Aspects of Nucleon-Nucleon Scattering Theory*

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1. INTRODUCTION

THIS article is not intended to be a comprehensive review but is concerned with a few selected aspects of the theory of nucleon-nucleon scattering. The bibliography is, therefore, incomplete from the point of view of covering the subject as a whole and even regarding those parts that are discussed in reasonable detail. The writer apologizes in advance for the omissions of references which from some viewpoints might have been logical to include.

Since the kinematics of the elastic nucleon-nucleon scattering problem is usually introduced nonrelativistically and the relativistic corrections are supplied later, it appeared desirable to introduce the phase shifts in such a way as to make the transition to the relativistic treatment immediate. This is done in Sec. 2. In Sec. 3 the relativistic kinematical relations are considered, especially for the spin orientation. The latter is treated in terms of the apparent nonrelativistic spin, referred to as the "spin" which is convenient for dealing with the problem without too many formal developments, and is closely related to the usual way of performing a phase-shift analysis. The rotation of the "spin" axis which occurs in the laboratory system if the "spin" is turned in a known way in the center of mass system is worked out in this connection. In Sec. 4 the scattering of charged nucleons by each other is discussed nonrelativistically in such a way as to facilitate the transition to the relativistic case. No attempt is made in this article to discuss the vast amount of experimental and theoretical material on precision p-p scattering experiments in the energy range from 400 keV to a few MeV. It nevertheless appeared desirable to briefly mention in Sec. 4 the position of these investigations in the field as a whole as it appears to the present reviewer, to supply a few references, and to mention the present status of corrections for vacuum polarization effects and of the as yet incompletely treated dynamic effects of electrons in the target molecules. In Sec. 5 the relativistic modifications of Coulomb scattering are considered with attention to the limitations of available treatments that represent a compromise between rigor and expediency in the approximate calculation of relatively small effects. In this section no attempt is made to include the effects of the anomalous parts of nucleon magnetic moments or of spin-orbit and spin-spin effects that arise from the Dirac part of the proton's magnetic moment. These matters are discussed in Sec. 7. In order to make the treatment self-contained and to make it easier to follow Sec. 7, equations for the calculation of the scattering matrix are collected in Sec. 6 in the notation used in phase parameter work at Yale, with an attempt at removing confusion which might arise on reading some of the original papers on account of a few misprints and oversights. The connection of the nucleon-nucleon scattering amplitudes with the calculation of differential cross sections and of the polarization parameter is pointed out in this section but the calculation of triple scattering parameters from the amplitudes is left out, since it is not essential for the understanding of succeeding parts of this article. In Sec. 7 a partial way out of an old difficulty of wave function distortion is described following some recent work in collaboration with Ruppel, with somewhat more emphasis on the assumptions made and on the limitations of the treatment than in the original paper. Although the magnetic moment effects are small, some of them are not always negligible and future work will have to take them into account. Section 8 discusses views regarding the nature of the nucleon-nucleon interaction. Among the items under discussion is the meaning of the word "potential." The necessity of exercising care is illustrated by means of a static model for the nucleon-nucleon interaction employing a nonquantized meson field and neglecting some relativistic effects as well as some effects of the nucleon-nucleon vacuum. This model shows that effects similar to those that can be produced by a hard repulsive core may be due to an effect of a transformation which reduces the radial differential equation to a standard form and has the effect of producing a node of the transformed function at the hard core radius. The object in reviewing this situation is mainly that of illustrating reasons for

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doubting literal interpretations of potential energy models and expressing doubt regarding plausible, but in reality unjustified, conclusions which have been drawn regarding the character of nucleon-nucleon interactions on insufficient evidence. From a similar viewpoint, the qualitative indications concerning the spin-orbit interaction which have been obtained in this model are briefly reviewed. At no point was it intended to suggest in the original work or here that this imperfect model represents the actual interaction in a precise sense, but it is believed that it shows the caution that must be used in identifying descriptions of phenomenological potentials with the actual situation. Toward the end of Sec. 8 the advantages of investigating the interaction from the outside in is discussed, the general viewpoint being that it is at present more important to ascertain the physical origin of the interaction than to have a complete quantitative treatment of all features of nucleon-nucleon scattering. This conclusion is followed up by considerations regarding the one-pion exchange interaction in Sec. 9 which is partly concerned with the extent to which classical mechanics is applicable to distant collisions, the differences in viewpoint of the "pole" and the "space localization" approaches. Section 10 is concerned with tests of charge independence, primarily from the viewpoint of the one-pion exchange interaction and in Sec. 11 some of the evidence concerning the two-pion exchange interaction is briefly reviewed. In Sec. 12 the possibly common origin of the spin-orbit interaction and of the repulsive core in an interaction of nucleons with vector mesons is discussed.

The selection of topics for the article has been partly on the basis of personal interest and partly influenced by what the writer believes to be especially important for the future development of the subject. A major omission is the lack of an account of such progress as has been achieved by means of dispersion relations[1] which are promising to add much to the quantitative treatment of the subject. This omission is perhaps justifiable at this particular time because the many resonances[2] that have been found indicate that the pion-pion interactions are still incompletely understood. This matter has of course a bearing on Sec. 12 as well. Since in the latter the limitations rather than the accomplishments of the vector meson hypothesis are under discussion, its inclusion appeared nevertheless appropriate.

2. INTRODUCTION OF PHASE SHIFTS

The presentation of the scattering theory for two colliding nucleons is usually made starting with the nonrelativistic theory and pointing out afterwards the modifications required by a relativistic approach. In the present review it appears desirable to attempt to reverse the procedure and to employ an approximate relativistic treatment to begin with. The notation used will be such as to make the transition to the nonrelativistic approximation immediate. In view of the close equality of the masses of the two nucleons, they will be taken as exactly equal.

On account of the conservation of energy and momentum which has not as yet been seriously challenged for a closed physical system, one may assume that there exists a Lorentz frame in which the total momentum of the two nucleons is zero. In the part of the configuration space for which the two nucleons may be described by a Schrödinger equation, one may therefore set

$$(\mathbf{p}_1 + \mathbf{p}_2)\psi = 0\,, \tag{2.1}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$ are, respectively, the momentum operators of the two nucleons and ψ is the wave function. Equation (2.1) involves the assumptions that for large internucleon separations r, the emissions of mesons and of photons produce negligible effects. For energies below the threshold of meson production and for sufficiently large distances between nucleons, the former assumption is justifiable. The second assumption is only an approximation, there being a finite probability of the emission of electromagnetic radiation even in the case of neutron-neutron collisions. On account of the largeness of the nucleon mass, the probability of such radiation is small, however, and

[1] M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. (New York) **2**, 726 (1957); M. L. Goldberger, M. T. Grisaru, and S. W. MacDowell, Phys. Rev. **120**, 2250 (1960); H. P. Noyes and D. Y. Wong, Phys. Rev. Letters **3**, 191 (1959); H. P. Noyes, Phys. Rev. **119**, 1736 (1960); G. F. Chew, *ibid.* **112**, 1380 (1958); D. Amati, E. Leader, and B. Vitale, Nuovo cimento **17**, 68 (1960); **18**, 409 (1960); A. D. Galanin, A. F. Grashin, B. L. Iaffe, and I. Ya Pomeranchuk, Nuclear Phys. **17**, 181 (1960); A. F. Grashin and I. Yu. Kobsarev, *ibid.* **17**, 218 (1960); N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **26**, 337 (1961); S. Furuichi, *ibid.* **27**, 51 (1962); Y. Hara, *ibid.* **26**, 627 (1961); **27**, 429 (1962); K. Yamamoto, *ibid.* **26**, 1014 (1961). No attempt to have the bibliography complete has been made.

[2] B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. Letters **7**, 178 (1961); A. Pevsner, R. Kraemer, M. Nussbaum, P. Schlein, T. Toohig, M. Block, A. Kovacs, and G. Meltzer, Aix en Provence Conference, 1961 (unpublished); D. D. Carmony and R. T. Van de Walle, Phys. Rev. Letters **8**, 73 (1962); N. Booth, A. Abashian, and N. Crowe, *ibid.* **7**, 178 (1961); D. Stonehill, C. Baltay, H. Courant, W. Finkinger, E. C. Fowler, H. Kraybill, J. Sandweiss, J. Sanford, and H. Taft, *ibid.* **6**, 624 (1961); A. R. Erwin, R. March, W. D. Walker, and E. West, *ibid.* **6**, 628 (1961); E. Pickup, D. K. Robinson, and E. O. Salant, *ibid.* **7**, 192 (1961). The latter papers contain complete reference lists. No attempt to have the bibliography complete has been made.

the assumption that errors arising from its neglect are not important will be made below.

In the cases of n-n and n-p scattering for sufficiently large internucleon distances r one deals with free waves, the spin-orbit and spin-spin interactions falling off sufficiently rapidly with r to make it possible in the case of elastic collisions to take their effects into account by means of phase shifts. On the other hand, for p-p scattering it is well known from discussions of the nonrelativistic problem that the Coulomb field cannot be taken care of that way. A device avoiding this difficulty has been introduced by Gordon[3] in his paper on the exact nonrelativistic solutions of the Schrödinger equation for Coulomb scattering. Gordon surrounds the scattering center by a sphere of large radius R carrying a uniform surface charge distribution which neutralizes the point charge and makes the space for $r > R$ field free. Free-particle wave functions may then be used for $r > R$ and the Rayleigh-Faxen-Holtsmark theory of scattering by central fields becomes applicable. The essential feature of the Gordon screening sphere device is that it reduces the Coulomb scattering problem to one with a known answer. It may be noted that R is made infinite eventually and that in this limit the physical predictions obtained from the solution become independent of R. It is thus not essential to have a definite physical model such as a screening sphere in order that Gordon's device should work. It suffices to have any mathematical modification of the original problem which makes the particles noninteracting for $r > R$. This is possible even in the relativistic case by multiplying the Coulomb potential e^2/r by a decaying function like $e^{-\alpha r}$. The physical interpretation of such a factor depends on the energy. Since, however, the decaying function will eventually be made equal to unity, the neglect of Lorentz contraction effects on the screening distribution cannot be essential. The device of modifying the problem in such a way as to have no interaction between particles at sufficiently large r is especially helpful in the discussion of the relativistic problem, because it makes it easier to remove the difficulties involved in the treatment of e^2/r to a later stage in the discussion.

For two noninteracting Dirac particles 1, 2, one can describe the state of the combined system by means of a 16-component wave function, which is the solution of

$$(H_1 + H_2)\psi = E\psi\,, \tag{2.2}$$

where H_1, H_2 are Dirac single-particle Hamiltonians for particles 1 and 2. In order to have an easy passage to the Pauli spin nonrelativistic description, it will be convenient to employ the original Dirac representation of the Dirac equation. In this the free particle equation has the form

$$(p_0 + \boldsymbol{\alpha}\cdot\mathbf{p} + \beta Mc)\psi = 0 \tag{2.3}$$

with

$$p_0 = -\frac{\hbar}{ic}\frac{\partial}{\partial t}\left(= \frac{E}{c}\right), \quad \mathbf{p} = \frac{\hbar}{i}\nabla\,, \tag{2.4}$$

and

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \alpha_4\,. \tag{2.5}$$

In the two last forms each element of the array representing a component of $\boldsymbol{\alpha}$ or of β is itself a 2×2 matrix. Through most of the consideration the exact form of the $\boldsymbol{\sigma}$ is immaterial but whenever a specific choice of representation will be needed it will be the same as originally used by Pauli, namely,

$$\sigma_x = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0, & -i \\ i, & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}. \tag{2.6}$$

In order not to complicate the notation, the symbol ψ is used both for the 16-component relativistic two-particle function and the 4-component one-particle function. In the latter case it is convenient to use two functions each with two components, namely,

$$\Psi = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.7}$$

For a free particle, these satisfy

$$\begin{aligned} (p_0 + Mc)\Phi + (\mathbf{p}\boldsymbol{\sigma})\Psi &= 0\,, \\ (p_0 - Mc)\Psi + (\mathbf{p}\boldsymbol{\sigma})\Phi &= 0\,. \end{aligned} \tag{2.8}$$

If the particle is in a positive energy state with an energy slightly above Mc^2, these equations give much larger values of the two components of Ψ than of Φ barring exceptional situations such as nodes in Ψ. For this reason the components ψ_3, ψ_4 will be referred to as "large" and ψ_1, ψ_2 will be similarly referred to as "small." Similarly, in the case of two particles it is convenient to break up the 16-component ψ into Ψ, ξ_1, ξ_2, Φ with Ψ "large" in both particles, ξ_1 small in 1 and large in 2, ξ_2 large in 1 and small in 2, Φ small in both. For noninteracting particles the equations

[3] W. Gordon, Z. Physik **48**, 180 (1928); cf., N. F. Mott, Proc. Roy. Soc. (London) **A118**, 542 (1928) for an independent and somewhat different treatment.

satisfied by these functions are, setting $c = 1$,

$$(E - 2M)\Psi + \pi_1\xi_1 + \pi_2\xi_2 = 0\,, \quad (c = 1)$$
$$\pi_1\Psi + E\xi_1 + \pi_2\Phi = 0\,, \tag{2.9}$$
$$\pi_2\Psi + E\xi_2 + \pi_1\Phi = 0\,,$$
$$\pi_2\xi_1 + \pi_1\xi_2 + (E + 2M)\Phi = 0\,,$$
$$\pi_i = (\mathfrak{d}_i\cdot\mathbf{p}_i) \quad (i = 1,2)\,. \tag{2.10}$$

The function Ψ plays a role similar to that of a nonrelativistic function for two particles each with spin 1/2 and is suitable for the classification of states in the usual spectroscopic terminology of $^1S_0$, $^3S_1$, $^1P_1$, $^3P_0, \cdots$ states. Disregarding for the moment the possibility of coupling between states with different L and the same J, every state ^sL_J, $s = 1,3$, is characterized by an asymptotic form of Ψ which consists of a linear combination of spin functions $^s\chi_m$, angular space functions $Y_{L,\mu-m}$ and a radial space function $\mathfrak{F}_{L,J}/r$ so that

$$\Psi = \mathcal{Y}_\mu^{L,J}\mathfrak{F}_{L,J}/r\,, \tag{2.11}$$

where

$$\mathfrak{F}_{L,J} \backsim \text{const} \times \sin\,(kr - L\pi/2 + \delta_J^L) \tag{2.12}$$

and

$$\mathcal{Y}_\mu^{L,J} = \sum_m \begin{pmatrix} L & , & J \\ \mu - m & , & m \end{pmatrix} Y_{L,\mu-m}\chi_m \tag{2.13}$$

is the linear combination of products of space angular functions and spin functions corresponding to total angular momentum J. From Eq. (2.11) on, the specialization to the barycentric coordinate system of Eq. (2.1) has been made. Equations (2.1), (2.2), and (2.13) have been written down in the same form as in nonrelativistic theory but it is readily seen that the significance of the symbols is not altered in relativistic theory. For the phase shifts δ_J^l this follows from the fact that the forms for Ψ, ξ_1, ξ_2, Φ that apply for zero phase shifts will involve cosines of the same argument (the phase) for ξ_1 and ξ_2 if Ψ contains the sine and therefore, according to the last Eq. (2.9), Φ contains the sine. The insertion of the phase shift in all the trigonometric functions will not affect the validity of the equations which will remain simply conditions on the amplitudes in the components of Ψ, ξ_1, ξ_2, Φ that multiply the sines and cosines of the changed phase. The phase shift has, therefore, the same significance relativistically as nonrelativistically, viz., as the addition to the asymptotic phase of all components.

Before discussing the significance of $\mathbf{L}$ and $\mathfrak{d}$ for the two-particle case, it will be useful to recall the somewhat similar situation for one particle in a central field. In that case, the usual spectroscopic classification of a term corresponds to the nonrelativistic interpretation of Ψ of Eq. (2.8). The transition to radial functions is, in fact, accomplished for these equations by setting

$$\Psi = \mathcal{Y}g(r)\,, \quad (\hat{r}\cdot\mathfrak{d})\Phi = i\mathcal{Y}f(r)\,, \tag{2.14}$$

where $\mathcal{Y}$ is the angular spin function corresponding to the spectroscopic classification of the level. The radial functions f and g satisfy the familiar radial equations

$$(p_0 + mc)f - \hbar[d/dr + (k + 1)/r]g = 0$$
$$(p_0 - mc)g + \hbar[d/dr + (1 - k)/r]f = 0\,, \tag{2.15}$$

where the numbers k are the eigenvalues of the operator

$$k = \rho_3[(\mathbf{L}\mathfrak{d}) + 1]\,,$$
$$(\rho_3 = \beta = \alpha_4\,, \quad \mathbf{L} = [\mathbf{r}\times\mathbf{p}]/\hbar)\,. \tag{2.16}$$

The $\mathfrak{d}$ in the last equation is the 4×4 matrix introduced by Dirac. The possibility of writing Ψ in the particular manner used in (2.14) originates in the fact that in the representation used ρ_3 is a diagonal matrix with the first two elements equal to 1 and the last two to -1. It follows from (2.16) that the eigenfunctions of k correspond to $(\mathbf{L}\cdot\mathfrak{d})$ with $\mathfrak{d}$ having the meaning of Pauli's 2×2 matrix vector. The correctness of the equations corresponding to the second equality in Eq. (2.14) follows from Dirac's original argument concerning k commuting with the central field Hamiltonian and can be verified by direct calculation.

But another way of seeing that $\Psi = \mathcal{Y}g(r)$ gives the correct J relativistically is to note that the first equality in (2.8) gives $\Phi = -(p_0 + Mc)^{-1}(\mathbf{p}\mathfrak{d})\Psi$ and that $\mathbf{J}$ commutes with $(\mathbf{p}\mathfrak{d})$. Therefore, the infinitesimal rotations for spins and coordinates together are represented by the equations with the same coefficients. It follows therefore that to every irreducible representation of the rotation group employing the two component Ψ as a basis, there corresponds the same irreducible representation employing as a basis the corresponding Φ and hence also the corresponding four component ψ.

Similarly, for two particles, the irreducible representations of the rotation group employing as a basis the four-component functions Ψ of Eq. (2.11) are seen to correspond to the same representations making use as a basis of the corresponding 16-component

function ψ. From (2.9), leaving aside unessential and obviously scalar factors, it is clear that

$$\xi_1 \propto (\boldsymbol{\sigma}_1^P \cdot \mathbf{p})\Psi\,,$$
$$\xi_2 \propto (\boldsymbol{\sigma}_2^P \cdot \mathbf{p})\Psi,\ \Phi \propto (\boldsymbol{\sigma}_1^P \cdot \mathbf{p})(\boldsymbol{\sigma}_2^P \cdot \mathbf{p})\Psi\,. \quad (2.17)$$

The last follows readily on eliminating ξ_1, ξ_2 from the four equations, and the character of ξ_1, ξ_2 then is obvious from the second and third equations. For definiteness a superscript P is used in the immediate context to distinguish the Pauli σ's from the Dirac ones. The total angular momentum operator is

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2\,, \quad (2.18)$$

with

$$\mathbf{J}_i = \mathbf{L}_i + \tfrac{1}{2}\,\boldsymbol{\sigma}_i\,, \quad (i = 1,2) \quad (2.19)$$

and $\mathbf{L}_i$ in units $\hbar$. Introducing

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2\,, \quad (2.20)$$

it is legitimate to write

$$\hbar\mathbf{L}_1 = [\mathbf{r}_1 \times \mathbf{p}]\,, \quad \hbar\mathbf{L}_2 = -[\mathbf{r}_2 \times \mathbf{p}]\,, \quad \mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2 \quad (2.21)$$

as long as the operations are in the center-of-mass system and only relative coordinates are involved in quantities other than ψ. Hence, according to (2.19), (2.20), and (2.21)

$$\mathbf{J} = \mathbf{L} + \tfrac{1}{2}\,(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\,, \quad (2.22)$$

with

$$\hbar\mathbf{L} = [\mathbf{r} \times \mathbf{p}]\,, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2\,. \quad (2.23)$$

The operator in (2.22) is for use with the 16-component ψ and the connection with rotations can be established by means of this operator. Since each $\boldsymbol{\sigma}_i$ has no matrix elements in the squares corresponding to one matrix element label being in the "small" and the other in the "large" group, and within the same group it has the same elements as $\boldsymbol{\sigma}_i^P$, it is convenient to introduce

$$\mathbf{J}^P = \mathbf{L} + \tfrac{1}{2}\,(\boldsymbol{\sigma}_1^P + \boldsymbol{\sigma}_2^P)\,, \quad (2.24)$$

which operates on Ψ, ξ_1, ξ_2, Φ separately. The infinitesimal rotations of Ψ may be represented by means of $\mathbf{J}^P$ and the different Ψ corresponding to different μ in (2.11) form the basis of an irreducible representation of the rotation group. Since

$$[\mathbf{J}^P, (\boldsymbol{\sigma}_i \cdot \mathbf{p})] = 0\,, \quad (2.25)$$

Eq. (2.17) shows that ξ_1, ξ_2, Φ transform under rotations like Ψ and hence each of them forms the basis of the same irreducible representation. Consequently, the 16-component function ψ does also. The nonrelativistic classification of L and J values according to properties of Ψ has, therefore, an exact relativistic meaning for J in terms of ψ. The relativistic function that corresponds to Ψ has therefore the same J as Ψ.

The above reasoning is not affected by the coupling of states with different L to give the same J. For the eigenphaseshift solutions the only difference for the asymptotic forms is that one deals with linear combinations of two functions $\mathcal{Y}_\mu^{L,J}$ rather than one such function. These functions belong to the same irreducible representation and hence from here on the previous reasoning applies. It is not actually necessary to use either the asymptotic region or the eigenphase-shift solutions for the proof because for any r the function Ψ is a linear combination of two terms like that in (2.11) but with different L. Since r is not affected by rotations, the linear combination forms again the basis of the representation.

Since the relativistic discussion of the problem is reducible to that of a nonrelativistic one, it is convenient to work in terms of the latter and to classify states in the language of nonrelativistic atomic spectroscopy. In describing the spin condition of the particles, it is similarly convenient to use the Pauli $\boldsymbol{\sigma}_i^P$. The expectation value of $\boldsymbol{\sigma}_i^P$ computed employing Ψ differs from that of the true Dirac $\boldsymbol{\sigma}_i$ computed employing ψ. The spin direction obtained in such a treatment is therefore not the direction of the physical spin. Nevertheless, it is a useful quantity because the primary usefulness of the specification of the spin lies in the enumeration of states. For moderate energies the difference in the two descriptions is not large. Whenever confusion is likely to arise regarding the meaning of the spin, the Pauli spin in the center of mass system will be referred to below as the "spin."

3. RELATIVISTIC KINEMATICS

It has been pointed out by Stapp[4] that a rotation of the spin direction results in a succession of three Lorentz transformations, the first consisting in going from the rest frame of the scattered particle to the center-of-mass system; the second from the center-of-mass system to the laboratory frame; the final transformation from the laboratory system to a new rest frame of the scattered particle. He gave an explicit formula for the sine of the angle of rotation and pointed out that the rotation takes place about an axis perpendicular to the scattering plane so that it should have no effect on double scattering and depolarization experiments since the polarization vector

[4] H. P. Stapp, Phys. Rev. **103**, 425 (1956).

is perpendicular to the scattering plane in these experiments. Stapp's results have been extended by Chao and Shirokov[5] to somewhat more general situations with a disagreement concerning a factor in Stapp's formula. Stapp defines the spin direction in terms of the space components of the four-dimensional pseudovector associated with the spin of a Dirac particle. A related discussion of transformations of the spin direction has been given by Wigner.[6] The origin of the rotation discussed by Stapp is a geometrical one and stems from the noncommutativity of rotations. In view of the simplicity of the treatment of the collision by means of the Pauli spins in the c.m. system, it appears worth while to supplement these considerations by means of an approach that may be less elegant but is directly related to the description of experiments.

Since questions of spin direction are of interest in applications to the calculation of the polarization of nucleon beams, it is necessary to discuss first the relationship between the correlated beams of particles moving in opposite directions in the center-of-mass system and the two-particle wave function.

An incident wave train of finite length will be considered in the center-of-mass system. The unavoidable small diffuseness at the beginning and the end will not enter the considerations because a definition of the head of the wave train within a few wavelengths is amply sufficient for actual measurement purposes. The incident wave train may be Fourier analyzed in terms of states with definite relative angular momenta and the Rayleigh-Faxen-Holtsmark scattering theory applied to each monochromatic component. When the scattered parts are superposed there is obtained a wave train of finite length diverging from the origin of the c.m. system. To any point in this outgoing wave packet there corresponds a definite value of $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the displacement vectors of particles 1 and 2 from the origin. This wave packet is in the space of relative coordinates of the two particles. If the monochromatic wave functions of relative motion are multiplied by appropriate monochromatic plane wave functions representing the motion of the center of mass, a more general type of packet results. In this latter case, the two colliding particles can be localized within definite domains and the conditions for space definition of the particles obtaining in an actual experiment may then be reproduced. The wave function in the c.m. system gives information directly only about the relative position of the two particles and their spin condition and if the momentum of the c.m. is taken to be exactly zero, neither the position of the c.m. nor that of either particle can be defined. Nevertheless, the wave function of relative motion determines the number of particles of either type that are scattered per unit solid angle in any specified direction. This follows from the wave packet construction localizing the two particles that has been just referred to. The explicit verification of these facts falls outside the scope of the present article, but the fact that a verification is possible may be found useful in picturing the process.

Localization of a proton within the radius of a Bohr hydrogenic orbit corresponds to an uncertainty $mc/137$ in its momentum which is smaller than the momentum of a 10-MeV proton by a factor of roughly $1/(4 \times 10^4)$. The Heisenberg uncertainty relation is seen not to interfere with the possibility of localizing the proton more than sufficiently well in space even though the momentum is reasonably well defined. For practical purposes of calculation, the localization in space is usually not considered since it suffices to calculate changes in the momentum direction. On the other hand, the observation of a particle's direction often involves the observation of its position. Some discussion of the consistency of the two views appeared appropriate therefore.

By means of a suitable slit system, a beam of scattered particles can be isolated from the wave packet just discussed and its direction can be confined to a narrow cone. The Fourier analysis of this beam consists of plane waves with nearly the same direction and the condition will be idealized by considering only one of the plane waves, thus making the definition of the scattered direction perfect. According to (2.8) if there are two mutually orthogonal solutions $\Psi^{(1)}$ and $\Psi^{(2)}$ so that

$$(\Psi^{(1)},\Psi^{(2)}) = 0 \tag{3.1}$$

and if $\mathbf{p}$ and E is the same for the two then

$$(\Phi^{(1)},\Phi^{(2)}) = 0\,, \quad (\psi^{(1)},\psi^{(2)}) = 0\,, \tag{3.2}$$

as follows from a short calculation. Here the $\Phi^{(i)}$, $\psi^{(i)}$ correspond to the states described by $\Psi^{(i)}$. To two orthogonal Pauli waves there are seen to correspond two orthogonal Dirac waves.

In the usual formalism of dealing with statistical mixtures by means of density matrices, it is necessary to introduce an orthonormal set of eigenfunctions. The functions needed are the four component Dirac functions ψ. Since, however, (3.1) follows from (3.2),

[5] Chou Kuang-Chao and M. I. Shirokov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 1230 (1958); Soviet Phys.—JETP, **7**, 851 (1958).

[6] E. P. Wigner, Revs. Modern Phys. **29**, 255 (1957), Appendix I.

the orthogonality condition may be equally well imposed on the two-component Pauli Ψ. The "meaning" of the "spin" is irrelevant for the possibility of specifying the properties of the statistical mixture and therefore it is possible to specify it completely by means of the Pauli functions. It is essential to note that the specification of the expectation values of $\mathfrak{d}^P$, $\langle\mathfrak{d}^P\rangle$, determines the statistical mixture of the Pauli functions.[7] The knowledge of the true spin condition is therefore unnecessary for the description of scattering experiments. If one speaks of $\langle\mathfrak{d}^P\rangle$ one does so only because this quantity determines how the beam should behave in scattering experiments rather than because $\langle\mathfrak{d}^P\rangle$ is of direct interest. The situation would, of course, be different if the experiments were directly concerned with the observation of the spin direction as would be the case if the directions of electrons and neutrinos in the β decay of the scattered neutrons were under investigation.

In order to discuss the rotation of the "spin," it is convenient to know how its direction changes as one goes from the laboratory system to the center-of-mass system. By the laboratory system one usually means a Lorentz frame of reference in which the target particle is at rest because this is very nearly the situation in actual experiments. The center-of-mass system will be referred to as K and the laboratory system as K'. The incident particle will be supposed to be moving along the positive x axis. The system K will be taken to move in this direction also and the transformation will be taken as involving only (x,t) and (x',t'), i.e., the simple translatory transformation will not be supplemented by a space rotation. The transformation has the usual form

$$x' = xch\epsilon + ctsh\epsilon \quad (ch\epsilon \equiv \cosh\epsilon, sh\epsilon \equiv \sinh\epsilon)$$
$$ct' = ctch\epsilon + xsh\epsilon\,. \tag{3.3}$$

The corresponding transformation of wave functions is[8]

$$\psi' = [\exp(-\alpha_x\epsilon/2)]\psi\,. \tag{3.4}$$

Since the rotation of the spin is small the rather good approximation of assuming the masses of projectile and target to be equal will be made. From (3.4) and (2.8) it follows that for the projectile

$$\Psi_f' = -\hat{s}\sigma_x\Phi_f + \mathfrak{c}\Psi_f = [(\hat{s}^2/\mathfrak{c})\sigma_x(\hat{p}_f\cdot\mathfrak{d}) + \mathfrak{c}]\Psi_f\,, \tag{3.5}$$

where

$$\mathfrak{c} = ch(\epsilon/2)\,, \quad \hat{s} = sh(\epsilon/2)\,. \tag{3.6}$$

The subscript f designates the value of a quantity in the final state. Linearizing (3.5),

$$\Psi_f' = \{\mathfrak{c} + (\hat{s}^2/\mathfrak{c})\cos\theta + i(\hat{s}^2/\mathfrak{c})(\xi\cdot\mathfrak{d})\}\Psi_f\,, \tag{3.7}$$

where

$$\xi = [1_x \times \hat{p}_f]\,. \tag{3.8}$$

In the usual convention the scattering angle θ in the center-of-mass system is between 0 and π and hence $\sin\theta$ is positive. Hence ξ, the absolute value of ξ, is $\sin\theta$. Specializing (3.7) to $\theta = 0$ there follows a similar relation between Ψ_i' and Ψ_i which can be used to express Ψ_f in terms of Ψ_i, since the relation of Ψ_f to Ψ_i is known through the knowledge of the scattering matrix in K. If, in going from i to f, the "spin" is turned in K through the angle Θ around the direction l in a counter clockwise sense when viewed from the positive l axis, then

$$\Psi_f = a[\exp(-i\sigma_l\Theta/2)]\Psi_i\,, \tag{3.9}$$

where a is independent of the spin coordinate. Substitution in (3.7) gives

$$\Psi_f' = \frac{a}{\mathfrak{c}^2 + \hat{s}^2}[\mathfrak{c}^2 + \hat{s}^2\cos\theta + i\hat{s}^2\sigma_\xi\sin\theta] \times [\exp(-i\sigma_l\Theta/2)]\Psi_i'\,. \tag{3.10}$$

This formula gives the transformation of the "spin" in the laboratory system K'. The last two factors on the right-hand side are conveniently combined into

$$(\Psi_i')^{tu} = [\exp(-i\sigma_l\Theta/2)]\Psi_i'\,, \tag{3.11}$$

which gives a wave function in K' corresponding to the "spin" having been turned in K' through the angle Θ when viewed from the positive direction of the l axis. Comparison of (3.11) with (3.9) shows that the l axis in (3.11) has to be used with the same direction cosines in K' as it was used in (3.9) which referred to a spin rotation in K. The superscript tu in Eq. (3.11) indicates that Ψ' has been changed to correspond to a turned "spin." The whole change in the "spin" direction corresponds to turning the "spin" direction obtained in K' by means of (3.11) and contained in $(\Psi_i')^{tu}$ in accordance with the first two factors in (3.10). This relation is of the form

$$\Psi_f' = (A + iB\sigma_\xi)(\Psi_i')^{tu}\,, \tag{3.12}$$

where

$$A = [a/(\mathfrak{c}^2 + \hat{s}^2)](\mathfrak{c}^2 + \hat{s}^2\cos\theta)\,,$$
$$B = [a\hat{s}^2/(\mathfrak{c}^2 + \hat{s}^2)]\sin\theta\,. \tag{3.13}$$

[7] L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952); L. Wolfenstein, Ann. Rev. Nuclear Sci. **6**, 43 (1956); G. Breit and J. S. McIntosh, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41, Part 1, p. 466, and references to original papers in these articles.

[8] P. A. M. Dirac, Proc. Roy. Soc. (London) **A117**, 610 (1928); **A118**, 351 (1928).

It follows from (3.12) that

$$\langle\sigma'_\xi\rangle = (\Psi'_f,\sigma_\xi\Psi'_f)/(\Psi'_f,\Psi'_f)$$
$$= ((\Psi'_i)^{tu},\sigma_\xi(\Psi'_i)^{tu})/((\Psi'_i)^{tu},(\Psi'_i)^{tu}) = \langle\sigma'_\xi\rangle^{tu} \quad (3.14)$$

which means that the rotation of $\langle\mathfrak{o}\rangle$ taking place in K' as a result of the transformation (3.12) is around the ξ direction. The primary meaning of this "direction" is again the purely formal one of employing the same direction cosines for ξ in K' as in K. It makes no difference however whether $\hat{p}_f$ or $\hat{p}'_f$ is used in (3.8) in the determination of ξ/ξ.

It is convenient to introduce a right-handed Cartesian coordinate system ξ, η, ζ. It then follows from (3.13) that

$$\langle\sigma'_\eta + i\sigma'_\xi\rangle_f = \langle\sigma'_\eta + i\sigma'_\xi\rangle_i^t e^{-i\Omega}, \quad (3.15)$$

where the expectation values are evaluated in K' employing Ψ'_f on the left and $(\Psi')^t$ on the right while

$$\sin\Omega = \frac{2\mathfrak{s}^2(\mathfrak{c}^2 + \mathfrak{s}^2\cos\theta)\sin\theta}{\mathfrak{c}^4 + \mathfrak{s}^4 + 2\mathfrak{c}^2\mathfrak{s}^2\cos\theta}. \quad (3.16)$$

The sense of rotation for $\Omega > 0$ is such as to turn the positive ζ axis towards the positive η axis. According to (3.15) the "spin" direction is turned by the scattering in K' through the same angle Θ as in K and, in addition, through the angle Ω. A positive Ω corresponds according to (3.15) to a left-handed rotation around the positive direction of ξ, i.e., in the same sense as the rotation of $\mathbf{p}_f$ towards $\mathbf{p}_i$. The quantities $\mathfrak{c}^2$, $\mathfrak{s}^2$ are obtainable from $ch\epsilon = E_p/Mc^2$, where E_p is the energy of one particle in the rest system. Thus

$$\mathfrak{s}^2 = (E_p - Mc^2)/(2Mc^2),$$
$$\mathfrak{c}^2 = (E_p + Mc^2)/(2Mc^2). \quad (3.17)$$

For $\theta = \pi/2$, Eq. (3.16) becomes

$$\sin\Omega = \frac{E_p^2 - (Mc^2)^2}{E_p^2 + (Mc^2)^2} \quad (\theta = \pi/2). \quad (3.18)$$

For nearly nonrelativistic energies

$$\sin\Omega \cong [(E_p - Mc^2)/Mc^2]\sin\theta. \quad (3.19)$$

According to (3.16) there is no relativistic "spin" rotation if $\theta = 0$. The "spin" direction of the incident particle p is therefore the same in K and in K'. The equations listed so far make it possible therefore to obtain the spin orientation of p in K if it is known in K' (no change) and to obtain the spin orientation of p after scattering in K' if the angle Θ through which $\mathfrak{o}$ is turned in K is known. The over-all change in the orientation of $\mathfrak{o}_p$ in the laboratory system K' is thus known in terms of Θ and Ω for one collision. For the next collision a new x axis is defined in the direction of the first scattering in K' and the process is repeated. If the target particles t are unpolarized in K' they are also unpolarized in K as shown in connection with Eqs. (3.1) and (3.2). The equations, so far, suffice therefore for the treatment of successive scatterings of p from unpolarized targets. Additional, though similar, considerations are required however for the polarization of the target particle t after recoil as well as for the treatment of polarized targets.

For t, combining the spinor transformation (3.4) with the relation between "large" and "small" components it is seen that in the initial state the "spin" direction is the same in K and K', in the sense that the direction cosines of $\langle\mathfrak{o}\rangle$ are the same in the two systems. On the other hand, the calculation described in Eqs. (3.4) to (3.16) applies also to the recoiling target particle provided $\mathbf{p}_f$ is interpreted as the final momentum of the target particle in K. The only changes involved in treating target rather than incident particles consist, therefore, in interpreting θ as the angle between the final momentum direction of t and the x axis and in a reversal of the direction of the ξ axis resulting from $\mathbf{p}_{ft} = -\mathbf{p}_{fp}$ in (3.8). If in the collision the meaning of θ is reserved for the scattering angle of the incident particle, then it is necessary to use everywhere

$$\theta_t = \pi - \theta \quad (3.20)$$

in place of θ. The transformation of the spin direction of target particles can thus be carried out by means of (3.16) and all questions concerning polarized targets or polarization of recoil particles can be treated.

For identical particles the distinction between the projectile and the target is impossible. The equations may be applied however without change. In fact, for the initial state the "spin directions" are the same in K and K' for both p and t, since the translatory Lorentz transformations involved are in the direction of the particle momenta, and hence the question of spin rotation does not enter the specification of the collision process in K. The standard calculation with antisymmetrized functions in K yields wave amplitudes for particles with specified "spin" orientations. When these amplitudes are transformed from K to K' by means of (3.4) the "spin" direction changes in the same way for the scattered and the recoiling target particles provided they have the same final direction. Equation (3.16) is thus directly applicable.

It has been tacitly assumed above that it is known when an incident beam is unpolarized in K. If this is not known the usual method of determining the polarization parameter P by observing the asymmetry of scattering in the second of two successive

scatterings cannot be carried out. Since the transition from K' to K does not change the "spin" orientation, an incident beam unpolarized in K' is unpolarized in K. The question resolves itself therefore into whether a beam produced by an ordinary accelerator is polarized. As an example one may consider the acceleration by a Van de Graaff machine which acts on the protons by means of an electrostatic field. A general electrostatic field gives polarization effects as has been shown by Mott in the case of electrons scattered by a nucleus. Two circumstances combine to make the expected effects negligible. One is that the angle between the electric field and the proton momentum is small. The other is that electric field used in the machine is relatively weak. The ratio of the term containing $[\mathbf{p} \times \boldsymbol{\delta}]$ to the term containing the Laplacian is of the order of magnitude

$$\frac{e}{\hbar c}\,\frac{\mathcal{E}p}{2Mck^2}\,\theta = \frac{\theta}{4}\,\frac{\hbar}{McD}\,\frac{v}{c}\,.$$

Here $\mathcal{E}$ is the electric field, v the proton velocity, θ the angle between the field and momentum directions, and D is the distance through which the field must act continuously in order to give the protons their final energy. The factor $\hbar/(McD) \leqq 2 \times 10^{-14}/D$ with D in centimeters and the effect is therefore negligible.

The transformation of directions of motion is conveniently calculated by means of[9]

$$\cos\theta = \frac{2 - (\gamma + 3)\sin^2\theta_l}{2 + (\gamma - 1)\sin^2\theta_l}, \tag{3.21}$$

where

$$\gamma = 1/(1 - v'^2/c^2)^{1/2} \tag{3.22}$$

and v' is the velocity of the incident particle in the laboratory system K'. The energies are related by

$$E'_p/(Mc^2) = 2[E_p/(Mc^2)]^2 - 1\,, \quad E'_p = Mc^2\,, \tag{3.23}$$

where E_p and E'_p are the relativistic energies of that particle p which is incident in the laboratory system. The differential cross section σ'_Ω has a direct geometrical meaning as accounting for the number of collisions per unit solid angle in K' for the incident particle. The velocity of the incident particle is v', that of the target particle zero, and the usual meaning of the cross section is applicable. The definition of the differential cross section σ_Ω in the c.m. system is made such that

$$\sigma'_\Omega d\Omega' = \sigma_\Omega d\Omega\,. \tag{3.24}$$

[9] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1933), especially p. 270.

The geometrical meaning of σ_Ω is not as unambiguous as that of σ'_Ω, but the convenience of the above definition is obvious. The ratio of solid angles is obtainable from (3.21) as

$$\frac{d\Omega'}{d\Omega} = \frac{2(\gamma + 1)^{1/2}}{[(\gamma - 1)^{1/2}\cos\theta + \gamma + 3]^{3/2}\,(1 + \cos\theta)^{1/2}}\,. \tag{3.25}$$

If a stream of nucleons with density ρ'_1 and velocity v'_1 is incident on another set of nucleons with density ρ'_2, then on transformation of densities and velocities it is found that

$$v'_1\rho'_1\rho'_2 = 2v_1\rho_1\rho_2\,, \tag{3.26}$$

substantiating the meaning of σ_Ω as introduced by means of (3.24). The velocity of the incident particle in K' is related to its velocity in K by

$$v' \equiv v'_1 = 2v_1/[1 + (v_1/c)^2]\,. \tag{3.27}$$

4. EFFECT OF THE COULOMB FIELD, VACUUM POLARIZATION, AND RELATED QUESTIONS

Postponing the consideration of relativistic electromagnetic effects, the Coulombian interaction between two protons may be taken into account approximately by inserting e^2/r for the potential energy in the wave equation for the four-component Ψ. If, following the temporary plan of neglecting relativistic interaction effects, then starting with (2.9) and inserting $E - (e^2/r)$ for E, eliminating ξ_1, ξ_2, and Φ, and discarding all but terms linear in e^2 as well as commutators with $1/r$, there results in the c.m. system the approximate equation

$$[1 + \Delta/k^2 - 2\eta/kr]\Psi = 0 \tag{4.1}$$

with

$$k = (E^2 - 4M^2c^4)^{1/2}/2\hbar c\,, \tag{4.2}$$

and

$$\eta = \frac{Ee^2/c}{2\hbar(E^2 - 4M^2c^4)^{1/2}} = \frac{e^2}{2\hbar v_1}\,. \tag{4.3}$$

Here v_1 is the velocity of particle 1 in the c.m. system. This value of the parameter η is not quite good enough in a relativistic treatment and will be improved on shortly. The value of k given by (4.2) is, on the other hand, the relativistic one, since for noninteracting particles the calculation is exact, and may also be written

$$k = p_1/\hbar = (E_1^2 - M^2c^4)^{1/2}/(\hbar c) \tag{4.4}$$

where, as before, subscript 1 designates the incident particle. Equation (4.1) has the same form as that

found in nonrelativistic treatments. A solution of it generally used[3] to describe the collision is

$$\Psi^c = e^{ikz-\pi\eta/2}\Gamma(1+i\eta)M(-i\eta,1;ik(r-z))$$
$$\backsim \left\{1+\frac{\eta^2}{ik(r-z)}+\cdots\right\}\exp\{i[kz+\eta\ln k(r-z)]\}$$
$$-\frac{\eta}{k(r-z)}\left\{1+\frac{(1+i\eta)^2}{ik(r-z)}+\cdots\right\}$$
$$\times\exp\{i[kr-\eta\ln k(r-z)+2\delta_0]\}\,, \qquad (4.5)$$

where M is the confluent hypergeometric function

$$M(\alpha,\beta;x) = 1+\frac{\alpha}{1!\beta}x+\frac{\alpha(\alpha+1)}{2!\beta(\beta+1)}x^2+\cdots \qquad (4.6)$$

and

$$\delta_L = \arg\Gamma(L+1+i\eta)\,. \qquad (4.7)$$

The asymptotic form in (4.5) is for large r. The entrance of the spin coordinates is ignored since (4.1) contains no spin interactions. Equation (4.5) represents a condition closely resembling that corresponding to the incidence of a plane wave e^{ikz} in the space of relative coordinates modified by outgoing waves. It has been shown by Gordon[3] that the wave function of Eq. (4.5) is, in fact, the limit of a wave function obtained in a modified problem obtained from that of Eq. (4.1) by modifying the potential energy e^2/r in such a way as to have it vanish beyond a certain distance $R+\Delta R$ and be exactly Coulombian for $r<R$. For the modified problem the requirement that Ψ be asymptotic to a specified incident plane wave modified by the addition of an outgoing wave defines the wave function. The way in which this limiting procedure leads to (4.5) may be understood in terms of the Rayleigh-Faxèn-Holtsmark method. Thus, at $r=R$ the function to substitute in place of the usual

$$F_L = (\pi\rho/2)^{1/2}J_{L+1/2}(\rho)\backsim\sin(\rho-L\pi/2),\ \rho=kr \qquad (4.8)$$

is

$$\mathfrak{F}_L^{GS}\backsim\exp[i(\delta_L+\epsilon_L-\eta\ln 2kR)]$$
$$\times\sin[kr-(L\pi/2)+\delta_L-\eta\ln(2kr)]\,. \qquad (4.9)$$

This form arises because the regular solutions of the differential equations for F_L^c, which stands for r times the radial function, and satisfies

$$\left[\frac{d^2}{d\rho^2}+1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right]F_L^c = 0 \qquad (4.10)$$

may be normalized so as to have the asymptotic form

$$F_L^c\backsim\sin(\rho-L\pi/2+\delta_L-\eta\ln 2\rho)\,. \qquad (4.11)$$

Consequently,

$$\delta_L-\eta\ln(2kR) \qquad (4.12)$$

is the phase shift at $r=R$. On the assumption that the transition from e^2/R to zero can be made sufficiently gradual in ΔR to make reflection effects at $r=R$ and $r=R+\Delta R$ negligible, and with the understanding that ϵ_L is the additional phase shift arising in ΔR, standard scattering theory yields (4.9). For any assigned L it is possible to make R large enough to make $L(L+1)/\rho^2$ in (4.10) negligible for this L and smaller ones. For such L all the ϵ_L are the same. Their common value will be called ϵ. Forgetting for the moment about the higher L for which this procedure is inapplicable, the usual scattering theory gives for $r<R$

$$\psi^{GS} = \textstyle\sum_L i^L(2L+1)P_L(\cos\theta)\mathfrak{F}_L^{GS}$$
$$= e^{i(\epsilon-\eta\ln 2kR)}\textstyle\sum_L i^L(2L+1)P_L(\cos\theta)e^{i\sigma_L}F_L^c/\rho$$
$$= e^{i(\epsilon-\eta\ln 2kR)}\Psi^c\,. \qquad (4.13)$$

The last equality has been shown to hold in reference 3. Since R is large and since the asymptotic form (4.9) cannot apply for L greater than approximately $(2\rho\eta)^{1/2}$, the sum requires modification at high L. This modification represents diffraction effects caused by the Gordon Sphere. However, by making R very large the value $L=(2kR\eta)^{1/2}$ can be made to exceed any preassigned L in (4.13). The diffraction effects are important only for small θ and are not important if $\theta\gg 1/kR$. Thus, by making R sufficiently large a definite limit is approached by

$$e^{-i(\epsilon-\eta\ln 2kR)}\psi^{GS}\,. \qquad (4.14)$$

The Gordon sphere procedure may be regarded as a device making it possible to define uniquely the meaning of Ψ^c in terms of a solution of the partial differential equation (4.1) with assigned boundary conditions. In the nonrelativistic approximation it could be argued besides that atomic electrons provide screening of the Coulomb field with approximately spherical symmetry and that the Gordon sphere provides a schematic model which is closer to reality than the model of the collision between two point charges. In particular, the divergence of the total scattering cross section when it is integrated over all solid angles for the Coulomb field is absent for finite R. This corresponds to the absence of scattering of classical particles when they pass outside the screening sphere.

Satisfying as these considerations are in the nonrelativistic theory, they appear more artificial in the relativistic problem. In the space of relative co-

ordinates the screening does not take place with spherical symmetry, an arbitrarily large amount of flattening being obtainable by increasing the energy. The justification for employing the customary solution Ψ^c becomes purely formal in this case. Furthermore, it has not been shown that the limit approached by Ψ is independent of the shape of the screening surface. On the other hand, it follows from (4.14) that properties of the screening surface have to be used in the definition of the quantity that approaches a definite limit. Fortunately, the effects of the Coulomb field on the scattering amplitude are not important at high energies. At energies of the order of 10 MeV or less, at which the Coulomb scattering is important, the Gordon sphere model represents actual conditions more faithfully. Estimates indicate that screening effects of protons by atomic electrons are important at very small angles but do not appreciably affect scattering within the usual experimental range.

The electrostatic potential caused by a proton screened by the average field of the charge distribution of an electron in its nonrelativistic ground state is

$$\varphi = (e/r)(1 + r/a)e^{-2r/a}, \quad a = \hbar^2/me^2 \quad (4.15)$$

where a is the Bohr radius. In first-order Born approximation this potential gives a scattering cross section for a collision with a bare proton which differs slightly from the Rutherford value. The factor by which the Rutherford cross section should be multiplied is a function of $(2/qa)^2$, where q is the momentum transfer. Its expansion in powers of this parameter is

$$1 - 2(2/qa)^4 + \cdots, \quad (4.16)$$

the second term in the expansion being zero. The correction caused by screening is insignificant in this approximation under most practical circumstances. For center of mass scattering angle $\theta = 10°$ at incident laboratory energy of 4 MeV an estimate employing the reduced mass for the collision of free protons in the calculation of q gives approximately $1–10^{-11}$ for the correction factor. Even if one regards the cancellation of the coefficient of $(2/qa)^2$ as accidental and therefore inapplicable to molecular hydrogen, the value $(2/qa)^2 \cong 2 \times 10^{-6}$ is too small to matter in ordinary experiments. The first Born approximation cannot be relied on, of course, at low energies even though it yields the Rutherford formula for $a \to \infty$. Nevertheless, at the higher energies Eq. (4.16) has a meaning and the estimates quoted indicate that the screening effect is not very important.

The interpretation of precision experiments[10] on proton-proton scattering in the energy range of a few MeV is too vast a subject for treatment in the present article. There is no doubt regarding its ultimate importance for the complete identification of different phase parameter fits and the establishment of their energy dependence. At one time it has been thought that a precise interpretation of low energy p-p data might be indispensable because of the possibility of furnishing a guide to the energy dependence of P-wave phase shifts which was expected to be of great value in disentangling the many possibilities of fitting data in the energy range 40–350 MeV. Accumulation of experimental material at these higher energies and the recent addition of some n-p polarization data at 16 and 24 MeV[11] as well as of p-p polarization data[12] from 27 to 97 MeV and especially at 37.0 and 52.5 MeV, has decreased the need for this information for the purpose mentioned. Through an accumulation of experimental and theoretical evidence, the general course of the energy dependence of the low L phase shifts appears to be converging to a unique answer[13] and it appears possible that, so far as the P waves are concerned, the precision data in the region of a few MeV will be of primary value in confirming information regarding types of phase parameter fits obtainable from somewhat higher energies. Nevertheless, information regarding P and D waves at low E will be of interest because the asymptotic low E forms[14–16] can then be verified and the identification of empirical fits be made more certain. The 400 keV to 9 MeV energy region is important in providing information regarding the energy dependence

[10] R. G. Herb, D. W. Kerst, D. B. Parkinson, and G. J. Plain, Phys. Rev. **55**, 998 (1939); J. Rouvina, Phys. Rev. **81**, 593 (1951); Blair, Freier, Lampi, Sleator, and Williams, Phys. Rev. **74**, 553 (1948); N. P. Heydenburg and J. L. Little *as quoted* by M. C. Yovits, R. L. Smith, Jr., M. H. Hull, Jr., J. Bengston, and G. Breit, Phys. Rev. **85**, 540 (1952); D. L. Cooper, D. H. Frisch, and R. L. Zimmerman, Phys. Rev. **94**, 1209 (1954). Reference to some of the later papers containing the most precise measurements will be made at a more suitable place in this article. The references in this footnote are intended to give a representative cross section of work done rather than to provide an exhaustive list.

[11] W. Benenson, R. H. Walters, and T. H. May, Phys. Rev. Letters **8**, 66 (1962).

[12] P. Christmas and A. E. Taylor (private communication); AERE-PR/NP1, Research Group Progress Report, Nuclear Physics Division.

[13] M. H. Hull, Jr., F. A. McDonald, H. M. Ruppel, and G. Breit, Phys. Rev. Letters **8**, 68 (1962); G. Breit, M. H. Hull, Jr., K. E. Lassila, K. D. Pyatt, Jr., and H. M. Ruppel, Phys. Rev. (to be published).

[14] G. Breit, H. M. Thaxton, and L. E. Eisenbud, Phys. Rev. **55**, 1018 (1939).

[15] E. P. Wigner, Phys. Rev. **73**, 1002 (1948). Mathematically rigorous considerations may be found in this paper.

[16] G. Breit, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41, Part 1, Sec. 42. Somewhat popularized versions of considerations of the preceding references may be found here.

of K_0, the phase shift of the $^1S_0$ state, and its asymptotic form at $E = 0$. This phase shift compared with corresponding information from low-energy n-p scattering gave the first indication of charge independence of nuclear forces. There have been many fluctuations of "expert" opinion regarding the accuracy with which charge independence holds on the basis of comparison of information regarding K_0 from p-p and n-p experiments. Since the one-pion exchange interaction can now be tested directly in higher energy experiments, and since the fundamental symmetry law[17] responsible for the near equality of corresponding phase shifts in n-p and p-p interactions is presumably concerned with meson-nucleon interactions, the comparison of the values of K_0, after due allowance for the Coulomb interaction, has lost some of its early interest as a matter of possibly fundamental significance which might be approached by relatively simple calculations. The pions in the p-p, p-n, and n-n systems find themselves in electromagnetic fields that differ appreciably at large r and slight violations of apparent charge independence are expected on such grounds alone. There is, besides, the difference in the masses of charged and neutral pions to consider, since it makes it important to formulate the statement regarding symmetry of pion-nucleon interactions in isospin rather carefully. An example of a possible, though arbitrary, formulation will be seen in connection with tests of charge independence later on in this article. It would not be surprising, however, if progress in the understanding of the nature of the pion perhaps along the general lines of Fermi and Yang[18] were to make present day attempts to deal with the problem appear quite puerile when the laws are more completely understood. But even though the $^1S_0$ state is not any more the main source of information regarding a fascinating symmetry, the elimination of extraneous effects such as vacuum polarization from the low-energy p-p data is, nevertheless, important not only because it is helpful in paving the way for better knowledge regarding P and D waves, but also because one may hope that the pion-nucleon interactions will eventually be understood sufficiently well to be able to test theoretical ideas by means of the $^1S_0$ state and perhaps even to ascertain the presence of other pion-nucleon interactions such as the participation of K mesons.

Vacuum polarization effects resulting in a modification of the Coulomb potential have been calculated by Serber,[19] Uehling,[20] and by Schwinger.[21] Improvements on these calculations which matter for high nuclear charges have been made by Wichmann and Kroll but are not believed necessary for $Z = 1$. Foldy and Eriksen[22] have estimated the effect of the vacuum polarization on the $^1S_0$ interaction and found that the length replacing the scattering length of n-p scattering is increased in absolute value by 3 parts in 370 as a result of correcting for the presence of the vacuum polarization effect in the data. They compared this result with Schwinger's comparison[23] of p-p and n-p scattering employing a Yukawa potential. The decrease in the derived strength of the p-p interaction increased the discrepancy in the two interactions that remained in Schwinger's calculation from 7 to 9% and the suggestion was made by Foldy and Eriksen that this increase in the two apparent interactions might provide room for the possibility that the magnetic moments of nucleons are associated with spatial distributions of magnetism. However, as shown by Salpeter,[24] the employment of the hard core for the $^1S_0$ interaction changes Schwinger's results regarding magnetic moment corrections very seriously and the vacuum polarization effect is still a small effect in a relatively large discrepancy. Further considerations for wells of other shapes may be found in the second paper by Foldy and Eriksen.[22] Eriksen, Foldy, and Rarita[25] estimated contributions to P-wave scattering caused by vacuum polarization and found that roughly half of the apparent P-wave effects indicated by the latest then available data[26] could be explained as due to the vacuum polarization. On the other hand, the long range, $\sim \hbar/mc$, of the vacuum polarization potential V_{vp} makes it necessary to calculate effects on partial waves with $L > 1$ as well, since they drop off slowly with L. Since in the energy range with E below about 5 MeV the nuclear P and D waves are small and since V_{vp} is small, the contribution to the scattering amplitude arising from all $L > 0$ may be performed in first order of V_{vp} employing the Coulomb wave as the unperturbed function. This contribution to the scattering amplitude gives effects on the differential cross

[17] C. Møller and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **17**, No. 8, (1940); **18**, No. 6 (1940); K. M. Watson and K. K. Brueckner, Phys. Rev. **83**, 1 (1951); K. M. Watson, *ibid.* **85**, 852 (1952); H. A. Bethe and F. de Hoffman, *Mesons and Fields, Vol. II, Mesons* (Row, Peterson & Company, Evanston, Illinois, 1954).

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[23] J. Schwinger, Phys. Rev. **78**, 135 (1950).

[24] E. E. Salpeter, Phys. Rev. **82**, 60 (1951).

[25] E. Eriksen, L. L. Foldy, and W. Rarita, Phys. Rev. **103**, 781 (1956).

[26] H. R. Worthington, J. N. McGruer, and D. E. Findley, Phys. Rev. **90**, 899 (1953).

section $\sigma(\theta)$ which have to be subtracted from the experimental value $\sigma_{exp}(\theta)$ if conclusions regarding P and D waves of nuclear force origin are to be drawn. On the other hand, the calculation of the effect of V_{vp} on the S wave is not essential for this purpose particularly because the adjustment of the phase shift K_0 to obtain the best fit to the data has to be gone through anyway. The problem of explaining the dependence of K_0 on E can thus be treated separately from that of the presence of nuclear force effects for $L > 0$, provided the effects of V_{vp} on $L > 0$ can be corrected for. This problem has been undertaken by Durand[27] and by de Wit and Durand. Durand has summed the contribution of V_{vp} to the scattering amplitude analytically for $L > 0$. He is concerned with the energy range 1.4–4.2 MeV. The angular distribution of $\Delta\sigma/\sigma$, the fractional change in the differential cross section, is found by him to have a maximum at smaller angles than the p-wave vacuum polarization effects of Eriksen, Foldy, and Rarita,[25] although it is otherwise similar to that effect. It is also found by Durand[27] that effects of similar magnitude result from the employment of the relativistic value of the parameter $\eta = e^2/\hbar v$ according to some work of the writer, discussed in the next section. The detection of P-wave effects on the basis of the angular dependence of the cross section such as have been attempted from time to time[28] is therefore a delicate matter. De Wit and Durand[29] found that about half of the apparent mean P-wave phase shift of the data, then in the literature, could be attributed to vacuum polarization scattering and from their analysis obtained some modifications of the nuclear S-wave energy dependence. There are also some improved vacuum polarization calculations by de Wit and Breit[30] which are especially concerned with the lower part of the energy range treated in reference 29.

The calculations of Durand and of de Wit and Durand have been used by Knecht, Messelt, Berners, and Northcliffe[31] in connection with their experiments at 1.397, 1.855, and 2.425 MeV over an angular range from 12° to 90° in the c.m. system. The total uncertainties are believed to be about ±0.1% at large angles and ±0.3% at the smallest angles. These data are believed to be superior to those in reference 26, some additional sources of experimental error having been located and eliminated. Employing relativistic kinematics and taking into account the other effects just discussed, they find it possible to fit the data practically as well employing a pure S-wave nuclear anomaly as by means of a combination of S-wave and mean P-wave nuclear anomalies. Most of the experimental points are reproduced within the experimental uncertainty by the fits. The fits are appreciably better employing vacuum polarization corrections than without them. The calculations reported on in reference 31 employ only a mean P wave and the effect of split P waves combined with a D wave has not been used. It is believed by the authors that the mean effective P wave may be said to be no greater than about 0.02° in absolute value at incident energies from 1.4 to 2.4 MeV on the basis of their experiments and analysis.

A variant of the arrangement for making vacuum polarization calculations has been proposed and used by Heller.[32] He combines the Coulomb potential and the vacuum polarization potential into one "electric" potential and introduces phase shifts with respect to the asymptotic phases of wave functions in that potential. There is a certain esthetic value in this modification but it does not differ essentially from the one already described. For $L > 0$ in the low-energy region in which the vacuum polarization is important there is practically no difference from the older arrangement because the vacuum polarization phase shift may be calculated neglecting the small nuclear phase shifts. The attention paid by Heller to effects on the $^1S_0$ phase shift is concentrated on the so-called "shape parameter", the values of which he correlates with potential wells of different shapes following Jackson and Blatt[33] who make use of unpublished considerations of Schwinger which are equivalent to an expansion in powers of E of the function called f by Breit, Condon, and Present.[34] In the latter

[27] L. Durand, III, Phys. Rev. **108**, 1597 (1957).

[28] H. H. Hall and J. L. Powell, Phys. Rev. **90**, 912 (1953) and independent contemporaneous but unpublished work of M. H. Hull, Jr., at Yale gave mean P-wave phase shifts for the data of reference 26 of about −0.1°. M. H. Hull, Jr., and J. Shapiro, Phys. Rev. **109**, 846 (1958) have shown that admitting P and D waves and taking into account the possibility that $^3P_0$, $^3P_1$, $^3P_2$ have different phase shifts δ_0, δ_1, δ_2, as in G. Breit, C. Kittel, and H. M. Thaxton, Phys. Rev. **57**, 255 (1940), a variety of fits to the same cross section data is possible. The special circumstance that enters is that the differences in values of δ_0, δ_1, δ_2 simulate an angular dependence of the same type as that caused by K_2, the phase shift of $^1D_2$. The paper of Hull and Shapiro is sometimes misunderstood. The fits giving large absolute values of the polarization parameter P have not been claimed by them to be probable but only admissible by the data.

[29] M. de Wit and L. Durand, III, Phys. Rev. **111**, 1597 (1958).

[30] M. de Wit and G. Breit (unpublished).

[31] D. J. Knecht, S. Messelt, E. D. Berners, and L. C. Northcliffe, Phys. Rev. **114**, 550 (1959). This work and that in reference 26 was done under the guidance of Professor R. G. Herb and had the benefit of his long experience in precision p-p scattering measurements.

[32] L. Heller, Phys. Rev. **120**, 627 (1960).

[33] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950); cf. pp. 96–97.

[34] G. Breit, E. U. Condon, and R. D. Present, Phys. Rev. **50**, 825 (1936).

reference the introduction of the function f does not depend on the existence of a potential, however, and the importance of the shape parameter is largely that of a fitting parameter since the meaning of a static potential is quite unclear and since even Schwinger's effective range integral is not applicable for a velocity dependent potential.[35]

The problem of ascertaining the presence of P and D waves in the low-energy experiments is complicated by the possible presence of dynamic effects of molecular electrons, because in the experiments the proton beam is made to pass through a hydrogen gas. If the collision could be treated by means of classical mechanics only minor effects would be expected because the smallness of m/M has as a consequence the relative smallness of the probable momentum transfers. The slightly inelastic collisions resulting in electron excitations by amounts of the order of the ionization potential have to be counted together with the elastic collisions if the Rutherford scattering formula is to retain validity. Such a convention is not objectionable however because the counters used in p-p scattering experiments do not have sufficient resolving power to separate the inelastic from the elastic processes. The quantum mechanical problem is more complicated because[36] whenever an electron is transferred to a higher level, a new "channel" is opened up, in the sense of nuclear reaction theory. Since the quantum theory must give the classical theory in the appropriate limit, there must be a compensation of the first nonvanishing order of contributions to σ from the inelastic channels by the cross product term of the second-order effect on the scattering amplitude of the coherent elastic channel with the zero-order amplitude. The relatively large probability of the inelastic processes is pointed out in the first of the notes listed in reference 36. The presence of an acceleration effect is also pointed out there. This is the effect of the recoil of the target nucleon being sufficient to leave the molecular or atomic electron behind and thus opening a new channel. In terms of total probabilities this effect is also non negligible. The acceleration effect is distinct from the Coulomb excitation effect of atomic or molecular electrons and should be present for n-p scattering. In the second of the two notes in reference 36, results of more systematic considerations employing an extension of the Hamilton-Jacobi differential equation to the treatment of quantum excitations of atoms and the fact that the Hamilton-Jacobi method gives an approximate solution of the wave mechanical problem are used to show that there is exact compensation of the type discussed above, if the Hamilton-Jacobi method is used and if some small effects caused by the energy of excitation are neglected. This compensation exists for Coulomb excitations of any order. The author has been helped in this work by the concurrent progress of the calculations of de Wit, Fischer, and Zickendraht.[37] These authors find exact compensation for monopole, dipole, and quadrupole Coulomb excitation provided the radial Coulomb functions of the p-p collision process are replaced by their asymptotic forms employing $kr - \eta \ln 2kr - (L\pi/2) + \arg \Gamma(L + 1 + i\eta)$ for the phase and 1 for the amplitude and also provided the high frequency parts of the integrands of the Coulomb excitation integrals are neglected. These approximations may be seen to correspond to those in the second note listed in reference 36.

Although, as a result of these calculations, a large degree of compensation is expected, the effects are not understood as well as is desirable, the nonvanishing part of the dynamic effects of molecular electrons not having been reliably estimated so far. The interpretation of the data in reference 31 is thus still not in a finished state even apart from the question of the split P wave. The effects before compensation are large and even though under the simplifying assumptions made[36,37] there is exact compensation, it has not been shown with certainty that a sufficient fraction of the effects before compensation does not remain.

5. RELATIVISTIC CORRECTIONS FOR THE ELECTROMAGNETIC INTERACTION

The previous section, through a review of the Gordon sphere argument, leaves some doubt concerning the propriety of the usual way of assuming the applicability of the partial wave expansion such as in Eq. (4.13) in the phase-shift analysis of p-p scattering data. It is believed, nevertheless, that the customary procedure is not seriously in error in the nonrelativistic limit, close to which the electric interactions are most important, being reasonably well founded. There is some similarity here with the electromagnetic interactions in atoms. The nonrelativistic interactions suffice for the understanding of many problems even though for high-energy collisions the nonrelativistic predictions are inapplicable. This precedent makes the incomplete and unsatisfactory state of the relativistic theory perhaps somewhat more palatable.

[35] G. Breit and M. C. Yovits, Phys. Rev. **81**, 416 (1951).
[36] G. Breit, Phys. Rev. Letters **1**, 200 (1958); **2**, 401 (1959).
[37] M. de Wit, C. R. Fischer, and W. Zickendraht, Proc. Natl. Acad. Sci. U.S. **45**, 1047 (1959).

Since a completely relativistic treatment of the whole interaction is impossible before the physical nature of the nucleon-nucleon interaction is understood better than it is now, the establishment of relativistic corrections to the electromagnetic interaction has at present only a limited value. Nevertheless, for collisions at sufficiently small angles the protons are principally interacting at a large distance from each other and electromagnetic effects are then the more important. It will be seen that it is possible to make use of the separation of nucleons from each other also in the case of collisions at somewhat larger angles provided the phase parameters are determined phenomenologically for the lower angular momenta $L\hbar$. The execution of this plan is aided greatly by the possibility of dealing with the more distant part of the meson exchange interaction in the one-pion exchange approximation which will be discussed in a later section in this article.

The problem of relativistic corrections to the electromagnetic effects in nucleon-nucleon scattering has been first treated in the literature by Garren[38] then by Breit,[39] by Ebel and Hull,[40] again by Garren,[41] by Breit,[42] and lately by Breit and Ruppel.[43] Garren's two papers follow essentially the same method which consists in the application of Møller's matrix element[44] to the collision between two charged Dirac particles and a generalization of the matrix element to the case of anomalous magnetic moments. Many of the answers provided by this work are formally correct. It did not furnish, however, a justification for using the connection between δ_L and η of Eq. (4.7) or a formal proof for the employment of the velocity of the incident proton in the laboratory system in the calculation of η. As seen from the second paper of Breit,[42] the application to the nucleon-nucleon phase-parameter analysis carried out by Garren[41] may be expected to be strongly affected by wave function distortion effects. Breit's first paper is mainly concerned with providing a treatment free of the indefiniteness of the Møller scattering matrix element caused by the divergence which it introduces and thus of providing partial justification for the employment of the δ_L and of the relativistic formula for η. The difficulties in employing the Møller matrix element or the equivalent manipulation with Feynman diagrams are connected with the divergence of the scattering cross section at small angles and the related absence of a solution of the Schrödinger equation asymptotic to a plane wave plus an outgoing wave. The closest one comes to such a solution is that of Mott-Gordon reproduced above in (4.5). If the nonrelativistic problem is treated in momentum space one can calculate the first-order effects. On account of the infinite small angle scattering, however, the calculation of higher order effects by an iteration procedure runs into divergence difficulties.[39] Since the usual employment of the Møller matrix element is a relativistic extension of the momentum space treatment, it is only natural that similar difficulties occur for it. The character of the divergence may be seen by considering the nonrelativistic equation

$$[(-\hbar^2/2\mu)\Delta + e^2/r - E]\psi = 0\,. \tag{5.1}$$

Converting this to momentum space by means of

$$\psi(\mathbf{r}) = h^{-3/2}\int C_{\mathbf{p}} e^{i(\mathbf{kr})} d\mathbf{p}\,, \quad \mathbf{p} = \mathbf{k}\hbar \tag{5.2}$$

there results

$$\left(\frac{p^2}{2\mu} - E\right) C_{\mathbf{p}} + \int V(\mathbf{p},\mathbf{p}') C_{\mathbf{p}'} d\mathbf{p}'\,, \tag{5.3}$$

with

$$V(\mathbf{p},\mathbf{p}') = \frac{e^2/(\pi h)}{(\mathbf{p}-\mathbf{p}')^2}\,. \tag{5.4}$$

Equation (5.3) can be used to obtain the first-order correction to

$$C_{\mathbf{p}}^{(0)} = \delta(\mathbf{p} - \mathbf{p}_0) \tag{5.5}$$

which corresponds to

$$\psi^{(0)} = \exp(i\mathbf{k}_0\mathbf{r}) \tag{5.6}$$

giving

$$(E_{\mathbf{p}} - E - i\epsilon) C_{\mathbf{p}}^{(1)} = -V(\mathbf{p},\mathbf{p}_0) \tag{5.7}$$

and

$$\psi^{(1)} = -\int V(\mathbf{p},\mathbf{p}_0)(E_{\mathbf{p}} - E - i\epsilon)^{-1} e^{i(\mathbf{kr})} d\mathbf{p}\,. \tag{5.8}$$

The small positive constant ϵ secures the absence of incoming waves in $\psi^{(1)}$. There follows the asymptotic behavior

$$\psi^{(1)} \backsim -4\pi^2\mu\hbar V(p\hat{r},\hat{p}_0) e^{ikr}/r\,. \tag{5.9}$$

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[38] A. Garren, Phys. Rev. **96**, 1709 (1956).
[39] G. Breit, Phys. Rev. **99**, 1581 (1955). There are some inconsistencies in Eqs. (17) to (17.6) of this paper dealing with coupled states. A more logical presentation of this matter may be found in Eqs. (8) to (8.7) of G. Breit, Ann. Phys. (N. Y.) **16**, 346 (1961).
[40] M. E. Ebel and M. H. Hull, Jr., Phys. Rev. **99**, 1596 (1955).
[41] A. Garren, Phys. Rev. **101**, 419 (1956).
[42] G. Breit, Phys. Rev. **106**, 314 (1957).
[43] G. Breit and H. M. Ruppel, Phys. Rev. **127**, 2123 (1962).
[44] C. Møller, Z. Physik **70**, 786 (1931); Ann. Physik **14**, 531 1932).

From (5.8) to (5.9) the manipulations used are the same as those for any $V(\mathbf{p},\mathbf{p}')$ but in order that the steps should have a meaning, it is necessary to require that $C_{\mathbf{p}}^{(1)}$ be square integrable and this is not the case in the present problem. A closer examination of (5.8) shows, furthermore, the character of the difficulty and its relationship to the appearance of waves in nearly the original direction. In fact, the integration over angles involved in (5.8) gives

$$\int (\mathbf{k}-\mathbf{k}_0)^{-2} e^{i(\mathbf{kr})} d\Omega_{\mathbf{k}} = (2\pi/kk_0) \sum_L i^L (2L+1) \times P_L(\hat{k}_0\hat{r}) Q_L((k^2+k_0^2)/(2kk_0)) F_L(kr)/(kr) , \tag{5.10}$$

where the Q_L are Legendre functions of the second kind. They contain terms in $\ln[(k+k_0)/(k-k_0)]$, the integration in (5.8) consequently brings in $\ln \epsilon$ so that $\psi^{(1)}$ diverges. Since

$$Q_L\left(\frac{k^2+k_0^2}{2k\,k_0}\right) = \tfrac{1}{2} P_L\left(\frac{k^2+k_0^2}{2k\,k_0}\right) \times \ln\left[\left(\frac{k+k_0}{k-k_0}\right)^2\right] + \cdots , \tag{5.11}$$

where the terms left out contain the P_L only, and since for $k = k_0$ the P_L in this formula becomes $P_L(1) = 1$, the terms in $\ln(k-k_0)$ reproduce the incident wave with an infinite coefficient. The scattered wave in first Born approximation is seen to contain the incident wave with an infinite amplitude. An iteration procedure is thus impossible.

By employing the device of the Gordon sphere it is clearly possible to circumvent this difficulty and by means of this device one can make first-order calculations quite readily. In fact, employing the first-order formula

$$K_L = -\int (V/E) F_L^2 d\rho \tag{5.12}$$

which gives the phase shift K_L in terms of the regular free field wave function F_L of Eq. (4.8) one finds[39]

$$\delta K_0 = -\eta[C + \ln(2kR) - C_i(2kR)] , \tag{5.13}$$

where C is the Euler constant $0.5772\cdots$ and $C_i(x)$ is the cosine integral. For $kR \gg 1$,

$$\delta K_0 \backsim -\eta[C + \ln(2kR) - (1/2kR)\sin(2kR)] . \tag{5.14}$$

Identifying this with the phase of the regular Coulomb function

$$\rho - L\pi/2 - \eta \ln(2\rho) + \mathfrak{d}_L , \quad \mathfrak{d}_L = \arg\Gamma(L+1+i\eta) \tag{5.15}$$

and setting $r = R$, $L = 0$ one has

$$\mathfrak{d}_0 = -C\eta \tag{5.16}$$

both from the direct comparison of the phases and from the formula for $\mathfrak{d}_0$ in terms of the Γ function. The "Coulomb phase shift" $\mathfrak{d}_0$ is thus given correctly by (5.12) to within the first order in η. Similarly $\mathfrak{d}_{L+1} - \mathfrak{d}_L$ is correctly represented to first order in η by (5.12) according to

$$2\eta \int_0^\infty (F_{L+1}^2 - F_L^2) d\rho/\rho = \eta/(L+1) \cong \arg\Gamma(L+2+i\eta) - \arg\Gamma(L+1+i\eta) . \tag{5.17}$$

The reason for considering effects of η to the first order rather than exactly is that an exact relativistic treatment is difficult while a first-order calculation is relatively simple.

This can be carried out making use of the result[45] according to which the operator

$$(e^2/r)[1 - (\boldsymbol{\alpha}_p\boldsymbol{\alpha}_t)] \tag{5.18}$$

gives energy changes caused by electromagnetic interactions of Dirac particles p and t correctly to first order in e^2 provided the exchange integrals $A_{st;ts}$ arising in the treatment of identical particles are modified through the inclusion of factors

$$\cos(2\pi r_{PP'}/\lambda_{st}) , \quad \lambda_{st} = c/|E_s - E_t| , \tag{5.19}$$

where s, t denote single particle states with energies E_s, E_t, and P, P' refer to variable points in the coordinate space over which the double integral representing $A_{st;ts}$ is taken. In the c.m. system there occurs a simplification which makes it possible to use the operator of Eq. (5.19) directly. This simplification is caused by the vanishing of the $(E_s - E_t)^2$ which occur in the power-series expansion of the cosine in (5.19). Calculations employing (5.18) have been carried out in reference 39. Use is made there of a general connection between phase shifts and energy changes which leads to the relation

$$\delta(\delta_J^L) = -2\frac{dk}{dE}\int_0^\infty (\psi^N, (\delta V)\psi^N)_{sa} d\mathbf{r} , \tag{5.20}$$

where δ_J^L is the phase shift for a state with orbital angular momentum $L\hbar$ and total angular momentum $J\hbar$, δV is the change in the potential energy responsible for the change in δ_J^L on the left side of the equation, subscript sa denotes the inner product extended over spin and angle variables only, while ψ^N is a

[45] G. Breit, Phys. Rev. **34**, 375, 553 (1929).

suitably normalized 16-component function for the two particles. This normalization is such as to make

$$\langle(\psi^N,\psi^N)_{sa}\rangle_r \backsim 1/(2r^2)\,, \tag{5.21}$$

with the understanding that $< >_r$ denotes the average over r through a region containing many oscillations of ψ^N. In view of (5.21) one has

$$\langle(\Psi^N,\Psi^N)_{sa}\rangle_r \backsim [1/(2r^2)] \times [\langle(\Psi^N,\Psi^N)_{sa}\rangle_r/\langle(\psi^N,\psi^N)_{sa}\rangle_r]\,. \tag{5.22}$$

Here Ψ^N is the "large-large" part of ψ^N. Since the right-hand side of the equation contains inside the bracket a ratio independent of normalization the equation provides the normalization condition for Ψ^N. Equation (5.20) applies as it stands only to the "uncoupled" states, i.e., to states for which there is only one J that corresponds to L. For its generalization to the case of "coupled states" reference may be made to reference 39. Setting $c = 1$ the normalization condition is satisfied by such Ψ^N that

$$\Psi^N \backsim [(E_p + M)/2E_p](\mathcal{Y}_\mu^{L,J}/r) \times \sin\,(kr - L\pi/2 + \delta_J^L)\,, \tag{5.23}$$

where the $\mathcal{Y}_\mu^{L,J}$ are spin-angular functions normalized in accordance with

$$(\mathcal{Y}_\mu^{L,J},\mathcal{Y}_\mu^{L,J})_{sa} = 1\,. \tag{5.24}$$

The origin of Eq. (5.20) is a simple consideration of the way in which a discrete state in a large quantizing sphere changes adiabatically when δV is introduced. In the interests of brevity this consideration is not being reproduced. Applying this method of calculating changes in phase shifts to the relativistic operator of Eq. (5.18) it is found that for the singlet system the relativistic treatment differs from the nonrelativistic one only in the replacement of the nonrelativistic k by its relativistic value as in (4.4) and that the nonrelativistic η has to be replaced by the relativistic quantity η_r in accordance with

$$2\eta \to \left(\frac{E_p^2 + p^2}{\hbar p E_p}\right) e^2 = 2\eta_r\,. \tag{5.25}$$

Application of the transformation equations at the end of the section on relativistic kinematics then gives

$$\eta_r = e^2/(\hbar v')\,, \tag{5.26}$$

with v' as in (3.27). This means that η_r may be calculated by forgetting about relativistic effects provided the calculation is made in the laboratory system.

The relativistic k may also be expressed[39,46] in a simple way as

$$k_r = [\tfrac{1}{2} M(E_{r,\mathrm{lab}} - Mc^2)]^{1/2}/\hbar\,, \tag{5.27}$$

where $E_{r,\mathrm{lab}}$ is the relativistic kinetic energy, including the rest mass energy of the incident nucleon in the laboratory system. In the case of incident protons

$$E_{r,\mathrm{lab}} - Mc^2 = Ve\,, \tag{5.28}$$

where V is the effective accelerating potential. The relativistic k is thus obtained automatically if it is calculated in the most naive and nonrelativistic fashion directly in the laboratory system. These relations follow from (4.2) and (3.23) on substituting $E = 2\,E_p$ in (4.2). The convenience of agreements in form with nonrelativistic formulas for η_r and k in the laboratory system is obvious. Caution has to be used of course to use exactly the quantities involved in (5.26) and (5.28).

The derivation of the formula for η_r has only limited validity. One of the limitations to which it is subject is the absence of appreciable *wave function distortion* by nucleon-nucleon forces of nonelectromagnetic origin. This limitation entered the considerations through the employment of the free-particle function F_L in Eq. (5.12). This assumption does not apply for $L = 0$ and 1 at practically all of the experimentally available energies but is increasingly better as the energy decreases and L increases. The whole relativistic treatment is thus not well justified except for small angle collisions. For these the main contributions to the Coulomb wave arise from high L, i.e., from distant collisions in classical analogy. The practical limitations imposed by the requirement of a clearly small wave function distortion are severe. This may be seen by a consideration of the asymptotic form for the Coulomb scattered wave

$$\Psi_{sc}^c \backsim -[\eta/k(r - z)]\exp\{i[kr - \eta\ln(2\rho\mathbf{s}^2) + 2\sigma_0]\} = -(\eta/2\mathbf{s}^2)(e^{i\Phi}/kr)\exp(-i\eta\ln\mathbf{s}^2)\,, \tag{5.29}$$

where

$$\Phi = \rho - \eta\ln 2\rho + 2\delta_0 \tag{5.30}$$

and

$$\mathbf{s} = \sin\,(\theta/2)\,. \tag{5.31}$$

The angle dependent factor

$$-\frac{\eta}{2\mathbf{s}^2}\exp(-i\eta\ln\mathbf{s}^2) = \sum_L (2L+1) \times P_L(\cos\theta)\,\frac{[e^{2i(\sigma_L-\sigma_0)} - 1]}{(2i)}\,. \tag{5.32}$$

[46] G. Breit and M. H. Hull, Jr., Nuclear Phys. **15**, 216 (1960).

For small η this gives the approximation

$$-(\eta/2\mathbf{s}^2)\exp(-i\eta\ln\mathbf{s}^2)$$
$$\cong \eta[3P_1 + 5(1+\tfrac{1}{2})P_2 + 7(1+\tfrac{1}{2}+\tfrac{1}{3})P_3 + \cdots]. \quad (5.33)$$

The approximation used above for individual terms implies that

$$2\eta(1+\tfrac{1}{2}+\cdots+1/L) \cong 2\eta\ln L \ll 1 \quad (5.34)$$

and breaks down when $2\eta\ln L \cong 1$. For $E \cong 150$ MeV this occurs at $L \sim 10^7$. The relative contributions of the P_L to the series may thus be estimated by means of (5.34). If $E = 150$ MeV so that $\eta \cong 0.0129$ and for $\theta = 10°$, $\eta\ln\mathbf{s}^2 = -0.063$. Hence, $\exp(-i\ln\mathbf{s}^2) \cong 1$, $1/(2\mathbf{s}^2) \cong 65$ while the sum of the first three terms in brackets in (5.33) is $\cong 23$ which is about 30% of the whole. The sum of the first two terms is about half this amount. At this energy the phase shifts for $L = 0$ and 1 are quite appreciable and those for $L = 2$ are not negligible. It is not justifiable therefore to neglect wave distortion effects on Ψ^c in this case. Other similar examples illustrating this situation may be found in reference 42. Since $\theta = 10°$ is among the smaller scattering angles at which measurements are usually made, the "distant collisions" concept of classical dynamics has to be handled with care and needs additional consideration.

The actual situation is not as bad, however, as the above estimates would indicate, the usual nonrelativistic procedure of introducing phase shifts in treating scattering anomalies in a Coulomb field being especially adaptable for a partial removal of the difficulty. In this procedure the asymptotic phase of a partial wave is referred to the corresponding phase of the regular function of a purely Coulombian field as a comparison standard and the excess of the actual phase over the Coulombian is called the phase shift. The contribution to the scattering amplitude is then usually evaluated by subtracting the Coulomb amplitude and adding it. The differences in the terms of partial wave expansions are expressed in terms of functions of phase shifts and the amplitude appears as a sum of such functions plus the Coulomb amplitude. The difference between this procedure and that of reference 39, which is reviewed from (5.13) to (5.25) above, is that in the nonrelativistic problem there is available a convenient exact solution of the Coulombian case on which nuclear effects are superposed, while in the relativistic treatment just reviewed, the nuclear effects are supposed available and the electromagnetic interaction is superposed on them by an approximate procedure.

For low L the phase has a combined origin. For high L the electromagnetic effect is the dominant one. It is not important whether for low L the phase is calculated in one or another order, especially if all that is wanted is a phenomenological description of scattering by means of phase shifts. For high L the only requirement is that the Coulomb phases are correctly represented. This requirement is approximately satisfied in Eqs. (5.12) to (5.17) supplemented by the employment of η_r. Although the considerations are good only to the first order in η, they agree with the exactly known nonrelativistic limit. Employment of nonrelativistic formulas for the phase (5.15) modified through the use of η_r may thus be expected to be accurate at both low and high E, since the effects become small at high E. The approximation is probably better than a simple interpolation formula as may be seen from the following argument.

The transition from the nonrelativistic η to η_r is caused in the first-order calculation by the employment of (5.18) rather than the simpler e^2/r, which has validity in all orders of nonrelativistic theory. The uncertainties come in, therefore, in higher orders of $\eta_r - \eta_{nr}$, where the subscript nr stands for nonrelativistic. In the lower part of the 0 to 350 MeV region the higher orders of η are important and the $\eta_r - \eta_{nr}$ effect is small. In the upper part the higher orders of η are not very important and $\eta_r - \eta_{nr}$ has an appreciable effect.

The change from η_{nr} to η_r may be explained directly from (5.18) by noting that neglecting spin effects one may make the replacement

$$1 - (\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2) \to 1 + c^2p_1^2/E_1^2, \quad (5.35)$$

where subscript 1 denotes that the quantity is evaluated for particle 1. This replacement is not exact but becomes so on the energy shell. The second term in (5.35) is, in the case of free particles, v_1^2/c^2. Hence, employing (4.3) and (3.27),

$$\eta_r = (e^2/2\hbar v_1)[1 + (v_1/c)^2] = e^2/\hbar v_1', \quad (5.36)$$

in agreement with (5.26). As r increases, the employment of $(e^2/r)[1 - (\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)]$ becomes more accurate because the interaction becomes weaker. Hence, the progressive phase shift term $-\eta\ln(2\rho)$ in (5.15) may be expected to be correctly represented provided η is replaced by η_r. This is also the case for the low L group and for the same reason. It is more difficult however to distinguish between the goodness of η and that of η_r for the calculation of $\delta_L - \delta_0$ and an exact relationship has not been established to the

writer's knowledge. However, a comparison with the exact solution for the scattering of a particle by a point charge shows that the employment of η_r for such calculations is justifiable. This comparison can be made at small scattering angles only. In this case, however, it is reasonably meaningful.[39] The energy of the recoiling proton is approximately Θ^2 times the incident energy, where Θ is the scattering angle in the laboratory system. For $\Theta = 1/6$ rad and $E = 300$ MeV the recoil proton has an energy under 10 MeV and scattering from it may therefore be approximated by the scattering from a fixed proton. For further details concerning this comparison reference may be made to reference 39. It is, of course, natural that η_r occurs in these calculations since v' in (5.26) is the velocity in the laboratory system.

The case of coupled states such as the ${}^3P_2 - {}^3F_2$ system has not been considered above in connection with the relativistic effects. Reference may be made in this connection to reference 39 and for a more consistent consideration of the transition from the effective energy matrix to the eigenphaseshifts to a newer paper.[47]

It is thus seen that the relativistic refinement has some foundation not only for the progressive part of the phase shift but also for the so-called Coulomb phase shifts δ_L (a term having little relationship to the physical situation), but that the difference between η_{nr} and η_r if used in (4.1) may not be considered as having been exactly justified especially for large-angle collisions. Fortunately, the relativistic features of Coulomb effects have proved so far to be important for small-angle collisions only.

6. SCATTERING MATRIX

The triplet part of the incident wave modified by the electromagnetic interaction is conveniently taken to have the form

$$\Psi^{0c} = \sum_\mu a_\mu \chi_\mu \psi^c \tag{6.1}$$

with ψ^c as on the right-hand side of (4.5), and with the relativistic η of (5.26), the employment of which will be understood without explicit mention from here on. For p-p scattering and, if the triplet to singlet coupling is neglected, also for n-p scattering, the scattered wave may be written as

$${}^3\Psi_{Sc} = \sum_{\mu,m} \chi_\mu S_{\mu,m} a_m / r \,. \tag{6.2}$$

The functions χ_μ are

$$\chi_1 = \alpha_1\alpha_2 \,, \quad \chi_0 = (\alpha_1\beta_2 + \alpha_2\beta_1)/2^{1/2} \,,$$
$$\chi_{-1} = \beta_1\beta_2 \,, \tag{6.3}$$

where α_i, β_i are single-particle spin functions corresponding, respectively, to spin orientations along the positive and negative directions of the quantization axis. Disregarding, for the present, coupling between states of the same J but different L, there are available[48] formulas for the calculation of the $S_{\mu,m}$ as follows:

$$k(S_{1,1} - S^c) = k(S_{-1,-1} - S^c) = \alpha_2 e^{i\Phi} = e^{i\Phi} \sum_L \tfrac{1}{2} e_{L0}[(L+2)Q_{L,L+1} + (2L+1)Q_{L,L} + (L-1)Q_{L,L-1}]P_L, \tag{6.4}$$

$$-kS_{0,1}e^{-i\varphi} = kS_{0,-1}e^{i\varphi} = -2^{-1/2}\alpha_1 \sin\theta e^{i\Phi}$$
$$= 2^{-1/2}e^{i\Phi} \sum_L e_{L0}[L(L+2)Q_{L,L+1} - (2L+1)Q_{L,L} - (L^2-1)Q_{L,L-1}] \sin\theta P'_L/[L(L+1)] \,, \tag{6.5}$$

$$kS_{-1,1}e^{-2i\varphi} = kS_{1,-1}e^{2i\varphi} = \alpha_3 \sin^2\theta e^{i\Phi}$$
$$= \tfrac{1}{2} e^{i\Phi} \sum_L e_{L0}[LQ_{L,L+1} - (2L+1)Q_{L,L} + (L+1)Q_{L,L-1}] \sin^2\theta P''_L/[L(L+1)] \,, \tag{6.6}$$

$$kS_{1,0}e^{i\varphi} = -kS_{-1,0}e^{-i\varphi} = 2^{-1/2}\alpha_4 \sin\theta e^{i\Phi} = 2^{-1/2}e^{i\Phi} \sum_L e_{L0}(Q_{L,L+1} - Q_{L,L-1}) \sin\theta P'_L \,, \tag{6.7}$$

$$k(S_{0,0} - S^c) = \alpha_5 e^{i\Phi} = e^{i\Phi} \sum_L e_{L0}[(L+1)Q_{L,L+1} + LQ_{L,L-1}]P_L \,, \tag{6.8}$$

$$e_{L0} = \exp(2i\sigma_{L,0}) \,, \quad \delta_{L,0} = \delta_L - \delta_0 \tag{6.9}$$

with Φ as in (5.30) and

$$S^c = -[\eta/(2k\mathbf{s}^2)] \exp[i(\Phi - \eta \ln \mathbf{s}^2)] \,, \tag{6.10}$$

[47] G. Breit, Ann. Phys. (N. Y.) **16**, 346 (1961).

[48] G. Breit and M. H. Hull, Jr., Phys. Rev. **97**, 1047 (1955). Through an oversight, the factor $[4\pi(2L+1)]^{1/2}$ has been omitted under the summation in Eq. (1.4) of this reference.

where $\mathbf{s}$ is as in (5.31), while

$$Q_{L,J} \equiv Q(\delta_J^L) , \quad Q(\delta) \equiv (e^{2i\delta} - 1)/(2i) \quad (6.11)$$

and δ_J^L is the phase shift for orbital angular momentum $L\hbar$ and total angular momentum $J\hbar$.

For identical particles the corresponding quantities are

$$S^{ac} = 2^{-1/2}(\eta/2k)[-\mathbf{s}^{-2} \exp(-i\eta \ln \mathbf{s}^2) + \mathbf{c}^{-2} \exp(-i\eta \ln \mathbf{c}^2)]e^{i\Phi} \quad (6.12)$$

$$S^a_{\mu,\nu} = 2^{1/2} S_{\mu,\nu} (\mu \neq \nu) \quad (6.13)$$

$$S^a_{\mu,\mu} - S^{ac} = 2^{1/2}(S_{\mu,\mu} - S^c) . \quad (6.14)$$

Here superscript a refers to antisymmetrization. For nonidentical particles the scattering matrix for singlet states is

$$k(s_{00} - S^c) = e^{i\Phi} \sum_L (2L + 1) e_{L0} P_L Q_L , \quad (6.15)$$

where

$$Q_L \equiv Q(K_L) \quad (6.16)$$

and K_L is the phase shift for a singlet state with orbital angular momentum $L\hbar$. For unpolarized nonidentical particles the differential collision cross section σ is

$$\sigma = \tfrac{1}{4}[|s_{00}|^2 + \sum_{\mu,\nu} |S_{\mu\nu}|^2] . \quad (6.17)$$

For identical particles the scattering matrix for singlet states is obtainable from

$$s^a_{00} - S^{sc} = 2^{1/2}(s_{00} - S^c)$$

$$S^{sc} = 2^{-1/2}(\eta/2k)[-\mathbf{s}^{-2} \exp(-i\eta \ln \mathbf{s}^2) - \mathbf{c}^{-2} \exp(-i\eta \ln \mathbf{c}^2)]e^{i\Phi} \quad (6.18)$$

and the differential cross section is

$$\sigma^a = \tfrac{1}{2}[|s^a_{00}|^2 + \sum_{\mu,\nu} |S^a_{\mu\nu}|^2] . \quad (6.19)$$

The superscript a indicates antisymmetrization. In the last equation the effect of recoils is included. In the presence of coupling between states with the same J but different L the values of the amplitudes $\alpha_1, \cdots, \alpha_5$ must be changed[49] by adding $\Delta\alpha_1, \cdots, \Delta\alpha_5$ having values as follows:

$$\Delta\alpha_1 = \Delta\alpha_4 = -2 \sum_l B_{l+1} \times [(l + 1)P'_{l+2} - (l + 2)P'_l] \quad (6.20)$$

$$\Delta\alpha_2 = -\sum_l (l + 1)(l + 2) B_{l+1}(P_{l+2} + P_l) \quad (6.21)$$

$$\Delta\alpha_3 = -\sum_l B_{l+1}(P''_{l+2} + P''_l) , \quad \Delta\alpha_5 = -2\Delta\alpha_2 , \quad (6.22)$$

where

$$B_{l+1} = \tfrac{1}{2} \mathfrak{I}_{l,l+2}[(l + 1)(l + 2)]^{-1/2} \times \exp[i\mathfrak{d}_{l,0}(\eta) + i\mathfrak{d}_{l+2,0}(\eta)] \quad (6.23)$$

and the notation for $\mathfrak{d}_{l,0}$ is the same as in (6.9). The $\mathfrak{I}_{l,l+2}$ is defined by the requirement that the interaction between colliding particles produce the changes

$$F_l \mathcal{Y}_\mu^{l,j} \to F_l \mathcal{Y}_\mu^{l,j} + \mathfrak{I}_{l,l} \mathcal{Y}_\mu^{l,j} H_l + \mathfrak{I}_{l,l+2} \mathcal{Y}_\mu^{l+2,j} H_{l+2}$$

$$F_{l+2} \mathcal{Y}_\mu^{l+2,j} \to F_{l+2} \mathcal{Y}_\mu^{l+2,j} + \mathfrak{I}_{l+2,l} \mathcal{Y}_\mu^{l,j} H_l + \mathfrak{I}_{l+2,l+2} \mathcal{Y}_\mu^{l+2,j} H_{l+2} , \quad (6.24)$$

where the two coupled states have orbital angular momenta $l\hbar$, $(l + 2)\hbar$ and total angular momentum $(l + 1)\hbar$. The value of the latter in units $\hbar$ is used as a subscript for B of Eqs. (6.20), (6.21), and (6.22). The quantities H_l in (6.24) are

$$H_l = G_l + iF_l , \quad (6.25)$$

where F_l and G_l are, respectively, the regular and irregular Coulomb functions. Besides the additions $\Delta\alpha_i$ the formulas for the α_i are understood to have every $Q(\delta^l_j)$ of the coupled pair of l, j values replaced by the corresponding diagonal element of the 2×2 matrix formed by the four $\mathfrak{I}$ in (6.24). Explicitly

$$Q_{l,l+1} \to \mathfrak{I}_{l,l}, Q_{l+2,l+1} \to \mathfrak{I}_{l+2,l+2} . \quad (6.26)$$

In (6.24)

$$\mathcal{Y}_\mu^{LJ} = \sum_m \begin{pmatrix} 1 , L \\ m , \mu - m \end{pmatrix} ; J \Big) Y_{L,\mu-m} \chi_m , \quad (6.27)$$

where the first factor is the Wigner addition coefficient with standard phase such as used by Condon and Shortley[50] and by Blatt and Weisskopf.[51] The five amplitudes α_i are not independent satisfying

$$\alpha_2 - \alpha_5 - \alpha_3 \sin^2\theta = (\alpha_1 + \alpha_4) \cos\theta \quad (6.28)$$

which follows from the time-reversal symmetry. The other Wolfenstein and Ashkin[52] relations are also satisfied by (6.4) to (6.7). The matrix

$$U = 1 + 2i\mathfrak{I} \quad (6.29)$$

[49] G. Breit, J. B. Ehrman, and M. H. Hull, Jr., Phys. Rev. **97**, 1051 (1955).

[50] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, New York, 1935).

[51] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

[52] L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952).

is unitary and symmetric[53] and is often used[54] as

$$U_J = \begin{pmatrix} (1-\rho_J^2)^{1/2}\exp(2i\theta_J^{J-1}) & , & i\rho_J \exp[i(\theta_J^{J-1}+\theta_J^{J+1})] \\ i\rho_J \exp[i(\theta_J^{J-1}+\theta_J^{J+1})] & , & (1-\rho_J)^{1/2}\exp(2i\theta_J^{J+1}) \end{pmatrix}, \tag{6.30}$$

a form very closely related to that occurring in the paper by Wigner.[53] It is also frequently used in the Blatt-Biedenharn[55] eigenvalue form

$$U = \begin{pmatrix} c_\epsilon^2 \exp(2i\delta_\alpha) + s_\epsilon^2 \exp(2i\delta_\beta) & , & c_\epsilon s_\epsilon[\exp(2i\delta_\alpha) - \exp(2i\delta_\beta)] \\ c_\epsilon s_\epsilon[\exp(2i\delta_\alpha) - \exp(2i\delta_\beta)] & , & s_\epsilon^2 \exp(2i\delta_\alpha) + c_\epsilon^2 \exp(2i\delta_\beta) \end{pmatrix}, \tag{6.31}$$

where

$$c_\epsilon = \cos\epsilon\,, \quad s_\epsilon = \sin\epsilon\,. \tag{6.32}$$

The parametrization used in (6.30) is closely related to the "nuclear bar" notation of Stapp, Ypsilantis, and Metropolis,[56] the symbols corresponding as follows:

$$\rho_J = \sin 2\bar{\epsilon}_J\,, \quad \theta_J^{J-1} = \bar{\delta}_{J-1}\,, \quad \theta_J^{J+1} = \bar{\delta}_{J+1}\,. \tag{6.33}$$

The relationship of (6.30) to (6.31) may be expressed as[54]

$$\begin{aligned} &\theta_J^{J-1} + \theta_J^{J+1} = \delta_\alpha + \delta_\beta\,, \\ &\tan(\theta_J^{J-1} - \theta_J^{J+1}) = (\cos 2\epsilon)\tan(\delta_\alpha - \delta_\beta)\,, \\ &\rho_J = (\sin 2\epsilon)\sin(\delta_\alpha - \delta_\beta)\,. \end{aligned} \tag{6.34}$$

The transition from nonidentical particles to identical ones can be stated either in terms of (6.12), (6.13), (6.14) or by means of

$$\alpha_i^a = 2^{1/2}\alpha_i\,, \quad (i = 1,2,3,4,5) \tag{6.35}$$

together with (6.12) and (6.18). The factor $2^{1/2}$ in the last formula arises as a product of $2^{-1/2}$ in the usual normalization of the antisymmetrized function and a factor 2 which comes about as a result of subtracting the negative of the odd angular functions.

According to the usual definition the polarization vector of a particle having spin $\frac{1}{2}$ will be used here in the sense of

$$\mathbf{P} = \ll\boldsymbol{\sigma}\gg\,, \tag{6.36a}$$

where $\boldsymbol{\sigma}$ is the Pauli spin vector and the double angular brackets indicate the statistical average of the quantum expectation value. No questions arise concerning this definition in nonrelativistic theory. According to Sec. 3 of this paper, the kinematical relativistic modifications are taken care of by transforming the directions of $\boldsymbol{\sigma}$ in making Lorentz transformations. The definition will therefore be used in terms of the "spin" rather than the true spin. There are available many good treatments of the statistical matrix of spin 1/2 particles[57] and no attempt at a logically deductive presentation of this subject will be made here. It will suffice to mention a few of the more essential facts.

The statistical matrix ρ has the general property that for any operator Y

$$\ll Y\gg = \mathrm{Tr}\,(\rho Y)/\,\mathrm{Tr}\,\rho\,. \tag{6.36b}$$

It is known that for spin 1/2 particles ρ is expressible in terms of $\ll\boldsymbol{\sigma}\gg$ by means of

$$\rho = \tfrac{1}{2}\{1 + \ll\sigma_x\gg\sigma_x + \ll\sigma_y\gg\sigma_y + \ll\sigma_z\gg\sigma_z\}\,\mathrm{Tr}\,(\rho). \tag{6.37}$$

This means that the spin properties of a statistical mixture of spin 1/2 particles are fully determined by $\mathbf{P}$. On scattering from a target the incident density matrix ρ^i is changed to a final density matrix ρ^f in accordance with

$$\rho^f = T\rho^i T^\dagger\,, \tag{6.38}$$

where $T^\dagger$ is the conjugate of the transposed transition matrix T. In this equation ρ and T refer to the scattered nucleon and to the target particle collectively. The scattering direction is supposed to have been specified. Denoting the unit matrix for the target spin states by 1_t the result of a single scattering of an unpolarized beam by an unpolarized target is to give a value of $\mathbf{P}$ after scattering equal to

$$\mathbf{P}_s = \mathrm{Tr}\,(T^\dagger 1_t \boldsymbol{\sigma} T)/\,\mathrm{Tr}\,(T^\dagger T)\,. \tag{6.39}$$

If nucleons with an initial polarization vector $\mathbf{P}_i$ are scattered by an unpolarized target, the differential

[53] E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **32**, 302 (1946).
[54] G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. **120**, 2227 (1960).
[55] J. M. Blatt and L. C. Biedenharn, Phys. Rev. **86**, 399 (1952); Revs. Modern Phys. **24**, 258 (1952).
[56] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957). This reference will be occasionally referred to as SYM.
[57] L. Wofenstein[7]; M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nuclear Sci. **10**, 291 (1960).

cross section for a given direction is

$$\sigma_\Omega = [2(2I_t + 1)]^{-1}\{\mathrm{Tr}\,(TT^\dagger) + (\mathbf{P}_t \cdot \mathrm{Tr}\,(T\boldsymbol{\sigma}_t T^\dagger))\}. \tag{6.40}$$

On the basis of these relations as well as other considerations, it has been shown by Wolfenstein and Ashkin,[7] that in a double-scattering experiment

$$\sigma_\Omega^{(2)} = [\sigma_\Omega^{(2)}]_0[1 + (\mathbf{P}_{s1} \cdot \mathbf{P}_{s2})]\,, \tag{6.41}$$

where the superscript 2 indicates the second scattering and both $\mathbf{P}_{s1}$ and $\mathbf{P}_{s2}$ have the same meaning as in (6.39), being defined as the values of $\mathbf{P}$ which arise as the result of scattering of an unpolarized beam by an unpolarized target. The energy of the unpolarized nucleons scattered from the unpolarized target nuclei used to define $\mathbf{P}_{s2}$ is supposed to be taken equal to the energy of polarized nucleons reaching target 2 in the double scattering experiment. The proof involves a justification of the interchange of T and $T^\dagger$ in (6.40) which in the case of a target with arbitrary spin makes use of the assumed invariance of the collision to time reversal in Wigner's sense.[58] It has also been shown by Wolfenstein and Ashkin that $\mathbf{P}_s$ of (6.39) is normal to the scattering plane. Writing

$$\mathbf{P}_{s1} = P_1\mathbf{n}_1\,, \quad \mathbf{P}_{s2} = P_2\mathbf{n}_2\,, \tag{6.42}$$

it follows that

$$\sigma_\Omega^{(2)} = [\sigma_\Omega^{(2)}]_0[1 + P_1P_2\cos\varphi]\,, \quad \cos\varphi = (\mathbf{n}_1 \cdot \mathbf{n}_2)\,, \tag{6.43}$$

where φ is the angle between the normals of the two scattering planes. In this form the relation is directly useful for measurements of P. The "polarization" P is a property of the nucleon and the target. It has an experimental significance in terms of a polarizing experiment to which (6.39) refers. It has also a meaning in terms of an analyzing experiment described by (6.40) and it is seen in (6.41) and (6.42) that P_1 and P_2 enter the result of a double-scattering experiment symmetrically. In a double-scattering experiment the mean spin direction is perpendicular to the incident momentum of the second scattering. This restriction makes the information derivable from experiment limited since the most general spin orientation of incident nucleons is not covered by it. In triple-scattering experiments the second scattering gives nucleon spin directions which are not necessarily perpendicular to the second scattering plane and information about these directions is obtainable by the third scattering which is thus used as an analyzer while the second scattering serves as a polarizer. The recent advances[59] made in techniques of producing polarized protons without the aid of scattering will make the first scattering unnecessary with an expected saving of beam intensity and consequent increase in accuracy.

The polarization P defined by (6.42) for either scattering can be calculated[49] for p-n scattering by means of

$$k^2(P_\sigma^y)_{p-n} = \tfrac{1}{2}\sin\theta\cos\varphi\,\mathrm{Im}\,\{\alpha_1\alpha_2^* - \alpha_1\alpha_3^*\sin^2\theta + \alpha_5\alpha_4^*\} \tag{6.44}$$

and for p-p scattering by means of

$$\begin{aligned} k^2(P_\sigma^y)_{p-p} &= 2\sin\theta\cos\varphi\,\mathrm{Im}\,\{\alpha_1(\alpha_2 + \alpha_c^T)^* - \alpha_1\alpha_3^*\sin^2\theta + (\alpha_5 + \alpha_c^T)\alpha_4^*\}, \\ \alpha_c^T &= \tfrac{1}{4}\eta[-\mathbf{s}^{-2}\exp(-i\eta\ln\mathbf{s}^2) + \mathbf{c}^{-2}\exp(-i\eta\ln\mathbf{c}^2)]\,, \end{aligned} \tag{6.45}$$

where the α_i are as in (6.4) to (6.8). On setting $\varphi = 0$ the scattering plane becomes the xy plane and P^y becomes P. In the derivation of (6.44) it was assumed that the total spin is a good quantum number. In (6.45) this assumption is practically irrelevant because the exclusion principle combined with parity and angular momentum conservation makes singlet-triplet coupling impossible. If charge independence is exact, there should be no such coupling for the n-p case either. Equation (6.44) may thus need some improvement. Similarly modifications in triple-scattering parameter calculations may have to be made.

According to Wolfenstein[60] one may express the

[58] E. P. Wigner, Göttinger Nachrichten **31**, 546 (1932).

[59] Proceedings of the International Symposium on Polarization Phenomena of Nucleons (Basel, 1960), in Helv. Phys. Acta, Suppl. VI, 1961; E. Baumgartner, L. Brown, P. Huber, H. Rudin, and H. R. Striebel, Phys. Rev. Letters **5**, 154 (1960); H. Rudin, H. R. Striebel, E. Baumgartner, L. Brown, and P. Huber, Helv. Phys. Acta **34**, 58 (1961); C. W. Drake, D. C. Bonar, R. D. Headrick, and V. W. Hughes, Rev. Sci. Instr. **32**, 995 (1961); C. W. Drake, D. C. Bonar, R. D. Headrick, and V. W. Hughes (to be published): J. M. Dickson and M. K. Craddock, International Conference on High Energy Accelerators, New York, September 1961 (unpublished).

[60] L. Wolfenstein, Phys. Rev. **96**, 1654 (1954); L. Wolfenstein, reference 7, Eq. (44).

matrix T (his M) for nucleon-nucleon scattering as

$$T(\mathbf{k},\mathbf{k}') = BS + C(\sigma_n^{(1)} + \sigma_n^{(2)})_n + \tfrac{1}{2} G[(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{K})(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{K}) + (\boldsymbol{\sigma}^{(1)}\cdot\mathbf{K}_+)(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{K}_+)]T + \tfrac{1}{2} H[(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{K})(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{K}) - (\boldsymbol{\sigma}^{(1)}\cdot\mathbf{K}_+)(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{K}_+)]T + N\sigma_n^{(1)}\sigma_n^{(2)}T\,. \tag{6.46}$$

Here $S = \frac{1}{4}[1 - (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})]$ and $T = \frac{1}{4}[3 + (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})]$ are, respectively, the singlet and triplet projection operators while the directions along which the spin components are taken are those of the previously defined $\mathbf{n}$ and of

$$\mathbf{K}_+ \equiv \mathbf{k}' + \mathbf{k}\,, \quad \mathbf{K} \equiv \mathbf{k} - \mathbf{k}'\,. \tag{6.47}$$

In this parametrization there enter only five angle dependent complex parameters. They satisfy however different symmetry requirements for isospin singlets and triplets. For the latter,[57,60] a change of θ to $\pi - \theta$ makes $(B,C,H) \rightarrow (B,C,H)$ and $(G,N) \rightarrow -(G,N)$. The oddness and evenness of the transformation is reversed for triplets. Another form frequently used is[61]

$$\sigma_\Omega \ll \sigma_f \gg / [\sigma_\Omega]_0 = [P + D(\ll\boldsymbol{\sigma}\gg_i\cdot\mathbf{n})]\mathbf{n} + [A(\ll\boldsymbol{\sigma}\gg_i\cdot\hat{k}) + R(\ll\boldsymbol{\sigma}\gg_i\cdot[\hat{n}\times\hat{k}])]\mathbf{s} + [A'(\ll\boldsymbol{\sigma}\gg_i\cdot\hat{k}) + R'(\ll\boldsymbol{\sigma}\gg_i\cdot[\hat{n}\times\hat{k}])]\hat{k}'\,. \tag{6.48}$$

Here $\hat{k}$, $\hat{k}'$ are, respectively, unit vectors along incident and outgoing directions in the laboratory system and

$$\mathbf{s} = [\mathbf{n}\times\mathbf{k}']\,. \tag{6.49}$$

These equations have the advantage of giving an immediate connection with the triple-scattering parameters D, R, A, A' but being referred to the laboratory system care must be taken regarding the relativistic spin rotation. Equation (6.48) applies with the above reservations to collisions with particles of any mass. The five coefficients D, A, R, A', R' are not independent. For infinitely heavy targets $R' = -A$; for two particles of equal mass $(A + R')/(A' - R) = \tan(\Theta/2)$. The parameter D measures depolarization, R the rotation of the spin direction in the scattering plane for incident transverse spin direction, A for incident longitudinal spin direction. For further information reference may be made to the literature quoted above.

7. MAGNETIC MOMENT INTERACTIONS

The phase shift method described in Sec. 6 also gives the magnetic moment interactions between protons if one neglects effects of the anomalous parts of the moments, i.e., it provides contributions to the scattering matrix having their origin in the Diracian parts of the moments. The assumption of the absence of appreciable wave distortion effects is inherent in this treatment as well as the previously quoted one of Garren[38,41] as has been mentioned in reference 39 and particularly in reference 42. Numerical examples of the seriousness of wave distortion effects caused by the relatively large participation of large L may be found in the last reference. This part of the situation is somewhat similar to that which has been discussed in connection with the relativistic generalization of the electrostatic interaction mentioned in the previous section. There is an important difference in the wave distortion effects in the two cases, however. For the "electrostatic" interaction, as previously pointed out, the wave distortion effect in the case of low L is important only if one is trying to account for the phase shifts from a fundamental theory. Otherwise, omission of their consideration is immaterial since the asymptotic phase itself is the only quantity of interest for a phase shift analysis. The effects of nucleonic magnetic moments are, on the other hand, not included in the conventional phase standardization and therefore the employment of a contribution to the scattering amplitude neglecting the distortion effects is not proper. The character of the error thus committed will be more concretely appreciated if it will be recalled that formally the magnetic moment effects of the Diracian protons are very much like those familiar in the theory of atomic spectra for electrons. Discarding some relativistic effects which appeared in the calculations and omitting the spin-spin interactions, the contributions to the scattering matrix are, in fact, derivable from an

[61] L. Wolfenstein, reference 7, Eq. (39).

interaction Hamiltonian

$$H' = (\mu_0^2/\hbar r^3)\{[(\mathbf{r}_1 - \mathbf{r}_2) \times (2\mathbf{p}_2 - \mathbf{p}_1)]\cdot\boldsymbol{\sigma}_1 + [(\mathbf{r}_2 - \mathbf{r}_1) \times (2p_1 - \mathbf{p}_2)]\cdot\boldsymbol{\sigma}_2\}\,, \quad (7.1)$$

where

$$\mu_0 = e\hbar/(2Mc) \quad (7.2)$$

is the nucleon Bohr magneton. It would obviously be improper to employ the corresponding interaction energy in atomic spectra neglecting the change in the wave function caused by the electrostatic interaction of the electrons either with the nucleus or with each other. On account of the presence of $1/r^3$ in H' the phase shifts caused by H', according to (5.12), contain the factor

$$\int_0^\infty (F_L^2/\rho^3)d\rho = 1/[2L(L+1)]\,. \quad (7.3)$$

The contributions of the phase shifts for different L to the scattering amplitude are, therefore, as in the series

$$\sum_L (2L+1)P_L(\cos\theta) \times (L, -1, -L-1)/[L(L+1)]\,, \quad (7.4)$$

the three numbers in parentheses corresponding, respectively, to $J = L+1$, L, $L-1$. The series for $J = L \pm 1$ is comparable to

$$\sum_1^\infty P_L(\cos\theta) = \left(\frac{1}{2\mathbf{s}}\right) - 1\,. \quad (7.5)$$

For $\theta = 10°$, $[1/(2\mathbf{s})] - 1 = 4.7$, while $P_1 + P_2 = 1.94$. The first two L contribute 41% of the whole sum. At bombarding energy of 150 MeV the first maxima of F_L fall at roughly $r = 2.0 \times 10^{-13}$ cm for $L = 1$, 2.9×10^{-13} cm for $L = 2$, and 3.7×10^{-13} cm for $L = 3$. For the lower L the maximum is well within the range of nuclear forces and since values of F_L at smaller r than those for the maximum matter also, appreciable distortion effects may be expected. No general reason for expecting compensation effects of the influence of different L appears to exist. The direct employment of the scattering amplitude caused by the spin-orbit interaction is thus unjustifiable.

Fortunately, a direct employment of this part of the scattering amplitude is not needed for the phenomenological phase parameter analysis. The usual procedure, the justification for which will be discussed in Sec. 9, is to divide the phase parameters into the low L and J and the high L and J groups. The phase parameters for the latter are used in the one-pion-exchange (OPE) approximation while those for the lower L are determined by adjustment of the phase shifts and coupling parameters to experimental data. It is, furthermore, assumed, in applying the OPE approximation, that the effects of the one-pion-exchange potential (OPEP) may be calculated neglecting wave function distortion caused both by the OPEP and by the penetration of the wave function towards values of r smaller than those for which the OPEP may be assumed to represent the interaction. It is consistent with this assumption to neglect wave function distortion in the calculation of the effects of the spin-orbit and spin-spin interactions in dealing with the OPE group. On the other hand, the validity of the phenomenological analysis is not impaired by omitting the contribution of H' to the phase parameters of the low L group. This simply means that the phase parameters determined include the effects of H'. These effects will have to be corrected for eventually, in obtaining the phase parameters that would be caused by mesonic interactions alone. It is probably premature however to be making these corrections, the understanding of interactions at small r being still in a rudimentary stage. On the other hand, the effect of terms like H' should, in principle, be taken into account for the OPE group since it is doubtless present in the observed scattering. This viewpoint has been recently pursued by Ruppel and Breit[43] and their treatment including the relevant parts of reference 42 is briefly reviewed below.

In the nonrelativistic approximation the sum of the effects of the Diracian and the anomalous (mesonic) parts of the magnetic moments of the protons is represented by

$$H' + H'' = -(\mu_0^2/\hbar r^3)(3 + 4\mu_a) \times \{[(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{p}_1]\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\}\,. \quad (7.6)$$

Quantities referring to the two protons 1 and 2 are distinguished here by subscripts 1 and 2, respectively, μ_a is the anomalous part of the proton moment, the $\boldsymbol{\sigma}$ are the Pauli spin vectors. If (7.6) is combined with the e^2/r term the scattering matrix becomes

$$S^{c''} = -\frac{\eta}{2k\mathbf{s}^2}\exp[i(\Phi - \eta\ln\mathbf{s}^2)] \times \left\{1 + (3 + 4\mu_a)\frac{i\hbar^2}{4M^2c^2}[\mathbf{k}_f \times \mathbf{k}_i]\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\right\} \quad (7.7)$$

in the case of nonidentical protons. Subscripts i and f refer to the initial and final states. The quantity S is used in the convention of Eq. (6.2). For unsymmetrized product wave functions $S^{c''}$ correspond to

changes in α_4 and α_1 as in

$$(\Delta\alpha_4)_0 = -(\Delta\alpha_1)_0 = -(3+4\mu_a)\frac{\hbar^2k^3}{2M^2c^2}S^ce^{-i\Phi}. \tag{7.8}$$

The subscript 0 indicates the neglect of the exclusion of the low L group and corresponds to the employment of the unsymmetrized quantity

$$\alpha_c^{un} = kS^ce^{-i\Phi} = -(\eta/2\mathbf{s}^2)\exp(-i\eta\ln\mathbf{s}^2), \tag{7.9}$$

the superscript *un* standing for "unsymmetrized." The exclusion of the low L group requires the application of the correction factor

$$f^{un} = 1 - 2\mathbf{s}^2\sum_1^{Lm}\frac{2L+1}{L(L+1)}P'_L(\mu), \tag{7.10}$$

according to

$$\Delta\alpha_i = f^{un}(\Delta\alpha_i)_0 \quad (i = 1,4). \tag{7.11}$$

For identical particles,

$$\Delta^a\alpha_i = f^a(\Delta\alpha_i)_0, \tag{7.12}$$

$$f^a = 1 - \frac{4}{\mathbf{s}^{-2}-\mathbf{c}^{-2}}\left(\sum_1^{Lm}\right)_{\text{odd }L}\frac{2L+1}{L(L+1)}P'_L(\mu). \tag{7.13}$$

The superscript a refers to "antisymmetrized." The convention in employing the symbol Δ^a is such as to make the quantity directly additive to one containing odd L alone. In this respect it is similar to that followed for α_c of (6.45). The superscript a is thus used in Δ^a in a slightly different way from that in α_i^a of (6.35), the intention being to use the Δ^a quantities directly in combination with Eqs. (6.4) to (6.8) employing sums over odd L only. Explicitly

$$\Delta^a\alpha(\theta) = \tfrac{1}{2}[\Delta\alpha_1(\theta) - \Delta\alpha_1(\pi-\theta)]. \tag{7.14}$$

By means of (7.12), (7.13) the nonrelativistic spin-orbit interaction contributions to S can be calculated for p-p scattering. The effect on the polarization P can be obtained from

$$\Delta\{k^2(P\sigma)\}_{p\text{-}p} = 2\,\text{Im}\{-(\alpha_2+\alpha_c^T)+\alpha_3\sin^2\theta-(\alpha_5+\alpha_c^T)\}\times\Delta^\alpha(\alpha_1\sin\theta) \tag{7.15}$$

combined with the values of $\Delta\sigma/\sigma$. The effect on P, when significant, is often more important than that on σ. In these cases (7.15) gives ΔP directly.

For n-p scattering the spin-orbit part of the Hamiltonian is

$$H'_{n\text{-}p} = -4\mu_n(\mu_0^2/r^3)(\mathbf{L}\cdot\boldsymbol{\delta}_n). \tag{7.16}$$

The part of S, which includes now singlets as well as triplets, arising from this perturbation is

$$S' = 4i\mu_n\mu_0^2\frac{e^{ikr}}{Mv^2\mathbf{s}^2}([\mathbf{k}_i\times\mathbf{k}_f]\cdot\boldsymbol{\delta}_n). \tag{7.17}$$

Resolving this quantity into parts symmetric and antisymmetric in the spins, the part of S' responsible for triplet scattering is singled out. For it the contributions to α_4 and $-\alpha_1$ are

$$(\beta)_0 \equiv (\Delta^{TT}\alpha_{4,n-p}) = -(\Delta^{TT}\alpha_{1,n-p})_0 = 4\mu_n\mu_0^2\frac{k^3}{Mv^2\mathbf{s}^2}. \tag{7.18}$$

Introducing

$$\beta = f^{un}(\beta)_0 \tag{7.19}$$

and

$$\alpha_0 = \textstyle\sum_L(2L+1)P_LQ(K_L), \tag{7.20}$$

it is found that

$$\Delta(k^2P\sigma)_n = (\beta/2)\,\text{Im}\,(2\alpha_2+\alpha_5+\alpha_0)\sin\theta, \tag{7.21}$$

$$\Delta(k^2P\sigma)_p = (\beta/2)\,\text{Im}\,(-2\alpha_3\sin^2\theta+\alpha_5-\alpha_0)\sin\theta. \tag{7.22}$$

The subscripts n, p indicate that the quantities are evaluated for neutrons and protons, respectively. The angle θ is the scattering angle of either particle in the center-of-mass system. Although $(\beta)_0$ is introduced through the triplet part of S', the calculation leading to (7.21), (7.22) makes use of the part of S' off-diagonal in singlets and triplets, namely,

$$S'^{(ST)} = \frac{(\beta)_0\sin\theta}{2^{1/2}k}e^{ikr}\begin{pmatrix}0&1&0&1\\-1&0&0&0\\0&0&0&0\\-1&0&0&0\end{pmatrix}, \tag{7.23}$$

where rows and columns are labeled in the order of magnetic quantum numbers, 0_0, 1, 0, -1 from top to bottom and from left to right, respectively. The 0_0 refers to the singlet states. The spin functions used for (7.23) are

$$\chi_0^0 = (\alpha_n\beta_p-\alpha_p\beta_n)/2^{1/2}, \quad \chi_1 = \alpha_n\alpha_p,$$
$$\chi_0 = (\alpha_n\beta_p+\alpha_p\beta_n)/2^{1/2}, \quad \chi_{-1} = \beta_n\beta_p, \tag{7.24}$$

where subscripts n, p refer to neutrons and protons, respectively. In n-p scattering the spin-orbit interaction couples states with the same J and parity but different isotopic spin T. In the main analysis of n-p data performed so far, it is assumed that the T = 0 and T = 1 states are uncoupled. This assumption is probably justified by the relative smallness of

the coupling caused by the magnetic spin-orbit interaction as well as by the relatively smaller accuracy of n-p scattering data as compared with that for p-p. While the general possibility that lack of charge independence may affect the requirements of n-p data analysis and increase the minimum number of independent experiments[57] needed for ascertaining the Wolfenstein parameters is well recognized, it is, nevertheless, instructive to note that lack of charge independence in the case of the spin-orbit effect caused by nucleon magnetic moments has among other consequences the coupling of singlets to triplets for the same J and L, a phenomenon familiar in the theory of atomic spectra.[50] As a consequence the $^3P_1$ phase shift obtained for $\mathrm{T} = 1$ from p-p data analysis is not, strictly speaking, applicable to n-p data analyses since for the n-p system there is a displacement of the $^3P_1$ phase shift arising from its coupling to $^1P_1$. These effects would presumably be most noticeable whenever the two phase shifts are close to each other. In the case of magnetic moment effects corrections for them can be applied by means of (7.23). The possibility of similar breakdowns of charge independence arising from mesonic effects remains, however. These may be stronger than the magnetic moment effects and may remain an unknown source of error in the n-p data analysis as long as it depends on the insertion of $\mathrm{T} = 1$ phase parameters from fits to p-p scattering data rather than on a self contained employment of n-p data.

There are of course other types of expected violations of charge independence which also make the employment of the $\mathrm{T} = 1$ parameters from p-p data not strictly applicable to the n-p case. Thus, for the OPE a phase shift

$$\delta(m_{\pi 0}), \mathrm{T} = 1, (p - p) \tag{7.25}$$

is not equal to[54]

$$2\delta(m_{\pi +}) - \delta(m_{\pi 0}), \mathrm{T} = 1, (n - p) \tag{7.26}$$

while the $PSps$ theory gives the expected phase shift for n-p scattering as in (7.26) in terms of the same functional symbol $\delta(m_\pi)$ as (7.25) gives it for the corresponding state of the p-p system. Strictly speaking, this and similar effects should also be taken into account for the phase parameters in the low L group (the "searched group" for p-p). Similarly, an assumption of a difference in the pion-nucleon coupling constants for charged and neutral pions on the $PSps$ theory violates charge independence, but, as in the case just considered, does not couple the $J = L$ states of different T to each other. As long, however, as the effect of the proton charge on the charged pions introduces a lack of symmetry with respect to the neutron and the nature of pionic contributions to the spin-orbit interactions has not been fully analyzed it is hard to be sure that there do not exist additional causes for the coupling of $J = L$ states with different T to each other. The changes in experimental quantities other than $P\sigma$ are implicitly contained in (7.17) supplemented by (7.19) for the n-p case and (7.12), (7.13), (7.14) in p-p scattering. For example, the changes in the cross sections are obtainable from

$$\begin{aligned}(\Delta\sigma/\sigma)_{n-p} = {} & [2\beta \sin^2\theta \,\mathrm{Re}\,(\alpha_4 - \alpha_1)]/[|\alpha_0|^2 \\ & + |\alpha_1|^2 \sin^2\theta + 2|\alpha_2|^2 + |\alpha_4|^2 \sin^2\theta \\ & + 2|\alpha_3|^2 \sin^4\theta + |\alpha_5|^2], \end{aligned} \tag{7.27}$$

$$\begin{aligned}(\Delta\sigma/\sigma)_{p-p} = {} & [2(\Delta^a\alpha_4) \sin^2\theta \,\mathrm{Re}\,(\alpha_4 - \alpha_1)]/[|\alpha_0 \\ & + \alpha_c^S|^2 + 2|\alpha_2 + \alpha_c^T|^2 + 2|\alpha_3|^2 \sin^4\theta \\ & + |\alpha_5 + \alpha_c^T|^2 + (|\alpha_1|^2 + |\alpha_4|^2) \sin^2\theta]. \end{aligned} \tag{7.28}$$

These formulas show that the cross section is insensitive to the magnetic moment spin-orbit interaction effect. Similarly, the polarization correlation coefficient $C_{nn} = \langle \sigma_{py}\sigma_{ny} \rangle$ is insensitive to $S'^{(ST)}$ of (7.23), mentioned more explicitly in reference 43.

The spin-orbit interaction calculations described above are nonrelativistic. The phase-shift method described in Sec. 5 is applicable to the relativistic calculation giving effects of the Dirac part of the nucleon magnetic moment. Such calculations have been carried out by this method[39] and the results are in agreement with those of Garren[41] who used Feynman diagrams. On account of the divergence in the treatment of Coulomb scattering which is inherent in the Feynman diagram method, the phase-shift method appears to be the more logical. In the calculations as carried out by Breit[39] or Ebel and Hull,[40] there is no certainty, however, that η_r is the right quantity to use in exp $(-i\eta \ln \mathbf{s}^2)$, the whole calculation being formally of relative order e^2. Other evidence indicates, however, that for small scattering angles which are the more important cases in the applications, η_r is more nearly correct than η_{nr}. A more thorough treatment of these questions has not appeared worthwhile because of the relative smallness of the spin-orbit and spin-spin effects. For the same reason the application of relativistic correction factors to the removal of the low L group from the magnetic spin-orbit effect has not been carried out exactly in reference 43. Instead, the relativistic correction factor applicable for the undistorted wave function effect has been used. An appreciable frac-

tional error is perhaps introduced by this approximation in those cases for which the removal of the low L group is relatively most important, i.e., for the larger scattering angles. In these cases, however, the whole magnetic moment spin-orbit effect is not large and from a practical viewpoint the approximation appears justifiable.

So far, effects of the anomalous part (non-Diracian) of the nucleon magnetic moments have not been explicitly treated by the phase-shift method. The close similarity of the expressions entering the calculations to those for the Diracian parts of the moments and the comparison with calculations in the absence of the Coulomb interaction may be used, however, as a justification of the Feynman diagram procedure for this case. There is little doubt regarding the phase-shift method giving the same result in the present problem. The unsymmetrized relativistic correction factor applicable to (7.10), when used to obtain (7.13) according to (7.14) gives on this basis

$$(f_r^{un})_{p\text{-}p} = \{2/[E_1(E_1+1)]\}\{2E_1+1+2E_1 \times (E_1+1)\mu_a - \mathbf{s}^2[E_1-1+2(E_1^2-1)\mu_a]\} \times (3+4\mu_a)^{-1} \quad (7.29)$$

for p-p scattering. Here E_1 is the energy of one of the protons in the center-of-mass system in Mc^2 units. For n-p scattering the relativistic correction factor is

$$(f_r)_{n\text{-}p} = 1 - [(E_1-1)/E_1]\mathbf{s}^2 , \quad (7.30)$$

which is obtainable from the p-p case by keeping the μ_a terms only.

The inclusion of nucleon magnetic moment effects reported on here presupposes that the electric currents responsible for the magnetic moments are concentrated in a volume of negligible dimensions. This simplifying assumption is obviously not strictly correct. The relativistic corrections should be regarded therefore as somewhat speculative and uncertain, the selection of possible covariant forms for the interaction being based on the point hypothesis. In a more complete theory the electromagnetic structure of nucleons will play a role and it is not known that its effects are smaller than the relativistic corrections. For distant collisions leading to small angle scattering the gradients of electric and magnetic effects produce smaller effects within the dimensions of the nucleon as compared with the values of the fields and the application of the results is more justifiable.

The nonrelativistic addition to the Hamiltonian arising from the magnetic spin-spin interaction is for the p-p case

$$H_{SS} = -(1+\mu_a)^2\mu_0^2 S_{12}/r^3 , \quad (7.31)$$

where

$$S_{12} = 3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})/r^2 - (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) . \quad (7.32)$$

The change in S^c caused by this addition is[43]

$$\Delta_{SS}S^c = \tfrac{1}{3}(1+\mu_a)^2\mu_0^2 q^2 S_{12}(\mathbf{q}) , \quad (7.33)$$

where

$$\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f . \quad (7.34)$$

The exclusion of the low L group can be carried out in this case also. The somewhat lengthy formulas may be found in reference 43. In the cases for which numerical applications have been examined, the spin-orbit effects are appreciably more important than those of the spin-spin interactions.

In Table I are shown values of the unsymmetrized correction factor of Eq. (7.10) for p-p scattering.

TABLE I. Values of f^{un}, the unsymmetrized correction factor for wave function distortion.

$\theta =$ L_m	5°	10°	20°	30°	150°	160°	170°	175°
1	0.99	0.98	0.91	0.80	−1.80	−1.91	−1.98	−1.99
2	0.98	0.94	0.77	0.51	2.24	2.65	2.91	2.98
3	0.97	0.89	0.59	0.19	−2.25	−3.15	−3.78	−3.94
4	0.95	0.82	0.38	−0.11	1.84	3.37	4.56	4.89
5	0.93	0.75	0.18	−0.32	−1.12	−3.31	−5.23	−5.80

Since in the phenomenological searches the OPEP group often starts at $L = 4$ or more, the correction factor is not negligible, although its importance decreases at very small angles. Figure 1 shows a comparison of fit YLAM with proton-proton polarization data at 147-MeV bombarding energy, together with the result of applying the correction for the magnetic moment spin-orbit effect. The value of L_m used was 3. Agreement with experiment[62] is seen to be improved[63] by employing the correction. This improvement is marked in the angular range 8° to 20° but at 6.2° and around 4° the agreement is worsened. Measurements at very small angles are difficult however. The figure shows that the omission of low L does not have a pronounced effect in the angular region in which improvement with experiment is most marked. On

[62] J. N. Palmieri, A. M. Cormack, N. F. Ramsey, and R. Wilson, Ann. Phys. (N. Y.) **5**, 299 (1958). In cases of two values for the same angle they were averaged.

[63] The first indication of improvement in the fit due to applying the magnetic moment spin-orbit interaction correction, which however did not include a consideration of the effect of the low L group or of some effects of the incident energy, was contained in a letter to the writer from Dr. E. H. Thorndike, then at Harvard. A more complete account may be found in footnote 14 of reference 43.

the other hand, it becomes appreciable at only slightly higher angles and its omission would be misleading in this angular region.

The relativistic correction factor for the unsymmetrized spin-orbit interaction effect has values 0.99 at $\theta = 5°$ to 20° and 0.98 at $\theta = 30°$ for p-p scattering

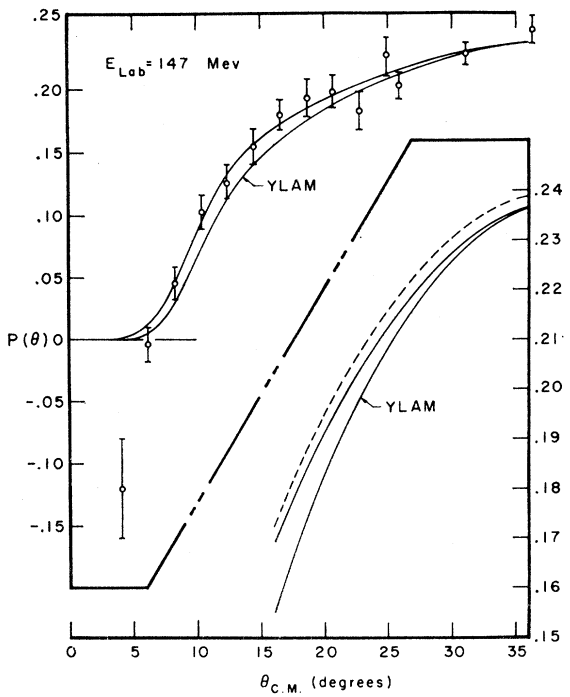

FIG. 1. Comparison of magnetic moment spin-orbit effect with experiment for p-p scattering at bombarding energy of 147 MeV. Original YLAM fit corresponds to lower graph. Inclusion of the magnetic moment spin-orbit effect gives the upper curve. The inset in the lower part of the figure shows on a magnified scale the curves just mentioned and a third dashed plot obtained if effect of the low L group is not subtracted. Inset shows that subtraction leaves an effect which is pronounced at small scattering angles only (distant collisions).

at 210-MeV bombarding energy and is 0.95 for $\pi - \theta$ and the same values of θ as above. The spin-spin effects computed for fit YLAM to p-p scattering at 147-MeV bombarding energy for scattering parameters P, R, A and D at $\theta = 30°$ are, respectively, 0.00016, −0.00057, −0.00009, 0.00052; for $\theta = 20°$ they are 0.00014, −0.00068, 0.00000, 0.00044. At $\theta = 10°$ and 5° they have even smaller values.

No definite indications of over-all improvement in p-p and n-p fits as a result of applying magnetic moment spin-orbit corrections have been found so far and no striking indications of improvement in n-p fits in special cases have been noticed. It appears, however, that extra care in securing high accuracy in low angle polarization measurements might be worthwhile because confirmation of the theoretically expected effect should add to or subtract from confidence in the phase parameters of mesonic origin and should thus be useful in their determination.

8. VIEWS REGARDING THE INTERACTION

a. Introductory Remarks

The main part of the interaction responsible for nucleon-nucleon scattering is understood much less satisfactorily than that between electrically charged particles. Even though quantum electrodynamics is not in a truly satisfactory state, its agreement with experiment in the nonrelativistic domain of the Bohr-Sommerfeld-Schrödinger-Dirac theory of the hydrogen atom extended by the more modern developments along the lines of Schwinger-Tomonaga-Feynman-Dyson is most striking. The strong guides arising from the excellence of agreement between theory and experiment are absent, however, in the nucleon-nucleon problem. Nevertheless, there is no doubt regarding the participation of the pion field in the interaction. But theoretical quantitative treatment of the interaction problem has been only partially successful. As is well known, the mathematical treatment is more difficult than in electromagnetism, at least partly because of the large value of the pion-nucleon coupling constant which interferes with the application of perturbation theory.

Traditionally physicists tend to describe interactions by means of potentials. This habit is rooted in the successful application of the concept in many fields of classical physics as well as in the nonrelativistic part of atomic quantum theory. It is, of course, natural to try to apply the same method to the nucleon-nucleon problem. No way has apparently been found so far, however, of obtaining a self-consistent treatment which can pretend to give an account of the low-energy class of phenomena such as the binding energy of the deuteron and low-energy scattering simultaneously with the higher energy phenomena. At least part of the difficulty is in the rather large absolute values of the effective potential which are arrived at in the fourth order of the $PSps$ theory and in the lack of convergence of the successive orders of perturbation theory for the problem as a whole. The large absolute value of the effective potential, if it is attractive, makes a nonrelativistic approach meaningless and makes it necessary to consider the effects of nucleon recoil caused by pion emission as well. There is, in other words, no *a posteriori* consistency in a nonrelativistic and nonrecoil meson theoretical calculation employing the

PSps theory and the same applies to practically all calculations that have been attempted on the problem.[64,65]

b. Meaning of the Word "Potential"

The word "potential" is used in a variety of ways. In some cases it is used in a sense closely related to that employed in classical mechanics. Such a potential is meant to be inserted in a Schrödinger wave equation which contains only the nucleon coordinates. This kind of potential has to be derived by elimination of the pion field from the original equations or by some equivalent procedure. Such procedures imply that a solution of the problem exists. But calculations by the Tamm-Dancoff method give no assurance of the existence of a solution if all states of the pion-nucleon system are to be brought in before the elimination. Furthermore, if no approximations are made, the calculation taking into account only a finite number of pions gives a nonlocal operator for the potential. The large value of the pion-nucleon constant gives any result obtained with a finite number of pions in intermediate states a questionable meaning. In view of the nonlocal character of the interaction, there is some question regarding the meaning of the subdivision of the range of distances of r into regions characterized by the applicability of potentials corresponding to one pion, two pion, and many pion exchanges, a division which has been used in much of the literature. Such a meaning could be a definite one had the potential been local.

The question of the proper normalization of the wave function is another troublesome point. The normalization integral to which the requirement of quadratic integrability should be applied is the sum of the normalization integrals over subspaces corresponding to different numbers of pions. The quadratic integrability of the Schrödinger equation with a potential cannot be assumed to imply similar integrability for the whole Fock space and a direct application of the Schrödinger equation in the usual manner is therefore questionable. Caution regarding this matter appears to be in order, especially because of the known divergence of the theory.

If the nonlocal potential is to be used in the above-mentioned sense and if it should in some way prove possible to justify the questionable points just mentioned, the potential would be expected to be directly applicable to phase-shift calculations because in the region of configuration space corresponding to large values of the internucleon distance r the removal of the difficulties would presumably assure the dominance of only one function in Fock space and this function would then be the proper ψ to use in the considerations of Sec. 2. But the function would not be directly useful for the calculation of interactions with other fields such as needed in special problems such as, e.g., the problem of the photodisintegration of the deuteron. The actual interaction with an external electromagnetic field involves, in fact, the whole wave function rather than one component of it in Fock space. On account of the complexity of these matters, as well as the inherent difficulty of treating them properly, there has been a tendency to disregard the distinctions which have just been emphasized. Whether this optimism will eventually be justified is difficult to predict.

In order to see more concretely what might be involved in the elimination of the field and what relaship the resultant Schrödinger equation has to the actual problem, a calculation has been carried out for a much simpler problem,[66] in the hope that it will indicate at least some of the characteristic features of the actual relationship. The quantized pion field was replaced by an unquantized one, some nonrelativistic approximations were made and some important effects of the nucleon-nucleon vacuum were omitted. The Hamiltonian had the usual form

$$H = H_a + H_b + \tfrac{1}{2}\int \sum_\alpha [(\nabla\varphi_\alpha \cdot \nabla\varphi_\alpha) + (\partial\varphi_\alpha/c\partial t)^2 + \mu^2\varphi_\alpha^2]d\mathbf{r} + (4\pi)^{1/2} g \times \sum_\alpha [\tau_\alpha^a \rho_2^a \varphi_\alpha(\mathbf{r}^a) + \tau_\alpha^b \rho_2^b \varphi_\alpha(\mathbf{r}^b)] . \quad (8.1)$$

Here $\alpha = 1, 2, 3$ is used to distinguish between the components of the isotopic spin in the space of which $\boldsymbol{\tau}^a$ and $\boldsymbol{\tau}^b$ are vectors; $\mu = m_\pi c/\hbar$ where m_π is the pion mass; ρ_2 is one of the three matrices ρ introduced by by Dirac, ρ_1 entering $\boldsymbol{\alpha} = \rho_1\boldsymbol{\delta}$, while $\beta = \rho_3$ and $\rho_3\rho_1 = i\rho_2$; H_a and H_b are Dirac-type single-particle Hamiltonians for particles a and b. From (8.1) there follows the usual equation for the D'Alembertian of the meson-field function φ_α. On replacing the D'Alembertian by the Laplacian the φ_α can be solved for. On substitution into (8.1) and removal of self-energy terms there is obtained a new Hamiltonian

$$H = -c(\boldsymbol{\alpha}^a \cdot \mathbf{p}^a) - \beta^a Mc^2 - c(\boldsymbol{\alpha}^b \cdot \mathbf{p}^b) - \beta^b Mc^2 - g^2(\boldsymbol{\tau}^a \cdot \boldsymbol{\tau}^b)\rho_2^a \rho_2^b e^{-\mu r}/r . \quad (8.2)$$

[64] L. Hulthén and M. Sugawara, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. **39**, Part 1, and references therein.

[65] S. S. Schweber, H. A. Bethe, and F. de Hoffman, *Mesons and Fields, Vol. I, Fields* (Row, Peterson & Company, Evanston, Illinois, 1957), and references therein.

[66] G. Breit, Phys. Rev. **111**, 652 (1958).

For singlets this equation simplifies to

$$\left\{\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + \frac{M}{\hbar^2}(E - 2Mc^2 - {}^1V_{\text{eff}})\right\}^1 \times u^+(r) = 0\,, \tag{8.3}$$

where

$$x = \mu r, \mu = m_\pi c/\hbar\,, \tag{8.4}$$

$$\Psi + \Phi = {}^1R_L^+(r)Y_{L,M_L}(\theta,\varphi)\,, \tag{8.5}$$

$${}^1R_L^+(r) = {}^1u^+(r)(E+F)^{1/2}/r\,, \tag{8.6}$$

$$F = (g^2/r)(\boldsymbol{\tau}^a\cdot\boldsymbol{\tau}^b)e^{-\mu r}\,, \tag{8.7}$$

and the effective potential is

$${}^1V_{\text{eff}} = -\frac{(E-2Mc^2)^2}{4Mc^2} + \frac{F^2}{4Mc^2} + \frac{m_\pi}{M}(m_\pi c^2) \times\left[\frac{3}{4}\left(1+\frac{1}{x}\right)\left(\frac{F}{E+F}\right)^2 - \frac{F}{2(E+F)}\right]. \tag{8.8}$$

Equation (8.3) is of the standard form often used for the product of r and the radial function. According to (8.6) the reduction to such a standard form involves a transformation from the original radial function ${}^1R_L^+$ to u_1^+. The relationship between ${}^1R_L^+$ and u_1^+ according to (8.7) is quite different at large and small r, indicating that the relationship between the function entering an equation with an effective potential and the original equation has to be treated carefully since an additional power of $1/r$ enters the normalization integral at small r.

In Eq. (8.8) the first term is the relativistic correction to the kinetic energy. The second term gives a repulsive potential which becomes large at small distances. It represents a soft core in the same sense as an infinite repulsive potential is referred to as a hard core. The last term shows a slight "velocity dependence" through the entrance of E. The shorter range part of this term is repulsive and qualitatively similar to $F^2/(4Mc^2)$. The longer range part of the last term is attractive for isotopic triplets (T = 1) and repulsive for isotopic singlets (T = 0). Since nucleon pair formation has been neglected, ${}^1V_{\text{eff}}$ cannot be expected to represent the whole effective potential but it may perhaps be noteworthy that a repulsive core arises essentially as a result of the distinction between ${}^1R_L^+$ and ${}^1u^+$.

For triplets the transformation corresponds to (8.6) is

$$\Psi + \Phi = Y_{L,M_L}(\theta,\varphi)\,{}^3u^+(r)[(E+F)/(E-F)]^{1/4} \times (E^2-F^2)^{-(1+S_{12})/12}\,. \tag{8.9}$$

The forms corresponding to (8.3) and (8.8) are lengthy and will not be reproduced here. One of the terms entering the equation on ${}^3u^+$ has the structure of a spin-orbit interaction

$$V_{LS} = \frac{\hbar^2(F/r)dF/dr}{M(E^2-F^2)}(\mathbf{L}\cdot\mathbf{S}) \backsim \frac{\hbar^2}{2M^2c^2}\frac{d(F^2/4Mc^2)}{rdr}(\mathbf{L}\cdot\mathbf{S})\,. \tag{8.10}$$

Although reminiscent of the Thomas term, it differs from it in sign at large r and may be referred to as an *anti-Thomas* term. The difference in the sign is caused by the pseudoscalar rather than scalar character of the field and coupling. For large r this term has the sign suggested by the shell theory of nuclear structure[67] and by the phenomenology of proton-proton scattering which is concerned with triplet odd terms only. It will be noted that the formula for V_{LS} gives a sign reversal when $F = E$ and the sign of the resultant effect depends therefore on the value of r at which $F = E$ is satisfied. Estimates made in reference 66 on the basis of the Gartenhaus potential[68] give a value of V_{LS} approximately $\frac{1}{4}$ of that obtained by Signell and Marshak[69] by fitting p-p scattering data. This disagreement may be partly due to the difficulty of identifying the $F^2/4\ Mc^2$ uniquely by comparison with the singlet-even potential. Another estimate employing the Lévy potential[70] in the Blatt-Kalos[71] modification gave somewhat more than needed for the phenomenological value. These comparisons with experiment are very qualitative and have never been meant in any other sense.[72] In the estimate just quoted the repulsive-core part of the potential was inferred from a graph of the Gartenhaus potential in one case and a fit to low energy data by Blatt and Kalos, which does not appear as relevant now, in the other. If one calculates more directly from (8.7) employing the approximate value of 14 for $g_0^2 = g^2/\hbar c$, the coefficient 17 in Eq. (12) of reference 66 for $F^2/4\ Mc^2$ becomes replaced by 101 and the calculated V_{LS} is about 0.60 of the empirical value of Signell and Marshak. Considering the uncertainties in the empirical V_{LS} caused by flexibility of other features of the potential, it is

[67] M.G. Mayer, Phys. Rev. **75**, 1969 (1949); **78**, 16 (1950); O. Haxel, J. H. D. Jensen, and H. E. Suess, *ibid.* **75**, 1766 (1949); Z. Physik **128**, 295 (1950).

[68] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).

[69] P. S. Signell and R. Marshak, Phys. Rev. **106**, 832 (1957); **109**, 1229 (1958).

[70] M. Lévy, Phys. Rev. **88**, 725 (1952).

[71] J. M. Blatt and M. H. Kalos, Phys. Rev. **92**, 1563 (1953).

[72] The intention of the writer has apparently been interpreted in the opposite sense by G. F. Chew, *Proceedings of the* 1958 *International Conference on High Energy Physics at CERN* (CERN, Scientific Information Service, Geneva, Switzerland), cf. especially p. 104.

difficult to exclude the possibility that the fit is fortuitous.

For large r, the spin-orbit interaction arising as the anti-Thomas term of $F^2/4\,Mc^2$ is 9 times larger for T = 0 than for T = 1. At the time of the publication[66] the phenomenological evidence regarding V_{LS} in T = 0 states was very incomplete but there were some indications in the work of Gammel and Thaler[73] that for T = 0 the V_{LS} is about 0.8 of that for T = 1. This appeared not to fit the factor 9 arising as the ratio of $(\boldsymbol{\tau}^a\cdot\boldsymbol{\tau}^b)^2$ in the two cases. On the other hand, the reversal of sign of V_{LS} expected at $F = E$ according to (8.10) was expected to occur at larger r for T = 0 than for T = 1 and hence no decision for or against Eq. (8.10) as the sole cause of V_{LS} was reached. According to present day phenomenology it is more probable that if V_{LS} is to be used as having the same sign for all r, then for T = 0 a sign opposite to that for T = 1 should be used. The possibility that (8.10) is directly related to major effects in the actual V_{LS}, if such exist, has not been excluded but it appears less probable now than previously.

The same calculations[66] have shown the presence of interaction terms of the type

$$T^p_{ab} = 6(\mathbf{S}\cdot\mathbf{p})^2 - 2\mathbf{S}^2\mathbf{p}^2\,, \tag{8.11}$$

$$\mathbf{S} = \tfrac{1}{2}\,(\boldsymbol{\sigma}^a + \boldsymbol{\sigma}^b)\,, \tag{8.12}$$

which has a structure similar to that of the tensor operator but with $\mathbf{p}$ replacing $\mathbf{r}$. An equivalent type of energy dependence was proposed by Marshak and Okubo[74] on the basis of invariance requirements (along the lines of Eisenbud and Wigner[75]). They showed that on the energy shell it can be replaced by the use of other potentials provided among these a related velocity dependent term involving $(\boldsymbol{\sigma}_1\cdot\mathbf{L})(\boldsymbol{\sigma}_2\cdot\mathbf{L})$ which has been proposed on the basis of invariance considerations by Puzikov, Ryndin, and Smorodinsky[76] is included.

As has been mentioned in reference 66 the very imperfect and only schematic model of a calculation eliminating the meson field shows among other nonstatic features that: (a) One may expect that the usual insertion of the electrostatic energy term in the radial equation involves an inaccuracy on account of the transformations (8.6) and (8.9) which have to be performed in order to obtain the standard form of the radial equation; this inaccuracy may be expected to affect the comparison of p-p and n-p interactions and therefore also the tests of charge independence; (b) effects similar to those of a repulsive core can arise not only in the manner discussed in connection with (8.8) but also on account of the appearance of nodes in ${}^3u^+$ with positions which play a role similar to core radii; (c) velocity dependence of the magnitudes of effective potentials and of core radii.

In principle, the proper employment of the Bethe-Salpeter[77] equation or the equivalent two-particle Green's function of Schwinger[78] should yield a complete answer to the problem of determining the probability of relative location of the nucleons. The calculations as they are usually carried out have not furnished the answer however. The principal reason is the inherent difficulty of the problem which has no such easy good starting point as the electrostatic approximation in atomic physics. In addition, it may be remarked that the calculations as they are frequently carried out,[79] not only contain a number of approximations but end up with an equation on four "large" components in momentum space. Since the Bethe-Salpeter equation is used for a pair (hole) theory, the possibility of localizing the particles is lost, as it should be. The momentum space function cannot be used, therefore, in the same way for deriving the probability amplitudes in coordinate space as though the particles were free. A "potential" derived in this manner is, therefore, not obviously related to the actual position of a particle and speculations regarding the internucleon distance at which a particular approximation to the original equation becomes inapplicable are necessarily only qualitative. The claims for the existence of repulsive cores which have been supposedly derived from the $PSps$ theory do not appear to be very convincing. It would indeed be strange if the Bethe-Salpeter-Schwinger approach could be carried farther than the Tamm-Dancoff method in questions dealing with the validity of the approximations at one or another distance because the troublesome questions of relative probabilities of different components in Fock space which are present in the Tamm-Dancoff method cannot be done away with. On the other hand, if the interest is only in the final scattered state, then formally the

[73] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291, 1337 (1957).
[74] R. E. Marshak and S. Okubo, Bull. Am. Phys. Soc. **3**, 11 (1958); S. Okubo and R. E. Marshak, Ann. Phys. (N. Y.) **4**, 166 (1958).
[75] L. Eisenbud and E. P. Wigner, Proc. Natl. Acad. Sci. U.S. **27**, 281 (1941).
[76] L. Puzikov, R. Ryndin, and J. Smorodinsky, Nuclear Phys. **3**, 436 (1957).
[77] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[78] J. Schwinger, Proc. Natl. Acad. Sci. U.S. **37**, 452 (1951).
[79] M. Lévy, Phys. Rev. **88**, 72 (1952).

problem should be soluble. But, for such a program a complete solution has not been carried out on account of the lack of convergence of the perturbation series.

In view of these difficulties and the possible entrance of other interactions such as that with K mesons, there appear to be practical advantages in investigations of the interactions at a large distance between the nucleons. Questions of localization of nucleons are then less important and there is then some hope of seeing—through comparison with experiment—whether pion exchange may be held primarily responsible for nucleon-nucleon interactions.

This viewpoint appears to have been first expressed in print by Taketani,[80] who outlined a program for the exploration of the nucleon-nucleon interaction based on dividing the range of values of r into regions; the outer being the one concerned with the one-pion exchange potential (OPEP), the intermediate region for which the two-pion exchange also matters, and a third inner region within which higher order exchanges may be appreciable. They and their coworkers[80] have attempted to demonstrate that the procedure had promise. The degree to which it is applicable inside the OPEP region is still not clear. This question, as well as that of the applicability of the OPEP, will be returned to below; but at this point it appears desirable to point out a difference in emphasis regarding the employment of the nucleon-nucleon potential which is, in part, suggested by the introduction of the OPEP.

There are two main objects in nucleon-nucleon scattering investigations. One is to ascertain the physical phenomena, such as meson exchange, that are responsible for the scattering and to find out as quantitatively as possible the underlying laws. Another is to provide workers on nuclear reactions, binding energies and nuclear spectroscopy with basic and useful information regarding nucleon-nucleon interactions. The first object can be contributed to by the investigation of special phases of the problem such as properties of distant collisions with a gradual extension of the explored interaction region to smaller distances. The second object is not so easily divided into parts and its achievement may have to be postponed. It is probable that the most valuable feature of the three region program will turn out to be the stimulation furnished by it for an exploration of the nucleon-nucleon interaction from the larger toward the smaller r.

A partial wave analysis of the wave function furnishes an approximate classification of effects according to values of the classical impact parameter and makes it possible to separate parts of the scattering amplitude which are not concerned with close collisions. It is natural to expect that the part of the scattering amplitude concerned with high angular momenta corresponding to the distant collisions can be expanded in powers of g^2 with less difficulty than the whole scattering amplitude. For a comparison of theory with experiment employing this part of the scattering amplitudes, an exact knowledge of the remaining part of the amplitudes may not be needed and there is evidence that this is in fact the case. From this viewpoint, a straightforward calculation of the scattering amplitude in powers of g^2 is of interest, because the parts of it which correspond to distant collisions can be used in comparisons with measurements. The quantity of main interest is then Dyson's S matrix for the over-all effect of the collision. Calculations made by means of it can be put in terms of a quantity, having the dimensions of a potential, of a type first introduced by Schwinger[81] in connection with quantum electrodynamics. In connection with nucleon-nucleon scattering, the word "potential" is used in this sense by Gupta.[82] A distinction between this type of potential and the more usual kind has to be drawn. The Schwinger-Gupta potential has no direct relationship to the two-body Schrödinger equation of the kind familiar from the Tamm-Dancoff method. This matter is discussed in detail by the author.[47] In the papers of Schwinger and Gupta the potential is introduced by means of an operator called K through

$$S = \frac{1 - iK/2}{1 + iK/2} \quad , \tag{8.13}$$

in a manner similar to that used by Heitler.[83] It is shown in reference 47 that the same ultimate objective can be achieved going from S to the phase shifts and phase coupling parameters directly and in some respects more efficiently. It is also shown that the knowledge of S allows the determinaton of a third kind of potential which gives the complex scattering amplitudes in the first Born approximation in a manner similar to that in which the Gupta-Schwinger potential gives their K. By means of a wave packet

[80] M. Taketani, S. Nakamura, and M. Sasaki, Progr. Theoret. Phys. (Kyoto) **6**, 581 (1951); J. Iwadare, S. Otsuki, R. Tamagaki, and W. Watari, *ibid.* **16**, 455 (1956); Suppl. Progr. Theoret. Phys. (Kyoto) **3**, 32 (1956); S. Otsuki, Progr. Theoret. Phys. (Kyoto) **20**, 171 (1958); R. Tamagaki, *ibid.* **20**, 505 (1958).

[81] J. Schwinger, Phys. Rev. **74**, 1439 (1948).

[82] S. N. Gupta, Phys. Rev. **117**, 1146 (1960).

[83] W. Heitler, Proc. Cambridge Phil. Soc. **37**, 291 (1941).

argument, it may be seen that the customary consideration of the adiabatic switching in and out of the field need not enter the considerations.[47] The S matrix may thus be used for discussions of physical rather than bare nucleons.

9. THE OPEP AND THE OPE INTERACTION

If one believes that in some sense there exists a Schrödinger equation containing only the coordinates of the two nucleons, then it is natural to speak of the potential that results if the exchange of only one pion at a time is considered and to call this potential the one-pion exchange potential (OPEP). If, on the other hand, faith is put only in the expansion of the scattering amplitudes in powers of g^2, then it is natural to speak of a one-pion exchange (OPE). There is no practical difference between these viewpoints in this low approximation. The OPEP viewpoint furnishes, in fact, the first power of g^2 in the expansion of the amplitudes if it is used in the first Born approximation. Believers in the ultimate superiority of the dispersion relations approach often prefer the OPE designation since no implications regarding the existence or the practicability of using a potential is involved in it. The emphasis is then made on the singularities which are contained in the OPE contribution to the amplitudes. In some of the more important applications these contributions have to be broken up into parts corresponding to different partial waves and the clear meaning of the singularity is then lost partly because different partial waves are used in different applications and partly because in p-p scattering analysis the poles do not show themselves in a pronounced manner. The omission of the low L group involves some consideration of the localization of effects of different L, at least to the extent of estimating the classical impact parameter and a comparison with a distance for which one would suppose the two-pion effect to be important. The injection of estimates of distances that control the admission of one or another partial wave into a sum is a departure from the purist dispersion-theoreticians's viewpoint which is that of dealing only with the analytic properties of the amplitudes. It appears fair to say, therefore, that there is also very little essential difference in the theoretical viewpoints leading to distinctions between the OPEP and OPE designations.

Neglecting electromagnetic effects the classical mechanics picture is not wholly inapplicable at energies in the middle of the experimentally investigated range. For an incident energy of 150 MeV the wavelength of relative motion is $\simeq 4.7 \times 10^{-13}$ cm and $k = 1.33 \times 10^{13}$ cm^{-1}. The classical turning point is then

$$r_{tp} = 0.75[L(L+1)]^{1/2} \times 10^{-13} \text{ cm}. \quad (9.1)$$

Working with Gauss error field free wave packets the relation between the wave packet spread $\Delta^t x$ at time t which was formed at time t with spread Δx is

$$(\Delta^t x)^2 = (\Delta x)^2 + \frac{\hbar^2 t^2}{(M/2)(\Delta x)^2}, \quad (9.2)$$

where $M/2$ is the reduced mass and the wave packet is in the space of relative coordinates. The relation of the Δx to the mean square deviations is

$$(\Delta^t x)^2 = 2\langle (x - \bar{x}_t)^2 \rangle, \quad (\Delta x)^2 = 2\langle (x - \bar{x})^2 \rangle. \quad (9.3)$$

Adjusting Δx to make $\Delta^t x$ a minimum at time $t = T$ and letting $\bar{x}_t = 0$,

$$\langle (x^2)_T \rangle = 2\langle x^2 \rangle = D\Omega/(2\pi) = D/k, \quad (9.4)$$

where D is the distance traveled during T. Since the potential contains $e^{-\mu r}$ as a factor, the force will decrease by about a factor $1/e$ if

$$(D^2 + r_{tp}^2)^{1/2} - D = 1/\mu. \quad (9.5)$$

Solving for D, substituting in (9.4), and making use of (9.1)

$$\frac{\langle (x^2)_T \rangle}{r_{tp}^2} = \frac{\{(k/\mu)^2 + 2[L(L+1)]^{1/2} k/\mu\}^{1/2}}{L(L+1)}. \quad (9.6)$$

For $E = 150$ MeV, $L = 3$ this gives $\langle (x^2)_T \rangle^{1/2}/r_{tp} = 0.56$ and for $L = 4$, $\langle (x^2)_T \rangle^{1/2} r_{tp} = 0.47$. According to (9.4) the corresponding ratios at $t = 0$ are 0.707 times these numbers. The wave packet thus remains definitely smaller than the distance r_{tp} during the important part of the collision and a classical picture of the collision has an approximate meaning. An association of L with the impact parameter r_{tp} may be expected, therefore, to have a meaning also. The wave packet in the above estimates cannot be made really small without making L large and it is, therefore, necessary to consider the problem of localization of effects quantum mechanically. This has been done by Breit and Hull[46] by employing the first-order perturbation formula for the calculation of phase parameters. The usual form of the OPEP

$$V^{(2)} = \frac{1}{3} g^2 \left(\frac{h\mu}{2Mc}\right)^2 (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \times \left[(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + S_{12}\left(1 + \frac{3}{x} + \frac{3}{x^2}\right)\right] \frac{e^{-\mu r}}{r} \quad (9.7)$$

was used. Here S_{12} is the usual tensor force operator while

$$\mu = m_\pi c/\hbar, \quad x = \mu r. \quad (9.7')$$

In Fig. 2 are shown values of the integrand of the first-order phase shift formula plotted against x for $L = 5$ and incident energy of 274 MeV. The lower curve is for the integrand of the term proportional to $(\mathfrak{d}_1 \cdot \mathfrak{d}_2)$ in (9.7). The upper curve has reference to the coefficient of S_{12}. The positions of the classical

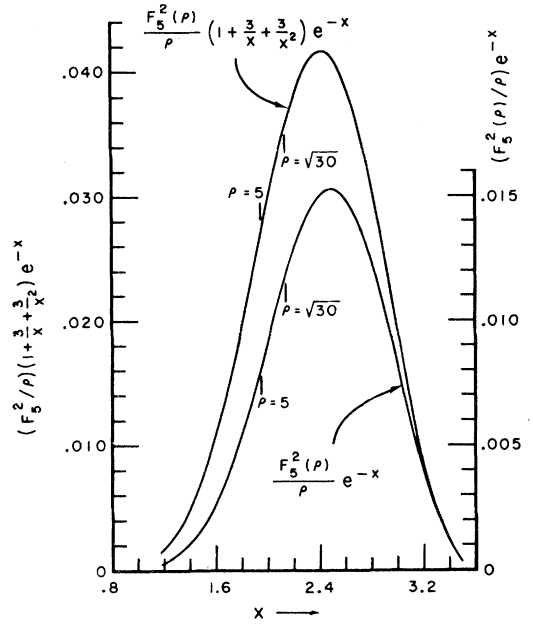

FIG. 2. Integrands of first-order formula giving phase parameters for $L = 5$. Upper curve is concerned with coefficient of S_{12}, the tensor operator. Lower curve gives integrand of coefficient of spin-spin part of interaction. The two integrands are seen to be localized in about the same region of x. Classical turning points in different conventions are marked by vertical dashes.

turning point are marked by $\rho = 5$ on a completely literal interpretation and $\rho = (30)^{1/2}$ employing the quantum mechanical interpretation of the centrifugal barrier. It is seen that the effect is reasonably well localized for this rather high L. Figure 3 shows a comparison of the values of F_L^2 for $L = 5$ with those for $L = 1$. In order to display the effects at the smaller ρ the graphs are drawn for $F_L^2(\rho)/\rho$. There are two scales for ρ for the two L. These have been adjusted such that $\rho = \rho_L = [L(L+1)]^{1/2}$ falls in the same position for the two L. The ordinate scales have been adjusted so as to make the curves intersect at approximately ρ_L. The sensitivity to the insertion of a potential is seen to fall off more rapidly as ρ decreases from ρ_L towards smaller values. This means that the assumption of localization of effects of the potential is a much better one for $L = 5$ than for $L = 1$, provided the values of ρ_L correspond for the two cases. The correspondence can be obtained approximately by starting with an E for which the OPEP is known to give reasonably good values and then lowering K_L and E simultaneously. This graph indicates that as L and E are lowered, the classical turning point can be made to remain the same but that for low L the definiteness of the low distance cutoff is decreased. There have been some experimental indications that the low E, low L values do not fit in with the OPEP as well as the high E, high L. On this basis, judging by the empirically indicated inapplicability of the OPEP for $L = 1$ at about 20 MeV, it appears probable[46] that for $r < 2.9 \times 10^{-13}$ cm there are interactions stronger than the OPEP. From the empirical applicability of the OPEP for $L \geqq 5$ in the majority of phase parameter searches it appeared that the OPEP is the main effect for $r > 2.7(5) \times 10^{-13}$ cm in one estimate and $r > 3.0(2) \times 10^{-13}$ cm in another. On the same basis, but employing $L \geqq 3$ at 275 MeV, the dominance of OPEP for $r > 1.7 \times 10^{-13}$ cm appeared probable. The diffuseness of the $L = 1$ sensitivity curve makes the localization of this interaction region difficult. Employing evi-

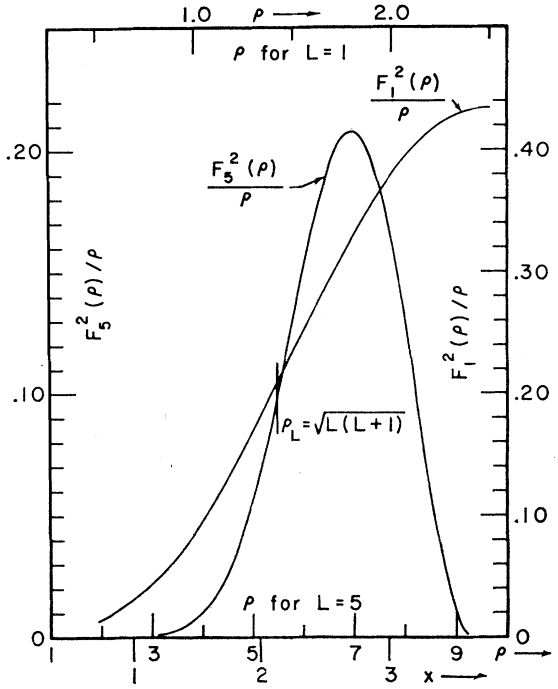

FIG. 3. The square of r times the radial function divided by $kr = \rho$ for $L = 1$ and 5. Point of inflection is marked ($\rho = \rho_L$), and the abscissa scales are adjusted to have it fall at the same abscissa for $L = 1$ and 5. Ordinate scales are adjusted so that the maxima of the two curves are at about the same level. The scale of x refers to graph for $L = 5$ and its relation to the scale of ρ corresponds to incident laboratory energy of 274 MeV. Localization of sensitivity function is more marked for $L = 5$ than for $L = 1$.

dence[46] from $L \geqq 3$ trials, it appears probable that this interaction sets in at $r < 1.6 \times 10^{-13}$ cm. The tentative conclusion was reached, therefore, that the OPEP is the main interaction for $r > 2.9 \times 10^{-13}$ cm and that for $r < 1.6 \times 10^{-13}$ cm some other effect is the main one.

In such estimates there is a chance of being misled by the change in $\mathfrak{F}_L$ caused by interactions at distances smaller then those at which the OPEP applies. Some related estimates of the distortion effects on $\mathfrak{F}_L$ caused by the OPEP itself have been made[46] with reasonably encouraging results but the question of wave distortion cannot be expected to receive a complete answer before more is known about the interaction at r below the OPEP range of values. Transcribing the equations in a form more directly related to Eq. (6.30) the phase parameters for the OPEP are[46]

$$\theta_J^{J-1} = -\frac{(g^2k/4E_p)}{2J+1}(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)(Q_J - Q_{J-1}), \quad (9.8)$$

$$\theta_J^{J+1} = -\frac{(g^2k/4E_p)}{2J+1}(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)(Q_{J+1} - Q_J), \quad (9.9)$$

$$\rho_J = 2(\delta^{L-1,L+1})_J = -2[J(J+1)]^{1/2}\frac{(g^2k/4E_p)}{2J+1} \times (\boldsymbol{\tau}_1\boldsymbol{\tau}_2)(Q_{J+1} + Q_{J-1} - 2Q_J) = 2[J(J+1)]^{1/2}(\theta_J^{J+1} - \theta_J^{J-1}), \quad (9.10)$$

$$\delta_L^L = -\frac{(g^2k/4E_p)}{2L+1}(\boldsymbol{\tau}_1\boldsymbol{\tau}_2) \times [LQ_{L+1} + (L+1)Q_{L-1} - (2L+1)Q_L], \quad (9.11)$$

$$K_L = \frac{g^2k\hat{a}^2}{8E_p}(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)Q_L\left(1 + \frac{\hat{a}^2}{2}\right), \quad \hat{a} = \frac{\mu}{k}. \quad (9.12)$$

Here the argument of the Q_L is the same throughout and

$$g^2 = \hbar c g_0^2 = 137.0\, e^2 g_0^2, \quad g_0^2 \cong 14. \quad (9.13)$$

In these formulas the relativistic effect of Eq. (4.7) of reference 46 has been included in Eqs. (2.4), (2.5), (2.6), (3.2), and (3.3) of reference 46. The derivation of this effect was made[46] using the previously discussed paper[66] as a starting point. It should be mentioned that in reference 46 the explanation concerning $(\delta^{L-1,L+1})_J$ in the first few lines of p. 220 should have had $U - 1 = 2i\mathfrak{z}$ in place of $\mathfrak{z}$. With this replacement it becomes obvious that $\rho_J = 2(\delta^{L-1,L+1})_J$ as in Eq. (9.10) above. It may also be mentioned that in Eq. (1.4) of reference 46 the numerator of the first fraction was meant to be $2L$ rather than $2L + 1$. Equations (9.8) to (9.12) are slightly more general through the inclusion of the isotopic spin effect to those published earlier by Cziffra, MacGregor, Moravcsik, and Stapp[84] who have obtained their equivalent for p-p scattering in another notation by first calculating the scattering amplitude relativistically and analyzing it into partial wave contributions. In the review article of MacGregor, Moravcsik, and Stapp[57] the formula for their α^J, a quantity closely related to the ρ_J used here, has a minus in place of a plus sign before the second term in curly braces.

The last two references emphasize the fact that contributions to the scattering amplitude have poles at

$$\cos\theta = \pm(1 + \hat{a}^2/2). \quad (9.14)$$

This fact is, in a sense, a jumping off place for some dispersion theoretical treatments of nucleon-nucleon scattering. It is not clear, however, that a dispersion theoretical approach can successfully replace the semiquantitative considerations regarding the inclusion of various L values in the OPEP group of phase shifts. If the grouping according to distance criteria has approximate sense, as it seems to have in the actual analysis, it would be surprising if the field theoretical starting point had no sense at all and if the localization of the OPEP were inapplicable. Strictly speaking the division of the phase parameters into the low and high L groups should not be made *ad hoc* in dispersion theory and should be replaced by a consideration of the coupled equations involved. Perhaps through a development[85] of such techniques, the space localization of effects will become unnecessary. At present, however, it still appears to be furnishing the simplest way of dealing with the question of division into the OPEP and non-OPEP groups. It also has defects connected with the always present possibility that a large interaction at a small distance and especially a resonance effect will make an estimate of wave distortion effects unreliable. But one would suppose that such phenomena will more seriously affect the program[80] of dealing with three regions than the question of a safe criterion for choosing the smallest L in the OPEP group. For the former, however, one could reasonably expect the description in terms of nucleon positions in the innermost region to be definitely inapplicable and to

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[84] P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959); cf. M. J. Moravcsik, P. Cziffra, M. H. MacGregor, and H. P. Stapp, Bull. Am. Phys. Soc. **4**, 49 (1959); M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **116**, 1248 (1959); cf. also A. F. Grashin, J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1717 (1959).

[85] D. Amati, E. Leader and B. Vitale, cf. reference 1 above.

have only partial applicability in the two-pion exchange region which could easily be affected by various resonances. In principle, the OPE region is affected by them also, but certainly not as strongly.

On account of the difference in the masses of charged and uncharged pions, the application of Eqs. (9.8) to (9.13) can be made directly only in p-p scattering. In this case the OPEP is caused by the exchange of neutral pions only and $m_{\pi 0}$ the mass of the neutral pion, has to be used throughout. In n-p scattering charged pions also participate. An easy consideration shows[54] that the contributions to $(U-1)/(2i)$, i.e., the quantities listed in (9.8) to (9.12), are obtainable from

$$(U-1) \rightarrow 2(U-1)_{m_{\pi^+}} - (U-1)_{m_{\pi^0}}, \quad (\mathrm{T}=1) \tag{9.15}$$

and

$$(U-1) \rightarrow \tfrac{1}{3}\left[2(U-1)_{m_{\pi^+}} + (U-1)_{m_{\pi^0}}\right], \quad (\mathrm{T}=0) \tag{9.16}$$

The calculations on n-p scattering have been carried out in this manner by Hull *et al.*[86] and in some tests of charge independence to be discussed presently. In such tests the employment of (9.15) and (9.16) presupposes the strict validity of the pseudoscalar theory with pseudoscalar coupling ($PSps$). The general form of the OPEP is the same for pseudoscalar and pseudovector coupling, in the ordinary developments. If the actual interaction should not be of the pure $PSps$ type, a change in Eqs. (9.8) to (9.13) is required and conclusions regarding charge independence will, in principle, be affected but Eqs. (9.15) and (9.16) will retain their form.

10. TESTS OF CHARGE INDEPENDENCE AND OF THE FORM OF THE ONE-PION EXCHANGE POTENTIAL

Charge symmetry of nuclear forces was postulated early in nuclear theory by Heisenberg.[87] Charge independence in the more complete sense was suggested by the interpretation[34,88] of the early proton-proton scattering experiments. These considerations have been concerned with the comparison of p-p and n-p interactions in $^1S_0$ states. Applications to nuclear structure in terms of the isotopic spin formalism[89,90] are well known. Unambiguous tests of charge independence employing data on nuclear structure proved difficult, but no contradictions with the hypothesis have been found and some confirmations are available.[91] Nucleon-nucleon scattering in the 0–340-MeV energy range offers possibilities of performing tests of the hypothesis which are free of the complexities that arise in nuclear structure investigations. Techniques of performing the tests depend on whether low or high L states are being considered. For low L only a few tests have been made apart from those concerned with the $^1S_0$ interaction. For the latter, the literature is extensive and no attempt to cover it will be made here. No definite indication of a discrepancy has been found for the $^1S_0$ state. Tests concerned with the low L group as a whole have also been made[92] without finding a definite difference between n-p and p-p nonelectromagnetic interactions in $\mathrm{T}=1$ states. These tests were made by first employing the $\mathrm{T}=1$ phase parameters obtained from p-p scattering in the n-p analysis, determining the $\mathrm{T}=0$ phase parameters from n-p data and then releasing the $\mathrm{T}=1$ phase parameters so as to obtain the best fit to n-p data. On account of the limited character of the latter, a full release has not been carried out, since the accuracy in any individual parameter would then be too low. Instead, all the $\mathrm{T}=1$ parameters of the non-OPEP group have been varied at once employing a linear variation between two of the better p-p fits, YLA and YR, available at the time. The difference in the value of the linearly varying parameter giving best agreement with experiment turned out small. A related procedure was to vary the parameters along the gradient in $\mathrm{T}=1$ subspace. For further details, reference may be made to the original paper.[92] The conclusion arrived at was that $\mathrm{T}=1$ phase parameters for the p-p and n-p cases were the same to within 0.03 of the standard deviation of the parameters. There is no reason to suspect the validity of charge independence for the

[86] M. H. Hull, Jr., K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **122**, 1606 (1961).

[87] W. Heisenberg, Z. Physik **77**, 1 (1932); **78**, 156 (1932); **80**, 587 (1932).

[88] G. Breit and E. Feenberg, Phys. Rev. **50**, 850 (1936); G. Breit and J. R. Stehn, *ibid.* **52**, 396 (1937); G. Breit, H. M. Thaxton, and L. Eisenbud, *ibid.* **55**, 1018 (1939).

[89] B. Cassen and E. U. Condon, Phys. Rev. **50**, 846 (1936). These authors showed how the hypothesis of charge independence can be formulated in terms of the isotopic spin formalism which was introduced by Heisenberg.[87]

[90] E. P. Wigner, Phys. Rev. **51**, 106, 947 (1937).

[91] D. H. Wilkinson and G. A. Jones, Phil. Mag. **44**, 542 (1953); D. H. Wilkinson, Phys. Rev. **90**, 721 (1953); Phil. Mag. **44**, 1019 (1953); A. B. Clegg and D. H. Wilkinson, *ibid.* **44**, 1269 (1953); D. H. Wilkinson and A. B. Clégg, *ibid.* **44**, 1322 (1953); **1**, 291 (1956); G. A. Jones and D. H. Wilkinson, *ibid.* **45**, 703 (1954); D. H. Wilkinson, *ibid.* **1**, 379 (1956). For n-n interaction cf. K. M. McVoy, Phys. Rev. **121**, 1401 (1961); and K. Ilakovac, L. G. Kuo, M. Petravic, and I. Slaus, *ibid.* (to be published).

[92] G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. Letters **4**, 79 (1960).

low L group from these tests. On the other hand, since the phase parameters were not accurately known, the tests did not prove the validity of charge independence for the non-OPEP group with great accuracy.

Other tests[92] were concerned with the comparison of values of g_0^2 obtainable from n-p data with those from p-p employing the OPEP group of phase parameters. Less directly related information had already been obtained in the work of Taketani *et al.*[80] who used the OPEP in less accurate adjustments to scattering and deuteron data. Cziffra, MacGregor, Moravcsik, and Stapp[84] adjusted g_0^2 to the 310-MeV group of p-p scattering data and found good agreement with expectation from pion-nucleon data having been the first to demonstrate the usefulness of the procedure in phase parameter analysis. The tests quoted earlier[92] appear to be the first in which the values of g_0^2 obtained from p-p and n-p scattering data are obtained employing comparable data treatments.

The method employed was as follows. In obtaining a fit to data such as YLAM for the p-p case, the value of the coupling constant g_0^2 was somewhat arbitrarily set at 14. The phase parameters of the low L group were changed by the gradient method so as to improve agreement with experiment employing procedures described by Breit *et al.*[54] The value of g_0^2 used in the calculation of the contributions of the OPEP group of phase parameters (*alias* the one-pion group) to the amplitudes and measured quantities was then varied, the weighted mean square deviation of the data called D in reference 54 was calculated, and the value of g_0^2 corresponding to the minimum of D was taken as the best g_0^2 for that fit. This procedure has been previously used in the case of the 310-MeV group of p-p data by Cziffra *et al.*[84] To obtain an error, estimate the parabolic curve giving the variation of D with g_0^2 was read to give values of g_0^2 corresponding to a change of D by $\pm 1/N$ from the minimum, with N standing for the number of measurements used in the analysis. The appreciable excess of D over unity is taken into account by multiplying the error estimate by $D^{1/2}$. A more elaborate method of dealing with the statistics of the problem was not used because there are probably some systematic sources of error present which may be more important than the statistical ones. For example, if the OPEP group includes phase parameters with a too low L, the g_0^2 determination will be subject to error. The values of g_0^2 corresponding to the best fits to data then available were[92] 13.5 ± 0.9 from p-p search YLAM employing $L \geqq 5$ for the OPEP group; 14.5 ± 0.7 from n-p search YLAN2M employing $L \geqq 5$ for the OPEP group and varying the T = 0 and T = 1 OPEP phase parameters together, 14.5 ± 1.1 varying the g_0^2 for T = 0 alone and keeping $g_0 = 14$ for T = 1. For fit YLAN1 varying g_0^2 for T = 0, 1 together the value of g_0^2 was obtained as 14.1 ± 0.8. Table I of reference 92 contains other not very different numbers for nonrelativistic one-pion treatment and pion masses approximated by $m(\pi^+)$. The values quoted here are for relativistic one-pion treatment with pion masses entering as in (9.15), (9.16) for n-p and $m(\pi_0)$ used for p-p cases.

The tests just quoted were made before the work on the published n-p fits[86] was completed. Various evidence points to fit YLAN3M as the more likely of the 6 fits described by Hull *et al.*[86] Therefore, the tests in reference 92 may not be regarded as final therefore. On the other hand, the essential agreement of values obtained from different p-p and n-p fits indicated marked stability of g_0^2 derived from data to the choice of fit.

This work was followed by an attempt to test the mathematical form of the OPEP by comparison with experiment.[93] The OPEP was separated into its spin-spin and tensor potential parts:

$$V^{(2)} = V^{(2)}_{(\sigma\sigma)} + V^{(2)}_T . \tag{10.1}$$

The phase parameters owing their origin to the two parts of the potential are then conveniently denoted by the same subscripts so that, referring to a phase parameter generically as δ,

$$\delta^{1\pi} = \delta^{1\pi}_{(\sigma\sigma)} + \delta^{1\pi}_T . \tag{10.2}$$

It was then supposed that, on account of some failure of the theory, the relative proportions of the two terms in (10.1) may differ from their theoretical value but that the mathematical form of each term is that given by theory. Since the value of g_0^2 giving best agreement with experiment is affected by employing the altered form of $V^{(2)}$ and since it is desirable to provide checks on the numerical work, the same objective was approached in two ways, a and b, corresponding to the introduction of two parameters q_a and q_b giving rise to changes in (10.2) having the forms

$$\delta_a = \delta^{1\pi} + q_a \delta^{1\pi}_{(\sigma\sigma)} = (g_0^2/14)[\delta_0 + q_a \delta^{1\pi}_{0(\sigma\sigma)}] , \tag{10.3}$$

$$\delta_b = \delta^{1\pi} + (14/g_0^2) q_b \delta^{1\pi}_{(\sigma\sigma)} = (g_0^2/14)\delta_0^{1\pi} + q_b \delta^{1\pi}_{0(\sigma\sigma)} , \tag{10.4}$$

arising from corresponding changes in (10.1). The

[93] G. Breit, M. H. Hull, Jr., K. E. Lassila, and H. M. Ruppel, Phys. Rev. Letters **5**, 274 (1960); Proc. Natl. Acad. Sci. U.S. **46**, 1649 (1960).

subscript 0 indicates that the quantity is calculated for $g_0^2 = 14$. For (10.3) the values of g_0^2 and of q_a are adjusted to give the best fit to experiment, for (10.4) g_0^2 and q_b are similarly treated. In the absence of inaccuracies, the values found for best fits should satisfy

$$g_0^2 q_a = 14\, q_b\,, \tag{10.5}$$

a relation useful for checking purposes. A third parameter q_c was introduced through

$$\delta_c = \delta^{1\pi} + (14/g_0^2) q_c \delta_c^{1\pi} = \delta^{1\pi} + q_c \delta_{c0}^{1\pi}\,. \tag{10.6}$$

Here q_c is the coefficient of

$$V_c^{(2)} = \frac{1}{3} g_0^2 \left(\frac{\hbar\mu}{2Mc}\right)^2 e^{-\mu r}/r\,, \quad g^2 = \hbar c g_0^2, \tag{10.7}$$

which is introduced on the right-hand side of (10.1). This potential would be obtained from the standard expression for the OPEP on setting $(\tau_1\tau_2) = 1$, $(\sigma_1\sigma_2) = 1$, and omitting S_{12}. It is $-1/3$ of the singlet even potential corresponding to the OPEP.

Deviations of the three q's from zero indicate a preference of the data for a modified $V^{(2)}$. The more elaborate tests involved simultaneous variations of g_0^2 and of one of the q's. These were preceded by simpler tests in which g_0^2 was kept at a fixed value and the probable q was determined by minimizing the weighted sum of the squares of deviations calculated from measured quantities. An example of this may be seen in Table II which is a reproduction of Table 1 of the second reference in reference 93.

TABLE II. Values of parameters q_a, q_b, q_c as determined from search YLAM of p-p scattering data.

Assumed g_0^2	q_a	q_b	q_c
12	-0.14 ± 0.18	-0.12 ± 0.15	0.19 ± 0.30
14	-0.21 ± 0.16	-0.22 ± 0.15	-0.09 ± 0.27
17	-0.32 ± 0.13	-0.36 ± 0.16	-0.48 ± 0.34

Taken literally these numbers indicate that the spin-spin and tensor parts as given by the OPEP are not quite in the ratio most favored by experiment. Since search YLAM is not perfect and since the OPEP group of phase parameters may be including, at some energies, parameters that show effects of two-pion exchange, the limits of error may be sufficiently underestimated to account for the discrepancy. The errors in the table are the purely statistical ones estimated by the same procedure as has been described for the g_0^2 determination.

Simultaneous adjustment of g_0^2 and the q to the same data gave values as in Table III which is a reproduction of Table 3 of reference 93.

TABLE III. Values of (q, g_0^2) pairs from p-p search YLAM.

Case	q	g_0^2
a	-0.24 ± 0.16	14.0 ± 1.3
b	-0.22 ± 0.17	14.1 ± 1.3
c	-0.04 ± 0.33	13.6 ± 1.4

These are similar to the values in Table II. If the error limits are estimated more conservatively on the basis of extreme values in the curvilinear parallelograms,[93] again with inclusion of $D^{1/2}$, they become $+0.26, -0.33$ for the first entry of the second column, ± 0.25, ± 0.64 for the second and third entries, and ± 1.4, ± 2.0, and ± 2.6 for the first, second, and third rows of the last column, respectively. The deviations from 0 fall within these error limits which may be too conservative estimates of the accuracy of the determinations. However, in view of the various uncertainties already mentioned and the additional one caused by the employment of a preliminary rather than final form of YLAM, no claim has been made for the reality of the deviations of the q's from zero.

Similar tests have been made employing n-p searches YLAN3 and YLAN2M with somewhat similar results. It is noteworthy that $q_c = -4.5 \pm 1.0$ was obtained for YLAN3 employing $T = 0$, 1 together when K_4, the singlet phase shift for $L = 4$, was included in the OPE group. This statistically significant variation from $q_c = 0$ was interpreted[94] as indicating that the two-pion exchange potential was probably operating in addition to the OPE. This explanation of the large q_c is in agreement with the fact that the ^3G phase-shift parameters change by amounts comparable to their OPEP values in the YLAN3 search.

Work is in progress on the form of the OPEP by methods similar to those described. It is more difficult to test the theoretical form in the sense used above, than to obtain values of g_0^2 for an assumed form. Judgment regarding the meaning of deviations which have been indicated by tests performed should therefore be reserved.

Somewhat related work has been done by Signell,[95]

[94] G. Breit, *Proceedings of the* 1960 *Annual International Conference on High Energy Physics at Rochester, N. Y.* (Interscience Publishers, Inc., New York, 1960), p. 674. The conference was held August 25–September 1, 1960. The remark was made on the basis of collaboration with co-authors of papers in.[92]

[95] P. Signell, Phys. Rev. Letters **5**, 474 (1960).

who varies the pion mass in the computation of the contribution to scattering of the OPEP group. He finds that solutions 1 and 2 of MacGregor *et al.*[84] do not show a minimum of the mean square deviation at the physically correct pion mass value and, in fact, do not show a pronounced minimum in the rather wide mass range investigated. The analysis under consideration is of the 310-MeV p-p Berkeley set of data. On the other hand, Signell finds that a pronounced minimum in approximately the correct position is obtained for the 95-MeV Harvard data.[96] He expresses the opinion that conclusions previously arrived at[84] on the basis of the 310-MeV data should be considered as uncertain in view of this test. It is possible, on the other hand, that the 310-MeV group of data is not sensitive to the pion mass quite apart from questions of reliability of conclusions reached. It would appear also that for small m_π the analysis would have to be modified by moving the minimum L admitted in the OPE group toward larger values. Doing so would decrease the statistical accuracy, however, and clear tests on the pion mass would therefore be difficult.

Another piece of evidence regarding the validity of the OPE interaction is provided by Riazuddin[97] who, employing dispersion relations, produces evidence to the effect[98] that only one of 4 different sets of ^3P phase shifts obtained by MacGregor[97] in his analysis of data below 40 MeV is consistent with experiment, and that this is the only set consistent with the OPE.

Recently, especially careful measurements of the n-p scattering cross section have been made at 350 MeV for the express purpose of determining the pion-nucleon coupling constant by the pole method.[99] As a result of fitting by means of backward and forward single-pion-exchange poles and a polynomial combined with the investigation of the effects of varying the meson mass, using an alternative analytical form, and including possible two-pion poles, the authors arrive at the value $g_0^2 = 14.3 \pm 1.0$. A preliminary analysis of the same data has been performed at Yale.[99a] Employing the 350-MeV cross-section data alone in a simultaneous variation of all the OPE phase parameters with $L > 4$, the value $g_0^2 = 14.6 \pm 0.9$ was obtained. For the same type of variation, employing all available data from 217 to 350 MeV, 14.5 ± 0.2 was the result. If K_4 is included, these two numbers become 14.5 ± 0.4 and 14.5 ± 0.3, respectively. Nuclear magnetic moment effects were included. Similar values were obtained for variations of T = 0 and T = 1 parameters separately. Employment of the published form[54] of YLAM with inclusion of newer data in the g_0^2 variations, but without researching the parameters in the low L (non-OPEP) group and including K_4 in the OPEP group and excluding nuclear magnetic moment effects, gave $g_0^2 = 13.7 \pm 0.8$; including the magnetic moment effects and excluding K_4 from the OPEP group the value changed to 14.7 ± 0.9; exclusion of magnetic moment effects in the last test changed the value to 15.5 ± 1.0. The value 14.7 ± 0.9 is presumably the most conscientiously obtained one among these, since there is evidence against inclusion of K_4 in the OPEP group. This value agrees well with those obtained in the just quoted n-p determinations of g_0^2. Since the latter are heavily weighted by charge exchange scattering, the evidence is for nearly the same value of the coupling constant for π^+ and π°. Experience shows, however, that g_0^2 determinations can be affected by improvements in searches for best values of parameters in the low L (non-OPEP) group. Some changes in the values listed may occur when YLAM is researched and when the researching of YLAN3M is improved. The conclusions regarding the universality of g_0^2 should therefore be regarded as preliminary.

11. THE TWO-PION EXCHANGE INTERACTION

The two-pion exchange (TPE) interaction is more difficult to treat theoretically than the OPE. It is also more difficult to select experimental data which have a reasonably direct bearing on the TPE. Some evidence for the presence of TPE effects has been obtained in the second reference in footnote 93 and rough estimates have been available at the time of the 1960 Annual International Conference on High Energy Physics at Rochester. They were referred to in the discussion that followed Lévy's summary of

[96] E. H. Thorndike and T. R. Ophel, Phys. Rev. **119**, 362 (1960); also references under footnote 62.

[97] Riazuddin, Phys. Rev. **121**, 1509 (1961); M. H. MacGregor, *ibid.* **113**, 1559 (1959).

[98] H. P. Noyes, *Proceedings of the Rutherford Jubilee International Conference in Manchester* (Heywood and Company Ltd., London, 1961), p. 77.

[99] A. Ashmore, W. H. Range, R. T. Taylor, B. M. Townes, L. Castillejo, and R. F. Peierls (to be published). The writer would like to express his indebtedness to the authors for supplying him with a preprint of this paper. The following additional references on determinations of g_0^2 by the pole method may be of interest: G. F. Chew[1]; P. Cziffra and M. J. Moravcsik, Phys. Rev. **116**, 226 (1959); N. S. Amaglobeli and Yu. M. Kazarinov, Soviet Phys.—JETP **10**, 1125 (1960); N. S. Amaglobeli, B. M. Golovin, Yu. M. Kazarinov, S. V. Medved, and N. M. Polev, *ibid.* **11**, 474 (1960); R. R. Larsen, Nuovo cimento **18**, 1039 (1960).

[99a] G. Breit, M. H. Hull, Jr., F. A. McDonald, and H. M. Ruppel, "Form of One-Pion Exchange Potential and Charge Independence of Nucleon-Nucleon Interaction" (submitted for the 1962 International Conference on High Energy Physics at CERN).

progress on nucleon and antinucleon processes. The evidence has been substantiated by further work.[100] The field theoretical calculation used is that of Gupta,[82] which is made on the *PSps* theory by a completely covariant procedure employing the method of auxiliary fields. It should be mentioned, however, that the TPE potential which is tabulated by Gupta and which is used in reference 99 is obtained only after some approximations in the evaluation of the general formulas. One of these consists in the omission of the spin-orbit interaction which is contained[101] in the general results, the discussion of which will be returned to presently. As has been mentioned in Sec. 8 of this review, the quantity referred to as the potential[82] is not the type of potential that is under discussion in calculations by the Tamm-Dancoff method but has[47] a more direct relationship to the scattering matrix. Considerations in references 82 and 47 show that including effects of order g^4, one may calculate with Gupta's potential employing the first order Born approximation. This was done[100] making use of field free functions, i.e., neglecting Coulomb effects. Making use of values of phase parameters obtained for fits YLAM to *p-p* and YLAN3M to *n-p* data in the comparison, the numbers indicated that the additions to the OPE estimated on the basis of the central part of the TPE potential were in semiquantitative agreement with the phenomenology, provided certain criteria were used regarding the applicability of the estimates. These were similar to the considerations with sensitivity curves referred to previously[46] and illustrated in Figs. 2 and 3 of this article. On this basis, on account of the relatively slow increase of the absolute value of the regular radial function $|F_L|$ with r, it is not to be expected that the TPE can hold for $L = 1$ in a clean form, its effects not becoming well localized at the larger r. On the other hand, better localization at the larger r is expected for larger L. These expectations are borne out on the whole by the numerical comparisons.[100] Two of the most striking cases may be seen in Fig. 4 for phase shift K_4 and in Fig. 5 for ${}^3\theta_3^p$. The dashed lines correspond to dispersion theoretical calculations of Galanin, *et al.*[1] and of Grashin and Kobsarev.[1] The lines marked O. P. and O. P. E. have been calculated from the OPEP alone. The dots surrounded by circles represent the result of adding the TPE effect to that of the OPE. There are also shown curves for the values of K_2 corresponding to the phenomenological fits YLAM and YRB1 and for values of ${}^3\theta_3^p$ curves corresponding to *n-p* fits YLAN1, YLAN3, and YLAN3M in Figs. 4 and 5, respectively. The error bars apply to YLAM and YLAN3M. The points marked TPE represent the result of superposing OPE and TPE effects and are not just explanatory of the designations used. In Fig. 5, it is striking that the OPE values have very little relation

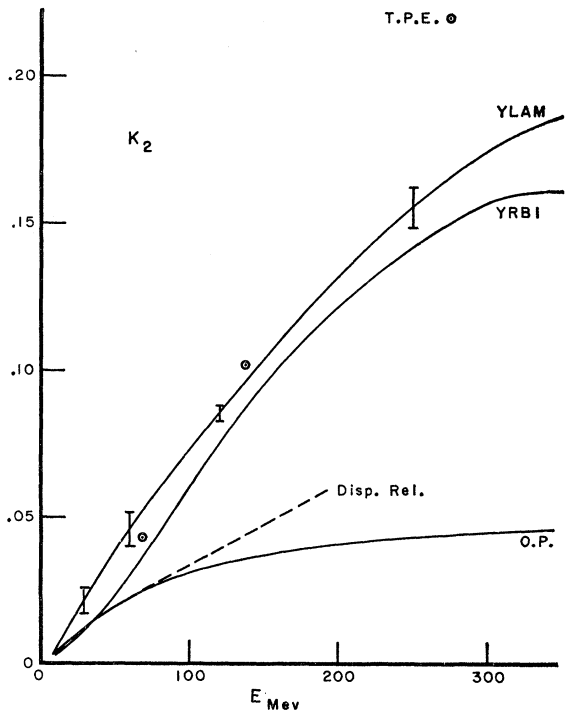

FIG. 4. Comparison of *p-p* phenomenological fits YLAM and YRB1 with sum of first-order effects of one-pion and two-pion exchange effects for the ${}^1D_2$ phase shift K_2. The three points designated as dots surrounded by circles are the computed sums. The error bars refer to YLAM. The O.P. curve is for the one-pion exchange alone. The dashed line corresponds to dispersion relation calculations of Galanin *et al.* (reference 1) and of Grashin and Kobsarev cited in reference 100.

to the phenomenological ones. This is connected with a result of Grashin's[102] who found that, in triplet states with total angular momentum $J = L + 1$, the energy dependence of the OPE phase shift is anomalously rapid at small energies. This may also be seen as a cancellation of dominant terms employing the formulas in reference 46. From the latter viewpoint, it is not surprising that the OPE has a sign opposite to the phenomenological one. It will be noted from Figs. 4 and 5 that for $L = 2$ the TPE gives an improvement over the OPE in the inter-

[100] G. Breit, K. E. Lassila, H. M. Ruppel, and M. H. Hull, Jr., Phys. Rev. Letters **6**, 138 (1961).
[101] S. N. Gupta, Phys. Rev. **122**, 1923 (1961).
[102] A. F. Grashin, J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1717 (1959) [translation: Soviet Phys.—JETP **9**, 1223 (1959)].

mediate energy range below ~150 MeV and that definite disagreement sets in above this energy range. This fits in qualitatively with the sensitivity curve considerations previously mentioned. For reasons already mentioned, not too much significance should be attached to the almost quantitative agreement

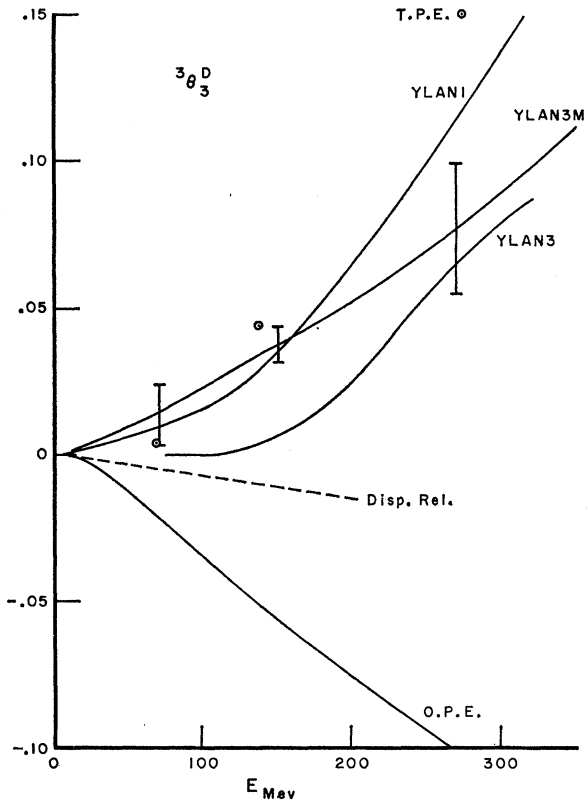

FIG. 5. Comparison of n-p phenomenological fits YLAN1, YLAN3, and YLAN3M with sum of first-order effects of one-pion and of central potential part of two-pion exchange effects for ${}^3\theta_3^D$. Error bars refer to YLAN3M. The O.P.E. curve is for the one-pion exchange alone. Otherwise as in Fig. 4.

seen in parts of the plots. On the other hand, the qualitative features appear significant. It may be mentioned in this connection that all the six phenomenological n-p fits of reference 86 fall above the axis in the ${}^3\theta_3^p$ comparison.

It will be noted that the comparisons between theory and experiment are being made here in terms of an expansion in powers of g^2 and that they do not appear to be discouraging. It is very unlikely that such an expansion can be carried through at all energies and for the whole interaction. But the indications appear to be that the shielding of the small r region by the centrifugal barrier is of appreciable help in securing a semiconvergent expansion in a limited energy region for sufficiently high L. The analogy to quantum electrodynamics naturally comes to mind. There also, the expansion in powers of the fine structure constant is under suspicion. Nevertheless, good agreement with experiment is obtained. There is, however, no comparison between the two cases regarding the quantitative character of the agreement.

A related investigation has been carried out by Feshbach, Lomon, and Tubis[103] who combine the boundary condition[104–107] method with the OPE and different forms[108] of the TPE potential. The inner region of small values of r is treated by the boundary condition method employing an energy-independent boundary condition. In these general features the work of Feshbach, Lomon, and Tubis (FLT) is similar to that of Saylor, Bryan, and Marshak[107] (SBM) and differs from it principally in two respects: (a) The latter authors employ only the Taketani-Machida-Ohnuma (TMO) potential[108] which is believed to have been improved on by later work;[108] (b) they compare their calculations with experimental data on p-p scattering at a few energies, while FLT compare their theoretical phase parameters with the phenomenological fit YLAM of reference 54 and the effective range representation of low-energy p-p data. These differences in approach give rise to some preferences between FLT and SBM. In connection with (a) there appears to be a definite preference for FLT and the work of SBM may be regarded, in this respect, as exploratory only. In fact, it originated in the observations that for the phenomenological potential devised by Bryan,[109] the interaction is especially strong between the hard-core radius, 0.38 $\hbar/(m_\pi c)$ in his calculations, and 0.6 $\hbar/(m_\pi c)$ and that for larger r the potential is comparatively weak resembling a combination of OPE and TPE effects. The employment of the boundary condition model with a potential tail appeared, therefore, as a natural

[103] H. Feshbach, E. Lomon, and A. Tubis, Phys. Rev. Letters **6**, 635 (1961).
[104] G. Breit and W. G. Bouricius, Phys. Rev. **74**, 1546 (1948); **75**, 1029 (1949).
[105] H. Feshbach and E. Lomon, Phys. Rev. **102**, 891 (1956). The considerations in the introduction to this reference regarding the bearing on meson theory and the relationship to hard core potentials appear to be very closely related to those in reference 104.
[106] A. M. Saperstein and L. Durand, III, Phys. Rev. **104**, 1102 (1956).
[107] D. P. Saylor, R. A. Bryan, and R. E. Marshak, Phys. Rev. Letters **5**, 266 (1960).
[108] M. Taketani, S. Machida, and S. Ohnuma, Progr. Theoret. Phys. (Kyoto) **6**, 683 (1951); **7**, 45 (1952); K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1032 (1953); A. Klein, Progr. Theoret. Phys. (Kyoto) **20**, 257 (1958); A. Klein and B. H. McCormick, *ibid.* **20**, 876 (1958).
[109] R. A. Bryan, Bull. Am. Phys. Soc. **5**, 35 (1960); Nuovo cimento **16**, 895 (1960).

attempt.[110] In connection with (b) there appear to be two aspects in the comparison. Since the p-p fit, YLAM has been made employing many data at a number of energies, while SBR employ more limited experimental material, since FLT go to special pains to produce a good reproduction of YLAM and the effective range parametrization of data below 9 MeV, their parametrization should again be preferable. On the other hand, some arbitrariness is involved in arriving at a phenomenological fit such as YLAM, especially in the employment of correction functions that are used to improve the fits. There would be some advantage, therefore, in employing semi-phenomenological models such as those used by SBR and FLT by making a final adjustment directly to the data. This point has been discussed in connection with some unpublished results by the dispersion theoretical treatment of K_0 and K_2 by Amati, Leader, and Vitale[1] and the Yale potential[111] by Breit.[112] This discussion indicates that even though YLAM is better than fit YRB1 from the statistical viewpoint, the shape of the (K_0,E) plot appears in some respects as the more probable. Such features which may become more important in the future can probably be better taken care of by employing such models as that used by FLT in a final direct adjustment to data. The adjustment would have to be much better, however, than the data reproduction of SBM because, e.g., the differential cross section at 90 MeV is consistently too high at large angles and at 39.4 MeV the Coulomb interference region is not well reproduced according to the first figure in reference 107. The work of FLT has been extended[113] to n-p scattering and improved results on p-p scattering have been obtained. For the latter, the verbal report[113] contained the definite conclusion of there being no need for pair suppression. The n-p fit used was YLAN3 of reference 86 which does not appear to be the most probable fit. The most definite information to this effect became known after the delivery of the paper.[114]

The employment of an energy independent boundary condition in combination with a theoretically derived potential is, of course, a somewhat arbitrary procedure. The question naturally arises as to whether it is sufficiently well founded to be able to draw definite conclusions regarding the physical situation. The usefulness in securing more certain fits to data which has been mentioned earlier also depends on the degree to which the physical picture employed is meaningful. If this picture is sufficiently far from the actual one, the parametrization of the energy dependence of phase parameters may, in fact, be occurring in terms of inappropriate functional forms. For these reasons, it appears desirable to understand more clearly what is involved in the boundary condition model as used by SBM and FLT.

The first application of the boundary condition model,[104] in connection with nucleon-nucleon scattering, was partly motivated by the desirability of demonstrating that the description of scattering by means of static potentials is not the only reasonable one. The energy independence of the homogeneous logarithmic derivative Y of the radial function was not a necessary feature of the fit obtained. In fact, the graphical procedure used in that paper indicates the possibility of employing an approximately linear energy dependence of that quantity. The hypothesis of $\partial Y/\partial E = 0$ was the simplest and happened to work. Some arguments indicating the plausibility of $\partial Y/\partial E = 0$ have been discussed in reference 104 and closely related ones in reference 105. These arguments do not show, however, that $\partial Y/\partial E = 0$ and barring special circumstances such a relationship is not the most natural for a general system. Some deviation from it would be expected at least at energies close to the meson production threshold. The separation of the whole range of r into two parts for one of which only TPE and OPE are sufficient, while the other is dominated by the presence of so many pions that at its boundary the reduced width of the system is zero, can be only a schematic description of the actual situation. It may be possible to show that once one moves out of the outer region of r in towards smaller distances, the many pion states increase in number so rapidly that the picture involving a rapid transition from one condition to the other does indeed follow from the meson theory. But there has apparently

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[110] The employment of a potential tail in combination with the boundary condition model has been previously used by Saperstein and Durand[106] at the author's suggestion. At that time the phenomenology of the subject was not sufficiently advanced to lead to conclusions of any certainty in this respect.

[111] K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962); cf. also T. Hamada, Progr. Theoret. Phys. (Kyoto) **24**, 1033 (1960). The joining to the low-energy region has been performed more thoroughly in the former of these two references.

[112] G. Breit, *Proceedings of the Rutherford Jubilee International Conference in Manchester* (Heywood and Company Ltd, London, 1961), p. 77, paper D 1/1. The matters mentioned in the text concerning the energy dependence of K_0 and K_2 are illustrated in Figs. 1 and 2 of the paper quoted. Figures 3 and 4 of that paper are essentially Figs. 4 and 5 of the present article.

[113] H. Feshbach, Bull. Am. Phys. Soc. **7**, 20 (1962), paper E 3.

[114] P. M. Patel, A. Carroll, N. Strax, and D. Miller, Phys. Rev. Letters (to be published).

been no clear demonstration, so far, that this is to be expected. An argument concerning $\partial Y/\partial E = 0$ corresponding to the limit of the causality requirements[115] has been verbally advanced, on the grounds that at the small radius at which the boundary condition is used it is reasonable to expect instant propagation across the sphere corresponding to that radius. If one really deals here with fundamental breakdowns of space-time concepts and if nucleon-nucleon scattering has helped to find them, the whole research is amply justified. But there are other unexplored ways of attempting to fit phenomenology which may also prove successful. The above reactions to the FLT and SMB work are not meant in a critical sense but are intended to bring out some of the points which appear to need further examination. Taken literally, the work of FLT appears to lend strong support to the belief that the TPE interaction has some physical reality.

No explicit introduction of the spin-orbit interaction is used by FLT. Its phenomenological equivalent is furnished by the specification of boundary conditions for phase parameters belonging to the same multiplet. The tail of the spin-orbit interaction is thus taken to have zero range.

It is not clear, however, that the spin-orbit interaction represented in this manner has a meaning in terms of a pair of functions corresponding to even and odd triplet states. There is, however, no real reason why this should be required.

An exact connection between the boundary condition model and an equivalent potential such as would be used in calculations of binding energies of light nuclei does not appear to have been established.

12. VECTOR MESONS AND THE NUCLEON-NUCLEON SPIN-ORBIT INTERACTION

One of the intriguing questions concerning nucleon-nucleon interactions is the possible presence of spin-orbit effects. At various times it has been thought that this kind of interaction is necessary for the explanation of the fine structure of nuclear levels and it has been used as one of the postulates of the shell-structure model[67] of nuclear structure. The interactions of nucleons in nuclei are not easily analyzed in terms of nucleon-nucleon interactions, however. Some promising explanations of parts of spin-orbit-like effects in nuclei have been given by Feingold and Wigner[116] in terms of second-order effects involving the two-nucleon tensor force. It is difficult, therefore, to try to draw unique conclusions regarding the spin-orbit potential from nuclear structure studies.

With the many uncertainties regarding the propriety of descriptions of nucleon-nucleon interactions, the introduction of an old fashioned spin-orbit potential term, V_{LS}, may be a too naïve procedure and there is, in fact, little definite evidence in its favor. Some indirect evidence has been provided by the large value of the polarization parameter P observed in p-p scattering. It may be shown[117] that the first-order effect of a tensor force potential can give rise only to $P = 0$. The proofs quoted do not rely on there being no appreciable distortion of the wave function by the nucleon-nucleon interaction except for that part of the distortion caused by the tensor-potential itself and go a little beyond what can be inferred from Wolfenstein's parametrization of the scattering matrix. On the other hand, for large tensor potentials $P \neq 0$ and a neat mathematical proof of the existence of V_{LS} does not seem possible. Nevertheless, on account of the vanishing of the S_{12} effects in first order, it is hard to account for the observed polarization employing static potentials except by introducing a V_{LS} term. The earliest evidence was from the ^3P terms effective in p-p scattering and on this basis the opinion was expressed at the Seattle Conference on Theoretical Physics that p-p scattering contains indications of V_{LS}.

The subsequent success of Signell and Marshak and of Gammel and Thaler in explaining some of the measurements on nucleon-nucleon scattering employing a V_{LS} appeared to indicate at first some relatively simple possibilities provided the purely phenomenological V_{LS} was accepted. Comparison of the potentials with more data gave a more pessimistic view especially regarding the former of the two quoted,[118] even after the range of V_{LS} was decreased so as to fit better with theoretical possibilities. A later fit of some of the p-p scattering data by means of a potential devised by Bryan[109] also relied heavily on the employment of a spin-orbit potential. The range parameter of the tail of this potential was appreciably smaller than the value expected on the supposition

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[115] E. P. Wigner, Phys. Rev. **98**, 145 (1955); G. Breit, *Encyclopedia of Physics, Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41, Part 1, Sec. 47 (β).

[116] A. M. Feingold and E. P. Wigner, Phys. Rev. **79**, 221 (A) (1950); A. M. Feingold, *ibid.* **101**, 258 (1956) are two of the earlier references.

[117] Appendix to reference 42. Another arrangement of the proof has been worked out by M. S. Wertheim in his Yale dissertation.

[118] M. H. Hull, Jr., K. D. Pyatt, Jr., C. R. Fischer, and G. Breit, Phys. Rev. Letters **2**, 264 (1959).

that V_{LS} was caused by nucleon-pion interactions, which corresponds[119] to an exponential factor

$$\exp(-2m_\pi cr/\hbar)\,. \tag{12.1}$$

Gammel and Thaler's V_{LS} was also of a shorter range than that corresponding to expectation for the pion-nucleon interaction. These results revived an old suggestion[120] that the nucleon-nucleon spin-orbit potential and the repulsive core could be explained by postulating interactions between nucleons and a vector meson field. Estimates regarding the vector meson mass which would correspond to the tail of Bryan's V_{LS} were then published,[121] together with considerations regarding the possible bearing of the nucleon-vector meson interaction on nucleon-antinucleon scattering, the electromagnetic form factors, and on the possibility that the static potentials would have better validity for nuclear physics applications than would be expected if only nucleon-pion interactions were involved. The spin-orbit potential to be expected was already available[120] and was such that it entered the two-nucleon Schrödinger equation in the combination

$$-J+\frac{\hbar}{4M^2c^2}\frac{dJ}{rdr}\times\{[\mathbf{r}\times(2\mathbf{p}_2-\mathbf{p}_1)]\cdot\boldsymbol{\sigma}_1-[\mathbf{r}\times(2\mathbf{p}_1-\mathbf{p}_2)]\cdot\boldsymbol{\sigma}_2\}\,, \tag{12.2}$$

$$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2 \tag{12.3}$$

where $-J$ is the central potential caused by the same interaction, $\mathbf{p}$, $\boldsymbol{\sigma}$ are, respectively, the momentum and Pauli spin vector of a particle, M the particle mass and quantities referring to the two particles are distinguished by subscripts 1 and 2. In the center-of-mass system this is equivalent to

$$-J-\frac{3}{2}\frac{\hbar^2}{M^2c^2}\frac{dJ}{rdr}(\mathbf{L}\cdot\mathbf{S})\,, \tag{12.4}$$

$$J=-(q^2/r)e^{-\kappa r}\,. \tag{12.5}$$

Here $\mathbf{S}=(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2$ is the total spin of the two particles. The estimates indicated a vector meson mass between 9 and 12 pion masses and repulsive cores of large magnitude and concentration at small r. Independently and from a different starting point, Sakurai[122] has dealt with the same problem. While in his first paper Sakurai employed a hasty reduction to an equivalent problem in the center-of-mass system which led to an omission of a factor 3 in the spin-orbit term, his later first-order covariant calculation does not have this defect. In the last of his papers just quoted,[122] he arrives at the estimate of the vector meson mass of between $3m_\pi$ and $4m_\pi$. A reexamination of these questions[123] showed that, employing hard-core phenomenological potentials, the wave function distortion caused by the potential is appreciable. On account of it, the first-order perturbation calculation may be expected to exaggerate the effect of V_{LS} by a factor of ~ 4 or more. The estimate of the range constant of V_{LS}, made employing first-order calculations, therefore appeared unreliable. It was also noticed[123] that the small masses of the vector meson lead to extensions of the repulsive potential into the region of large r which are hard to reconcile with other evidence regarding the long-range part of the nucleon-nucleon potential. In the same note,[123] the Gammel-Thaler spin-orbit interaction was examined with the conclusion that it corresponds to a vector meson mass of about $4m_\pi$ and preliminary results of collaboration with Hull, and Lassila were mentioned. According to these, the Yale data fits spoke in favor of a V_{LS} with a fall-off steepness intermediate between that of Gammel-Thaler and of Bryan, indicating therefore a mass between $4m_\pi$ and the larger estimates. It should be mentioned that on the basis of electromagnetic form factors, Nambu[124] proposed a neutral vector meson with isospin 0 and a mass between 3 and 4 pion masses. Sakurai's proposal was partly motivated by considerations related to those of Yang and Mills[125] in a general attempt of relating possible vector meson interactions between nucleons and hyperons to the isospin, the baryon number and the hypercharge and of obtaining a deeper view of relationships in strong interactions.

In the course of the work on the Yale potential[111] attempts were made to represent V_{LS} in a form corresponding to (12.4) and (12.5). According to this work a meson mass greater than 7 m_π appears improbable but a mass of 6 m_π is apparently reasonably likely. It is of interest that the empirical requirement is[111]

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[119] M. L. Goldberger, Y. Nambu, and R. Oehme[1]; G. Breit.[66]

[120] G. Breit, Phys. Rev. **51**, 248 (1937); **53**, 153 (1938); cf. discussion following Eq. (17.8) of the first of these two references.

[121] G. Breit, Proc. Natl. Acad. Sci. U.S. **46**, 746 (1960); a preliminary report on this paper was read at the 1960 annual meeting of the National Academy of Sciences in Washington, D. C.

[122] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960); Nuovo cimento **16**, 388 (1960); Phys. Rev. **119**, 1784 (1960).

[123] G. Breit, Phys. Rev. **120**, 287 (1960).

[124] Y. Nambu, Phys. Rev. **106**, 1366 (1957).

[125] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

that the sign of V_{LS} depends on T and that for T = 1 (triplet-odd states) the sign is the same as required by shell structure, i.e., $v_{LS} < 0$ if the interaction potential is written as

$$V_{LS} = v_{LS}(\mathbf{L}\cdot\mathbf{S})$$

while for triplet even states $v_{LS} > 0$. The meson mass fitting was done for T = 1 because the most definite indications for the presence of V_{LS} has come from these states. It is noteworthy that a continuation[126] of the previously quoted calculations of Gupta yields this relationship of signs for the spin-orbit part of the two-pion exchange potential. For triplet-odd (T = 1) states his calculations give an effect much smaller in absolute value than needed phenomenologically, for triplet-even states (T = 0) an appreciably larger one. Qualitatively, these relations form a consistent picture if one considers the actual V_{LS} as compounded of the neutral vector meson and the TPE effects. On this view V_{LS}^{-} is largely a neutral vector meson effect, the TPE effect being small. On the other hand, the neutral vector meson hypothesis gives the wrong sign for v_{LS}^{+}, while the TPE effect has the correct sign and is too large. The sum of the TPE and neutral vector meson effects appears to have the correct sign ($v_{LS}^{+} > 0$) and to be of roughly the right magnitude as has been mentioned.[112] At that time, prepublication news concerning the discovery[2] of the ω meson by Maglić, Alvarez, Rosenfeld, and Stevenson became available to the Conference through Noyes and it appeared reasonable to speculate on the possibility that the ω meson might be the neutral vector meson previously postulated. Its mass of $\sim 5.7 m_\pi$ happens to fit the calculations with the Yale potential mentioned earlier. The long mean life of this meson and its isospin fit in with the requirements.

On the other hand, the theory of the V_{LS} is not quantitative and it is not certain that it provides the ultimate explanation, as will next be discussed. The following matters appear relevant. (a) If the vector meson is to provide a combined explanation of the repulsive core and of the spin-orbit interaction the calculations with phenomenological potentials must be revised. These employ hard cores while the vector meson interaction implies a soft core. Furthermore, the shape of the phenomenological v_{LS} plot against r is not correctly given by the shape expected on the neutral vector meson hypothesis as has been assumed. (b) The presence of the soft core having its origin in interactions of other than pionic origin makes it necessary to revise the pion theoretical calculations[82,126] which are applicable only in their absence. (c) Strictly speaking, the "potentials" following from the neutral vector meson hypothesis and from the pion interaction with the latter as treated by Gupta are quantities of different types. The former is, in fact, more closely related to a potential that may be used in a Schrödinger two-body equation while the latter has a more direct relationship to the scattering matrix.

In addition to these unsettled questions, there are other doubtful points, perhaps of a more fundamental character. (A) If the neutral vector meson coupling is caused by a nucleonic charge q as in (12.5), then an attraction between a nucleon and an antinucleon is to be expected as an extension of the Dirac-Oppenheimer electron-positron pair theory. This attraction appears to fit observation regarding nucleon-antinucleon scattering in a qualitative manner.[121,122] But it is also known that in nucleon-antinucleon collisions pions are copiously produced and there is various evidence to the effect that the ω mesons do not account for all of the pion production. The general picture suggested by experiment is that if vector mesons are produced in N-$\overline{N}$ collisions, they interact strongly and intimately with the pion field. On this view, the employment of a fixed κ in (12.5) would be a decided oversimplification since a strong interplay between N-$\overline{N}$ pairs, pions and vector mesons would be expected to take place. (B) On account of the interactions of nucleons with pions and the vector mesons, as well as the smearing out of the location of the nucleon by pair theory effects, the nucleon may not be treated as a point and (12.5) may be supposed to be erroneous for this reason also. The empirically established existence of electromagnetic form factors[127] indicates directly the existence of such phenomena. They, in turn, have to be explained by the interaction with the vector field. Qualitative considerations along these lines are possible but a consistent picture must consider the situation self-consistently and there is no knowledge at present regarding the quantitative modifications that would result if a self-consistent treatment of electromagnetic form factors, repulsive cores, and spin-orbit effects were carried through. (C) If the effects just mentioned under (A) were followed through one would be happy to find that the N-$\overline{N}$ system can be bound through the action of the vector meson field to form a pion[128] along the lines of the Fermi-Yang model,[18] a view which is especially attractive on

[126] S. N. Gupta, Phys. Rev. **122**, 1923 (1961).

[127] R. Hofstadter, F. Bumiller, and M. R. Yearian, Revs. Modern Phys. **30**, 482 (1958) and references therein.

[128] Y. Fujii, Progr. Theoret. Phys. (Kyoto) **21**, 232 (1959).

account of the possibilities of extensions to strange particles pointed out by Sakata as an extension of the Fermi-Yang model. It is not known that the Fermi-Yang pion model can be made to work making use of vector mesons rather than weak interactions. If it can, the whole neutral vector meson view would receive encouragement. The possibility of accounting for the observed relationship between the observed pion masses is not necessarily absent because it may be possible to use tensor coupling of the mesons to the nucleons. But if there were a satisfactory model of this type, the virtual N-$\overline{N}$ pairs produced by the vector mesons would be producing virtual pions with an even larger probability than they do on the ordinary pion theory. Even if the Fermi-Yang model should not work out, the intimate interaction of the three fields would be expected to be present under the complicated conditions of nucleon-nucleon interactions. Since Fermi and Yang found that the pion-nucleon interaction indicated by their approximate considerations was a linear combination of pseudoscalar and pseudovector couplings, a similar situation is likely to arise if neutral vector mesons are used as the cement. The OPE would not be affected but there would be effects on the TPE and all of the quantitative attempts of treating V_{LS} would have to be revised. (D) The discovery of resonant states other than the ω meson[2] indicates that pions can interact in many ways and that therefore the direct application of the *PSps* theory may be improper.

The different potentials designed to fit N-N scattering data have been recently subjected to tests in calculations of the binding energy of H^3 by Blatt, Derrick, and Lyness.[129] The binding between the three nucleons turned out too weak for the potentials giving the best reproduction of scattering data.[111] In terms of the discussion in Sec. 8, this situation is not surprising from a general point of view. These potentials are good only as ways of parametrizing the collision matrix. Their derivation does not assure their applicability in a Schrödinger equation, except for the one purpose for which they have been designed. In other words, they are good only on the energy shell of the two-particle system while matrix elements off the shell are also needed for the H^3 problem.

While it is easy to explain away the difficulty on such general grounds, it is desirable to find the specific cause of the disagreement. An attempt to do so has been made in[121] with reference to the work of Derrick and Blatt. At that time the calculations with the Gammel-Thaler potential were mainly under discussion. The emphasis now has to be put rather on the newer potentials. Among the arguments mentioned previously, the last one having to do with nucleon-vector meson couplings of the "tensor" type still appears to have validity. This type of coupling could affect the spin-spin and the tensor parts of nucleon-nucleon interactions. A change in the proportions of these effects could change the qualitative aspects of the H^3 calculations. In fact, as pointed out by Derrick and Blatt,[129] the relative largeness of tensor force effects is the main reason for obtaining too weak binding in the triton. The meson theoretical potentials and their relatives appear to rely too much on their tensor part in giving binding in the deuteron and too little on the spin dependence of nuclear forces proposed many years ago by Wigner to be in agreement with the binding energies of light nuclei. In addition, it may be noted that the resonant states which have been found and which can disintegrate into pions will have some tendency of absorbing the virtual pions surrounding nucleons and thus possibly of decreasing the OPE interaction for the smaller internucleon distances. This effect should decrease the strength of the tensor part of the interaction. A fit to the deuteron binding energy will then necessitate an increase in the spin dependence of nuclear interactions giving an effect in the desired direction. This matter has apparently not been treated quantitatively.

Reference may be made to the articles by Phillips,[130] by MacGregor, Moravcsik, and Stapp,[57] Moravcsik and Noyes,[131] and by Gammel and Thaler[132] for reviews of aspects of the subject only touched on or unmentioned in the present article.

Note added in proof. This paper submitted for publication on June 22, 1962. Material, the existence of which was learned afterwards, has not been covered in it, even when this might have been possible in proof.

SUMMARY

Selected aspects of nucleon-nucleon scattering are critically reviewed. An introductory section explains the motivations. In Sec. 2 phase shifts are treated relativistically in a manner similar to the usual non-

[129] J. M. Blatt, G. H. Derrick and J. N. Lyness, Phys. Rev. Letters **8**, 323 (1962); cf. also G. H. Derrick and J. M. Blatt, Nuclear Phys. **8**, 310 (1958).

[130] R. N. J. Phillips, Repts. Progr. in Physics **22**, 562 (1959).

[131] M. J. Moravcsik and H. P. Noyes, Ann. Rev. Nuclear Sci. **11**, 95 (1961).

[132] J. L. Gammel and R. M. Thaler, *Progress in Elementary Particle and Cosmic Ray Physics*, (North-Holland Publishing Company, Amsterdam, 1960), Vol. 5, Chap. II, p. 99.

relativistic one. Section 3 is concerned with relativistic kinematics, Sec. 4 with nonrelativistic treatments and with a brief survey of low-energy p-p scattering. Section 5 deals with relativistic corrections to Coulomb scattering, Sec. 6 with the form of the nucleon-nucleon scattering matrix, and Sec. 7 with the effects of nucleon magnetic moments. General views regarding nucleon-nucleon interactions are surveyed in Sec. 8, and the one-pion exchange interaction in Sec. 9. Tests of charge independence are reviewed in Sec. 10 and evidence concerning two-pion exchange in Sec. 11. Attempts to explain the still somewhat hypothetical spin-orbit potential and repulsive core by means of heavy mesons strongly interacting with nucleons are discussed in Sec. 12.

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