(e) Let $B(M,N)$ be the maximum for $\Psi^\dagger F^\dagger F\Psi$. (A16) implies

$$\Psi^\dagger F^\dagger F\Psi \leqq X^2\beta^2 B(M-2,N) + 4\alpha_1^2(1-X^2) + \beta^2(1-X^2)B(M-2,N-2) + 4\alpha_1(1-X^2)^{1/2}X\beta[B(M-2,N)]^{1/2}, \quad \text{(A17)}$$

where

$$\beta = (1-2\alpha_1^2)^{1/2} \geqq 0, \quad X = (\phi_{00}^\dagger\phi_{00})^{1/2} \geqq 0.$$

(f) We can now prove by induction that for even M and N

$$B(M,N) = N(M-N+2)/M \quad \text{(A18)}$$

as follows: Substitute (A18) into the right-hand side of (A17) and maximize the resultant expression with respect to α_1 and X. After some straightforward algebra, one finds the only maximum of the right-hand side of (A17) at

$$X^2 = (M-N)/M, \quad \alpha_1^2 = 1/M,$$

where it assumes the value of $B(M,N)$ in (A18). The induction is then easily completed. Q.E.D.

It is clear from the above that the only maximum of $\Psi^\dagger F^\dagger F\Psi$ with F having the form (A8) is obtained when

$$\alpha_1 = \alpha_2 = \cdots = M^{-1/2}. \quad \text{(A19)}$$

Furthermore, each pair of states (1,2), (3,4),$\cdots$ is never occupied singly. For such a problem it is easy to see that we can define $M/2$ sets of Pauli spin matrices so that

$$F = M^{-1/2}\sum_{M/2}(\sigma_i^x + i\sigma_i^y).$$

Thus, $$F^\dagger F = M^{-1}[(\sum \boldsymbol{\sigma})^2 - (\sum \sigma^z)^2 - 2\sum \sigma^z]. \quad \text{(A20)}$$

The condition that the total number of particles is N is

$$\sum \tfrac{1}{2}(1-\sigma^z) = N/2. \quad \text{(A21)}$$

Equations (A20) and (A21) show that there is only one largest eigenvalue for $F^\dagger F$ consistent with (A21).

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Effective Interactions and Coupling Schemes in Nuclei*

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I. INTRODUCTION

SHELL-model calculations of nuclear-energy levels can be carried out only by using *effective interactions* between the nucleons. The interaction between *free* nucleons is highly singular and leads to strong short-range correlations between them. Shell-model wave functions contain no such correlations and therefore do not furnish an exact description of nuclear states. Still, under certain conditions, these functions can be used for energy calculations. To do this, it is necessary to introduce the effect of the short-range correlations into the interaction Hamiltonian. Under favorable conditions this modification results in the replacement of the free-nucleon interaction by a reaction matrix or effective interaction.

The derivation of the effective interaction in finite nuclei has not yet been carried out. It is, therefore, impossible to know in advance whether the shell model can be used for the calculation of nuclear energies. The only way to find the answer to this question is by trying to carry out such calculations. In the past, many such attempts have been made. The main difficulty has been the lack of information about the effective interaction to be used. In the last few years an approach which avoids this difficulty has been used. The effective nuclear interaction was determined in several cases from the experimental energies. The consistency of the shell-model description of these cases was checked as follows. If the effective interaction between nucleons is, indeed, a two-body interaction, its matrix elements in n-

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particle configurations are linear combinations of the matrix elements in two-particle configurations. The restriction to two-body (effective) interaction thus imposes certain conditions on nuclear energies. If these conditions are satisfied by the experimental energies, we conclude that the shell model may be used for the calculation of the energies considered. In such cases we obtain a reliable set of matrix elements of the effective interaction. These can be used to predict other energies. On the other hand, only if the experimental data do not satisfy these conditions is it possible to conclude that the specific shell-model wave functions used do not give an adequate description. It often happens that energy levels can be consistently given by the shell model and yet cannot be reproduced with simple minded phenomenological interactions.

The very idea of an effective interaction involves configuration admixtures. A hard core in the interaction potential admixes to any given shell-model configuration many configurations that may lie arbitrarily high. The combined effect of these configuration mixtures is to give an effective interaction from which the hard core has been eliminated. The effect of the short-range correlations described by these configuration admixtures can therefore be replaced by a modification of the two-body interaction. This is, indeed, the case if the admixtures are small and the perturbation series is convergent. It is also important to realize that this is the case only if the excited configurations differ from one another by the quantum numbers of two, four, six, etc., particles. The matrix elements of the interaction which appear in these expansions involve configurations that differ by the quantum numbers of two nucleons. Due to momentum conservation this is the only type of excitation that may occur in infinite nuclear matter. For finite nuclei, however, another type of configuration admixtures is possible which involves single particle excitations. Later on these will be considered in detail.

Thus, configuration interaction may be replaced in many cases by a modification of the two-body interaction. In our approach it is impossible to recognize the contributions of the various configurations. However, there may be cases in which the effect of the configurations admixed may vary from nucleus to nucleus and even be different for the various states of the same nucleus. In such cases, configuration interaction will have to be considered explicitly. It is clear that the inclusion of configuration interaction in such cases makes sense only if the zero-order approximation gives, at least, a rough agreement. The question whether configuration interaction is important or not is not a matter of belief or arguments. In the case of every level, it is decided by the ratio of the nondiagonal matrix elements to the distance of the corresponding perturbing levels. The effect may be negligible for certain levels (e.g., certain ground states) and more important for others.

In trying to determine matrix elements of the effective interaction from the experimental data it is imperative to use wave functions which are as simple as possible. Only then can one hope to extract useful and reliable information from the available data. If the wave functions used are complicated (e.g., involve several configurations in the most general form) there may be too many matrix elements of the effective interaction. The few available experimental data will then not determine all these theoretical parameters. Even if good agreement is obtained by a "reasonable" choice of the undetermined parameters, it cannot be called quantitatively significant. Good agreement can be obtained only if the data are consistent with the theory. In other words, if several theoretical parameters reproduce accurately many more experimental data.

The simplest possible wave functions are those given by the jj-coupling shell model. A jj-coupling configuration has generally less states than the corresponding LS-coupling configuration. Any intermediate coupling calculation involves several jj-coupling configurations and the interactions between them. The prevalent use of jj coupling is not because its functions give the most accurate description of nuclear states. Even though it is only an approximation, jj coupling is used in order to be able to obtain quantitative agreement. In some cases the agreement obtained is quite spectacular.

One such case is the relation between the jj-coupling configurations in Cl^{38} and K^{40}. In ${}_{17}Cl_{21}^{38}$ there is one $1d_{3/2}$ proton outside closed shells and one $1f_{7/2}$ neutron outside closed shells (including the closed $1d_{3/2}$ neutron shell). In ${}_{19}K_{21}^{40}$ there are three $1d_{3/2}$ protons and one $1f_{7/2}$ neutron outside the same closed shells. The $d_{3/2}f_{7/2}$ configuration of Cl^{38} should have four states with $J = 2, 3, 4, 5$. The K^{40} configuration is that of a $d_{3/2}$ proton *hole* (one $d_{3/2}$ proton missing from closed shells) and one $f_{7/2}$ neutron and therefore also has four states with $J = 2, 3, 4, 5$. The $d_{3/2}\, f_{7/2}$ interaction energy in the four-nucleon configuration is a linear combination of the interaction energies in the two-nucleon configuration of Cl^{38} and vice versa. The kinetic energy is the same in all states of a given configuration. Also, the mutual interaction of the three $d_{3/2}$ protons in

K^{40} is the same in all states since only one antisymmetric state, with $J = 3/2$, is available for these protons. Thus, starting from the K^{40} spectrum, the level spacings in Cl^{38} can be calculated.[1,2] The results of such a calculation are presented in Fig. 1 and compared with the experimentally measured level scheme of Cl^{38}.

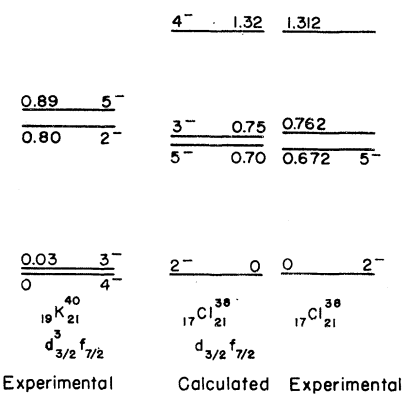

FIG. 1. Related configurations in Cl^{38} and K^{40}.

The good agreement obtained in Fig. 1 justifies the assumptions as well as the configuration assignments. The only essential assumption is that of jj coupling and effective two-body interactions between nucleons. The results indicate not only that a d proton hole and a f neutron hole couple in jj coupling (a single-particle or single-hole wave function is the same in jj coupling and LS coupling). They indicate that jj coupling also holds in the d^7 configuration of Cl^{38}. Only then do we have a closed $d_{5/2}$ shell (and $s_{1/2}$ shell) and a single $d_{3/2}$ proton. Another important result is that the matrix elements of the two-body effective interaction change very little by going from Cl^{38} to K^{40}. This also happens to be the case when larger groups of nuclei are considered. This fact greatly simplifies things and enables the achievement of quantitative agreement in many cases. The example described above is so simple and the agreement is so good, that it could serve as a useful check for phenomenological potentials. Potential interactions that give rise to very strong configuration interactions in all cases would probably fail to give good agreement for both Cl^{38} and K^{40}.

In the following, the information obtained from nuclei about the effective interaction will be presented. The general properties of this interaction will be discussed as well as their implications on the coupling scheme in nuclei.

II. INTERACTIONS BETWEEN IDENTICAL NUCLEONS

In this section the matrix elements of the effective interaction between identical nucleons will be considered. We shall thus look at configurations with either protons or neutrons outside closed shells. The low-lying levels belong, generally, to a configuration in which all extra protons or neutrons outside closed shells are in one j orbit. We thus consider j^n configurations of n such identical particles.

As a concrete example for such configurations, let us consider the oxygen isotopes beyond O^{16}. In O^{16} both protons and neutrons are in closed shells. It is really difficult to excite such a structure as evident from the height of the first excited state—6 MeV above the ground state. The next nucleus is O^{17} with one neutron outside closed shells. In the ground state this neutron occupies the $1d_{5/2}$ orbit as evident from the spin, parity, and magnetic moment of O^{17}. Excited states of O^{17} can now be obtained by raising the extra neutron into a higher single-particle orbit. In this way the following states are obtained: $1/2^+$ at 0.87 MeV (presumably the $2s_{1/2}$ orbit), $7/2^-$ at 3.85 MeV (presumably the $1f_{7/2}$ orbit), and $3/2^+$ at 5.08 MeV (presumably the $1d_{3/2}$ orbit). We take these energies as effective single-neutron energies in the central field due to the O^{16} core to be used in shell-model calculations. As mentioned above, no calculation of these energies nor any explanation of their order has yet been given. It should be remarked that the $d_{5/2}$-$d_{3/2}$ splitting (due to effective single-neutron spin-orbit interaction) is rather large and even larger than the $d_{5/2}$-$f_{7/2}$ splitting. Various statements have been made about the separations of major oscillator shells from each other. Here is an example of how schematic such statements are. As more $d_{5/2}$ nucleons are added, their stronger interaction with a $d_{3/2}$ nucleon lowers its position relative to the $f_{7/2}$ orbit. Eventually, the standard order of these two orbits is reached. The order of single-nucleon levels depends critically on the occupied orbits and may differ considerably from any schematic picture. Other cases of relative movements of single-nucleon levels will be considered later.

Other levels in O^{17} which also involve excitations of nucleons from one oscillator shell to the next are the $1/2^-$ level at 3.06 MeV and the $3/2^-$ level at 4.55 MeV. These levels presumably contain excitations of single nucleons from the $1p_{1/2}$ and $1p_{3/2}$ orbits, respectively. The order of these hole levels is reversed as compared to the single nucleon $1p_{3/2}$ and

[1] S. Goldstein and I. Talmi, Phys. Rev. **102**, 589 (1956).
[2] S. P. Pandya, Phys. Rev. **103**, 956 (1956).

$1p_{1/2}$ levels. It is seen that the presence of the single $1d_{5/2}$ neutron makes this type of excitation much lower in energy as compared with O^{16}. This "softening of the core" is probably due mainly to the gain in $d^2_{5/2}$ pairing energy. We shall not discuss these levels further, since they involve excitations of both protons and neutrons from the closed shells.

In O^{18} there are two neutrons outside the O^{16} core and the probable low configuration is $d^2_{5/2}$. This configuration has antisymmetric states with $J = 0, 2, 4$. Indeed a 2^+ state lies at 1.98 MeV and a 4^+ state at 3.55 MeV. Another 0^+ state lies at 3.63 MeV and another 2^+ state at 3.92 MeV. We interpret these higher states as due to the $2s^2_{1/2}$ configuration and $d_{5/2}\, s_{1/2}$ configuration, respectively. The $s^2_{1/2}$configuration has a higher single neutron energy ($2 \times 0.87 = 1.74$ MeV), as well as a lower pairing energy.[3] The $d_{5/2}\ s_{1/2}$ configuration has a higher single-neutron energy, and an interaction energy much smaller than the $d^2_{5/2}$ pairing energy. Naturally, there may be some interaction between the two 0^+ states as well as between the two 2^+ states. In first approximation we ignore this interaction, part of which may still appear in the matrix elements of the effective interaction. Thus, making this assumption about the O^{18} levels, we can make use of them in obtaining information on the matrix elements of the effective interaction in the $1d^2_{5/2}$ configuration. Only when more experimental data are incorporated will it be possible to check the consistency of the approach. To achieve this we have to go to the $d^3_{5/2}$ configuration in O^{19}.

The interaction energies in the $d^3_{5/2}$ configuration (whose antisymmetric states have $J = 5/2,\ 3/2,\ 9/2$) are linear combinations of those in the $d^2_{5/2}$ configuration. The coefficients in these linear combinations involve coefficients of fractional parentage. However, for $j = 5/2$ a closed formula exists for the interaction energies in the j^n configuration. These energies are given by[4,5]

$$V(j^n,J,v) = \tfrac{1}{2}\, n(n-1)a + \left(J(J+1) - \tfrac{35}{4}n\right)b + \tfrac{1}{2}\,(n-v)(8-n-v)c\,. \qquad (1)$$

The coefficients a, b, and c in (1) depend on the nature of the interaction. If the two-particle interaction energies, $V_J = V(j^2,J,v)$, are given, a, b, and c can be calculated by equating V_J to the expression (1) with $n = 2$. The quantum number v is the *seniority* of the state considered. For $J = 0$ ($n = 2,4,6$) it is $v = 0$, for $J = 2,4$ ($n = 2,4$) we have $v = 2$, for $J = 5/2$ ($n = 1,3,5$), $v = 1$, and for $J = 3/2,\ 9/2$ ($n = 3$) it is given by $v = 3$. With the help of (1), the $(5/2)^3$ interaction energies can readily be expressed in terms of the V_J. These expressions are demonstrated graphically in Fig. 2. The level spacings of the $(5/2)^2$ and $(5/2)^3$ configurations are plotted there, relative to the $J = 0$ and $J = 5/2$ states, respectively, as a function of a certain parameter which characterizes the interaction. The $V_0 - V_2$ spacing is fixed throughout. At the left end of the diagram b is equal to zero and $V_2 = V_4$. At the right end, c is equal to zero and the energies have the $J(J+1)$ dependence. Between these two limits $V_0 - V_4$ is extrapolated linearly. It is seen how for $b = 0$, the 3/2 and 9/2 levels of the $(5/2)^3$ configuration coincide since the energy depends only on the seniority. For $c = 0$, these levels are well separated and the 3/2 state even becomes the ground state.

The interaction for which $b = 0$ (as well as $a = 0$) is called a "pairing interaction." It has been used recently as a phenomenological interaction (also for $j > 5/2$). The spacing between the $J = 0$, $v = 0$ ground state and the degenerate $v = 2$ ($J = 2,\ 4,\ 6\ldots$) levels is referred to as the "energy gap." If this were to give a good description of nuclear spectra the $J = 4$ and $J = 2$ levels in O^{18} were very close in energy. Unless we want to use different forces for different levels, we must conclude that the "pairing interaction" is very different from the actual effective interaction in nuclei. We also see that the δ force (which is sometimes described as the physical basis for using the "pairing interaction") fails to give the observed large spacing between the $J = 2$ and $J = 4$ levels.

The inadequacy of the "pairing interaction" (as well as the δ force) is demonstrated much more clearly in the $d^3_{5/2}$ configuration. In the $b = 0$ limit, the 3/2 and 9/2 states should be degenerate and lie two-thirds of the "energy gap" above the 5/2 ground state. In O^{19} there is a $3/2^+$ level, 0.1 MeV above the $5/2^+$ ground state. This level most probably belongs to the $d^3_{5/2}$ configuration and has a very small admixture of the $d^2_{5/2}\, d_{3/2}$ configuration. This is evident from its very small separation from the 5/2 ground state (compare the 5 MeV $d_{5/2} - d_{3/2}$ splitting in O^{17}), from its not showing any stripping (which could go only by a $d_{3/2}$ neutron capture) and the much attenuated $M1$ transition to the ground state.[5] Thus, the energy gap seems to be nonexistent in O^{19}. The $9/2^+$ level in O^{19} has not yet been identified. The only other level known in O^{19}, below 3 MeV is a $1/2^+$ level

[3] I. Talmi and I. Unna, Nuclear Physics **30**, 280 (1962).
[4] G. Racah, *L. Farkas Memorial Volume* (Research Council of Israel, Jerusalem, 1952), p. 294.
[5] I. Talmi and I. Unna, Annual Rev. Nuclear Sci., **10**, 353 (1960).

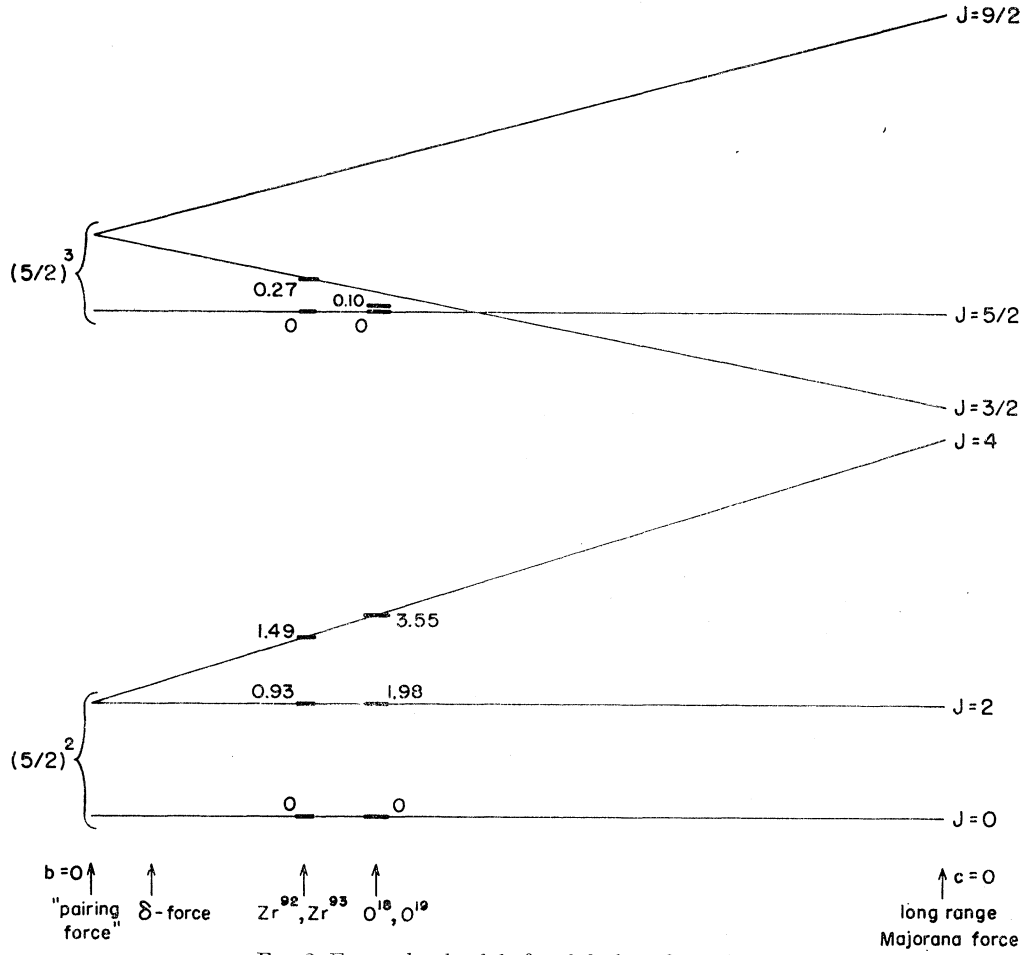

FIG. 2. Energy levels of $d_{5/2}{}^2$ and $d_{5/2}{}^3$ configurations.

at 1.47 MeV, presumably due to the $d^2_{5/2}$ $s_{1/2}$ configuration.

On the other hand, if we take the matrix elements of the effective interaction from O^{18}, the $3/2^+$ level is predicted to lie 0.3 MeV above the ground state which is in reasonable agreement with experiment. A much better agreement is obtained in the case of the $2d^n_{5/2}$ configurations in the Zr isotopes.[6] The position of the $3/2^+$ state in $_{40}Zr^{93}_{53}$ is calculated from the Zr^{92} levels at 0.26 MeV. The experimental $3/2^+$ level lies 0.27 MeV above the $5/2^+$ ground state. In Zr^{94} with the $d^4_{5/2}$ configuration, which is complementary to the $d^2_{5/2}$ configuration, the positions of the $J = 2$ and $J = 4$ levels above the ground state (at 0.92 and 1.47 MeV, respectively) are almost exactly the same as in Zr^{92} (at 0.93 and 1.49 MeV, respectively). This agreement is, indeed, better than in the oxygen isotopes where in O^{20} the $J = 2$ level is 1.68 MeV above the ground state as compared with 1.98 MeV in O^{18} (the $J = 4$ level has not yet been identified). The experimental level spacings in the oxygen and zirconium isotopes are marked in Fig. 2.

The occurrence of low lying states with $J = j - 1$ (and $v = 3$) in j^3 configurations is by no means confined to O^{19} or the $d^3_{5/2}$ configurations. In the $1f_{7/2}$ neutron shell (between Ca^{41} and Ca^{48}) and the $1f_{7/2}$ proton shell (Sc^{49} to Ni^{56}) low-lying $5/2^-$ states occur in $f^3_{7/2}$ configurations as well as in the complementary $f^{-3}_{7/2}$ ($\equiv f^5_{7/2}$) configurations. These states show no stripping and decay by very slow $M1$ transitions. Their positions agree well with that calculated from the $f^2_{7/2}$ configurations (occurring in even nuclei).[7] The spacings between the $J = 2,4,6 (v = 2)$ levels relative to the $J = 2 - J = 0$ spacings are much bigger than predicted by the δ force (and are cer-

[6] I. Talmi, Phys. Rev. (to be published).

[7] I. Talmi, *Proceedings of the Rehovoth Conference on Nuclear Structure*, edited by H. J. Lipkin (North-Holland Publishing Company, Amsterdam, 1958), p. 31.

tainly far from being degenerate as suggested by the "pairing interaction"). As a result, the calculated $J = 5/2 - J = 7/2$ separation is much smaller than the $J = 0 - J = 2$ separation and agrees well with the experiment. In the $1g^n_{9/2}$ configurations a very low-lying $7/2^+$ level is observed whose position can be calculated very well from the $g^2_{9/2}$ spectrum in in neighboring nuclei.[8] Some level schemes, taken from experiment, are given in Fig. 3, exhibiting the

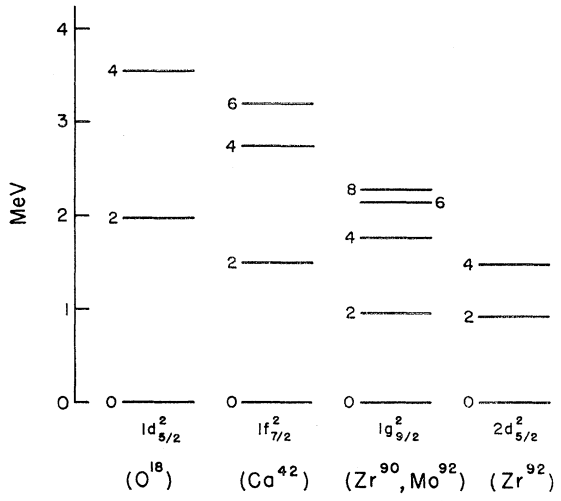

FIG. 3. Energy levels of j^2 configurations.

actual behavior of the matrix elements of the effective interaction and the degree of departure from the "pairing interaction."

The level schemes presented in Fig. 3 are taken from specific nuclei, but only in cases where there are enough experimental data to check the validity of the description in terms of j^n configurations. Although more consistent values for the energy levels can be obtained by least-squares fits, they do not change the general features of Fig. 3. All these spectra deviate appreciably from those obtained from short-range potentials. In fact, if one tries to obtain the experimental level spacings by using a Majorana potential (which gives in the long-range limit a $J(J+1)$ behavior of level spacings), its range has to be much bigger than the "reasonable value" usually adopted in such calculations (which is obtained from the pion mass).

Comparing the spectra of different nuclei, we see that there is a decrease of the level spacings as a function of mass number. This corresponds to a decrease of the interaction energy in the $J = 0$ state (pairing energy) to be discussed later on. This decrease is obviously due to the fact that in larger nuclei the outer nucleons are farther apart. It is also seen that this decrease is less pronounced for the states with higher l values (and no radial nodes) like $1f_{7/2}$ and $1g_{9/2}$. Unfortunately, not enough level spectra are known for a quantitative study of this effect.

The preceding discussion was concerned with level spacings. The energies of ground states will now be considered. For $j > 5/2$ no simple closed expression such as (1) is available for interaction energies. However, things are simpler if we consider only the *average* interaction energies in groups of states with the same seniority v in the j^n configuration. These averages are given by the following expression,[4,5] which is even simpler than (1),

$$V(j^n,v) = \tfrac{1}{2}n(n-1)a + \tfrac{1}{2}(n-v)(2j+3-n-v)b\,. \qquad (2)$$

The coefficients a and b in (2) are given by the special case of that expression for $n = 2$. In that case we obtain

$$\begin{aligned} a + (2j+1)b &= V(j^2,0) = V_0 \\ a &= V(j^2,2) = \overline{V}_2 \\ &= \sum_{J>0 \text{ even}} (2J+1)V_J \Big/ \sum_{J>0 \text{ even}} (2J+1) \\ &= \frac{1}{(j+1)(2j-1)} \sum_{J>0 \text{ even}} (2J+1)V_J \end{aligned} \qquad (3)$$

or explicitly

$$\begin{aligned} a &= \overline{V}_2 \\ b &= (V_0 - \overline{V}_2)/(2j+1)\,. \end{aligned} \qquad (4)$$

The formula (2) refers to the interaction energy of a group of actual eigenstates, if the interaction energy is diagonal in the seniority scheme (this is always the case for a two-body interaction and $j \leq 7/2$). In the general case it represents the average expectation value of the interaction in the group of states with given seniority.

In particular, there is only one state with $v = 0$ (namely, a $J = 0$ state) in the j^n configuration if n is even, and only one $v = 1$ state (a $J = j$ state) if n is odd. Thus, (2) can be directly applied to the interaction energies of ground states of j^n configurations of identical nucleons. Substituting $v = 0$ or $v = 1$ in (2), a simpler expression results as follows:

$$\begin{aligned} V(j^n,\text{g.s.}) &= \tfrac{1}{2}n(n-1)(a-b) + [\tfrac{1}{2}n](2j+2)b \\ &= \tfrac{1}{2}n(n-1)\alpha + [\tfrac{1}{2}n]\beta\,. \end{aligned} \qquad (5)$$

[8] I. Talmi and I. Unna, Nuclear Phys. 19, 225 (1960).

The $[n/2]$ in (5) is the step function which is equal to $n/2$ if n is even and $(n-1)/2$ if n is odd ($[x]$ is the largest integer not exceeding x). The coefficients α and β in (5) are obtained from (4) to be

$$\alpha = \frac{2(j+1)\overline{V}_2 - V_0}{2j+1} \quad \beta = \frac{2(j+1)}{2j+1}(V_0 - \overline{V}_2) . \tag{6}$$

The general formula (5) is very simple indeed. It contains only a term quadratic in n and a pairing term. This simplicity is a direct result of the seniority scheme. Equation (5) holds for any two-body interaction whether central or noncentral, local or nonlocal.

The total energy of the n nucleons outside closed shells also contains, in addition to the mutual interaction (5), a sum of n equal single-particle energies. Each of these energies, to be denoted by C, is the sum of the kinetic energy of a single j nucleon and its interaction with the closed shells. The total binding energy of the nucleus also contains the energy of the closed shells. In order to separate these two parts, we subtract from the binding energy of the nucleus considered the binding energy of the nucleus with no j nucleons outside closed shells. The remaining energy is given according to our assumptions by

$$\text{B.E.}(j^n) - \text{B.E.}(n=0) = nC + \tfrac{1}{2}n(n-1)\alpha + [\tfrac{1}{2}n]\beta . \tag{7}$$

It should be realized that any changes linear in n of the binding energy (B.E.) of the closed shells are absorbed into the first term in (7). Similarly, any quadratic changes of B.E.$(n=0)$, as well as linear changes in C, are absorbed into the second (quadratic) term in (7). Actually, some of these changes can be expressed as due to configuration interaction and, as such, are included, by definition in the effective interaction (5).

It is now possible to check the consistency of the model by trying to find, in a given shell, constants C, α, and β that will reproduce accurately the experimental energies. This procedure will be meaningful if there are more than three experimental energies. In that case, the theoretical parameters are determined by a least-squares fit, so that they best reproduce the experimental energies. When these best values are introduced into (7), the calculated values are obtained. These should be compared with the experimental energies to check the agreement. An example of such an analysis, for the proton and neutron $f_{7/2}^n$ configuration, is given in Table I.[9] For the sake of convenience, the binding energies are taken to be positive numbers. This convention will also be used in the following. The agreement obtained is seen to be very good and statistically significant.

TABLE I. Binding energies of $1f_{7/2}^n$ configurations (in MeV).

Nucleus	Binding energy[a] Experimental	Calculated	Nucleus	Binding energy[b] Experimental	Calculated
$_{20}Ca_{21}^{41}$	8.36	8.38	$_{21}Sc_{28}^{49}$	...	9.69
$_{20}Ca_{22}^{42}$	19.83	19.86	$_{22}Ti_{28}^{50}$	21.78	21.72
$_{20}Ca_{23}^{43}$	27.75	27.78	$_{23}V_{28}^{51}$	29.82	29.86
$_{20}Ca_{24}^{44}$	38.89	38.80	$_{24}Cr_{28}^{52}$	40.34	40.32
$_{20}Ca_{25}^{45}$	46.31	46.26	$_{25}Mn_{28}^{53}$	46.90	46.90
$_{20}Ca_{26}^{46}$	56.72	56.82	$_{26}Fe_{28}^{54}$	55.75	55.80
$_{20}Ca_{27}^{47}$	...	63.81	$_{27}Co_{28}^{55}$	60.85	60.81
$_{20}Ca_{28}^{48}$	73.95	73.93	$_{28}Ni_{28}^{56}$	...	68.17

[a] From these the binding energy of Ca^{40} was subtracted.
[b] From these the binding energy of Ca^{48} was subtracted.

Equation (7) is, in fact, a mass formula covering a limited region. It very accurately reproduces the parabolas of the mass surface. Furthermore, the pairing term is a natural result of the seniority scheme and is not added by making *ad hoc* assumptions. The fact that the semi-empirical mass formula also has similar terms can now be justified on the basis of the shell model. The appearance of such terms in the semi-empirical mass formula *is not* the basis of (7). The expression (7) is an exact mathematical result that holds for any interaction *in the seniority scheme.*[10] Another feature of (7), not shared by the semi-empirical mass formula, is the fact that the coefficients C, α, and β are constants. The values of these parameters depend, however, on the shell considered and change discontinuously from one shell to the next. In the semi-empirical mass formula these discrete jumps are replaced by a smooth variation with mass number. This way a formula that covers a greater region is obtained, yet the detailed agreement offered by (7) is lost.

The theoretical parameters C, α, and β that give the best fit in the case of the neutron $f_{7/2}^n$ configurations are given (with the statistical errors) by

$$C = 8.38 \pm 0.05 \quad \alpha = -0.23 \pm 0.01$$
$$\beta = 3.33 \pm 0.12 \text{ MeV} . \tag{8}$$

[9] I. Talmi, Phys. Rev. **107**, 326 (1957).

[10] Occasionally one wonders whether the existence of such a simple formula should not have been kept as a trade secret. The good agreement in Table I, had it been obtained by the lengthy procedure of using tables of coefficients of fractional parentage, Racah coefficients, etc., would seem much more impressive.

For the proton $f^n_{7/2}$ configurations the corresponding quantities are

$$C = 9.69 \pm 0.04 \quad \alpha = -0.78 \pm 0.01$$
$$\beta = 3.11 \pm 0.09 \text{ MeV} . \qquad (9)$$

In both cases the quadratic term is repulsive (negative in our convention) and the pairing term is attractive (positive). Comparing the sets (8) and (9), the values of C should not be expected to be equal. In the case of the protons, C also contains the electrostatic repulsion of one $f_{7/2}$ proton and the protons in closed shells. This strong repulsion is more than compensated for by the interaction of the $f_{7/2}$ proton with the closed $f_{7/2}$ neutron shell. The $f_{7/2}$ neutrons in the Ca isotopes do not have this extra interaction. Similarly, the values of α and β for the protons should also be reduced by the Coulomb interaction in comparison with the neutron values. This is, indeed, the case for the values of α and β in (8) and (9). The value of the Coulomb pairing energy[11] is equal, with the quoted errors, to the differences in β between (8) and (9). The change in α is, however, bigger than expected, and may reflect changes in the effective interaction, due to the different closed shells in the two cases as mentioned in the discussion following (7).

The only parameter that is related to excited states is the coefficient β of the pairing term. It is simply related, by (6), to the position of the center of mass of the $v = 2$ $J = 2, 4, 6 \ldots$ levels, $\overline{V}_2$, above the $J = 0$ ground state. In the Ca^{42} spectrum (Fig. 3) this center of mass lies 2.7 MeV above the ground state. This value is in fair agreement with the values 2.96 ± 0.11 MeV and 2.76 ± 0.08 MeV obtained from (8) and (9), respectively. A similar analysis of the neutron $2d^n_{5/2}$ configurations beyond Zr^{90} also gives good agreement between (7) and experiment.[6] The theoretical parameter β obtained from this analysis is equal to 1.50 MeV. The value of $V_0 - \overline{V}_2$ obtained from this is $1.50\,(6/7) = 1.29$ MeV. This agrees very well with the value obtained from the $2d^2_{5/2}$ excited states (Fig. 3). The center of mass of the $J = 2$ and $J = 4$ levels lies in Zr^{94} at 1.27 MeV and in Zr^{92} at 1.29 MeV above the ground state.

The energy parameters C, α, and β obtained from this type of analysis in various configurations are presented in Table II. The behavior of the pairing parameter β as a function of mass number is very similar to the trends of the energy spacings of Fig. 3. There is a decrease with mass number which is less pronounced for higher l values (and no radial nodes).

TABLE II. Parameters of the effective interaction in j^n configurations of identical nucleons (in MeV).

Orbit	Single nucleon energy C	Coefficient of quadratic term α	Coefficient of pairing term β
neutron $1d_{5/2}$ [a]	4.2	−0.1	3.8
neutron $1f_{7/2}$	8.38	−0.23	3.33
proton $1f_{7/2}$	9.69	−0.78	3.11
proton or neuton $1g_{9/2}$ [a]	...	...	2.2
neutron $2d_{5/2}$	7.28	−0.19	1.50

[a] Tentative values.

The energy parameter α turns out to be repulsive (negative) in all cases considered. This is required by the saturation properties of nuclear energies. As mentioned earlier, the quadratic term is responsible for the fact that nuclei to which more and more nucleons of the same kind are added become less and less stable. Dealing with energies of actual nuclei, this term must be repulsive. We shall make further comments on this point but first take notice of a simple yet interesting fact. Due to (7), as well as to the corresponding expression for both protons and neutrons outside closed shells to be given later, nuclei with magic proton or neutron numbers have no extra binding energy as compared to preceding even nuclei. Moreover, the occurrence of the repulsive term makes such nuclei have relatively less binding energy than some preceding even nuclei. This is clearly demonstrated by considering the separation energy of the last nucleon. This separation energy is given by C for $n = 1$, is higher by the value β for even n than for odd n, due to the pairing term, and is decreasing linearly with n, due to the quadratic term. Magic nuclei are more stable than others, only because the nuclei beyond them have much lower separation energies. Stability is not a property of a single nucleus but depends on its neighbors. A nucleus is unstable only if another nucleus is available to which it can decay. Magic nuclei are thus more stable relative to nuclei beyond them. Only the total binding energy or the separation energy is a good objective measure for the amount of binding. The behavior discussed above is clearly demonstrated for the neutron $1f_{7/2}$ shell[9] in Fig. 4 and for the neutron $2d_{5/2}$ shell[6] in Fig. 5. The separation energies are plotted against n and are seen to lie on two parallel straight lines. The separation energy of the first nucleus beyond the closed shells is also plotted, so that the sharp drop can be visualized. The drop in Fig. 4 is much bigger

[11] B. C. Carlson and I. Talmi, Phys. Rev. 96, 436 (1954).

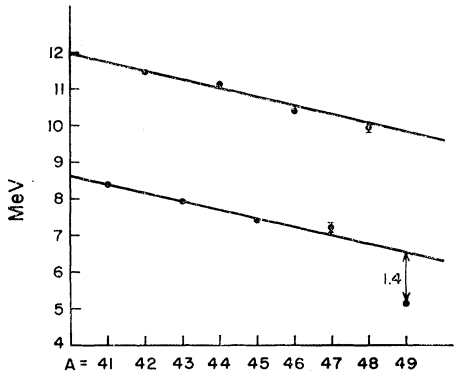

FIG. 4. Neutron separation energies in Ca isotopes.

than the drop in Fig. 5 in agreement with the fact that 28 is a magic number, whereas only a subshell is being closed at $N = 56$.

This behavior of the separation energies should be contrasted with the atomic case. There, the corresponding quantity, called ionization energy, has sharp peaks at the rare gases and sharp drops beyond them. The reason for this behavior is that the ionization energy of the *neutral* atoms is being plotted against Z. As Z increases, the central field due to the nucleus increases and this effect is only partially compensated by the mutual repulsion of the electrons (the "screening" is not very effective within a shell). In an atom of an alkali metal the last electron must occupy another orbit due to the Pauli principle, where its kinetic energy is higher and where it is effectively screened by the other electrons. The increase in ionization energy is not due to the "pairing energy of the electrons" (as stated somewhere). The interaction of the electrons is *repulsive* and therefore they occupy states of highest seniority where the pairing energy is *minimum*. Had the electron pairing energy been important, it would have decreased the ionization energy of atoms with closed electron shells. In nuclei, due to saturation, there is no rise in the separation energy, only the drop at the beginning of a new shell is present. It is true that nuclei with closed shells have more binding energy than calculated with the semi-empirical mass formula. This fact, however, reflects only a shortcoming of that formula in predicting details of the mass surface.

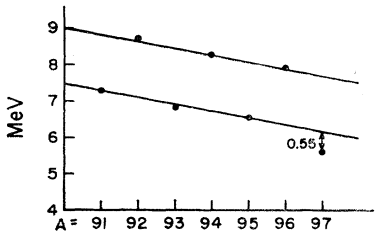

FIG. 5. Neutron separation energies in Zr isotopes.

Let us turn back to the theoretical expression (5). Occasionally statements are made that the interaction energy in the seniority scheme is roughly proportional to n. Therefore, when more nucleons are put into the shell, another coupling scheme takes over for which the energy is roughly proportional to n^2. Looking at (5), we see that the first statement is not very accurate. The interaction energy in the seniority scheme may well have a quadratic term in n, even for interactions for which the seniority is a good quantum number. In any case, if an interaction reproduces the experimental energies, *the quadratic term must be repulsive*. As is well known, there is no group of nuclei where the binding energy increases like n^2. Therefore, the argument that a term proportional to n^2 becomes bigger than a term which goes like n for larger n cannot be applied to actual nuclei in spite of its great simplicity. As to the question of the interaction being diagonal in the seniority scheme, the lowest j value for which it can be checked is $j = 9/2$ (for $j \leq 7/2$, any two-body interaction is diagonal in the seniority scheme). The experimental information on the $g_{9/2}^2$ level spacings shows that the effective interaction is also practically diagonal in the seniority scheme in this case.[8]

Other misleading statements concern the range of the interaction in nuclei. It is alleged that it is the long-range components of the nuclear forces that give rise to the coupling scheme which eventually becomes that of independent nucleons moving in a deformed potential. Actually, the range of the potential is not sufficient to specify its behavior. The exchange character is equally important. For example, any potential multiplied by the spin operator $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$ gives rise to an odd-tensor interaction which is diagonal in the seniority scheme and has the pairing property.[12] Thus, irrespective of the range, the quadratic term vanishes for such an interaction. In the long-range limit it leads to energy spacings determined by a $J(J+1)$ term. Although these spacings look very different from those due to a short-range force, the energy in the ground state is just the sum of pairing energies.

In a recent paper,[13] a criterion for the nuclear forces to have a long range, and even a criterion for the existence of the shell model, is given. The total interaction energy in a closed shell of identical nucleons divided by $2j+1$ is denoted by D. In terms

[12] G. Racah and I. Talmi, Physica **18**, 1097 (1952).
[13] S. Kahana, Nuclear Phys. **31**, 315 (1962).

of this D and $V_0 = \langle j^2 J = 0 | V | j^2 J = 0 \rangle$, that criterion is

$$D > V_0 . \tag{10}$$

The ratio V_0/D is considered a measure for the length of the range. Using (5) for $n = 2j + 1$, we obtain the following expression for the quantity D

$$D = \tfrac{1}{2}(2j\alpha + \beta) = \tfrac{1}{2} V_0 + \tfrac{1}{2}(2j - 1)\alpha . \tag{11}$$

In the case of the δ force, as well as for every interaction with the pairing property, the parameter α vanishes [in this case $2(j + 1)\overline{V}_2 = V_0$], and therefore $V_0/D = 2$. In all cases of interactions that reproduce nuclear energies, α is negative whereas V_0 is positive. Thus, for any such interaction $V_0/D > 2$, and certainly, $V_0 > D$. If the condition (10) had made any sense, it would mean that the saturation properties of the nuclear interaction (which determine the relative sign of α and V_0) would destroy the shell model. It is the current opinion, though, that these saturation properties help establish the shell model. If we still like to use the loose terms "short-range" and "long-range" forces, we see that our results imply that the long-range components of the force (which give rise to the quadratic term) are repulsive within a j shell of identical nucleons.[14]

To summarize, we repeat that within the j^n configuration of identical nucleons the only attractive term in the interaction energies of ground states is the pairing term. The quadratic term is repulsive. Thus, in the ground state of the j^3 configuration, there is a repulsion between one nucleon and the other two. The only attraction in this case comes from the pairing term. This effect is by no means confined to identical nucleons in the same j orbit. The effective nucleon interaction between a j' neutron and a pair of neutrons in the $J = 0$ state of the j^2 configuration is always repulsive. This is also a result of the known fact that adding to a stable nucleus many nucleons of the same kind makes it less stable. This repulsive interaction is given by the simple expression

$$V(j^2(0)j') = V(j'^2(0)j) = 2\sum_{|j-j'|}^{j+j'} (2J + 1) V(jj'J)/(2j + 1)(2j' + 1). \tag{12}$$

The individual interaction energies $V(jj'J)$ need not all be repulsive, but their combination (12) is. It turns out that in actual cases the $V(jj'J)$ are small in their absolute value.[15,5] The absolute value of (12) is a few tenths of MeV in light nuclei and decreases for higher mass numbers.

This summarizes the properties of the effective interaction between identical nucleons. It does not seem probable that the interaction of many of them will lead to an average attractive interaction which will be stronger than the attractive pairing energy in the seniority scheme. In fact, nuclei where either protons or neutrons are in closed shells do not show the characteristic properties due to deformation. The deviations from the seniority scheme and spherical nuclei seem to be connected with the interaction between protons and neutrons outside closed shells. This point will be considered in detail in the next section. Before going into it, we shall summarize the little information available on configuration interaction.

As mentioned earlier, configuration interaction where two nucleons are excited, can be incorporated, under certain conditions, into the effective two-body interaction. Thus, if this interaction can be treated in perturbation theory, there is no simple way to distinguish between the interactions with the various configurations. There are, however, cases where configuration interaction of this type has a strong effect on certain energy levels. This occurs whenever two unperturbed states with the same value of J lie close in energy. If there is an appreciable nondiagonal matrix element of the interaction between them, the two states get strongly admixed. Configuration interaction can also be treated by the method of effective interactions. Also, nondiagonal matrix elements of the effective interaction can be taken as free parameters to be determined from the experimental data rather than from Yukawa or Gaussian phenomenological potentials. The only difficulty is that, in most cases, many nondiagonal elements may be present, so that many experimental data are required for a quantitative analysis.

A simple case where this could be carried out involves the $2p_{1/2}$ and $1g_{9/2}$ orbits. In this case there is only one possible nondiagonal element which connects the $J = 0$ states of the $p^2_{1/2}$ and the $g^2_{9/2}$ configurations. Enough experimental data are known, so that energy levels of these configurations could be treated and fair agreement obtained.[8] We shall not go into the details of that analysis which are presented in reference 8. The nondiagonal element turns out to be about 0.7 MeV. On the other hand, the

[14] Without realizing this fact, it would seem that the pairing force, given by (2) with $a = 0$, which has a repulsive quadratic term and gives $V_0/D = 2j + 1$, has a range shorter than the zero range δ force.

[15] I. Unna and I. Talmi, Phys. Rev. 112, 452 (1958).

spacing of the unperturbed levels in this particular case is of the order of 1 MeV. Thus, fairly large energy shifts are obtained. In Zr^{90} the interaction between the $p^2_{1/2}$ and $g^2_{9/2}$ configurations affects only the positions of the two 0^+ states, the spacings between the levels with $J = 2, 4, 6, 8$ are unchanged. In Mo^{92} the interaction between the $p^2_{1/2}\, g^2_{9/2}$ and the $g^4_{9/2}$ configurations affects the $J = 0$ state of the $p^2_{1/2}g^2_{9/2}$ configuration and the $v = 0$, $J = 0$ state of the $g^4_{9/2}$ configuration. It also shifts the $J = 2, 4, 6, 8$ in the $p^2_{1/2}g^2_{9/2}$ configuration and the $v = 2$, $J = 2, 4, 6, 8$ states in the $g^4_{9/2}$ configuration. However, all $J = 2, 4, 6, 8$ states of the $p^2_{1/2}g^2_{9/2}$ configuration, having $v = 2$, are pushed down by the same amount due to the interaction with the $g^4_{9/2}$ configuration. We therefore expect to find the same level spacings of the $J = 2, 4, 6, 8$ levels in Zr^{90} and Mo^{92}. This is actually the case.[16] The relative positions of the various $J = 0$ states give information about the nondiagonal element of the effective interaction. Here is an example of how configuration interaction strongly affects only certain levels (with $J = 0$) and does not change the spacings of other levels.

Another case where configuration interaction affects only certain states is found in the $f^n_{7/2}$ spectra, considered above. The ground-state energies agree very well with the description in terms of $f^n_{7/2}$ configurations both for proton and neutron configurations. Also, most of the excited states agree very well with this description. In particular, the position of the first excited $5/2^-$ state in $f^3_{7/2}$ or $f^5_{7/2}$ configurations is practically the same in all possible four nuclei. Furthermore, this experimental position agrees with that calculated from the $f^2_{7/2}$ spectra. On the other hand, the position of the $3/2^-$ level in Ca^{43} is only 0.59 MeV above the ground state, much lower than the calculated position of about 1 to 1.2 MeV. It is not difficult to find the perturbing state in this case. In Ca^{41} the single neutron $2p_{3/2}$ level is only 1.95 MeV above the $1f_{7/2}$ ground level. In Ca^{43} the $3/2^-$ ground state of the $f^2_{7/2}p_{3/2}$ configuration may be 2 MeV or even less (due to the $f_{7/2} - p_{3/2}$ interaction) above the $7/2^-$ ground state. Thus, these two $3/2^-$ states (of the $f^3_{7/2}$ and $f^2_{7/2}p_{3/2}$ configurations) may be quite close in energy and, as a result, interact strongly. The admixture of the $f^2_{7/2}p_{3/2}$ configuration in the 0.59-MeV state can be inferred from its large width for $l_n = 1$ stripping (the other perturbing state, with a larger width, is at 2.05 MeV). This could be contrasted with the unperturbed position of the $5/2^-$ state which shows no $l_n = 3$ stripping. In that case, the $1f_{5/2}$ level in Ca^{41} is believed to be higher than 6 MeV above the $1f_{7/2}$ ground state. No quantitative analysis of this effect has yet been carried out. In other cases it occurs only to a lesser degree. In V^{51}, for instance, the $3/2^-$ level is at 0.93 MeV. The smaller deviation from the calculated value is most probably due to the higher energy of the perturbing state. The single nucleon energy depends rather strongly on which of the shells are closed. In Sc^{49}, with one proton outside the Ca^{48} closed shells, there is experimental evidence indicating that the $2p_{3/2}$ level is 3.1 MeV above the $1f_{7/2}$ ground state (as compared to 1.95 MeV in Ca^{41}).

[16] R. van Lieshout, S. Monaro, G. B. Vingiani, and H. Morinaga, Bull. Am. Phys. Soc. **7**, 342 (1962).

III. PROTONS AND NEUTRONS OUTSIDE CLOSED SHELLS

Disregarding the repulsive Coulomb energy, nuclei are most stable when the number of protons is equal to the number of neutrons. This fact, in itself, is sufficient to prove that on the average the interaction between a proton and a neutron is attractive. In the detailed analysis, based on the shell model, this fact is clearly demonstrated. Yet occasionally the statement is made that the proton-neutron interaction is weak. As stated above, this is manifestly not true. Apart from the pairing energy, the only interaction which, on the average, is strong and attractive is the interaction between protons and neutrons. In this section we shall first discuss the available information on the proton-neutron interaction and then see how it can lead to deviations from jj coupling and from spherical symmetry of nuclei.

First, we consider nucleons (both protons and neutrons) in the same j orbit. There are very few cases where experimental information is available on the excited levels of such a configuration. Some of these cases will be considered later, but first we treat energies of ground states—binding energies. The matrix elements of the interaction energy in the j^n configuration are linear combinations of the interaction energies of the j^2 configuration. The coefficients of these linear combinations involve coefficients of fractional parentage and depend on the values of J, T (total isospin, for charge-independent interactions) and other quantum numbers of the states considered. However, if we consider in the j^n configuration, the expectation values of the interaction energy, averaged over a group of states with the same value of T and the same quantum numbers of the seniority scheme, a great simplification occurs. Such average interaction energies are simple linear combinations of the average interactions in the j^2 configuration. In the j^2 configuration the $T = 1$,

$J = 0$ state is characterized by $v = 0$, $t = 0$, where v is the seniority and t the reduced isospin.[17] The other states with $T = 1$, which have $J = 2, 4, 6, \ldots, 2j - 1$ have seniority $v = 2$ and $t = 1$. The states with $T = 0$ and $J = 1, 3, 5, \ldots, 2j$ also have $v = 2$ but a different reduced isospin, $t = 0$. The average interaction energies in the j^n configuration are thus linear combinations of V_0, $\overline{V}_2$ which is defined by (3) and

$$\overline{V}_1 = \sum_{J \text{ odd}} (2J+1)V_J \Big/ \sum_{J \text{ odd}} (2J+1) = \frac{1}{(j+1)(2j+1)} \sum_{J \text{ odd}} (2J+1)V_J \,. \quad (13)$$

The average interaction energies in a group of states of the j^n configuration having the same values of T, v, and t can be expressed in a form analogous to (2). This expression is given by[4,5]

$$\tfrac{1}{2}n(n-1)a' + (T(T+1) - \tfrac{3}{4}n)b' + Q(n,v,t,T)c' \,. \quad (14)$$

The eigenvalues of the seniority operator Q are given by

$$Q(n,v,t,T) = \tfrac{1}{4}(n-v)(4j+8-n-v) - T(T+1) + t(t+1) \,. \quad (15)$$

The parameters a', b', and c' are simple linear combinations of V_0, $\overline{V}_2$, and $\overline{V}_1$. Using (14) for the case $n = 2$ we obtain

$$a' = \frac{3\overline{V}_2 + \overline{V}_1}{4} \,, \; b' = \frac{\overline{V}_2 - \overline{V}_1}{2} \,, \; c' = \frac{V_0 - \overline{V}_2}{2j+1} \,. \quad (16)$$

Equation (14) gives only average interaction energies, but in certain simple cases it is more specific. In any j^n configuration, for any given value of T, there is only one state which has lowest seniority. These states have $J = 0$, $v = 0$, $t = 0$ if n is even, and $J = j$, $v = 1$, $t = 1/2$ if n is odd. Putting these values of v and t in (14), we directly obtain the interaction energies in these states. The resulting expressions turn out to be much simpler than (14), in the same way that (5) is simpler than (2). However, before applying these expressions to the calculation of nuclear energies, we must discuss the question whether the seniority is a good quantum number in the present case of both protons and neutrons outside closed shells. For identical nucleons, with $T = n/2$, the seniority is a good quantum number for $j \leq 7/2$ irrespective of the nature of the two-body interaction. In the general case, considered here, this is true only for $j \leq 3/2$. In the case of maximum T, the available information on the $1g_{9/2}^n$ configuration indicates that the seniority is a good quantum number even in that case. No decisive experimental evidence about this question exists for $d_{5/2}^n$ or $f_{7/2}^n$ configurations of protons and neutrons. However, there are several indications that the effective interaction may not be diagonal in the seniority scheme. Whereas, the interaction in the $J = 0$ state is by far the strongest of the $T = 1$ states, there are $T = 0$ states with as strong and even stronger interaction. Thus, the expectation values of the interaction in states with lowest seniority may not be spaced far from those of states with the same T and J and higher seniority. Nevertheless, we try to see whether approximating ground states by states with lowest seniority gives a good description of binding energies.

For the cases $v = 0$, $t = 0$ and $v = 1$, $t = 1/2$, the expression (14) simplifies into

$$V(j^n,T,\text{g.s.}) = \tfrac{1}{2}n(n-1)(a' - \tfrac{1}{2}c') + [T(T+1) - \tfrac{3}{4}n](b' - c') + [\tfrac{1}{2}n](2j+2)c' \quad (17)$$

where $[n/2]$ has the same meaning as in (5). The binding energy due to the n nucleons in the j orbit will be obtained by adding to (17) their kinetic energy and interaction with the closed shells as well as the Coulomb energy (C. E.) of the j protons. The electrostatic energy can be calculated by using a simple model (e.g., harmonic oscillator potential) or better be taken from mirror nuclei. The other energies are given as a sum of n equal single-nucleon energies C. We thus obtain for the difference in binding energies of the given nucleus and the nucleus with no j nucleons outside closed shells the simple expression

$$\text{B.E.}(j^n,T) - \text{B.E.}(n=0) = nC + \tfrac{1}{2}n(n-1)a + (T(T+1) - \tfrac{3}{4}n)b + [\tfrac{1}{2}n]c + \text{C.E.} \quad (18)$$

The constants a, b, and c which appear in (18) are linear combinations of V_0, $\overline{V}_2$, and $\overline{V}_1$. They are given explicitly by

$$a = \frac{[(6j+5)\overline{V}_2 + (2j+1)\overline{V}_1 - 2V_0]}{4(2j+1)} \,,$$
$$b = \frac{[(2j+3)\overline{V}_2 - (2j+1)\overline{V}_1 - 2V_0]}{2(2j+1)} \,, \quad (19)$$
$$c = \frac{2(j+1)}{2j+1}(V_0 - \overline{V}_2) \,.$$

It is now possible to see whether sets of values of

[17] B. H. Flowers, Proc. Roy. Soc. (London) **A212**, 248 (1952).

the constants C, a, b, and c can be found which accurately reproduce the experimental binding energies. It turns out that (18) gives a fair description of binding energies which, however, is not as good as in the case of only identical nucleons outside closed shells.[18] The agreement obtained for the $1d_{3/2}$ shell is demonstrated in Table III. The experimental energies are compared with the values obtained from (18) by using the best values of C, a, b, and c.[18]

TABLE III. Binding energies of $1d_{3/2}^n$ configurations (in MeV).

Nucleus	Binding energy[a] Experimental	Calculated	Nucleus	Binding energy[a] Experimental	Calculated
$_{16}S_{17}^{33}$	8.65	8.66	$_{18}Ar_{17}^{35}$	19.69	19.58
$_{16}S_{18}^{34}$	20.05	20.04	$_{18}Ar_{18}^{36}$	34.96	34.83
$_{16}S_{19}^{35}$	27.07	27.12	$_{18}Ar_{19}^{37}$	43.79	43.96
$_{16}S_{20}^{36}$	36.97	36.97	$_{18}Ar_{20}^{38}$	55.54	55.82
$_{17}Cl_{16}^{33}$	2.42	2.38	$_{19}K_{18}^{37}$	36.88	37.00
$_{17}Cl_{17}^{34}$	13.75	13.77	$_{19}K_{19}^{38}$	48.88	48.86
$_{17}Cl_{18}^{35}$	26.46	26.29	$_{19}K_{20}^{39}$	61.96	61.85
$_{17}Cl_{19}^{36}$	35.03	35.10	$_{20}Ca_{19}^{39}$	54.32	54.44
$_{17}Cl_{20}^{37}$	45.39	45.23	$_{20}Ca_{20}^{40}$	70.29	70.17

[a] From these the binding energy of S^{32} was subtracted.

Equation (18) is a mass formula which covers a limited region (where the j orbit is being filled). The appearance of a symmetry term and a pairing term in (18) gives a shell-model basis for such terms in the semi-empirical mass formula. The parameter a is very small in all cases analyzed. The parameter b in the symmetry term is large and repulsive (negative). It thus causes states with lower values of T to have lower energy (greater binding). In the case of identical particles, where $T = n/2$, the coefficient of $n(n-1)/2$ becomes $a + b/2$ and is therefore repulsive. The parameter c in the pairing term is large and attractive as in the case of identical nucleons. We also see that in the present case, nuclei with magic proton and neutron numbers do not have much extra binding as compared with preceding even-even nuclei with $T = 0$. Such magic nuclei are more stable because the nuclei beyond them have less binding. In order to reproduce the data, the parameters C, a, b, and c should be fixed for each j shell. These parameters are constant within each subshell, but change discontinuously from one shell to the other. In the semi-empirical mass formula, which covers a larger region, these jumps are replaced by a continuous variation of the parameters with mass number. This is the reason why mass formulas do not work properly for light nuclei. It is not that these nuclei are less regular than others. Their masses are given well by (18) with constant coefficients. However, only for heavier nuclei where the discrete changes in the constants become smaller, can a smooth formula give a good description.

The properties of the interaction between protons and neutrons in the same j orbit are also present in the interaction between protons and neutrons in different orbits. The interaction between a j proton and a j' neutron is strong and attractive. This clearly follows from the saturation properties of nuclear energies, even when protons and neutrons occupy different shells. It is also borne out by the more detailed analysis in cases where it is possible.[5,15] The states where there are m j protons and n j' neutrons, outside closed shells have definite isospin provided the j-neutron shell is filled. The interaction between a j proton and a j' neutron, in this case, is given by

$$V(jj'J) = \tfrac{1}{2}\left[V(jj'T = 1J) + V(jj'T = 0J)\right]. \quad (20)$$

The interaction between a pair of j protons and a j' neutron is given by Eq. (12), where $V(jj'J)$ is now defined by (20). The interaction energy $V(j^2(0)j') = V(jj'^2(0))$ is strong and attractive. Its strength depends on the specific j and j' orbits involved. As a result, when a certain proton shell is being filled, the relative positions of the single-neutron orbits may change considerably.

An example of this effect is presented in Fig. 6. In O^{17}, as already mentioned, the ninth neutron is in the $1d_{5/2}$ orbit in the ground state. The single $2s_{1/2}$ neutron orbit is 0.87 MeV above it. Removing two $1p_{1/2}$ protons leads to C^{15} where the order of these levels may be different. Information about the $p_{1/2}$-proton $d_{5/2}$-neutron and $p_{1/2}$-proton $s_{1/2}$-neutron interaction can be obtained from other nuclei in this region. From this information it is possible to predict

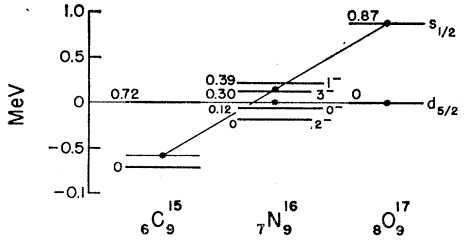

FIG. 6. Single $1d_{5/2}$ and $2s_{1/2}$ neutron levels.

[18] I. Talmi and R. Thieberger, Phys. Rev. **103**, 718 (1956).

the order of the $5/2^+$ and $1/2^+$ levels and their spacing in C^{15}.[5,15] In Fig. 6 this information is obtained from the level spacings in N^{16}.[19] The 2^- and 3^- levels in that nucleus are assigned the $p_{1/2}$-proton $d_{5/2}$-neutron configuration, whereas the 0^- and 1^- levels are taken to belong to the $p_{1/2}$-proton $s_{1/2}$-neutron configuration. In view of (12) it is clear that the predicted relative position of the C^{15} levels is obtained by *linear* extrapolation from the $5/2^+$ and $1/2^+$ levels of O^{17} through the *centers of mass* of the 2^- and 3^- levels and the 0^- and 1^- levels in N^{16}, respectively. The result is that the ground state of C^{15} is predicted to be a $1/2^+$ state with a $5/2^+$ state 0.58 MeV above it. This result is obtained also from the more detailed calculation[15,5] and is in good agreement with the experimentally observed order of levels and their spacing (0.72 MeV).

An analogous case which is more amusing concerns the order of the single $1p_{1/2}$ and $2s_{1/2}$ neutron levels. In C^{13} the ground state has spin $1/2^-$ which indicates that the last neutron is in the $1p_{1/2}$ orbit. The position of the $2s_{1/2}$ orbit, which is in a higher oscillator shell, is given by the first excited $1/2^+$ state at 3.09 MeV. If two $p_{3/2}$ protons are removed from C^{13}, the Be^{11} nucleus is obtained. It is plausible that the interaction of a $p_{3/2}$ proton with a $p_{1/2}$ neutron will be stronger than that with a $s_{1/2}$ neutron. Thus, the level spacing in Be^{11} may be quite different from that in C^{13}. The experimental information on the interactions involved can be obtained from the 1^+, 2^+ and 2^-, 1^- levels in B^{12}. A linear extrapolation from the C^{13} levels, through corresponding centers of mass of B^{12} levels, shows[19] that the ground state of Be^{11} is expected to have spin 1/2 and positive parity (as suggested by experimental data[20]) the $1/2^-$ state is predicted to lie 0.2 MeV above the $1/2^+$ ground state. Subsequent experiments confirmed this prediction.[21] The ground state of Be^{11} was verified to have spin $1/2^+$ and an excited state was found 0.32 MeV above it.[22]

It is worthwhile to mention that the linear extrapolation in Figs. 6 and 7 is not "systematics of even-parity states in nuclei of odd mass" in this region. It is based on the shell-model theoretical expression (12). The values of $V(jj'J)$ were determined directly from the experimental data. The same result would

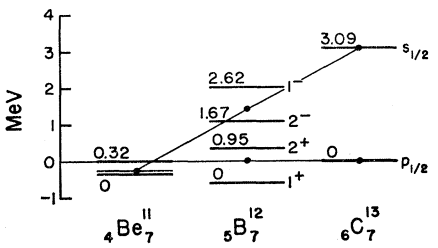

FIG. 7. Single $1p_{1/2}$ and $2s_{1/2}$ neutron levels.

have been obtained in a more "fashionable" way if a phenomenological interaction would have been first introduced to reproduce the B^{12} level spacings. The procedure described here only demonstrates that such a phenomenological interaction is not necessary in the present shell-model calculation. Several other cases of movements of single nucleon levels were recently published.[23]

The strong proton-neutron interaction gives rise to other effects which can be observed in the structure of nuclear spectra. In nuclei with protons and neutrons outside closed shells, it is possible to define a scheme of states in which the protons are in a definite state with spin J_p and the neutrons in a definite state with spin J_n. The total spin J of the states is obtained by coupling the given J_p to the given J_n. However, such states will not be eigenstates of the nuclear Hamiltonian, even if the protons and neutrons are in different shells. The proton-neutron interaction will generally admix states with different values of J_p and J_n (provided the total spin J is the same). The states described here belong to the same configuration, and the admixtures discussed are not configuration admixtures. These admixtures are due to the fact that protons and neutrons interact even if they are in different orbits. The matrix elements which give rise to such admixtures are linear combinations of the proton neutron interaction energies $V(jj'J)$. If all the $V(jj'J)$, with different values of J, are equal, these matrix elements vanish. Thus, the existence of nondiagonal matrix elements in the scheme characterized by J_p and J_n is directly related not to the strength of the proton neutron interaction but rather to the spread of $V(jj'J)$. From Fig. 1 it can be seen that for $j \equiv 1d_{3/2}$ and $j' \equiv 1f_{7/2}$ the various $V(jj'J)$ differ widely in energy.

The extent to which the admixtures discussed above influence energies of nuclear states also depends, of course, on the difference of the diagonal elements with different values of J_p and J_n of the

[19] I. Talmi and I. Unna, Phys. Rev. Letters **4**, 469 (1960).
[20] D. H. Wilkinson and D. E. Alburger, Phys. Rev. **113**, 563 (1959).
[21] P. F. Donovan, J. V. Kane, R. E. Pixley, and D. H. Wilkinson, Phys. Rev. **123**, 589 (1961).
[22] S. Hinds, A. E. Litherland, R. Middleton, and P. J. Pullen, *Proceedings of the Kingston Conference on Nuclear Structure*, edited by D. A. Bromley and E. W. Vogt (University of Toronto Press, Toronto, 1960), p. 436.
[23] B. L. Cohen and P. Mukherjee, Bull. Am. Phys. Soc. **7**, 347 (1962) and B. L. Cohen, Phys. Rev. (to be published).

energy matrix. An even group of identical nucleons in the lowest state has $J_p = 0$ and therefore $J = J_n$. The next higher state with this value of J can generally be obtained only with $J_p = 2$ which is considerably higher than the ground state. It is, therefore, clear that such admixtures will not be important for low-lying levels in odd-even nuclei. On the other hand, such admixtures can affect very strongly level spacings in odd-odd nuclei. This can clearly be seen from the following example.

In K^{42} the lowest proton configuration is that of a $d_{3/2}$ hole, $J_p = 3/2$, as in K^{40}. The neutrons are in the $f_{7/2}^3$ configuration. The ground state of the $f_{7/2}^3$ configuration has $J_n = 7/2$ but another state, with $J_n = 5/2$, lies only 0.37 MeV above it. Thus, the states with $J = 2,3,4$ of K^{42} are expected to have rather large admixtures of states with $J_n = 7/2$ and $J_n = 5/2$. The matrix elements of the $1d_{3/2}$-proton $1f_{7/2}$-neutron interaction are well known from the Cl^{38} –K^{40} analysis. Therefore, the energy matrices can be written down and diagonalized. The results are quite instructive.[24] Whereas the ground state of K^{40} has $J = 4$, the lowest diagonal element of the $d_{3/2}^{-1} f_{7/2}^3$ configuration has $J_n = 7/2$ and $J = 3$. However, diagonalization of the matrices leads to the lowest eigenvalue obtained for a state with $J = 2$, which agrees with the experimental value. The ground state turns out to be an almost equal admixture of states with $J_n = 7/2$ and $J_n = 5/2$ (states with other values of J_n are higher and do not contribute much). This example shows that no schematic coupling rule for spins of odd-odd nuclei can work in all cases. The matrix elements due to the proton-neutron interaction are of great importance in the determination of spacing and order of levels in odd-odd nuclei. It may be worthwhile to mention that in the special case of one proton (or neutron) with $j = 1/2$, the states characterized by different values of J_n (or J_p) are eigenstates of the proton-neutron interaction.[6]

The proton-neutron interaction gives rise also to configuration admixtures. Some of these will be discussed in detail later on. It may be worthwhile to repeat that the importance of configuration interaction is determined in every case by the distance of the perturbing states in addition to the magnitude of the matrix element connecting them. Thus, configuration interaction may be important even in configurations with $1d_{3/2}$ protons and $1f_{7/2}$ neutrons, in spite of the excellent agreement obtained for the Cl^{38} and K^{40} spectra. Consider, for instance K^{40}. The lowest possible proton configuration that could be admixed to the ground $d_{3/2}^{-1} f_{7/2}$ configuration lies more than 2.5 MeV above it, as seen from the K^{39} spectrum. Similarly, looking at Ca^{41} we see that the single $2p_{3/2}$-neutron level lies about 2 MeV above the $1f_{7/2}$ ground state. The $J = 2,3$ states of the $d_{3/2}^{-1} p_{3/2}$ configuration could be admixed to the corresponding states of the $d_{3/2}^{-1} f_{7/2}$ ground configuration of K^{40}. Even if the nondiagonal matrix elements would be of order $V \sim 0.2$ MeV the energy shifts V^2/Δ would turn out to be about 20 keV. Such shifts are consistent with the deviations between calculated and experimental level spacings in Fig. 1.

Consider, however, the ${}_{18}Ar_{21}^{39}$ nucleus. The ground state has $J = J_n = 7/2$ and mostly $J_p = 0$ with some admixture of the state with $J_p = 2$. The state with $J_p = 2$ lies about 2 MeV above the $J_p = 0$ state as observed in Ar^{38}. Other states, with $J_p = 2$, are also possible with $J = 3/2$, 5/2, 7/2, 9/2, and 11/2. They all should lie roughly 2 MeV above the ground state (their positions can be computed from the $d_{3/2}^{-1} f_{7/2}$ interaction). On the other hand, a $3/2^-$ state which belongs to the $d_{3/2}^{-2} p_{3/2}$ configuration (mostly with $J_p = 0$) is also expected to lie about 2 MeV above the Ar^{39} ground state. A nondiagonal matrix element of 0.2 MeV would strongly admix these two $3/2^-$ states of the $d_{3/2}^{-2} f_{7/2}$ and $d_{3/2}^{-2} p_{3/2}$ configurations and cause large energy shifts. An effect of this type is actually observed in Ar^{39}.

The general effect of configuration interaction in states where both protons and neutrons are outside closed shells is to cause large deviations from jj coupling. The fact that jj coupling works better for identical nucleons (or rather for high T values) has been observed by several authors. Both by those who used some simple phenomenological interactions[25,26,27] as well as by those who used the method of effective interactions.[15,5] We shall consider now a simple example in which it is possible to see the reason for this behavior.[28] Some of the features of this case may be more general and may apply to other cases.

Let us consider two nucleons outside the closed shells of O^{16}. It was mentioned earlier that $T = 1$ states (e.g., in O^{18}) can be consistently described in

[24] S. Goldstein and I. Talmi, Phys. Rev. **105**, 995 (1957).

[25] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955).

[26] M. G. Redlich, Phys. Rev. **99**, 1427 (1955).

[27] J. P. Elliott and A. M. Lane, *Encyclopedia of Physics*, edited by S. Flügge (Springer Verlag, Berlin 1957), Vol. 39, p. 241.

[28] Many of the following considerations were carried out in collaboration with Dr. I. Unna. A detailed description of the example considered here is presented in his Ph.D. Thesis, 1962 (unpublished).

terms of the $d^2_{5/2}$ configuration. It is instructive to see what is the situation for $T = 0$ states (in F^{18}). It turns out that jj coupling cannot reproduce the experimental data. With pure $d^n_{5/2}$ configurations Al^{26} would have the complementary configuration, $d^{-2}_{5/2}$, to that of F^{18} and therefore the same level scheme. The separation between the $T = 1$, $J = 0$, and $J = 2$ levels is about the same in Al^{26} and F^{18} (the $J = 4$ level has not been identified in Al^{26}). Also, the relative positions of the $T = 0$, $J = 5$ and $T = 1$, $J = 0$ levels are fairly close in both nuclei. The relative positions of the $T = 0$, $J = 3$ level in F^{18} and Al^{26} are not so similar. But most of all, the position of the $T = 0$, $J = 1$ levels is drastically different in these two nuclei. The lowest 1^+ level in Al^{26} lies about 1 MeV above the ground state, whereas in F^{18} the ground state has $J = 1$ and is about 1 MeV below the $J = 5$ or $J = 0$ levels. The position of the $J = 1$ state cannot be explained in terms of interaction with only the $s^2_{1/2}$ configuration. The F^{18} ground state has a considerable amount of the $d^2_{5/2}$ configuration (as evident from its beta decay to O^{18}). The next higher 1^+ state in F^{18} is only 1.7 MeV above the ground state. If these two states are strong admixtures of only the $d^2_{5/2}$ and $s^2_{1/2}$ configurations, the unperturbed position of the $J = 1$ state of the $d^2_{5/2}$ configuration would be much lower as compared to Al^{26}. It seems as if these two lowest $J = 1$ states (in F^{18}) are both pushed down by a higher $J = 1$ state. A reasonable configuration that could have such a $J = 1$ state is the $d_{5/2}\, d_{3/2}$ configuration.

What is the difference in the behavior of the $T = 1$, $J = 0$ and the $T = 0$, $J = 1$ levels? The only configurations that could admix to the $J = 0$ state of the $d^2_{5/2}$ configuration are those in which both nucleons are excited into a higher orbit, i.e., the $s^2_{1/2}$ or $d^2_{3/2}$ configurations. In the following we shall consider only the $d_{3/2}$ orbit since the discussion in that case can be carried out in a more general form. The difference in energy between the $d^2_{5/2}$ and the $d^2_{3/2}$ configurations, due to the spin-orbit interaction, is about 10 MeV. The inclusion of the mutual interaction modifies this difference and may introduce a nondiagonal matrix element connecting the $J = 0$ states of these two configurations.

There is no simple way to determine the matrix elements of the interaction in the present case. For the sake of illustration we shall consider a simple central interaction. In that case we expand both $d^2_{5/2}$ $J = 0$ and $d^2_{3/2}$ $J = 0$ states in terms of LS-coupling wave functions. The matrix elements of the mutual interaction, both diagonal and nondiagonal, are then linear combinations of the interaction energies in the various states of the d^2 configuration. Using the appropriate expansion coefficients,[29] we find in this case

$$\begin{aligned}
\langle d^2_{5/2} J = 0 | V | d^2_{5/2} J = 0\rangle &= \tfrac{3}{5} V(d^2\,{}^1S) + \tfrac{2}{5} V(d^2\,{}^3P)\\
\langle d^2_{3/2} J = 0 | V | d^2_{3/2} J = 0\rangle &= \tfrac{2}{5} V(d^2\,{}^1S) + \tfrac{3}{5} V(d^2\,{}^3P)\\
\langle d^2_{5/2} J = 0 | V | d^2_{3/2} J = 0\rangle &\\
&= (\sqrt{6}/5)[V(d^2\,{}^1S) - V(d^2\,{}^3P)]\,. \qquad (21)
\end{aligned}$$

We now assume that $V(d^2\,{}^1S)$ is much bigger in absolute value than $V(d^2\,{}^3P)$. In that case we can compute $V(d^2\,{}^1S)$ from the actual interaction energy V_0 in the $J = 0$ state of the O^{18} $d^2_{5/2}$ configuration. Using the value of V_0 obtained from experiment,[3] we obtain $V(d^2\,{}^1S) \sim 6$ MeV. Therefore, due to (21), the separation between the two interacting $J = 0$ levels becomes 11 MeV which is considerably bigger than the nondiagonal element which is about 3 MeV under our simplifying assumptions.

The situation is quite different in the case of the $T = 0$, $J = 1$ state. The mutual interaction would admix to the $d^2_{5/2}$ $J = 1$ state not only the $J = 1$ state of the $d^2_{3/2}$ configuration but also the $J = 1$ state of the $d_{5/2}\, d_{3/2}$ configuration. This latter state is separated only by 5 MeV from the $d^2_{5/2}$ state due to the spin-orbit interaction. Even this 5-MeV splitting is considerably reduced upon introduction of the mutual interaction. The interaction matrix is given in analogy with (21), by

	$d^2_{5/2}$	$d_{5/2}d_{3/2}$	$d^2_{3/2}$	
$d^2_{5/2}$	$7V(^3S) + 14V(^1P) + 4V(^3D)$	$\sqrt{7}(4V(^3S) - 2V(^1P) - 2V(^3D))$	$-\sqrt{14}(V(^3S) - 3V(^1P) + 2V(^3D))$	
$d_{5/2}d_{3/2}$	$\sqrt{7}(4V(^3S) - 2V(^1P) - 2V(^3D))$	$16V(^3S) + 2V(^1P) + 7V(^3D)$	$-\sqrt{2}(4V(^3S) + 3V(^1P) - 7V(^3D))$	$\times \frac{1}{25}$
$d^2_{3/2}$	$-\sqrt{14}(V(^3S) - 3V(^1P) + 2V(^3D))$	$-\sqrt{2}(4V(^3S) + 3V(^1P) - 7V(^3D))$	$2V(^3S) + 9V(^1P) + 14V(^3D)$	(22)

We obtain some idea about the numerical values of the elements of the interaction matrix (22) by making the drastic assumption of spin independence of the interaction, i.e., $V(^3S) = V(^1S)$, $V(^1P) = V(^3P)$, and $V(^3D) = V(^1D)$. Under this assumption $V(^3D)$ can be taken from the energy of the $J = 2$ state of the

[29] G. Racah, Physica 16, 651 (1950).

$d_{5/2}^2$ configuration in O^{18} [we also ignore $V(^3F)$]. We thus neglect $V(^1P)$, take $V(^3S)$ to be about 6 MeV, and for $V(^3D)$ we obtain a value of about 3 MeV. We then find that the interaction in the $J = 1$ state of the $d_{5/2}\ d_{3/2}$ configuration is 4.7 MeV which is 2.5 MeV stronger than that in the corresponding state of the $d_{5/2}^2$ configuration. The difference between the unperturbed states is thus reduced to 2.5 MeV only. For the nondiagonal matrix element connecting these two states we obtain the value of 1.9 MeV. As a result, these two states get strongly admixed. On the other hand, the interaction in the $J = 1$ state of the $d_{3/2}^2$ configuration is the same as in the $d_{5/2}^2$ configuration. The nondiagonal element between these two states turns out to be smaller than 2 MeV as compared to the 10-MeV difference. The nondiagonal matrix element connecting the $J = 1$ states of the $d_{5/2}\ d_{3/2}$ and the $d_{3/2}^2$ configurations is negligible (it is less than 0.2 MeV as compared to a 7.5-MeV energy difference). This structure of the interaction matrix is primarily due to the predominant interaction in the S state of the d^2 configuration. This feature leads to the lowering of the $d_{5/2}\ d_{3/2}\ J = 1$ state and the existence of a large matrix element connecting it with the $J = 1$ state of the $d_{5/2}^2$ configuration. Thus, the large admixture of the $d_{5/2}\ d_{3/2}$ configuration is rather insensitive to other details of the effective interaction.

The difference between the $T = 1$, $J = 0$ and $T = 0$, $J = 1$ states can be seen also in the LS-coupling scheme. The energy matrix for $T = 1$, $J = 0$ is[29]

	^1S	^3P
^1S	$V(^1S)$	$-a\sqrt{6}$
^3P	$-a\sqrt{6}$	$V(^3P) - a$

, (23)

where a is defined by the single-nucleon spin-orbit interaction $a(\mathbf{l}\cdot\mathbf{s})$. A 5-MeV splitting between the $d_{5/2}$ and $d_{3/2}$ orbits yields $a = 2$ MeV. Using this value of a, as well as the value given above of $V(^1S)$, we find that the difference between the diagonal elements of this matrix is 8 MeV. However, the nondiagonal element connecting them is about 5 MeV and gives rise to large admixture between these two states. The resulting function is rather close to the jj-coupling wave function of the $d_{5/2}^2$ configuration.

The energy matrix for the $T = 0$, $J = 1$ states is[29]

	^3S	^1P	^3D
^3S	$V(^3S)$	$a\sqrt{2}$	0
^1P	$a\sqrt{2}$	$V(^1P)$	$-a\sqrt{\frac{7}{2}}$. (24)
^3D	0	$-a\sqrt{\frac{7}{2}}$	$V(^3D) - \frac{3}{2}a$

Although the energy difference between the ^3S state and ^1P state is only 6 MeV, the nondiagonal element between them is much smaller than in (23). It is only 2.8 MeV and does not admix enough ^1P into the lower ^3S state as required for a $d_{5/2}^2$ wave function.

Certain features of the configuration interaction for two nucleons are also present in the case of several nucleons. Starting from states of the $d_{5/2}^n$ configuration with maximum T, the $d_{5/2}^{n-1}\ d_{3/2}$, $d_{5/2}^{n-2}\ d_{3/2}^2$, . . . configurations could also contribute to them. However, the matrix elements between states of the $d_{5/2}^n$ and $d_{5/2}^{n-1}\ d_{3/2}$ configurations are not expected to be large. The reason is that these elements are linear combinations of two nucleon matrix elements $\langle d_{5/2}^2 T\ J|V|d_{5/2}\ d_{3/2}\ T\ J\rangle$. Since we consider states with maximum isospin, T must be equal to 1 in the two-nucleon matrix elements. Therefore, J can be equal to 0, 2, or 4. However, $J = 0$ cannot be obtained by coupling $d_{5/2}$ and $d_{3/2}$ nucleons. We thus see that the nondiagonal elements between the $d_{5/2}^n$ and $d_{5/2}^{n-1}\ d_{3/2}$ configurations, with maximum isospin, do not contain the large interaction energy $V(d^2\ ^1S)$. If only $V(^1S)$ is different from zero, these nondiagonal elements all vanish. Thus, the configurations that are expected to be admixed to the $d_{5/2}^n$ configuration of identical nucleons differ by the quantum numbers of two, four, . . . nucleons. Such configurations lie at least 10 MeV higher, and thus are not expected to be strongly admixed. Moreover, their contributions can probably assume the form of a modification of the effective two-body interaction. As a result, jj coupling is a good approximation for identical nucleons.

If, however, the states considered do not have maximum isospin, the T in the matrix elements $\langle d_{5/2}^2\ T\ J|V|d_{5/2}\ d_{3/2}\ T\ J\rangle$ can be $T = 0$, and for $J = 1$ the matrix element will contain the large value of $V(d^2\ ^3S)$. Even if only $V(^3S) = V(^1S)$ is different from zero, large admixtures of the low-lying $d_{5/2}^{n-1}\ d_{3/2}$ configuration are expected, leading to a breakdown of jj coupling. From the point of view of LS coupling, this behavior means that the spin-orbit interaction is less effective in states which do not have maximum isospin. It has been argued that this is due to larger differences in interaction energies of the unperturbed LS-coupling states for lower isospin values. The simple example discussed above shows that this is not the case. With the choice of the interaction made above, the difference in interaction energy between the ^3S and ^1P states for $T = 0$ is exactly equal to that between the ^1S and ^3P states for $T = 1$. The different results are due to the non-

diagonal matrix element of the spin-orbit interaction being larger for the $T = 1$ than for the $T = 0$ states, as well as to the larger overlap between the ^1S and $d_{5/2}^2$ $J = 0$ wave functions. Also in the case of three d nucleons it seems that the difference between $T = 3/2$ and $T = 1/2$ states is mainly due to the matrix elements of the spin-orbit interaction.

The main difference between states with maximum isospin and other states seems to be that the latter are appreciably admixed with configurations obtained by single-nucleon excitations. The energy shifts due to such excitations *cannot* be absorbed into the effective *two-body* interaction. This type of configuration interaction does not describe correlations of two nucleons. Instead, it involves the single-nucleon wave functions. We can try to ascribe the change in the wave function to a modification of the single-nucleon wave functions. To do this we must construct wave functions which are linear combinations of single-nucleon functions with different values of j, e.g., $d_{5/2}$ and $d_{3/2}$. In other words, we must give up the spherical symmetry of the central field. In the case discussed here we start with single-nucleon wave functions

$$\psi(d_{5/2},m) + \alpha\psi(d_{3/2},m) , \qquad (25)$$

where m is the z projection of j. We now put several nucleons in the orbits (25), characterized by m, and project, from the product of their functions, wave functions with given values of J. In this way configuration admixtures as those described above are obtained. As long as α is small (compared to 1) the deformed potential well is only a mathematical device for building wave functions. If, however, α becomes bigger and more single-nucleon wave functions are added to (25), the situation may change. It may happen that some of the projected functions may be described as due to one definite intrinsic function in the deformed potential well. If this is the case, the deformed potential has a physical meaning and the eigenstates of the nucleus form rotational bands.

Thus, the proton-neutron interaction may give rise, in favorable cases, to deformed nuclei. The deviations from the spherical shell model that lead to this situation involve admixtures of configurations obtained by single-nucleon excitations. The possibility of such excitations is a direct consequence of the finite size of nuclei. In infinite nuclear matter no such excitations nor deformations are possible.

As mentioned earlier, the "short-range components" of the nuclear interaction are described as giving rise to the seniority scheme. The "long-range components" are described as field producing. It is amusing to realize that the *same* interaction, with large expectation values $V(^1S) = V(^3S)$ (which can be described as the pairing interaction in the d^n configuration), gives rise to both coupling schemes in our example. For states with maximum isospin, it only gives rise to excitations of nucleon pairs with $J = 0$. On the other hand, in other states the same interaction leads to deformed nuclei and thus acts like a "field-producing interaction." It seems that the nature of the nuclear states considered is much more important in the determination of the coupling scheme than the details of the nuclear interaction. It is true that we dealt above only with excitations of $d_{5/2}$ nucleons into $d_{3/2}$ orbits. However, the behavior discussed may well depend on rather general features of the nuclear interaction. Thus, excitations into other orbits may show the same features, although they cannot be described in terms of simple interaction as was the case above.

The different coupling scheme of the low-lying states in O^{19} ($T = 3/2$) and F^{19} ($T = 1/2$) may explain the unfavored beta decay of O^{19}. In the extreme case of pure jj coupling, this transition, as well as many others, would have been superallowed. In Wigner's supermultiplet theory such a transition occurs between states of different supermultiplets and should therefore be forbidden. Although such transitions have "normal" allowed log ft values they are much attenuated in comparison with mirror and other favored transitions. It seems that this very attractive feature of the supermultiplet theory may be still preserved in spite of the presence of strong spin-orbit interaction. Consider a situation in which the $J = 5/2$ state of F^{19} belongs to a well-defined (1/2, 1/2, − 1/2) supermultiplet. Even if the $J = 5/2$ state in O^{19} has a pure jj-coupling $d_{5/2}^3$ configuration, the transition would still be forbidden. Without spin-orbit interaction the $J = 5/2$ ground state of O^{19} would belong to the (3/2,1/2,1/2) supermultiplet. The spin-orbit interaction admixes to it states of the (3/2,3/2,3/2) supermultiplet but it cannot admix to it states of the (1/2,1/2, − 1/2) supermultiplet (these have isospin $T = 1/2$). If we now expand this $J = 5/2$ state of F^{19} in terms of jj-coupling wave functions, we obtain

$$\alpha\psi(d_{5/2}^3) + \beta\psi(d_{5/2}^2 d_{3/2}) + \gamma\psi(d_{5/2}d_{3/2}^2) + \delta\psi(d_{3/2}^3) . \quad (26)$$

The decay of the $d_{5/2}^3$ $J = 5/2$ state of O^{19} can proceed only to the first two terms of this expansion (it is a single-nucleon transition). Thus, even if the actual values of γ and δ are much smaller than in the supermultiplet theory, the transition would still be forbidden. If, however, due to the spin-orbit inter-

action, the value of β relative to that of α is smaller than in the supermultiplet wave function, the transition will be allowed but will not be favored.

Putting this result in another way, we see that the deviations from jj coupling for the $T = 1/2$ state largely attenuate the matrix element for beta decay. For the supermultiplet wave functions the rate of the decay has a stationary value. Therefore, slow decays, as in C^{14}, are incompatible with the supermultiplet theory. On the other hand, rates of beta decay are very sensitive to slight deviations from jj coupling. In the example presented here, the deviation from pure $d_{5/2}$ configurations is in the right direction and causes a large decrease in the rate of the transition considered.

Configuration interaction in its most general form including the $1d_{3/2}$ and $2s_{1/2}$ orbits in addition to $1d_{5/2}$ orbit is very complicated. It involves the diagonalization of large matrices but this is not the real difficulty. Such a calculation involves more than 50 matrix elements of the interaction. In the method of effective interactions, these must all be taken as free parameters to be determined by the experimental data. Thus, very many experimental energies will be required, many more than known to belong to such configurations. The jj-coupling approximation does not work very well in this region, as mentioned earlier in connection with the F^{18} spectra. Also, the F^{19} and Ne^{20} energies do not lend themselves to such a description. On the other hand, there is evidence that the collective model may have some validity in the d, s shell. Rotational bands have been identified in Mg^{25} (and Al^{25}) and even in Ne^{20}.

It is therefore of interest to see whether shell-model calculations could be carried out using the coupling scheme obtained from a deformed potential well. Such calculations were recently carried out.[30] The single-nucleon wave functions were taken as a linear combination of $1d_{5/2}$, $1d_{3/2}$, and $2s_{1/2}$ wave functions with $m = \pm 1/2$. This orbit may contain up to four nucleons. The actual wave functions were obtained by taking such product functions and projecting from them the components with definite values of J (and T). The resulting wave functions have configuration admixtures of a very restricted form. Thus, the matrix elements of the mutual two-body effective interaction can be expressed in terms of a rather small number of two-nucleon matrix elements. These latter elements, as well as the coefficients of the linear combinations in the single-nucleon wave functions, are taken as free parameters. They are determined by the experimental energy levels taken from O^{17}, F^{18} (and O^{18}), F^{19} (and Ne^{19}), and Ne^{20}. There are many more experimental energies than theoretical parameters and this gives a check on the coupling scheme used. The agreement obtained is very good and much better than that obtained before with simple phenomenological interactions. The wave functions obtained involve large admixtures of configurations with $1d_{3/2}$ and $2s_{1/2}$ nucleons. It may be significant that when the $T = 1$ levels of F^{18} (and O^{18}) are excluded the agreement is greatly improved.[31] The root mean square deviation is then reduced to half its value. This is a good measure for the improvement since the r.m.s. deviation takes into account the fact that the number of data is reduced. This is an example of how the method of effective interactions can give quantitatively significant results in a region which is not well described by jj coupling.

It is remarkable that Ne^{20}, with only four nucleons outside the closed shells of O^{16}, has properties associated with deformed nuclei. On the other hand, the oxygen isotopes seem to be fairly well described by the spherical shell model. The example considered above may indicate how this can be understood from the properties of the effective nuclear interaction. The discussion given above is far from being general or complete. It is only hoped that it can serve as an indication for the possibility of understanding the relations between the various coupling schemes which occur in actual nuclei.

[30] I. Unna, *Proceedings of the Rutherford Jubilee Conference*, edited by J. B. Birks (Academic Press Inc., New York, 1961), p. 273.

[31] I. Unna (to be published).