 will no longer hold and so we could look for R's such that (3.8) would be satisfied.

To determine the operator $R(C_{ss'})$ we make use of the theory of Racah tensors,[18] which applies to the pseudospin as it obeys the commutation rules of angular momentum. First we define the polynomial

$$V_\nu^f \equiv [(f+\nu)!]^{1/2}\,[(f-\nu)!(2f)!]^{-1/2}\,C_{21}^{f-\nu}P_{h_1h_2h_3\sigma t\tau}\,, \tag{3.9}$$

where $f=\frac{1}{2}(h_1-h_2)$.
If we apply the operator F_m [which from (2.6) can be thought of as a differential operator in the a_{ms}^+] to the polynomial V_ν^f, we obtain

$$F_mV_\nu^f = [f(f+1)]^{1/2}\,\langle f1\nu m|f\nu+m\rangle V_{\nu+m}^f\,, \tag{3.10}$$

and so V_ν^f is clearly a Racah tensor[18] of order f and projection ν. Second, we define the operator

$$\begin{aligned} W_{\nu''}^{f''} = {}&(-1)^{f''-\nu''}[(f''-\nu'')!(f''+\nu'')!]^{-1/2}\\ &\times C_{31}^{f''-\nu''}C_{32}^{f''+\nu''}, \end{aligned} \tag{3.11}$$

with $f''=\frac{1}{2}(h_1-h_3)$.
From the commutation relations (2.12) we have

$$\begin{aligned} [F_m,W_{\nu''}^{f''}] = {}&[f''(f''+1)]^{1/2}\\ &\times\langle f''1\nu''m|f''\nu''+m\rangle W_{\nu''+m}^{f''}\,, \end{aligned} \tag{3.12}$$

and so $W_{\nu''}^{f''}$ is a Racah tensor of order f'' and projection ν''.

We construct now the polynomial

$$\begin{aligned} \mathcal{P}_{h_1h_2h_3\sigma t\tau}^{h_1'h_2'h_1''} &= \sum_{\nu''\nu}\{\langle f''f\nu''\nu|f'f'\rangle W_{\nu''}^{f''}V_\nu^f\}\\ &\equiv R(C_{ss'})P_{h_1h_2h_3\sigma t\tau}\,, \end{aligned} \tag{3.13}$$

where $R(C_{ss'})$ is the operator

$$\begin{aligned} R(C_{ss'}) = \sum_{\nu''\nu}\{&\langle f''f\nu''\nu|f'f'\rangle(-1)^{f''-\nu''}\\ &\times[(f''-\nu'')!(f''+\nu'')!]^{-1/2}\,[(f+\nu)!]^{1/2}\\ &\times[(f-\nu)!(2f)!]^{-1/2}\,C_{31}^{f''-\nu''}C_{32}^{f''+\nu''}C_{21}^{f-\nu}\}\,. \end{aligned} \tag{3.14}$$

Clearly from the procedure of construction $\mathcal{P}$ is an eigenpolynomial of $\mathbf{F}^2,F_0$ with eigenvalues $f'(f'+1),f'$ and its degree in the creation operators of particles 1,2,3 is h_1',h_2',h_1'', respectively, so that $\mathcal{P}$ is an eigenpolynomial of the operators (3.8) with the same eigenvalues as Π. Two polynomials $\mathcal{P}$ that differ in any of their indices are orthogonal. This is obvious for the indices $h_1',h_2',h_1'',\sigma,t,\tau$ as they are eigenvalues of the Hermitian operators $C_{11},C_{22},C_{33},S,\mathbf{T}^2,T_3$, respectively. For the indices h_1,h_2,h_3 it is proved in Appendix B.

The Wigner coefficients we are looking for are given by the scalar product

$$\begin{aligned} &(d\mathcal{P}_{h_1h_2h_3\sigma tt}^{h_1'h_2'h_1''},\,\Pi_{\sigma't't'',\sigma tt}^{h_1'h_2'h_1''})\\ &\qquad\equiv\langle h_1'h_2'\sigma't',h_1''t'')h_1h_2h_3,\sigma t\rangle\,, \end{aligned} \tag{3.15}$$

where d is the normalization coefficient of the $\mathcal{P}$'s given by (B.9) of Appendix B, and we have taken $\tau=t$ on both sides of the scalar product as both $\mathcal{P}$ and Π are part of a basis for an irreducible representation of the SU_2 group associated with isospin, and so their scalar product is independent[21] of τ.

To evaluate the scalar product (3.15), we first notice that from the explicit expression[22] of $\langle f''f\nu''\nu|f'f'\rangle$ and the definitions of f'',f,f', we could write $R(C_{ss'})$ of (3.14) as

$$\begin{aligned} R(C_{ss'}) = e\sum_{\alpha=0}^{h_1-h_2}&(h_1-h_2-\alpha)![\alpha!(h_1-h_1'-\alpha)!]^{-1}\\ &\times C_{31}^{h_1-h_1'-\alpha}C_{32}^{h_2-h_2'+\alpha}C_{21}^{\alpha}\,, \end{aligned} \tag{3.16}$$

where e is given by

$$\begin{aligned} e = {}&[(h_1'-h_2'+1)!(h_1-h_1')!]^{1/2}\,[(h_1-h_2'+1)!\\ &\times(h_2-h_2')!(h_1'-h_2)!(h_1-h_2)!]^{-1/2}\,. \end{aligned} \tag{3.17}$$

Using the commutation relations (2.12) we can write R in the form

$$\begin{aligned} R(C_{ss'}) = eC_{21}\textstyle\sum_\alpha&\{(h_1-h_2-\alpha)!\\ &\times[\alpha!(h_1-h_1'-\alpha)!]^{-1}C_{31}^{h_1-h_1'-\alpha}C_{32}^{h_2-h_2'+\alpha}\\ &\times C_{21}^{\alpha-1}\}+e\textstyle\sum_\alpha\{(h_1-h_2-\alpha)!\\ &\times(h_2-h_2'+\alpha)[\alpha!(h_1-h_1'-\alpha)!]^{-1}\\ &\times C_{31}^{h_1-h_1'-\alpha+1}C_{32}^{h_2-h_2'+\alpha-1}C_{21}^{\alpha-1}\}\,. \end{aligned} \tag{3.18}$$

Now taking this form of the R in (3.13) and introducing the corresponding $\mathcal{P}$ in the scalar product (3.15), we see that the first summand in (3.18) gives no contribution to the scalar product as the operator C_{21} can be passed to the right-hand side of (3.15) as C_{12} and from (3.5c) $C_{12}\Pi=0$. For the second sum-

[21] E. P. Wigner, reference 1, p. 115.
[22] E. P. Wigner, reference 1, p. 192.

mand in (3.18) we could once again make the same type of development and continuing the process we finally obtain

$$\langle h_1'h_2'\sigma't',h_1''t'')h_1h_2h_3,\sigma t\rangle = de\{(h_1'-h_2)!\\ \times(h_1-h_2'+1)![(h_1-h_1')!(h_1'-h_2'+1)!]^{-1}\}\\ \times(C_{31}^{h_1-h_1'}C_{32}^{h_2-h_2'}P,\Pi)\,, \tag{3.19}$$

where Π is given by (3.4), P symbolizes

$$P_{h_1h_2h_3\sigma tt}(\Delta_m^1,\Delta_m^2,\Delta_m^3),$$

and we have made use of the identity[23]

$$\sum_{\alpha=0}^{h_1-h_1'}\frac{(h_1-h_2-\alpha)!(h_2-h_2'+\alpha)!}{\alpha!(h_1-h_1'-\alpha)!} = \frac{(h_2-h_2')!(h_1'-h_2)!(h_1-h_2'+1)!}{(h_1-h_1')!(h_1'-h_2'+1)!} \tag{3.20}$$

Again, making use of the explicit expression[22] for $\langle t't''\tau'\tau''|tt\rangle$ we could write the Π in (3.19) as

$$\Pi = g\sum_{\beta=0}^{2t'}\{(2t'-\beta)!(-1)^\beta[\beta!(t''-t+t'-\beta)!]^{-1}\\ \times[(\mathfrak{C}_{\bar{1}}{}^1)^\beta P'](\Delta_1^3)^{t''+t-t'+\beta}(\Delta_0^3)^{h_1''-2t''}\\ \times(\Delta_{\bar{1}}^3)^{t''-t+t'-\beta}\}\,, \tag{3.21}$$

where P' stands for $P_{h_1'h_2'0\sigma't't'}(\Delta^1{}_m,\Delta^2{}_m)$, and g is

$$g = \{[(2t+1)!(t''+t'-t)!]^{1/2}\\ \times[(t''+t'+t+1)!(-t''+t'+t)!\\ \times(t''-t'+t)!(2t')!(h_1''-2t'')!]^{-1/2}\}. \tag{3.22}$$

Using the commutation rules (2.11) we could, by a similar analysis to (3.18), express Π as sum of two terms, in which the first contains the operator $\mathfrak{C}_{\bar{1}}{}^1$ applied to a polynomial. The contribution of the first term to the scalar product is zero as $\mathfrak{C}_1{}^{\bar{1}}$ can pass to the left-hand side as $\mathfrak{C}_1{}^{\bar{1}}$ and as this commutes with $C_{ss'}$, we get $\mathfrak{C}_{\bar{1}}{}^1P = 0$ as $\tau = t$ for P. The second term can again be developed in the same way and continuing this procedure and making use of an identity similar to (3.20), we obtain, after substituting for P,P' their explicit expression (2.31), that

$$\langle h_1'h_2'\sigma't',h_1''t'')h_1h_2h_3,\sigma t\rangle = bb'deg\,\frac{(h_1'-h_2)!(h_1-h_2'+1)!(-t''+t'+t)!(t''+t'+t+1)!}{(h_1-h_1')!(h_1'-h_2'+1)!(t''+t'-t)!(2t+1)!}\\ \times\left(C_{31}^{h_1-h_1'}C_{32}^{h_2-h_2'}(\Delta_1^1)^k(\Delta_0^1)^l(\Delta_{10}^{12})^n(\Delta_{1\bar{1}}^{12})^p(\Delta_{10\bar{1}}^{123})^{h_3}\,,\,(\Delta_1^1)^{k'}(\Delta_0^1)^{l'}(\Delta_{10}^{12})^{n'}(\Delta_{1\bar{1}}^{12})^{p'}(\Delta_1^3)^{t''-t'+t}(\Delta_0^3)^{h_1''-2t''}(\Delta_{\bar{1}}^3)^{t''+t'-t}\right). \tag{3.23}$$

In (3.23) b,b' are the normalization coefficients for P,P' obtained in (A18), k,l,n,p are given by (2.30) and k',l',n',p' are

$$k' = \tfrac{1}{3}h_1' - \tfrac{2}{3}h_2' + \tfrac{1}{2}\sigma' + t'\,, \tag{3.24a}$$

$$l' = \tfrac{2}{3}h_1' - \tfrac{1}{3}h_2' - \tfrac{1}{2}\sigma' - t'\,, \tag{3.24b}$$

$$n' = -\tfrac{1}{3}h_1' + \tfrac{2}{3}h_2' - \tfrac{1}{2}\sigma' + t'\,, \tag{3.24c}$$

$$p' = \tfrac{1}{3}h_1' + \tfrac{1}{3}h_2' + \tfrac{1}{2}\sigma' - t'\,, \tag{3.24d}$$

where from (2.36) $3\sigma'$ is congruent modulo 3 with $(h_1'+h_2')$ and $2t'$ is congruent modulo 2 with $[\sigma' + \tfrac{2}{3}(h_1'+h_2')]$.

[23] E. Netto, *Lehrbuch der Combinatorik* (Chelsea Publishing Company, New York), pp. 250, 251.

The evaluation of the scalar product in (3.23) is sketched in Appendix C, and from the discussion given there and the explicit forms for b,b',d,e,g, we obtain for the Wigner coefficient of the SU_3 group in the $SU_3 \supset SU_2$ chain, the expression

$$\langle h_1'h_2'\sigma't',h_1''t'')h_1h_2h_3,\sigma t\rangle = \left[\frac{(h_1'-h_2)!(h_1-h_2'+1)!(h_1-h_1')!(h_2-h_2')!}{(h_1'-h_2'+1)!(h_1-h_2)!}\right]^{1/2}\\ \times\left[\frac{(-t''+t'+t)!(t''+t'-t)!(t''-t'+t)!(t''+t'+t+1)!(h_1''-2t'')!}{(2t')!(2t+1)!}\right]^{1/2}\\ \times\left[\frac{(h_1'-h_3+1)!(h_2'-h_3)!}{(h_1-h_3+1)!(h_2-h_3)!}\right]^{1/2}\left[\frac{(h_1-h_3+2)!(h_2-h_3+1)!h_3!}{(h_1+2)!(h_2+1)!}\right]^{1/2}\\ \times\left[\frac{(k+n+1)!(k+l+1)!n!p!}{(p+k+n+1)!(n+k+l+1)!k!l!}\right]^{1/2}\left[\frac{(k'+n'+1)!(k'+l'+1)!n'!p'!}{(p'+k'+n'+1)!(n'+k'+l'+1)!k'!l'!}\right]^{1/2}\\ \times\sum_{rs}\sum_{uv}\sum_{ab}\{(-1)^{t+t''-t'-h_1+h_1'+r+s+v}(h_2-h_2'-u+v)!(h_3+u-v)!$$

$$\times (h_2' - p + u - v)!(p - u + v)!(a + b)!(h_1 - l' - r - s - a - b)!(l' + r + a)!(s + b)!$$
$$\times [r!(h_1 - h_1' - r - s)!s!(t'' + t' - t - s)!(h_1'' - 2t'' - r)!(h_2 - h_2' - u)!u!$$
$$\times (n - h_2 + h_2' + u)!(p - u)!(h_3 - v)!v!(h_2 - h_2' - h_1'' + 2t'' - u + v + r)$$
$$\times (h_3 - t'' - t' + t + u - v + s)!(n' - a)!a!(p' - b)!b!(p' - p + u - v + a)!(p - p' - u + v + b)!]^{-1}\}, \tag{3.25}$$

where the summations extend over all values of the variables for which the factorials in the denominator are not infinite.

Note added in proof. The present derivation of a particular class of Wigner coefficients for SU_3 could be extended straightforwardly to general Wigner coefficients of unitary groups of arbitrary dimension if the above proofs are modified at two points.

(1) The fact that the polynomials (2.17) form a basis for an irreducible representation of SU_3 can be proved, instead of by Schur's lemma, by Racah's theorem (reference 19, p. 37) concerning the uniqueness of the highest-weight polynomials in an irreducible representation of semi-simple Lie groups. Highest weight would mean in this case that polynomials of the type (2.17) satisfy

$$\mathcal{C}_m{}^m P = h_m' P,\ \mathcal{C}_m{}^{m'} P = 0,\ m > m' . \tag{3.26a,b}$$

It is easily seen that conditions (3.26b) imply that the Z in (2.17) is a constant and so there is only one polynomial of the type (2.17) of highest weight, with

$$h_1' = h_1,\quad h_0' = h_2,\quad h_{\bar{1}}' = h_3 . \tag{3.27}$$

(2) The polynomial function $R(C_{ss'})$ in (3.16) could be derived in a more general fashion by first remarking that because of (2.14) R could, without loss of generality, be written as

$$R(C_{ss'}) = \sum_{\gamma\beta\alpha} A_{\gamma\beta\alpha} C_{31}^{\gamma} C_{32}^{\beta} C_{21}^{\alpha} . \tag{3.28}$$

Now applying Eqs. (3.5 a,b,c) to RP, we immediately obtain

$$\gamma = h_1 - h_1' - \alpha,\quad \beta = h_2 - h_2' + \alpha, \tag{3.29}$$

so that the coefficient in (3.28) could be written as A_α. Equation (3.5d), applied to RP, will then lead to a recurrence relation for A_α which gives precisely (3.16).

IV. COEFFICIENTS RELATING THE POLYNOMIALS IN THE CHAIN $SU_3 \supset R_3$ WITH THOSE IN THE CHAIN $SU_3 \supset SU_2$

In this section we will obtain the coefficients for the expansion of the polynomial $\mathrm{P}_{h_1h_2h_3q\lambda\lambda}$ of (2.43) in terms of the polynomials $P_{h_1h_2h_3\sigma t\tau}$ of (2.32). These coefficients will be needed in the next section for the determination of the collective wave function.

We first notice from (2.38) that $\mathcal{L}_0 = 2T_3$ and therefore in our expansion we can restrict ourselves to $\tau = \frac{1}{2}\lambda$. Furthermore $(\Delta_{10\bar{1}}^{123})^{h_3}$ appears both in (2.32) and (2.43) so we could eliminate it. Using the notation (2.18), the coefficients we are interested in are given by the scalar product

$$(P_{\kappa_1\kappa_20\sigma t\lambda/2}, \mathrm{P}_{\kappa_1\kappa_20q\lambda\lambda}) \equiv \langle \sigma t(\kappa_1\kappa_2)q\lambda\rangle . \tag{4.1}$$

The polynomial $P_{\kappa_1\kappa_20\sigma t\lambda/2}$ is defined in terms of the operator $(\mathcal{C}_{\bar{1}}{}^1)^{t-\lambda/2}$ acting on the polynomial $P_{\kappa_1\kappa_20\sigma tt}$ of (2.31). Passing this operator to the right-hand factor of the scalar product (4.1), our coefficient becomes

$$\langle \sigma t(\kappa_1\kappa_2)q\lambda\rangle = [(t + \tfrac{1}{2}\lambda)!]^{1/2}[(t - \tfrac{1}{2}\lambda)!(2t)!]^{-1/2} \times (P_{\kappa_1\kappa_20\sigma tt}, (\mathcal{C}_1{}^{\bar{1}})^{t-\lambda/2}\mathrm{P}_{\kappa_1\kappa_20q\lambda\lambda}) . \tag{4.2}$$

From the commutation relations

$$[\mathcal{L}_1, \mathcal{C}_1{}^{\bar{1}}] = 0,\ [\mathcal{L}_0, \mathcal{C}_1{}^{\bar{1}}] = 2\mathcal{C}_1{}^{\bar{1}} , \tag{4.3}$$

we see that the right-hand factor of the scalar product in (4.2) is a polynomial associated with the partition $[\kappa_1\kappa_2]$ and of angular momentum $2t$, we can write therefore

$$(\mathcal{C}_1{}^{\bar{1}})^{t-\lambda/2}\mathrm{P}_{\kappa_1\kappa_20q\lambda\lambda} = \sum_{q'} A_{qq'}^{\kappa_1\kappa_2\lambda t}\mathrm{P}_{\kappa_1\kappa_20q'2t2t} . \tag{4.4}$$

Applying the operator $(\mathcal{C}_1{}^{\bar{1}})^{t-\lambda/2}$ to the polynomial (2.43) and keeping in mind that

$$\mathcal{C}_1{}^{\bar{1}}\Delta_1^1 = \mathcal{C}_1{}^{\bar{1}}\Delta_{10}^{12} = \mathcal{C}_1{}^{\bar{1}}w_+ = 0 ,$$
$$\mathcal{C}_1{}^{\bar{1}}s = -2(\Delta_1^1)^2,\ \mathcal{C}_1{}^{\bar{1}}v = -2(\Delta_{10}^{12})^2 , \tag{4.5}$$

we easily obtain

$$A_{qq'}^{\kappa_1\kappa_2\lambda t} = \frac{(-2)^{t-\lambda/2}(t - \frac{1}{2}\lambda)![\frac{1}{2}(\epsilon - \lambda) - q]!q!}{(t - \frac{1}{2}\lambda - q + q')!(q - q')![\frac{1}{2}(\epsilon - 2t) - q']!q'!} \tag{4.6a}$$

where

$$\epsilon = \kappa_1 \text{ if } \kappa_1 - \lambda \text{ is even },$$
$$\epsilon = \kappa_1 - 1 \text{ if } \kappa_1 - \lambda \text{ is odd }. \tag{4.6b}$$

Instead of using the form (2.43) for the polynomial $P_{\kappa_1\kappa_2 0 q' 2t 2t}$, we notice from (2.42) that v can be expressed as [8,16]

$$v = (\Delta_1^1)^{-2}[(\Delta_{10}^{12})^2\, s - w_+^2]. \tag{4.7}$$

Expanding v^q in (2.43) and grouping terms, we can write

$$P_{\kappa_1\kappa_2 0 q' 2t 2t} = (\Delta_1^1)^{-\kappa_2+2t} \times \sum_\rho \{E_\rho^{q'} (\Delta_{10}^{12})^{\kappa_2-\rho} s^{(\kappa_1-2t-\rho)/2} w_+^\rho\}, \tag{4.8}$$

where ρ takes only even values if $\kappa_1 - 2t$ is even and in that case

$$E_\rho^{q'} = \binom{q'}{\frac{1}{2}\rho} (-1)^{\rho/2}, \tag{4.9a}$$

and ρ takes only odd values if $\kappa_1 - 2t$ is odd and in that case

$$E_\rho^{q'} = \binom{q'}{\frac{1}{2}\rho - \frac{1}{2}} (-1)^{(\rho/2)-1/2}. \tag{4.9b}$$

Substituting (4.8) and (4.4) in the scalar product (4.2), we obtain

$$\langle \sigma t(\kappa_1\kappa_2) q\lambda \rangle = \sum_{q'} \sum_\rho [(t + \tfrac{1}{2}\lambda)!]^{1/2} [(t - \tfrac{1}{2}\lambda)!(2t)!]^{-1/2} A_{qq'}^{\kappa_1\kappa_2\lambda t} E_\rho^{q'} b$$
$$\times \left((\Delta_1^1)^k (\Delta_0^1)^l (\Delta_{10}^{12})^n (\Delta_{1\bar{1}}^{12})^p, (\Delta_1^1)^{-\kappa_2+2t} (\Delta_{10}^{12})^{\kappa_2-\rho} s^{(\kappa_1-2t-\rho)/2} w_+^\rho \right). \tag{4.10}$$

We can now exchange the order of the summation in ρ and q', make use of the identity[23]

$$\sum_{q'} [(a-q')!q'!(b-c+q')!(c-q')!]^{-1} = (a+b)![(a+b-c)!a!b!c!]^{-1}, \tag{4.11}$$

and expand the powers of s, w_+ on the right-hand side and of $\Delta_{10}^{12}, \Delta_{1\bar{1}}^{12}$ on the left-hand side, to obtain finally

$$\langle \sigma t(\kappa_1\kappa_2) q\lambda \rangle = [(t + \tfrac{1}{2}\lambda)!]^{1/2} [(t - \tfrac{1}{2}\lambda)!(2t)!]^{-1/2} (-2)^{t-\lambda/2} b \sum_\rho E_\rho^q \Bigg[\{[\tfrac{1}{2}(\kappa_1 - \lambda - \rho)]!/[\tfrac{1}{2}(\kappa_1 - 2t - \rho)]!\}$$
$$\times \sum_{\nu\pi\alpha\beta} \binom{n}{\nu} \binom{p}{\pi} \binom{\frac{1}{2}(\kappa_1 - 2t - \rho)}{\alpha} \binom{\rho}{\beta} (-1)^{\nu+\pi+\alpha+\beta} 2^\alpha$$
$$\times \Big((\Delta_1^1)^{k+n+p-\nu-\pi} (\Delta_0^1)^{l+\nu} (\Delta_{\bar{1}}^1)^\pi (\Delta_1^2)^{\nu+\pi} (\Delta_0^2)^{n-\nu} (\Delta_{\bar{1}}^2)^{p-\pi},\ (\Delta_1^1)^{-\kappa_2+2t+\alpha+\beta} (\Delta_0^1)^{\kappa_1-2t-2\alpha-\beta} (\Delta_{\bar{1}}^1)^\alpha (\Delta_{10}^{12})^{\kappa_2-\beta} (\Delta_{1\bar{1}}^{12})^\beta \Big) \Bigg]. \tag{4.12}$$

The scalar product in (4.12) can be evaluated using Eq. (C6) of Appendix C. There appear some Kronecker deltas that are automatically satisfied, and one $\delta_{\alpha+\beta,p}$ that allows us to eliminate the α index in the summation. The explicit expression for our coefficient becomes then

$$\langle \sigma t(\kappa_1\kappa_2) q\lambda \rangle = \left[\frac{(t + \frac{1}{2}\lambda)!}{(t - \frac{1}{2}\lambda)!(2t)!} \frac{(k+n+1)!(k+l+1)!n!p!}{(p+k+n+1)!(n+k+l+1)!k!l!} \right]^{1/2} (-2)^{t-(\lambda/2)+p}$$
$$\times \sum_{\rho\nu\pi\beta} \{E_\rho^q [\tfrac{1}{2}(\kappa_1 - \lambda - \rho)]!\rho!(\kappa_2 - \beta)!(\nu+\pi)!(2t+p-\nu-\pi)!(l+\nu)!$$
$$\times [2^\beta (n-\nu)!\nu!(p-\pi)!(\tfrac{1}{2}\kappa_1 - t - \tfrac{1}{2}\rho - p + \beta)!(p-\beta)!(\rho-\beta)!(\kappa_2 - n - \beta + \nu)!(-p+\pi+\beta)!]^{-1}\}, \tag{4.13}$$

where E_ρ^q takes the same values as in (4.9) and the summation is extended to all non-negative values of the variables for which the factorials in the denominator are not infinite.

V. COLLECTIVE WAVE FUNCTIONS

In reference 7 we showed that long-range correlations between particles, when acting within a shell of the harmonic oscillator, could be represented by the quadrupole-quadrupole interaction

$$Q^2 = \Gamma - \tfrac{1}{3}H^2 - \tfrac{1}{2}\mathcal{L}^2 , \qquad (5.1)$$

where $H,\mathcal{L}^2$ are given by (2.25) and (2.39), respectively, and Γ is the Casimir operator for the three-dimensional unitary group and is defined by [8,16]

$$\Gamma = \sum_{m,m'} \mathcal{C}_m{}^{m'} \mathcal{C}_{m'}{}^{m} . \qquad (5.2)$$

As the collective motions[7,24] are associated with these long-range correlations, we could describe them by the Hamiltonian[8]

$$\mathcal{H} = H - (Q^2/\mathcal{G}) , \qquad (5.3)$$

where $\mathcal{G}$ is an arbitrary constant.

In reference 8 we obtained the integrals of motion, and the corresponding eigenvalues, associated with the Hamiltonian $\mathcal{H}$. In this section we shall indicate how we can use the coefficients (3.25) and (4.13) to determine the eigenfunctions. We shall discuss, in detail, the eigenfunctions for a system of three particles and then indicate how the analysis can be generalized to n particles.

Let us assume that, in the absence of the Q^2 interaction, the three particles are in energy levels of the harmonic oscillator characterized by the quantum numbers ν_1,ν_2,ν_3. These quantum numbers will not be changed by the interaction since the operator Q^2, being a function of $\mathcal{C}_m{}^{m'}$, will commute with the operators $C_{ss}(s=1,2,3)$ that give the degrees ν_1,ν_2,ν_3 of the polynomials in particles 1,2,3.

We start by considering a polynomial in the creation operators Δ_m^1, Δ_m^2 of particles 1,2 of the form

$$[(f'+\nu')!]^{1/2}[(f'-\nu')!(2f')!]^{-1/2} \times C_{21}^{f'-\nu'} P_{h_1'h_2'0\sigma't'\tau'}(\Delta_m^1,\Delta_m^2) , \qquad (5.4)$$

where P is given by (3.1) and

$$f' = \tfrac{1}{2}(h_1' - h_2'), \nu' = \tfrac{1}{2}(\nu_1 - \nu_2) , \qquad (5.5)$$

with h_1',h_2' restricted by the relation

$$h_1' + h_2' = \nu_1 + \nu_2 . \qquad (5.6)$$

From the discussion in Sec. II.1, the polynomials (5.4) clearly constitute a basis for an irreducible representation of SU_3 characterized by the partition $[h_1'h_2']$. Furthermore, the polynomial (5.4) is of degree ν_1 in Δ_m^1 and of degree ν_2 in Δ_m^2.

With the third particle we associate a polynomial

$$P_{\nu_3 00\sigma''t''\tau''}(\Delta_m^3) , \qquad (5.7)$$

of the type (3.2) in which $h_1'' = \nu_3$. From (5.4), (5.7), and the Wigner coefficient (3.25), we construct the wave function

$$\begin{aligned}|\nu_1\nu_2\nu_3,h_1'h_2',h_1h_2h_3,\sigma t\tau\rangle &\equiv [(f'+\nu')!]^{1/2}[(f'-\nu')!(2f')!]^{-1/2} \\ &\times C_{21}^{f'-\nu'} \sum_{\sigma't't''} \{\langle h_1'h_2'\sigma't',\nu_3t'')h_1h_2h_3,\sigma t\rangle \\ &\times \sum_{\tau'\tau''} \langle t't''\tau'\tau''|t\tau\rangle P_{h'_1h'_20\sigma't'\tau'}(\Delta_m^1,\Delta_m^2) \\ &\times P_{\nu_300\sigma''t''\tau''}(\Delta_m^3)\}|0\rangle , \end{aligned} \qquad (5.8)$$

where

$$h_1 + h_2 + h_3 = \nu_1 + \nu_2 + \nu_3, \sigma = \sigma' + \sigma'', \quad \sigma'' = 2(t'' - \tfrac{1}{3}\nu_3) , \qquad (5.9a,b,c)$$

with the last relation coming from (3.3).

The set of eigenfunctions (5.8) associated with σ,t,τ form a basis for an irreducible representation of U_3 characterized by $[h_1h_2h_3]$. They will therefore be eigenfunctions of the Casimir operator Γ with eigenvalues[16]

$$\gamma = h_1^2 + h_2^2 + h_3^2 + 2(h_1 - h_3) . \qquad (5.10)$$

While this result is of a general group-theoretical nature[19] it can also be seen from the fact that by construction (5.8) will be equal to (3.13) if in the R of (3.14) we replace the projection f' by ν'. As the Γ is a function of $\mathcal{C}_m{}^{m'}$ it commutes with $R(C_{ss'})$ and it can be applied directly to a polynomial of the form (2.32), which was shown previously[8,16] to be an eigenpolynomial of Γ with the eigenvalue (5.10). While (5.8) is an eigenfunction of Γ and H, it is not an eigenfunction of $\mathcal{L}^2$ or Q^2. To construct eigenfunctions of $\mathcal{L}^2$ and Q^2 we use (4.13) and the α_q^ω of (2.47) to define

$$\begin{aligned}|\nu_1\nu_2\nu_3,h_1'h_2',h_1h_2h_3,\omega\lambda\lambda\rangle &= \sum_q \sum_{\sigma,t} \\ &\times \{\alpha_q^\omega\langle\sigma t(h_1-h_3,h_2-h_3)q\lambda\rangle \\ &\times |\nu_1\nu_2\nu_3,h_1'h_2',h_1h_2h_3,\sigma t\,\tfrac{1}{2}\,\lambda\rangle . \end{aligned} \qquad (5.11)$$

The set of eigenfunctions (5.11), together with those obtained from them by applying powers of $\mathcal{L}_{\bar{1}}$ as in (2.45), still form a basis for an irreducible representation of U_3 characterized by $[h_1h_2h_3]$, but now the rows are labeled by $\omega\lambda\mu$. The wave function (5.11) is then an eigenfunction of the Hamiltonian $\mathcal{H}$ for the case of three particles, and from its construction it is also an eigenfunction of the integrals of motion of $\mathcal{H}$ that were obtained in reference 8.

To construct the collective wave functions for n particles, we start with a set of polynomials of $n-1$ particles that is a basis for an irreducible representa-

[24] B. R. Mottelson in *Proceedings of the International Conference on Nuclear Structure, Kingston, Canada,* edited by D. A. Bromley and E. W. Vogt (University of Toronto Press, Toronto, Ontario, 1960), p. 525–540.

tion of SU_3 characterized by $[h_1'h_2']$ and is labeled by the rows $\sigma't'\tau'$. We then take for the nth particle a polynomial of the type (5.7) of degree ν_n in Δ_m^n. The construction of the eigenfunction for n particles then follows exactly the same steps (5.8), (5.11).

Once we obtain the eigenfunctions of $\mathcal{H}$, we could give them any desired symmetry properties, e.g., we could antisymmetrize them if the spin part of the wave functions corresponds to all spins up. It is possible, though, to start by characterizing the wave function by a given representation of the permutation group,[16] and then requiring it to be an eigenfunction of $\mathcal{H}$. This procedure will be discussed elsewhere.

VI. APPLICATION TO ELEMENTARY PARTICLES

We shall limit the discussion in this section to the SU_3 symmetry theory of the Sakata model as given by Ikeda *et al.*[9] The operators a_{ms}^+ of the previous sections can now be thought of as creation operators of particles $s = 1,\cdots,n$, where these particles can be found in three states, proton for $m = 1$, Λ for $m = 0$, and neutron for $m = \bar{1}$. Clearly, the isospin is then associated with a unitary transformation of the proton-neutron states of the form (2.19), and so its components are given by the operators (2.20). The baryonic number operator is given by the H of (2.25), since on applying it to states in which each particle appears once, its eigenvalue h equals the number of particles. The strangeness operator is

$$-\mathcal{C}_0{}^0 \qquad (6.1)$$

as it gives zero when applied to neutron or proton states and $\bar{1}$ for a Λ state. If we denote by $\mathcal{S}$ the eigenvalue of (6.1), i.e., the strangeness quantum number, then it is related to the σ of (2.24) and (2.26c) by

$$\mathcal{S} = \sigma - \tfrac{1}{3}h\,, \qquad (6.2)$$

and from (2.36a), $\mathcal{S}$ is an integer. Finally, the charge operator is clearly $\mathcal{C}_1{}^1$ as it gives 1 when applied to a proton state and zero for a neutron or Λ state; from (2.20) and (2.25) we have the well-known relation[9]

$$\mathcal{C}_1{}^1 = \tfrac{1}{2}[H + (-\mathcal{C}_0{}^0)] + T_3\,. \qquad (6.3)$$

The strangeness $\mathcal{S}$ and isospin t associated with a given representation $[h_1h_2h_3]$ of U_3 are obtained from (2.30) if we use (6.2). The relations (2.30) hold for positive h's as only creation operators are involved. Yet as these relations depend on the differences $h_1 - h_3, h_2 - h_3$ it is clear that they would still hold if we subtract an integer from all the h's, i.e., if we translate the base line of the partitions[25] as illustrated in Fig. 3, for this would affect the representation of U_3 but not of SU_3. Therefore, for any set of integers $h_1 \geqslant h_2 \geqslant h_3$, the values of $\mathcal{S}$, t are restricted by

$$\begin{aligned} &\tfrac{1}{2}(h_1 - h_2 - h_3 - \mathcal{S}) - t \geqslant 0\,,\\ &\tfrac{1}{2}(h_1 + h_2 - h_3 + \mathcal{S}) - t \geqslant 0\,,\\ &\tfrac{1}{2}(h_1 - h_2 + h_3 + \mathcal{S}) + t \geqslant 0\,,\\ &\tfrac{1}{2}(-h_1 + h_2 - h_3 - \mathcal{S}) + t \geqslant 0\,, \end{aligned} \qquad (6.4a)$$

with $2t$ being congruent modulo 2 to $(h + \mathcal{S})$. From the integers h_1,h_2,h_3 in (6.4a) we can form the positive integers $\kappa_1 \geqslant \kappa_2 \geqslant 0$ of (2.18) and the integer h of (2.15). The first two characterize the irreducible representation of SU_3 whose rows are labeled by $\mathcal{S},t$ and $\tau = t\cdots - t$, and the last h is the baryonic number which now can take negative as well as positive values. Combining Eqs. (6.4a) by pairs we obtain the following bounds for $\mathcal{S}$ and t

$$-h_1 \leqslant \mathcal{S} \leqslant -h_3, \quad 0 \leqslant t \leqslant \tfrac{1}{2}(h_1 - h_3)\,. \qquad (6.4b)$$

With the help of the Wigner coefficients (3.25) we could now construct the polynomial functions in a_{ms}^+, of first order in each particle, that would be a basis for an irreducible representation of U_3. Again we are restricted to creation operators and so at first sight it would seem that we could construct the wave functions of hypernuclei associated with representations of U_3, but not of the other hyperons or mesons for which annihilation operators are also needed. To avoid this restriction we notice that under the unitary unimodular transformation (2.4) of a_{ms}^+, the vector

$$\bar{a}^m \equiv \sum_{m'm''} \epsilon^{mm'm''} a_{m'1}^+ a_{m''2}^+ = \tfrac{1}{2} \sum_{m'm''} \epsilon^{mm'm''} \Delta_{m'm''}^{12} \qquad (6.5)$$

transforms as

$$\begin{aligned} \bar{a}^m \rightarrow & \sum_{\bar{m}} \Big[\tfrac{1}{2} \sum_{m'm''} \sum_{\bar{m}'\bar{m}''} \epsilon^{mm'm''} \epsilon_{\bar{m}\bar{m}'\bar{m}''} U_{m'}^{\bar{m}'} U_{m''}^{\bar{m}''}\Big] \bar{a}^{\bar{m}} \\ &= \sum_{\bar{m}} (U^\dagger)_{\bar{m}}^{m} \bar{a}^{\bar{m}}\,, \end{aligned} \qquad (6.6)$$

and therefore it has the transformation properties

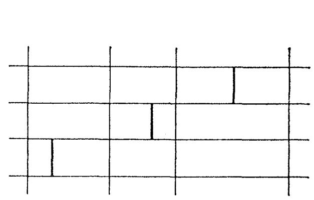

FIG. 3. Different representation of U_3 are given by the three integers corresponding to the lengths from the vertical base line to the short lines. When we change the base line, we change the representation of U_3 but not of SU_3, which depends only on the differences of these integers.

[25] I am indebted to Professor G. Racah for this remark.

(2.7) of the annihilation operator $a^{m}{}_{s}$. We can now write the polynomials associated with a basis for a representation $[h_1'h_2']$ of SU_3 in the form

$$P_{h_1'h_2'0\sigma't't'} = (-1)^{h_2'} b(a_1^+)^{k'}(a_0^+)^{l'}(\bar{a}_1)^{n'}(\bar{a}_0)^{p'}, \quad (6.7)$$

where we suppressed the particle index $s = 1$ in a_{m1}^+, and used (2.2) and (6.5). The exponents k',l',n',p' are given by (3.24) and it is trivial to see that under the substitution

$$h_1' \to h_1',\ h_2' \to h_1' - h_2',\ \sigma' \to -\sigma',\ t' \to t', \quad (6.8)$$

the exponents k',l' are interchanged with n',p'. Therefore a representation which in the particle picture is denoted by $(h_1'h_2',\sigma't'\tau')$, in the antiparticle or hole picture must be denoted by $\{h_1'h_1' - h_2', -\sigma't'\tau'\}$ and vice versa.

The results of the previous paragraph justify the use of the Wigner coefficients (3.25) for the construction, in terms of particles as well as antiparticles, of wave functions that are a basis for irreducible representations of U_3. For example, in the particle picture $(11,\sigma't'\tau')$, where from (2.30) σ',t' are restricted to $\sigma' = \frac{2}{3}, t' = 0$ and $\sigma' = -\frac{1}{3}, t' = \frac{1}{2}$, corresponds in the antiparticle picture to $\{10, -\sigma't'\tau'\}$. If we want then to construct a basis for a representation of U_3 from an antiparticle and a particle we would need the coefficients

$$\langle 11\sigma't',1t'')h_1h_2h_3,\sigma t\rangle, \quad (6.9)$$

where $[h_1h_2h_3]$ is restricted to [21] or [111]. More generally, we could construct the wave function for a system of n particles by starting with a wave function for $n-1$ of them in the particle or antiparticle picture, depending on whether the n'th is a particle or an antiparticle, and combining the products of the wave functions with the coefficients (3.25).

VII. FRACTIONAL PARENTAGE COEFFICIENTS IN THE p SHELL

In Sec. III we obtained the Wigner coefficients for the SU_3 group when the rows of the representation are labeled by $\sigma t\tau$. Clearly we could have carried out a similar analysis when the rows are labeled by $\omega\lambda\mu$, in which case the Wigner coefficients associated with the development of Fig. 1 could be denoted by

$$\langle h_1'h_2'\omega'\lambda',h_1''\lambda''\}h_1h_2h_3,\omega\lambda\rangle, \quad (7.1)$$

where ω'' is not indicated as for the single-row representation it is not independent of[8] h_1'', λ''. The coefficient (7.1) could be obtained from (3.25) if we made use of (2.47) and (4.13) to transform the rows $\sigma t\tau$ into $\omega\lambda\mu$.

The fractional parentage coefficients in the p shell are clearly given by (7.1) when we take $h_1'' = 1$, $\lambda'' = 1$. These fractional parentage coefficients are well known,[13] and so the present analysis provides only a different procedure for their derivation. It is interesting to note though that the analysis of Sec. III can be generalized to unitary groups in $2l+1$ dimensions as the explicit expression for the polynomials that are a basis for an irreducible representation $[h_1\cdots h_{2l+1}]$ of U_{2l+1} has already been obtained.[16] The analysis developed in this paper could then, in principle, be generalized to derive the fractional parentage coefficients in a shell of angular momentum l, or for that matter, in a mixture of shells.

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APPENDIX A

Normalization Coefficients of the Polynomial P.

In this Appendix we determine the normalization coefficient b of the polynomial (2.31). Denoting by P_0 the polynomial

$$P_0 \equiv (\Delta_1^1)^k(\Delta_0^1)^l(\Delta_{10}^{12})^n(\Delta_{1\bar{1}}^{12})^p(\Delta_{10\bar{1}}^{123})^{h_3} \equiv (\Delta_{10\bar{1}}^{123})^{h_3}P_{klnp}, \quad (A1)$$

where k,l,n,p are given by (2.30), we see that b^{-2} is given by

$$b^{-2} = (P_0,P_0) = \langle 0|P_0^+P_0|0\rangle. \quad (A2)$$

In the determination of (A2) we shall use throughout the fact (2.6), so that the adjoint operator of Δ_m^s can be interpreted as a differential operator, i.e.,

$$\Delta_m^{s+} = (\partial/\partial\Delta_m^s). \quad (A3)$$

From (A1), (A2) we obtain

$$(P_0,P_0) = \langle 0|(\Delta_{10\bar{1}}^{123+})^{h_3-1}P_{klnp}^+\{\Delta_{10\bar{1}}^{123+}P_0\}|0\rangle, \quad (A4)$$

where $\Delta_{10\bar{1}}^{123+}$ can be written as

$$\Delta_{10\bar{1}}^{123+} = \Delta_{10}^{12+}\Delta_{\bar{1}}^{3+} + \Delta_{\bar{1}1}^{12+}\Delta_0^{3+} + \Delta_{0\bar{1}}^{12+}\Delta_1^{3+} \quad (A5)$$

with

$$\Delta_{mm'}^{12+} = \Delta_m^{1+}\Delta_{m'}^{2+} - \Delta_{m'}^{1+}\Delta_m^{2+}. \quad (A6)$$

To evaluate the curly bracket in (A4) we notice that

$$\Delta_{\bar{1}}^{3+}(\Delta_{10\bar{1}}^{123})^{h_3} = h_3(\Delta_{10\bar{1}}^{123})^{h_3-1}\Delta_{10}^{12}\,, \text{ and cycl}\,. \quad (A7)$$

From (A7) and (A6) we obtain

$$\begin{aligned}\{\Delta_{10\bar{1}}^{123+}(\Delta_{10\bar{1}}^{123})^{h_3}P_{klnp}\} &\equiv \{\quad\} \\ &= h_3 \sum_{m,m'} \Delta_m^{1+}\Delta_{m'}^{2+}[(\Delta_m^1\Delta_{m'}^2 - \Delta_{m'}^1\Delta_m^2)P_{-1}]\,, \quad (A8)\end{aligned}$$

where

$$P_{-r} = (\Delta_{10\bar{1}}^{123})^{h_3-r}P_{klnp}\,. \quad (A9)$$

Using the commutation rules (2.5) and the definitions (2.10), as well as Eqs. (2.14), we obtain

$$\begin{aligned}\{\quad\} &= h_3(C_{11}+3)(C_{22}+2)P_{-1} \\ &= (h_1+2)(h_2+1)h_3P_{-1}\,, \quad (A10)\end{aligned}$$

and so

$$(P_0,P_0) = (h_1+2)(h_2+1)h_3(P_{-1},P_{-1})\,. \quad (A11)$$

Repeating this process for P_{-1}, etc., we obtain finally

$$\begin{aligned}(P_0,P_0) = [(h_1+2)!(h_2+1)!h_3!]\,[(h_1-h_3+2)! \\ \times (h_2-h_3+1)!]^{-1}\,(P_{klnp},P_{klnp})\,. \quad (A12)\end{aligned}$$

We can now write

$$(P_{klnp},P_{klnp}) = (P_{klnp-1},\Delta_{1\bar{1}}^{12+}P_{klnp})\,. \quad (A13)$$

Using (A6) and (A3) we easily see that

$$\Delta_{1\bar{1}}^{12+}\{(\Delta_{1\bar{1}}^{12})^pP_{kln0}\} = p(p+k+n+1)P_{klnp-1}\,. \quad (A14)$$

Substituting in (A13) and repeating the process p times, we obtain finally

$$\begin{aligned}(P_{klnp},P_{klnp}) &= [p!(p+k+n+1)!] \\ &\times [(k+n+1)!]^{-1}(P_{kln0},P_{kln0})\,. \quad (A15)\end{aligned}$$

The scalar product on the right-hand side of (A15) can be evaluated in a similar way by making use of

$$\Delta_{10}^{12+}P_{kln0} = n(n+k+l+1)P_{kln-1\,0}\,, \quad (A16)$$

and so we obtain

$$\begin{aligned}(P_{kln0},P_{kln0}) &= [n!(n+k+l+1)!] \\ &\times [(k+l+1)!]^{-1}(P_{kl00},P_{kl00}) \\ &= n!(n+k+l+1)!\,[(k+l+1)!]^{-1}k!l!\,, \quad (A17)\end{aligned}$$

where from (A3) the last scalar product gives $(k!l!)$.

Combining all of the previous results we obtain the normalization coefficient as

$$\begin{aligned}b &= \left[\frac{(h_1-h_3+2)!(h_2-h_3+1)!}{(h_1+2)!(h_2+1)!h_3!}\right]^{1/2} \\ &\times \left[\frac{(k+n+1)!}{p!(p+k+n+1)!}\right]^{1/2} \\ &\times \left[\frac{(k+l+1)!}{n!(n+k+l+1)!k!l!}\right]^{1/2}. \quad (A18)\end{aligned}$$

APPENDIX B

Orthonormalization of the Polynomials $\mathcal{P}$

In this Appendix we determine the normalization coefficient d of the polynomial $\mathcal{P}$ of (3.13), and we prove that two polynomials $\mathcal{P}$ differing in any of the indices h_1,h_2,h_3 are orthogonal.

Denoting by $\mathcal{P}^{h_1'h_2'}$ the polynomial

$$\mathcal{P}^{h_1'h_2'} = e^{-1}R(C_{ss'})P\,, \quad (B1)$$

where e is given by (3.17), R by (3.16) and P is a shorthand notation for the polynomial (2.32), we see that d^{-2} is given by

$$d^{-2} = e^2(\mathcal{P}^{h_1'h_2'},\mathcal{P}^{h_1'h_2'})\,. \quad (B2)$$

As $\mathcal{P}^{h_1'h_2'}$ satisfies the equation

$$C_{12}\mathcal{P}^{h_1'h_2'} = 0\,, \quad (B3)$$

we shall use the form (3.18) of R to reduce (B2) to the form

$$\begin{aligned}d^{-2} = e^2[(h_1'-h_2)!(h_1-h_2'+1)!][(h_1-h_1')! \\ \times (h_1'-h_2'+1)!]^{-1}\,(C_{31}^{h_1-h_1'}C_{32}^{h_2-h_2'}P,\mathcal{P}^{h_1'h_2'})\end{aligned} \quad (B4)$$

To evaluate (B4) we pass one of the C_{31} to the right-hand side as C_{13}, and notice that from the definition (B1), the commutation relations (2.12), and Eqs. (2.14) we obtain

$$C_{13}\mathcal{P}^{h_1'h_2'} = (h_1'-h_3+2)\mathcal{P}^{h_1'+1\,h_2'}\,. \quad (B5)$$

Substituting (B5) in (B4) and repeating the analysis h_1-h_1' times, we see that the scalar product in (B4) reduces to

$$(h_1-h_3+1)![(h_1'-h_3+1)!]^{-1}(C_{32}^{h_2-h_2'}P,\mathcal{P}^{h_1h_2'}) \quad (B6)$$

Now from the definition (B1) we see that

$$\mathcal{P}^{h_1h_2'} = (h_1-h_2)!C_{32}^{h_2-h_2'}P\,. \quad (B7)$$

Substituting (B7) in (B6) and again making use of the commutation rules (2.12) and of (2.14), we obtain for the scalar product in (B6)

$$(h_1 - h_2)!(h_2 - h_2')!(h_2 - h_3)![(h_2' - h_3)!]^{-1}. \quad \text{(B8)}$$

Combining all of the previous results, we obtain for the normalization coefficient

$$d = [(h_1' - h_3 + 1)!(h_2' - h_3)!]^{1/2} \times [(h_1 - h_3 + 1)!(h_2 - h_3)!]^{-1/2}. \quad \text{(B9)}$$

We now consider two polynomials $\mathcal{P}^{h_1'h_2'}, \bar{\mathcal{P}}^{h_1'h_2'}$ defined by (B1) where the first is characterized by the indices h_1,h_2,h_3 for the polynomial P of (2.32) while the second is characterized by the indices $\bar{h}_1,\bar{h}_2,\bar{h}_3$. Without loss of generality we can assume $\bar{h}_1 \geqslant h_1$. By a reasoning entirely similar to the one that takes us from (B2) to (B6), we prove that the scalar product

$$I \equiv (\mathcal{P}^{h_1'h_2'}, \bar{\mathcal{P}}^{h_1'h_2'}), \quad \text{(B10)}$$

is proportional to

$$(C_{32}^{h_2-h_2'}P, \bar{\mathcal{P}}^{h_1h_2'}), \quad \text{(B11)}$$

where $\bar{\mathcal{P}}^{h_1h_2'}$ has still the general form (3.13) as $h_1 \leqslant \bar{h}_1$. From (2.12) we see that C_{12} and C_{32} commute and so we could use the form (3.18) of $\bar{\mathcal{P}}^{h_1h_2'}$ to reduce (B11) to

$$I \propto (C_{32}^{h_2-h_2'}P, C_{31}^{\bar{h}_1-h_1}C_{32}^{\bar{h}_2-h_2'}\bar{P}), \quad \text{(B12)}$$

where $P,\bar{P}$ are given by (2.32) with indices $h_1h_2h_3$ and $\bar{h}_1\bar{h}_2\bar{h}_3$, respectively. As

$$[C_{13}, C_{32}^{h_2-h_2'}] = (h_2 - h_2')C_{32}^{h_2-h_2'-1}C_{12}, \quad \text{(B13)}$$

we see that $I = 0$ if $\bar{h}_1 > h_1$. For $\bar{h}_1 = h_1$, we could again use (2.12) to prove that $I = 0$ if $\bar{h}_2 \neq h_2$. Finally as

$$h_1 + h_2 + h_3 = h_1' + h_2' + h_1'' = \bar{h}_1 + \bar{h}_2 + \bar{h}_3, \quad \text{(B14)}$$

we conclude that $\mathcal{P}^{h_1'h_2'}, \bar{\mathcal{P}}^{h_1'h_2'}$ are orthogonal if they differ in any of the indices h_1,h_2,h_3.

APPENDIX C

Evaluation of Scalar Product in (3.23)

The evaluation of the scalar product (3.23), which is necessary for the determination of the Wigner coefficient (3.25), is a long but essentially straightforward process. Here we shall indicate only a few basic steps.

First we pass the operator $C_{31}^{h_1-h_1'}$ on the left-hand side of (3.23) to the right as $C_{13}^{h_1-h_1'}$. Then we notice that C_{13} will act only on products of powers of Δ_m^3 and as

$$C_{13}\Delta_m^3 = \Delta_m^1, \quad \text{(C1)}$$

we get on the right-hand side of (3.23) linear combination of terms of the same type as those already present plus a factor of the form $(\Delta_1^1)^{c'}$.

On the left-hand side of (3.23), $C_{32}^{h_2-h_2'}$ acts only on the product $(\Delta_{10}^{12})^n(\Delta_{1\bar{1}}^{12})^p$, and as

$$C_{32}\Delta_{1m}^{12} = \Delta_{1m}^{13}, \; m = 0, \bar{1}, \quad \text{(C2)}$$

we get on the left-hand side of (3.23) linear combinations of terms of the same type plus factors of the form $(\Delta_{10}^{13})^e(\Delta_{1\bar{1}}^{13})^f$. Now developing $\Delta_{10\bar{1}}^{123}$ as

$$\Delta_{10\bar{1}}^{123} = (\Delta_1^1)^{-1}(\Delta_{10}^{12}\Delta_{1\bar{1}}^{13} - \Delta_{1\bar{1}}^{12}\Delta_{10}^{13}), \quad \text{(C3)}$$

we see that the scalar product in (3.23) becomes a linear combination of scalar products of the form

$$\begin{aligned}&((\Delta_1^1)^a(\Delta_0^1)^b(\Delta_{10}^{12})^c(\Delta_{1\bar{1}}^{12})^d(\Delta_{10}^{13})^e(\Delta_{1\bar{1}}^{13})^f,\\ &(\Delta_1^1)^{a'}(\Delta_0^1)^{b'}(\Delta_{\bar{1}}^1)^{c'}(\Delta_{10}^{12})^{d'}(\Delta_{1\bar{1}}^{12})^{e'}(\Delta_1^3)^{f'}(\Delta_0^3)^{g'}(\Delta_{\bar{1}}^3)^{h'}),\end{aligned} \quad \text{(C4)}$$

where from (3.8) we must have the degrees of all particles the same on both sides, so that among other relations we get

$$f' + g' + h' = e + f. \quad \text{(C5)}$$

To determine (C4) we pass the terms Δ_m^3 from the right- to the left-hand side, and notice that in (C4)

$$(\Delta_1^{3+})^{f'}(\Delta_0^{3+})^{g'}(\Delta_{\bar{1}}^{3+})^{h'}(\Delta_{10}^{13})^e(\Delta_{1\bar{1}}^{13})^f, \quad \text{(C6a)}$$

is equivalent to

$$(-1)^{f'}\binom{e}{g'}\binom{f}{h'}f'!g'!h'!(\Delta_1^1)^{e+f-f'}(\Delta_0^1)^{e-g'}(\Delta_{\bar{1}}^1)^{f-h'}, \quad \text{(C6b)}$$

where use was made of (A3) and (C5). The scalar product (C4) then becomes proportional to

$$\begin{aligned}&((\Delta_1^1)^{a+e+f-f'}(\Delta_0^1)^{b+e-g'}(\Delta_{\bar{1}}^1)^{f-h'}(\Delta_{10}^{12})^c(\Delta_{1\bar{1}}^{12})^d,\\ &(\Delta_1^1)^{a'}(\Delta_0^1)^{b'}(\Delta_{\bar{1}}^1)^{c'}(\Delta_{10}^{12})^{d'}(\Delta_{1\bar{1}}^{12})^{e'}).\end{aligned} \quad \text{(C7)}$$

We develop the $\Delta_{10}^{12}, \Delta_{1\bar{1}}^{12}$ on the right-hand side of (C7) in terms of Δ_m^1, Δ_m^2. Now passing the powers of Δ_m^2 to the left-hand side, and using a formula analogous to (C6), the scalar product (C7) is reduced to one in which we have only powers of Δ_m^1 on both sides, which can be immediately evaluated using (A3).

In this way the scalar product (3.23) can be evaluated and the explicit expression appears in (3.25).