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The Conditions for a Quantum Field Theory to be Relativistic

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THE BASIC PROBLEM

ONE can build up a quantum field theory by working from a classical action principle. If one takes the action to be Lorentz invariant, the classical theory must be relativistic. With such an action, by following a standard method, one can put the classical equations of motion into the Hamiltonian form. They then refer to the concept of a state at a certain time, which is a nonrelativistic concept, so they are no longer manifestly relativistic. Still, one knows that they must be relativistic in their content, since they follow entirely from Lorentz-invariant assumptions.

When one passes over to the quantum theory one makes new assumptions—for example, one assumes an order for noncommuting factors in a product—and the condition that the original action shall be Lorentz invariant is no longer sufficient to ensure that the theory shall be relativistic. We must make a special investigation to see what is required for a quantum field theory to be relativistic.

At the basis of quantum mechanics we have the principle of superposition of states. We must consider the states as embedded in space-time. Taking space-time to be flat, it is subject to a group of operations of translation and rotation, the inhomogeneous Lorentz group (IHLG). These operations can be applied to the states to give other states, with the result that the states provide a representation of the IHLG, and the laws of quantum mechanics further require that the representation shall be a unitary one. Thus, each dynamical system in quantum theory corresponds to a unitary representation of the IHLG.

An alternative picture is to suppose each state to be referred to a system of coordinates (rectilinear and orthogonal) and to be described by a wave function with reference to these coordinates. If we change the system of coordinates by applying to it a translation or rotation, we get a different wave function. The various wave functions obtained in this way for one particular state again provide a unitary representation of the IHLG; in fact, the same representation as before. This second picture is the one needed for the development of the theory that will be given here.

If one worked out all unitary representations of the IHLG, one would obtain the theory of all dynamical systems, according to the accepted laws of quantum mechanics.

The problem of working out all unitary representations of the IHLG has been dealt with by Wigner,[1] taking the mathematical point of view that two representations are equivalent if they are connected by a unitary transformation. He decomposes the representations into their irreducible constituents and finds that the irreducible constituents provide theories of elementary particles with various spins. This work does not lead to any interaction between particles.

To bring in interaction, one must depart from the point of view of looking at two representations as equivalent if they are connected by a unitary transformation, a point of view which involves looking upon all unitary transformations as trivial. To a physicist, some unitary transformations are trivial, whereas others (for example, the S matrix) are far from trivial, so a physicist cannot look upon two representations connected by a unitary transformation as necessarily equivalent. With this broader point of view, the problem of obtaining all unitary representations of the IHLG is wide open. We shall here be concerned with discussing those representations that can serve as quantum field theories.

THE INTEGRATION PROCESS

A powerful way of obtaining representations of a group is to work from the infinitesimal operators. With the IHLG there are ten independent infinitesimal operators, the translations P_μ and the rota-

[1] E. Wigner, Ann. Math. **40**, 149 (1939).

tions $M_{\mu\nu} = -M_{\nu\mu}$ ($\mu,\nu = 0,1,2,3$). They satisfy the commutation relations

$$[P_\mu,P_\nu] = 0 \quad [P_\mu,M_{\rho\sigma}] = g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho$$
$$[M_{\mu\nu},M_{\rho\sigma}] = -g_{\mu\rho}M_{\nu\sigma} + g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\nu\sigma}M_{\mu\rho}. \tag{1}$$

After one knows the effect of each of the infinitesimal operators on a wave function, one needs a process of integration to determine the effect of a finite translation or rotation. If the representation is to correspond to a dynamical theory, this process of integration must play the role of the integration of the dynamical equations of motion.

In the usual form of dynamics one starts with an initial state at a certain time, with respect to some observer, and the integration of the equations of motion leads to the state at another time, with respect to the same or a different observer. To connect this form of dynamics with the representation theory, we express the dynamics entirely in terms of the state at the time $x_0 = 0$, and consider how the state at $x_0 = 0$ changes when the system of coordinates is changed. For a change which shifts the hyperplane $x_0 = 0$ in space-time we get a new state, and the integration of the dynamical equations of motion gives the connection between this new state and the initial one. This integration corresponds to an integration of infinitesimal group operators involving P_0 and M_{r0}. A change in the system of coordinates which does not shift the hyperplane $x_0 = 0$ does not lead to a new state, but merely refers the initial state to different coordinates x_1, x_2, x_3, which is a trivial change. This corresponds to an integration of infinitesimal group operators involving only P_r and M_{rs} ($r,s = 1,2,3$).

Thus, we see that we require a representation of the IHLG with a special property, namely, the integration with respect to P_r, M_{rs} must be trivial. The trivial operators form a subgroup. There is, of course, a disturbance of four-dimensional symmetry involved in picking out this subgroup. It does not arise from any lack of symmetry in the fundamental laws of nature, but only in our way of looking at them.

One could set up a form of dynamics corresponding to representations for which some different subgroup of operators are trivial. The various alternatives have been discussed by the author.[2] They have not been pursued very far and will not be considered here.

[2] P. A. M. Dirac, Revs. Modern Phys. **21**, 392 (1949).

LOCAL VARIABLES

We have to consider how to get a representation of the IHLG for which the integration with respect to P_r, M_{rs} is trivial. We can do this by working in terms of field quantities that are localized in the three-dimensional space $x_0 = 0$, so that the way in which any of them is changed by a change in the coordinate system x_1, x_2, x_3 is immediately evident.

The laws expressing how a field quantity V, located at the point x_1, x_2, x_3, is affected by the infinitesimal operators P_r, M_{rs} are

$$[V,P_r] = V_{,r} \tag{2}$$
$$[V,M_{rs}] = x_rV_{,s} - x_sV_{,r} + \chi_{rs}\,, \tag{3}$$

where $V_{,r}$ means $\partial V/\partial x^r$ or $-\partial V/\partial x_r$. The additional term χ_{rs} in (3) gives the effect of the rotation of axes on V. It depends on the tensor or spinor character of V and can easily be worked out in any particular case. Thus, if V is a scalar, χ_{rs} is zero. If V is a component A_t of a vector ($t = 1,2,3$), then

$$\chi_{rs} = -\delta_{rt}A_s + \delta_{st}A_r\,. \tag{4}$$

One can check this by taking (3) with V a scalar and χ_{rs} zero, differentiating it with respect to x_t and putting $V_{,t} = A_t$. If V is a component of a four-component spinor ψ describing an electron with spin matrices α_r, one finds

$$\chi_{rs} = \tfrac{1}{4}\,(\alpha_r\alpha_s - \alpha_s\alpha_r)\psi\,. \tag{5}$$

These laws serve to fix the infinitesimal operators P_r, M_{rs} in terms of the field quantities, as is illustrated by the examples below.

The field quantities are usually assumed to satisfy local commutation relations, so that the commutator (or anticommutator) of any two of them, located, respectively, at the points x and x' in the three-dimensional space, vanishes unless x and x' are very close, and then involves $\delta(x - x')$ and its derivatives. Under these conditions P_r and M_{rs} are of the form

$$P_r = \int K_r d^3x$$
$$M_{rs} = \int (x_rK_s - x_sK_r)d^3x\,, \tag{6}$$

where K_r is a localized field quantity having the physical meaning of momentum density. It can easily be worked out in any particular case.

If the field quantities are expressed in terms of canonical pairs ξ,η satisfying

$$[\xi,\eta'] = \delta(x - x')\,,$$

then

$$K_r = \sum \eta\xi_{,r} + \theta_{rs,s}\,, \tag{7}$$

where the sum is over all the canonical pairs and θ_{rs} is built up from contributions for all the nonscalar canonical field quantities. One sees that (7) leads to (2) with V equal to any ξ or η, and it also leads to (3) provided

$$[V,\theta'_{rs} - \theta'_{sr}] = -\chi_{rs}\delta(x - x') \,. \tag{8}$$

From this formula one can work out the contribution to θ_{rs} of any kind of nonscalar canonical field quantities. Thus, for canonical vectors A_r, B_s satisfying

$$[A_r,B'_s] = \delta_{rs}\delta(x - x')$$

the contribution is

$$\theta_{rs}(\text{vector}) = -A_rB_s \,, \tag{9}$$

and for a four-component spinor field satisfying

$$\psi_a\psi'^*_b + \psi'^*_b\psi_a = \delta_{ab}\delta(x - x') \quad (a,b = 1,2,3,4)$$

it is

$$\theta_{rs}(\text{spinor}) = -i\hbar\tfrac{1}{8}\,\psi^*(\alpha_r\alpha_s - \alpha_s\alpha_r)\psi \,. \tag{10}$$

We see that the momentum density depends only on the geometrical nature of the various field quantities occurring in the theory. It is merely a sum of contributions arising from the various basic field quantities, without interaction terms. Any physical interaction between the fields does not show itself here.

The momentum density must itself satisfy (2), (3), and (4), so that

$$[K_t,P_r] = K_{t,r}$$
$$[K_t,M_{rs}] = x_rK_{t,s} - x_sK_{t,r} - \delta_{rt}K_s + \delta_{st}K_r \,. \tag{11}$$

These equations combined with (6) immediately lead to those of the commutation relations (1) for which μ, ν are restricted to 1, 2, 3.

THE ENERGY DENSITY

To complete the representation we must obtain the remaining infinitesimal operators P_0, M_{r0} to satisfy (1). We may do this by introducing a suitable localized field quantity U and setting

$$P_0 = \int Ud^3x \quad M_{r0} = \int x_rUd^3x \,. \tag{12}$$

U has the physical meaning of energy density and must be chosen so as to make P_0, M_{r0} satisfy the correct commutation relations.

We take U of course to be a three-dimensional scalar. Thus it satisfies

$$[U,P_r] = U_{,r}$$
$$[U,M_{rs}] = x_rU_{,s} - x_sU_{,r} \,.$$

This leads to

$$[P_0,P_r] = 0 \quad [M_{t0},P_r] = \delta_{rt}P_0$$
$$[P_0,M_{rs}] = 0 \quad [M_{t0},M_{rs}] = -\delta_{rt}M_{s0} + \delta_{st}M_{r0} \,.$$

So the new infinitesimal operators P_0, M_{r0} satisfy the correct commutation relations with the previous ones P_r, M_{rs}.

It only remains to secure that they satisfy the correct commutation relations with one another. This gives us some conditions of the second degree in U, namely,

$$\iint x_r[U,U']d^3xd^3x' = \int K_rd^3x \tag{13}$$

$$\iint x_rx'_s[U,U']d^3xd^3x' = -\int (x_rK_s - x_sK_r)d^3x \,. \tag{14}$$

The commutator $[U,U']$ involves $\delta(x - x')$ and its derivatives, say,

$$[U,U'] = a\delta(x - x') + b_r\delta_{,r}(x - x') + c_{rs}\delta_{,rs}(x - x') + d_{rst}\delta_{,rst}(x - x') + \cdots \,. \tag{15}$$

The coefficients a, b_r, c_{rs}, d_{rst}, . . may be taken to be functions only of the field point x, since if field quantities at x' did occur in them, one could eliminate these field quantities by suitable transformations. The fact that $[U,U']$ is antisymmetric between x and x' imposes some relations on the coefficients. Interchanging x and x' in (15), we get

$$\begin{aligned}[U',U] &= a\delta - b'_r\delta_{,r} + c'_{rs}\delta_{,rs} - d'_{rst}\delta_{,rst} + \cdots \\ &= a\delta - (b_r\delta)_{,r} + (c_{rs}\delta)_{,rs} - (d_{rst}\delta)_{,rst} + \cdots \\ &= (a - b_{r,r} + c_{rs,rs} - d_{rst,rst}\cdots)\delta \\ &\quad + (-b_r + 2c_{rs,s} - 3d_{rst,st}\cdots)\delta_{,r} \\ &\quad + (c_{rs} - 3d_{rst,t}\cdots)\delta_{,rs} + \cdots \,. \end{aligned} \tag{16}$$

Putting (15) plus (16) equal to zero, we get

$$2a - b_{r,r} + c_{rs,rs} - d_{rst,rst}\cdots = 0 \tag{17}$$

$$2c_{rs,s} - 3d_{rst,st}\cdots = 0 \tag{18}$$

$$2c_{rs} - 3d_{rst,t}\cdots = 0 \,, \tag{19}$$

and so on. Equation (18) is a consequence of (19).

Equation (17) shows that

$$a = \alpha_{r,r} \,,$$

where

$$2\alpha_r = b_r - c_{rs,s} + d_{rst,st}\cdots \,. \tag{20}$$

Equation (18) shows that $c_{rs,s}$ can be expressed in terms of second derivatives, and hence from (20)

$2\alpha_r - b_r$ can be expressed in terms of second derivatives. It follows that

$$\int x_s(2\alpha_r - b_r)d^3x = 0\,. \tag{21}$$

We now have

$$\int [U,U']d^3x' = a = \alpha_{t,t}$$

and

$$\int x_s'[U,U']d^3x' = x_s\alpha_{t,t} - b_s\,.$$

Substituting these results in (13) and (14), we get

$$\int K_r d^3x = \int x_r\alpha_{t,t}d^3x = \int \alpha_r d^3x$$
$$= \tfrac{1}{2}\int b_r d^3x \tag{22}$$

from (20), and

$$\int (x_sK_r - x_rK_s)d^3x = \int x_r(x_s\alpha_{t,t} - b_s)d^3x$$
$$= \int (x_r\alpha_s + x_s\alpha_r - x_rb_s)d^3x\,.$$

With the help of (21) this gives

$$\int (x_sK_r - x_rK_s)d^3x = \tfrac{1}{2}\int (x_sb_r - x_rb_s)d^3x\,. \tag{23}$$

An obvious solution of the conditions (22) and (23) is

$$b_r = 2K_r\,. \tag{24}$$

The general solution is

$$b_r = 2K_r + \beta_{rs,s} \tag{25}$$

for some function β_{rs} which satisfies

$$\int (x_s\beta_{rt,t} - x_r\beta_{st,t})d^3x = 0$$

or

$$\int (\beta_{rs} - \beta_{sr})d^3x = 0\,, \tag{26}$$

that is, the antisymmetric part of β_{rs} must be a divergence.

Any energy density U with the commutation relation (15), with b_r satisfying (25) and (26), leads to infinitesimal operators P_μ, $M_{\mu\nu}$ satisfying all the commutation relations (1).

In the problem of finding such a U we may get some help from the classical theory. We may take any Lorentz-invariant action principle, and the U that it leads to will certainly satisfy the required conditions to the first order in $\hbar$. With luck we may then be able to arrange that it shall satisfy the conditions accurately.

EXAMPLE: THE BORN-INFELD ELECTRODYNAMICS

The Born-Infeld electrodynamics[3] is a modified Maxwell electrodynamics, the difference being important only for strong fields and leading to a finite total energy for a point singularity in the field. Expressed in the Hamiltonian form, the Born-Infeld electrodynamics involves vector field quantities **B** and **D**, the magnetic and electric inductions, satisfying

$$[B_r,D_s'] = \epsilon_{rst}\delta_{,t}(x - x')\,, \tag{27}$$

where ϵ_{rst} is antisymmetric and $\epsilon_{123} = 1$. The energy density, obtained from a Lorentz-invariant action, is of the form

$$U = \{1 + \mathbf{B}^2 + \mathbf{D}^2 + (\mathbf{B}\times\mathbf{D})^2\}^{1/2}\,. \tag{28}$$

One sees at once that U satisfies (15) with all the coefficients vanishing except a and b_r. In the classical limit b_r must have the correct value (24), which in the present case is $\epsilon_{rst}B_sD_t$.

With the quantization of the theory, we are faced with the commutator of B_r and D_s at the same point. According to (27) it should be $\epsilon_{rst}\delta_{,t}(0)$, a quantity which is somewhat ambiguous, but which it would be reasonable to count as zero on symmetry grounds. If we do this, the difficulties disappear. The order of the factors **B** and **D** in (28) then does not matter and the coefficient b_r must have the correct value also in the quantum theory.

The stronger conditions needed for quantizing the theory with respect to curvilinear coordinates are not fulfilled.[4] So the Born-Infeld electrodynamics can be quantized in agreement with special relativity, but not in agreement with general relativity.

CONCLUSION

The foregoing work analyzes the conditions that the infinitesimal operators P_μ, $M_{\mu\nu}$ have to satisfy, so far as concerns their commutation relations. The conditions are necessary, but they are not sufficient for a satisfactory quantum field theory. With a given solution of the conditions, one may find that, owing to the infinite number of degrees of freedom, one cannot integrate the equations of motion to get the effect of a finite motion of the hypersurface $x_0 = 0$

[3] M. Born and L. Infeld, Proc. Roy. Soc. (London) **A144**, 425 (1934).
[4] P. A. M. Dirac, Proc. Roy. Soc. (London) **A257**, 32 (1960).

on a wave function, all attempts to do so leading to divergent integrals. Under these circumstances, one does not really have a representation of the IHLG at all.

Physicists have constructed a method for handling the divergent integrals, the renormalization method, which is successful in many cases in leading to well-defined physical results that can be compared with experiment. However, the method can be considered only as a stopgap, since it does not follow logically from established physical laws. A satisfactory solution of the problem, if it is to conform to accepted basic laws of quantum mechanics, really must lead to a representation of the IHLG, and the renormalization technique does not do this.

Since the difficulties arise entirely from the integration of the infinitesimal operators, one can avoid them by short-circuiting the integration process, not using at all the infinitesimal operators, and dealing only with those integrals that are needed for comparison with experiment. A great deal of work has been done on these lines in recent years. However, it is unlikely that such work can lead to a complete theory. A process of integration seems to be an essential feature of dynamical theory.

It appears that the most hopeful prospect is to try to get representations of the IHLG in some more general way, still working from the infinitesimal operators and using a process of integration for the finite ones. There are several places where one might try to generalize the method that has been followed here. For example, one might depart from the local character of the energy density, or the local commutation relations for the basic field quantities. If all such attempts fail, one would have to consider altering the fundamental laws of quantum mechanics, which would involve finding some new general principle to replace the present use of representations of the IHLG.

SUMMARY

A quantum field theory in agreement with special relativity can be built up from the infinitesimal operators of translation and rotation. These operators are expressible in terms of a momentum density and an energy density. The momentum density is determined by the geometrical properties of the fields concerned. The energy density has to satisfy commutation relations for which certain conditions hold, given by Eqs. (15), (25), and (26).

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Geophysical Consequences of Dirac's Hypothesis

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SECTION 1

QUITE a revolution is going on at present concerning basic ideas of geophysics, geology, and paleoclimatology. Numerous authors have contributed to the new revolutionary picture of our Earth and its history that is now developing. Some of the contributions are newly detected empirical facts, others are new interpretations of things long known, and some of the ideas of the new theory have been conceived independently by several different authors. Among those who have contributed to the new ideas are Binge, Brill, Carey, Dicke, Egyed, Ewing, Fisher, Gamow, ter Haar, Heezen, Ivanenko, Jordan, Kirillow, Neumann, Sagitov, Teller, and Tharp.

This new understanding of the development of the earth is closely related to a generalization of the physical theory of gravitation. This started from Dirac's idea (1937) that the "constant" of gravitation, $\kappa = 8\pi G/c^2$, might in reality be a variable quantity. The generalization of Einstein's theory of gravitation that is generated by assuming κ to be a scalar field variable has been explored mathematically by the author since 1943. It was studied also by Einstein and Bergmann, Thiry, Dicke, and in connection with the author's work by Ludwig, Cl. Mueller, Heckmann, Fricke, Gressmann, Schuecking, Ehlers, Kundt, Pauli, Fierz, Just, Figueras, and Brill.

This generalized theory of gravitation, based on Dirac's hypothesis that κ is diminishing with time, approximately inversely proportional to the age of the universe, is meaningful in cosmology and astronomy, in connection with double stars, origin of