to the nucleon. Therefore, we have plotted the ω spectrum for events with momentum of the secondary-proton greater than 550 MeV/c (Fig. 10). The zeta is still not apparent.

We are indebted to the Bubble Chamber Group and the Cosmotron Department of the Brookhaven National Laboratory, for generous technical assistance in producing the bubble chamber pictures for this study, to the technicians of the Yale high-energy group for scanning the pictures and measuring the interactions, and to the other physicists of the Yale high-energy group for many stimulating discussions.

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Approximation Formulas for Nonrelativistic Bremsstrahlung and Average Gaunt Factors for a Maxwellian Electron Gas

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1. INTRODUCTION

THE radiation emitted during encounters of electrons with ions in a fully ionized gas of low density is called bremsstrahlung. The term "free-free transitions," which is customary in the astrophysical literature, refers to the same process. This radiation has proved to be of importance in cosmic-ray studies, in radio astronomy, in gas discharges, in thermonuclear experiments, and in other physical and astrophysical problems. The wide range in energy and frequency met in these applications necessitates different approximation formulas. This fact, combined with the fact that the relevant literature is spread over a long time interval, from Kramers (1923)[1] to now, makes it difficult to obtain a clear survey of the literature for practical purposes.

The aim of this paper is to collect in a concise manner the most important approximations that permit the rapid calculation of numerical results in the entire energy and frequency ranges with an accuracy of one percent.

The discussion, which can be naturally divided into a part dealing with incoming electrons of one velocity only (Secs. 2–5) and a part dealing with a gas having a Maxwellian velocity distribution (Secs. 6–9), is limited to nonrelativistic energies, i.e., energies smaller than 10^4 eV or temperatures lower than 10^8 °K, approximately. The restriction to low energies means that the formulas for dipole radiation suffice. Screening effects, arising from penetration of the electron into the electron shell of an atom (for very low energies), have also been omitted [cf., Guggenberger (1957), Hettner (1958)].

Theory and numerical data for bremsstrahlung at higher (relativistic) energies were reviewed by Heitler (1954), J. Stickforth (1961), and in great detail by Koch and Motz (1959). The latter authors state that "The nonrelativistic cross-section formulas derived in the dipole approximation by Sommerfeld with Coulomb wave functions have a complicated form with hypergeometric functions and are difficult to evaluate," and omit from their review a discussion of this topic, which is the exclusive topic of the present paper. Hence, only one formula [their 3 BN(a), our (12)] is common to the two papers.

In Sec. 2, rigorous formulas for electrons of one velocity are reviewed. In Sec. 3, a general division is made of the energy and frequency ranges into regions where different basic assumptions hold. A list of approximation formulas with their domains of validity is given in Sec. 4. In Sec. 5, we discuss the numerical results that are available in the literature.

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[1] See the list of references at the end of this paper.

The formulas for the bremsstrahlung by a Maxwellian distribution of electrons are given in Sec. 6. The explicit integration over the Maxwellian distribution and the domains of validity of the results for the free-free transitions are presented in Sec. 7. In Sec. 8, the contributions from the free-bound transitions are given. Finally, in Sec. 9, the combined free-free and free-bound continuum is discussed.

2. RIGOROUS FORMULAS FOR ELECTRONS OF ONE VELOCITY

Let a gas contain N_i fixed ions of charge $+Ze$ and N_e electrons of charge $-e$, traveling in random directions but all with the same speed v, the radiation emitted is then isotropically distributed. We call $\epsilon(\nu)$ the energy emitted per cm^3 per sec per sec^{-1} frequency bandwidth, and $\epsilon(\nu)/(4\pi)$ the same per steradian. The general formula is

$$\frac{\epsilon(\nu)}{4\pi} = \frac{8Z^2e^6}{3m^2vc^3} N_iN_eL \, . \tag{1}$$

Here, c is the velocity of light, m the mass of the electrons, and L is a function of two variables, namely, the energy $E = \frac{1}{2}mv^2$ of the electrons before the encounter and the frequency ν, which is emitted. The factor

$$g = \pi^{-1}\sqrt{3}L \, , \tag{2}$$

which measures the deviation of L from the classical short-wave limit $\pi/\sqrt{3}$ (see below), is traditionally called the Gaunt factor. Usually, L is expressed in terms of two parameters η and η', where η is a modified quantum number (Sommerfeld number) defined by the energy value of the initial state $E = Z^2\chi_0/\eta^2$ (analogous to the energy $E = -Z^2\chi_0/n^2$ for a bound state), whereas η' is the corresponding number for the final state. $\chi_0 = me^4/(2\hbar^2)$ is the ionization energy of hydrogen. The parameters η and η' have been defined positive for our case of attracting charges.

The parameter η may variously be expressed as

$$\eta = Z(\chi_0/E)^{1/2} = Ze^2/(\hbar v) = Z\alpha c/v$$
$$= Z/(ka_0) = kp_1 \, ,$$

where $h = 2\pi\hbar$ = Planck's constant, c = velocity of light, α = fine structure constant, a_0 = Bohr radius, $k = mv/\hbar$ = wave number of free electron, $p_1 = Ze^2/(mv^2)$ = impact parameter for a classical orbit in the form of an orthogonal hyperbola.

The frequency is expressed in terms of η and η' by

$$\nu = Z^2\nu_0(\eta^{-2} - \eta'^{-2}) \, ,$$

where $\nu_0 = \chi_0/h$ = frequency of the hydrogen ionization limit. Other combinations occurring in the theory are

$$x = -4\eta\eta'/(\eta' - \eta)^2 \qquad \text{(negative)},$$
$$1 - x = \{(\eta' + \eta)/(\eta' - \eta)\}^2$$

and

$$\xi = \eta' - \eta \qquad \text{(positive)}.$$

The maximum frequency emitted by free-free radiation for a given velocity v is

$$\nu_g = \tfrac{1}{2} mv^2/h = Z\nu_0/\eta^2 \, .$$

The rigorous quantum mechanical expression for L derived by Sommerfeld and Maue (1935) and given by Sommerfeld (1939) is

$$L(\eta,\eta') = \frac{\pi^2 x (d/dx)\{|F(i\eta,i\eta',1;x)|^2\}}{(e^{2\pi\eta} - 1)(1 - e^{-2\pi\eta'})} \, , \tag{3}$$

where $F(\alpha, \beta, \gamma; x)$ is the hypergeometric function

$$F(\alpha,\beta,\gamma;x) = 1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots \, .$$

This series converges only for $|x| < 1$, i.e., $\nu/\nu_g > 0.9706$, so that in almost all applications the analytic continuation for $|x| > 1$ has to be used (see e.g. Magnus and Oberhettinger (1948), Erdélyi [(1953), Vol. I, Ch. 2]).

Menzel and Pekeris [(1935), Eq. (1.22)] state without proof that another rigorous formula is

$$L(\eta,\eta') = \frac{\pi^2\eta\eta'}{(e^{2\pi\eta} - 1)(1 - e^{-2\pi\eta'})} \frac{|\Delta(i\eta,i\eta')|}{\eta' - \eta} \, , \tag{4}$$

where

$$\Delta(i\eta,i\eta') = \{F(-i\eta + 1, -i\eta',1;x)\}^2 - \{F(-i\eta' + 1, -i\eta,1;x)\}^2 \, .$$

In fact, they introduce the Sommerfeld numbers η and η' in a formula derived by McLean [(1934), Eq. (26)] for the bound-bound transitions, substituting $x = i\eta$ and $n' = i\eta'$ and normalizing properly.

Formal proofs of the identity of these results were given by H. C. Brinkman (unpublished) and by Grant (1958), to which paper we also refer for a clear review of the quantum mechanical derivation.

3. APPROXIMATION FORMULAS—A PHYSICAL DISCUSSION

The fact that L, or the Gaunt factor, depends on two parameters, makes it possible to display the domains of validity of various approximation formulas in a two-dimensional plane. Throughout this section and the next, reference is made to Fig. 1, where frequency is plotted vs energy of the incoming electron, both in logarithmic scales.

The borderlines of our diagram are based on the following considerations:

(a) The free-free and free-bound domains are divided by the diagonal AOF, equation $\nu = \nu_g$. The rigorous formula for L at this short-wave limit is simple [see below, Eq. (5)]. However, the values for $x = 1$, i.e., in points where $\nu = 0.9706\,\nu_g$, may strongly deviate from this formula.

(b) The energy limit, below which the nonrelativistic treatment is permitted, is given approximately by the horizontal line HF, equation $\eta = 0.03$, or E = 13.6 keV. Approximations valid at and above this limit have been discussed, e.g., by Heitler (1954), by Wandel, Hesselberg Jensen, and Kofoed-Hansen (1959), by Koch and Motz (1959), and by Stickforth (1961).

(c) The left border of the diagram, HLA, should be put at the frequency where the theory based on binary encounters begins to fail. This frequency depends on the density and will be discussed below.

Within the domain defined by these border lines, two order-of-magnitude dividing lines may be drawn:

(d) The horizontal line $L'O$, equation $\eta = 1$, or $E = 13.6$ eV, may be said to separate approximately the classical domain (below) from the quantum mechanical domain (above). Well above this line, the Born approximation for the electron wave function holds, i.e., the electron passes along the ion with little change in direction or phase, even in the orbit of lowest angular momentum. Classically, this corresponds to an almost rectilinear orbit.

Well below this line, the classical orbit which corresponds to the lowest angular momentum is nearly a parabola. Classically, a parabolic orbit gives negligible dipole radiation at low frequencies. Hence, the existence of a quantum mechanically defined lower impact parameter is unimportant and the approximation formulas for L suitable in this domain do not contain h.

(e) The sloping line ZO, equation $\nu = \nu_k$, where

$$\nu_k = v/(4\pi p_1) = mv^3/(4\pi Ze^2) = Z^2\nu_0/\eta^3 = \nu_g/\eta\ ,$$

separates, approximately, the domain in which nearly-straight-line encounters give the major contribution to the emitted energy (above ZO) from the domain in which the major contribution comes from the nearly-parabolic encounters (below ZO). Here, we use the term "nearly-straight-line" encounter for an encounter with $p > p_1$, total deviation $< 90°$, and we use the term "nearly-parabolic" encounter for an encounter with $p < p_1$, total deviation $> 90°$, in the classical model.

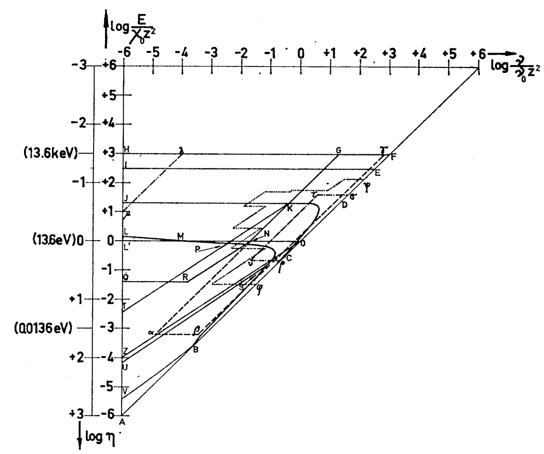

FIG. 1. Domains where different approximation formulas are valid within a 1% error in L and domains covered by numerical data.

A Note On The Density Dependence

It is clear that the concept of independent encounters fails, if we approach the plasma frequency

$$\nu_p = \{N_e e^2/(\pi m)\}^{1/2},$$

so that all formulas reviewed in this paper hold only for $\nu \gg \nu_p$. However, it is not obvious that this condition should be sufficient to make L independent of the density, and hence make the emission per unit volume strictly proportional to the square of the density.

The correctness of this assumption may be made plausible by referring to the Born approximation, Eq. (12). If the derivation in made *ab initio*, using classical theory and nearly-straight-line encounters, the result reads

$$L = \int dp/p = \ln(p_a/p_b)\ ,$$

where p_a is the upper impact parameter and p_b the lower impact parameter effectively contributing to the integration. When this formula was first used in radio astronomy, some confusion arose, which was soon clarified Burckhardt, Elwert, and Unsöld, (1948)].

Figure 2 illustrates the situation. There is a choice of two lower impact parameters, $p = 1/k$, corresponding to the electron of lowest angular momentum and $p = p_1$, corresponding to an orthogonal hyperbola. The upper value should be used. The ratio is η, and we shall, for simplicity, assume that $\eta < 1$, so that $p_b = 1/k$.

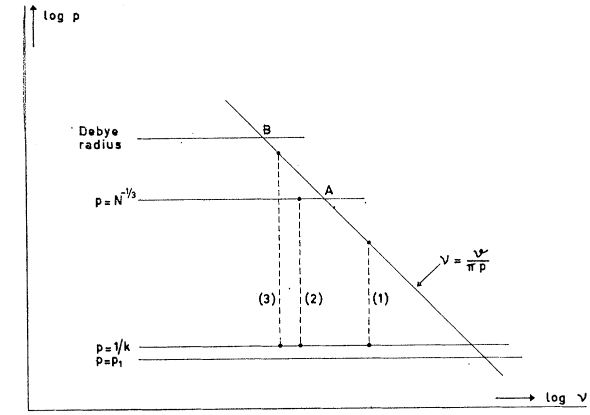

FIG. 2. Explanation of density cutoff in the Born approximation.

Further, there is a choice of two upper impact parameters. A distant encounter is slow, duration p/v, so that the highest Fourier component of the emitted radiation is of the order of $\nu = v/(\pi p)$. This gives at low density $p_a = v/(\pi\nu)$. The resulting Born approximation, illustrated by the dotted line 1 in Fig. 2, is $L = \ln [kv/(\pi\nu)]$, in agreement with (12). It was also argued that p_a could not become higher than $N^{-1/3}$. This gives the situation (2) with $L = \ln (kN^{-1/3})$, which would apply to any frequency to the left of A in Fig. 2.

However, it has since become generally accepted, although not strictly proven, that the cutoff parameter in problems of this kind should be the Debye radius $p = [k'T/(4\pi N_e e^2)]^{1/2}$, where k' is the Boltzmann constant. For frequencies between A and B in the figure this restores the original formula, see line 3 of Fig. 2. By order of magnitude, we may put $k'T = mv^2$ and find that the frequency of point B does not contain the velocity and equals $2\nu_p$.

The same argument may be repeated if $\eta > 1$, which means that the lower impact parameter is p_1. Hence, we may conclude that probably no density corrections to L are needed for any frequency satisfying $\nu \gg \nu_p$.

4. APPROXIMATION FORMULAS—A LIST OF THE VALIDITY DOMAINS

More than twenty approximation formulas for $L(\eta,\eta')$ may be found in the literature. Most of them (not all!) have a domain in Fig. 1, in which the approximation is quite good. The approximate position of this domain follows from the assumptions made in the derivation. In this section we have set the task, however, to determine, by comparison with rigorous numerical results, as well as possible in what precise domain of Fig. 1 each approximation formula is correct within a one percent error.

The formulas are consecutively numbered (5) to (20) and are divided into groups according to the basic assumption made in the derivation.

First Group, Formulas (5)–(10), Short-Wavelength Limit

In the short-wavelength limit, $\nu = \nu_g$, $x = 0$, $\eta' = \infty$, $\xi = \infty$, the rigorous formula assumes a simple form.

(a) Entire Line AOF

For $\nu = \nu_g$, η arbitrary, Biedenharn and Thaler [(1956), Eq. (26)] and Guggenberger (1957) give

$$L = 2\pi F_0(-\eta,2\eta)F_0'(-\eta,2\eta)\ . \tag{5}$$

The values of the Coulomb wave function, $F_0(-\eta,\rho)$, and its derivative with respect to ρ, $F_0'(-\eta,\rho)$, have been tabulated by Abramowitz and Rabinowitz (1954) for $\rho = 2\eta$.

(b) Subcase of (a). Line AO

For $\eta > 1$

$$L = \frac{\pi}{\sqrt{3}}\left\{1 + \frac{0.1728}{\eta^{2/3}} - \frac{0.0496}{\eta^{4/3}} - \frac{0.0172}{\eta^2} + \cdots\right\}. \tag{6}$$

The error is 0.2% in point O. The exact value in O is $L = 2.0024$ (this approximation: 2.0061).

(c) Subcase of (b). Line AB

For $\eta > 70$ we have within 1%

$$L = \pi/\sqrt{3} = 1.8138\ . \tag{7}$$

This formula [Kramers (1923), Eq. (71)] is known as the "soft approximation."

(d) Subcase of (a). Line DF

For $\eta < 0.2$

$$L = \frac{8\pi^2\eta^2}{e^{2\pi\eta} - 1}\left\{1 + \frac{10}{3}\eta^2 + \frac{196}{45}\eta^4 + \cdots\right\}. \tag{8}$$

(e) Subcase of (d). Line EF

For $\eta < 0.055$

$$L = \frac{8\pi^2\eta^2}{e^{2\pi\eta} - 1}\ . \tag{9}$$

(f) Subcase of (a). Line CF

For $\eta < 1.7$ the formulas (5), (8), and (9) can be replaced by the simpler expression

$$L = 2(1 - e^{-2\pi\eta})\ . \tag{10}$$

This formula is found, if one takes the short-wave limit $\eta' \to \infty$ of the Born-Elwert approximation [formula (11)] and can be verified directly by comparison with formula (8), if one expands

$$2(1 - e^{-2\pi\eta})(e^{2\pi\eta} - 1) = 8\pi^2\eta^2 \times \left(1 + \frac{\pi^2}{3}\eta^2 + \frac{2\pi^4}{45}\eta^4 + \cdots\right).$$

(g) Subcase of (e)

For $\eta < 0.003$, the preceding formula would be approximated within 1% by $L = 4\pi\eta$. However, this brings us outside the nonrelativistic domain, so that this simple formula is never a good approximation.

Second Group, Formulas (11)–(13), High Electron Energies, Born Approximation

Application of the Born approximation to the electron wave functions automatically confines the domain of validity to the upper part of Fig. 1. The Born-Elwert approximation [formula (11)] is a formula derived by Elwert from the Sommerfeld formula by taking only the first two terms in an expansion of the hypergeometric function in powers of $1/x$. This approximation has a larger domain of validity than the straightforward Born approximation [formula (12)]. The Sommerfeld approximation [formula (13)] is essentially the first term of an expansion in powers of η.

(h) Domain CFHJKC

The formula

$$L = \frac{\eta'}{\eta} \frac{1 - e^{-2\pi\eta}}{1 - e^{-2\pi\eta'}} \ln \frac{\eta' + \eta}{\eta' - \eta} \tag{11}$$

(known in the literature as Born-Elwert or Sommerfeld-Elwert formula) has been derived by Elwert (1939) with the restriction $\xi \ll 1$. Elwert also conjectured that this restriction would not be necessary

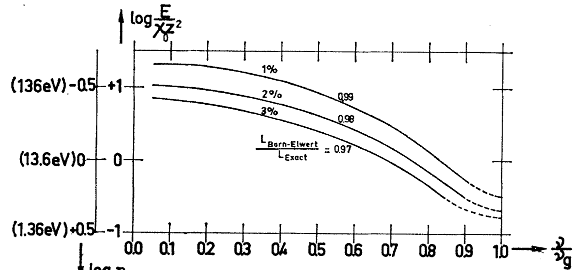

FIG. 3. Boundary of the domain of validity of the Born-Elwert formula for low electron energies.

and that the approximation could be used in the whole above-mentioned domain.

This conjecture is verified with a fair degree of certainty by Fig. 3 which shows the loci for 1, 2, and 3% error. The full-drawn parts of the curves were taken from Berger's exact numerical data (see Sec. 5), the end points at $\nu = \nu_g$ from formula (5) and the dashed curves show the probable interpolation. The 1% curve corresponds to the curve KC in Fig. 1. In the limit $\eta' \to \infty$, one obtains formula (10).

(i) Subcase of (h). Domain GHJKG

The following formulas are known in the literature as the Born approximation [Sauter (1933), Heitler (1954), Sec. 25, Eq. (18)]. They have been first derived by Gaunt (1930) and Maue (1932).

$$L = \ln \frac{\eta' + \eta}{\eta' - \eta} = \ln \frac{4\nu_g}{\nu} = \tfrac{1}{2} \ln |x| = \ln \frac{2}{1 - \eta/\eta'} = \ln \frac{2\eta}{\xi}. \tag{12}$$

The five expressions are not identical, but they are all suitable only in the long-wavelength limit and have very nearly the same domain in which they are correct within 1%.

(j) A Further Subcase of (h)

The formula

$$L = 4\pi\eta/(1 - e^{-2\pi\eta'})$$

would be correct within 1%, if $\eta < 0.002$ and $\eta/\eta' < 0.15$ and, hence, does not have any domain of validity in Fig. 1. The same is true for the formulas $L = 4\pi\eta$ and $L = 2\eta/\eta'$, which follow from it under even more extreme conditions.

(k) Domain EFHIE

The older formula of Sommerfeld (1931) holds for $\eta < 0.055$

$$L = \frac{4\pi^2\eta\eta'}{(e^{2\pi\eta} - 1)(1 - e^{-2\pi\eta'})} \ln \frac{\eta' + \eta}{\eta' - \eta}. \tag{13}$$

This formula may be considered as a modification of the Born approximation. In the limit $\eta' \to \infty$ one obtains formula (9).

Third Group, Formulas (14)–(16), Long-Wavelength Expressions

Several attempts have been made to utilize the fact that η and η' are nearly equal in the long-wavelength limit, in order to derive simplified expressions for L.

Such attempts are interesting only if they are not limited to very small or very large η, and, hence, may serve to bridge the gap between the domain of the Born approximation and the classical domain. Two such formulas are discussed here [(14) and (15)]. In the limit of $\eta \ll 1$, they reduce to the Born approximation (12). In the opposite limit, (15) reduces exactly and (14) reduces very nearly to the classical long-wavelength expression (16).

(*l*) *Domain GHQRNG*

For $\eta < 1$, $\xi < 0.01\eta$ (i.e., $\nu/\nu_g < 0.02$) and for $1 < \eta < 5$, $\xi < 0.01$, Kummerer (1957) obtained the formula

$$L = \frac{1}{1 - \pi(\eta' - \eta)} \left\{ \ln \frac{\eta' + \eta}{\eta' - \eta} - \rho \right\}, \quad (14)$$

where

$$\rho = \chi \left\{ \tfrac{1}{2} - (\eta' - \eta)^2 \left(\ln \frac{\eta' + \eta}{\eta' - \eta} \right)^2 \right\}$$

and

$$\chi = \ln(1 + \eta^2) + \eta^2/(1 + \eta^2) + 1/6 + (\eta^2 - 1)/[6(1 + \eta^2)^2] .$$

Values of $\frac{1}{2}\chi$ have been tabulated in Table I ($0 \leqslant \eta \leqslant 5$). For $\eta < 0.2$, we may omit ρ and the first factor of L, whereby the Born approximation is recovered. It is not useful to omit ρ and keep the first factor, for this hardly extends the domain of validity over that of the Born approximation.

(*m*) *Domain GHTKG*

Let $p_0 = \nu\eta/(2\nu_g)$. For $\nu/\nu_g \ll 1$ and $p_0 \ll 1$ (or $\xi \ll 1$) Elwert (1948) has derived, starting from the Sommerfeld formula,

$$L = \ln(2\eta/\xi) - \Phi(\eta) = \ln(4\nu_g/\nu) - \Phi(\eta) , \quad (15)$$

where

$$\Phi(\eta) \equiv C + \mathrm{Re}\,[\Gamma'(i\eta)/\Gamma(i\eta)] = \textstyle\sum_{k=1}^{\infty} \eta^2/[k(k^2 + \eta^2)]$$

and $\Gamma'(z) = (d/dz)\Gamma(z)$ represents the derivative of the gamma function $\Gamma(z)$ and $C = 0.5772\ldots$ is the Euler constant. This formula has been given also by Huby and Newns [(1951), Eq. (23)] and Alder, *et al.* [(1956), Eq. (IIE.66b)]. For $\eta \ll 1, \Phi(\eta) = 0$, this is again the Born approximation (12) and for $\eta \gg 1$, $\Phi(\eta) = C + \ln\eta$, the Kramers formula [see formulas (16) and (19)]. A table of the function $\Phi(\eta)$ in the interval $0 \leqslant \eta \leqslant 5$ is given below (Table I).

TABLE I shows that for very long wavelengths, where $\rho = \frac{1}{2}\chi$, the corrections to the Born approximation given by (14) and (15) are almost identical. The validity of (14) extends towards somewhat shorter wavelengths as the correction is made dependent on η'. On the other hand, the asymptotic expression for very low velocities in (14) is not quite exact, especially if ξ is not sufficiently small.

η	$\frac{1}{2}\chi$ [see (14)]	$\Phi(\eta)$ [see (15)]	$0.5772 + \ln\eta$ [see (16)]
0.0	0	0	$-\infty$
0.1	0.0124	0.0119	−1.7254
0.2	0.0482	0.0465	−1.0322
0.3	0.1039	0.1005	−0.6268
0.4	0.1745	0.1693	−0.3391
0.5	0.2549	0.2483	−0.1159
0.6	0.3406	0.3330	+0.0664
0.7	0.4280	0.4199	+0.2205
0.8	0.5146	0.5063	+0.3541
0.9	0.5989	0.5906	0.4719
1.0	0.6799	0.6718	0.5772
1.1	0.7572	0.7494	0.6725
1.2	0.8306	0.8231	0.7595
1.3	0.9002	0.8929	0.8396
1.4	0.9661	0.9591	0.9137
1.5	1.0287	1.0218	0.9827
1.6	1.0880	1.0813	1.0472
1.7	1.1444	1.1379	1.1078
1.8	1.1981	1.1916	1.1650
1.9	1.2492	1.2428	1.2191
2.0	1.2981	1.2917	1.2704
3.0	1.6913	1.6850	1.6758
4.0	1.9749	1.9686	1.9635
5.0	2.1961	2.1898	2.1867
∞	$0.5833 + \ln\eta$	$0.5772 + \ln\eta$	$0.5772 + \ln\eta$

(*n*) *Domain TPML*

In the low-energy (classical) limit ($\eta \gg 1$), formula (15) reduces to

$$L = \ln[4\nu_g/(\zeta\eta\nu)] = \ln(4\nu_g/\nu) - (C + \ln\eta) = \ln(2/\xi) - C , \quad (16)$$

where $\ln\zeta = C = 0.5772\cdots$ is the Euler constant. The values of $C + \ln\eta$ have been tabulated in Table I for $0 \leqslant \eta \leqslant 5$. The domain of validity is limited at the lower end, because the formula is essentially based on straight-line encounters.

Fourth Group, Formulas (17)–(20), Low-Energy Encounters, Classical Theory

(*o*) *Domain ABSPML*

For slow electrons, i.e., $\eta \gg 1$, we obtain the classical approximation. The de Broglie wavelength $\lambda = h/(2\pi mv)$ must be small compared with the "size" of the Coulomb field $Ze^2/(mv^2)$, which means $\eta \gg 1$. Taking in the exact quantum mechanical Sommerfeld formula for L the limit $\eta \to \infty$ and ξ finite, one finds (see e.g. Alder *et al.* [(1956), Eq. (IIE.58)], Biedenharn [(1956), Eq. (13)]),

$$L = (\pi^2/4) i\xi H_{i\xi}^{(1)}(i\xi) H_{i\xi}^{(1)\prime}(i\xi) , \quad (17)$$

where the derivative of the Hankel function of the first kind is denoted by

$$H_n^{(1)'}(z) = (d/dz)H_n^{(1)}(z) \,.$$

This expression may be obtained also by directly calculating the matrix elements for the bremsstrahlung with the WKB approximation for the Coulomb wave functions (see e.g. Alder *et al.* (1956), Secs. IIB.6 and IIE.5). Wentzel [(1924), Eq. (36)] derived this formula from classical radiation theory, using the correspondence principle.

(*p*) *Kramers' Formula*

Kramers [(1923), footnote on P. 860] derived from classical radiation theory for a hyperbolic orbit the formula

$$L(\eta,\nu) = (\pi^2/4)ip_0H_{ip_0}^{(1)}(ip_0)H_{ip_0}^{(1)'}(ip_0) \,, \tag{18}$$

where

$$p_0 \equiv 2\pi\nu Ze^2/(mv^3) = \nu\eta/(2\nu_g) \,.$$

A complete derivation of this formula is given by Landau and Lifshitz (1959). Apparently, this classical formula—where the influence of the radiation loss on the velocity of the particle has been neglected—must by symmetrized by replacing p_0 by $\xi = \eta' - \eta$. This symmetrization has been discussed by Wentzel (1924) (cf. case *o*), Ter-Martirosyan (1952), Alder, *et al.* [(1956), Sec. IIA.5], and Biedenharn and Thaler (1956). The quantities p_0 and ξ are related by

$$\begin{aligned} p_0 &= \xi\{1 - 3\xi/(2\eta') + \xi^2/(2\eta'^2)\} \\ &= \xi\eta(1 + \eta/\eta')/(2\eta') \,, \end{aligned}$$

so that the unsymmetrized Kramers formula can be used only if

$$\xi/\eta' \ll 1 \quad \text{or} \quad \nu/\nu_g \ll 1 \,.$$

(*q*) *Subcase of* (*p*)

In the domain *TPML*, where $\xi \approx p_0 \ll 1$, we obtain the expression that is usually called "the Kramers formula"

$$L = (1 + \pi p_0) \ln [2/(\zeta p_0)] \,, \tag{19}$$

where $\ln \zeta = C = 0.5772 \cdots$ is the Euler constant. This formula coincides with formula (16), because of the restriction $p_0 \ll 1$. In this domain the ratio of the exact quantum mechanical formula and the classical limit can be derived from formulas (15) and (16) [Biedenharn and Thaler (1956), Eq. (25)]

$$\frac{L_{\text{exact}}}{L_{\text{classical}}} = 1 + \frac{\ln \eta - \Phi(\eta) + C}{\ln (2/\xi) - C} \,.$$

With the use of this formula the upper 1% boundary of the classical domain can be determined and it is seen that for extremely small values of ξ (i.e., outside Fig. 1) the boundary is given by $\log \eta = 0.01 \log (\nu/\nu_0) - 0.25$. The fact that this line (*LM* in Fig. 1) somewhat deviates from the horizontal direction has no practical importance.

(*r*) *Subcase of* (*o*); *Domain ABSU*

The classical limit has the asymptotic expansion for large values of ξ [Watson (1958), p. 247]

$$L = \frac{\pi}{\sqrt{3}}\left\{1 + \frac{0.21775}{\xi^{2/3}} - \frac{0.01312}{\xi^{4/3}} + \cdots\right\}. \tag{20}$$

As has been shown by Grant (1958), this formula equals formula (17) within 1% for values of ξ as low as 1. For $\xi \to \infty$ the "soft approximation" (7) is recovered.

5. NUMERICAL RESULTS AND THEIR DOMAINS

The approximation formulas given in the last section do not suffice to compute L for all values of η and ν, as can be seen from Fig. 1. They must be supplemented with the results of numerical calculations based on the exact Sommerfeld formula. The following tables and diagrams can be found in the literature.

I. Domain $\rho\sigma\tau\upsilon$

Kirkpatrick and Wiedmann [(1945), Table II and Fig. 7] have computed L for about 60 points with values of η from 0.16 to 2.14. They employ the strict Sommerfeld formula as transformed by Weinstock (1942). Moreover, they find for each point by planimeter integration one further integral needed to compute the polarization of the emitted radiation.

Their quantities $W \times 10^{50}$, V/Z^2 and ν/ν_0 are numerically equal to our $38.45\eta^2L$, $1/(22.07\eta^2)$ and ν/ν_g, respectively. They claim an accuracy better than one percent.

II. Domain left of $\varphi\,\psi$

Berger (1957) has made machine computations on the basis of Sommerfeld's formula in this domain, with the exclusion of a region in the vicinity of the short-wave limit. The ordinate $W_sE/(Ad\nu)$ of his Figs. 1 and 2 and the ordinate $W_s/(Ad\nu)$ of his Fig. 3 are numerically equal to our $1.378\ L$ and $\pi^{-2}\eta^2L$, respectively. The abscissa of all his figures equals ν/ν_g. The computation refers to 327 incident electron energies and to about 50 frequencies per incident energy, so that a fine grid is available for interpolation. The short-wave boundary of his domain is given by $\nu/\nu_g = 0.70$ near φ and by $\nu/\nu_g = 0.999$ near ψ.

These results show resonance phenomena for energies near the ionization energy χ_0 of hydrogen and $\nu/\nu_g \lesssim 0.1$. We have not been able to decide whether these are brought about by a physical resonance or by a spurious effect in the numerical computations. No such resonance phenomena are seen in the curves of Karzas and Latter (see below). Berger claims that the error in the results is determined solely by the uncertainty of the fundamental atomic constants.

III. Domain $\alpha\beta\gamma G$

Grant [(1958), Table I] transformed the numerical results computed by Thaler, Goldstein, McHale, and Biedenharn (1956) for *like* charges into Gaunt factors, $g = \pi^{-1}\sqrt{3}L$, for *unlike* charges. The accuracy claimed is a few tenths of a percent. Approximately 200 tabulated values cover the area defined by $0.001 \leqslant \eta \leqslant 40$ and $1.01 \leqslant \eta'/\eta \leqslant 1.8$. His definitions of η, η', and ξ differ in sign from ours. The table is completed in a small triangle near β (see Fig. 1) by values computed from the classical formula (17), which differ in these regions from the exact values by less than 0.4%. Again, the underlying material is a table computed for like charges by Alder and Winther (1956). Tables based on the Born-Elwert approximation [our Eq. (11)] and on our Eq. (20) are also given.

IV. Domain $AF\lambda\mu$

Karzas and Latter [(1957), Figs. 1 and 2] give values of the Gaunt factor over almost the entire domain of Fig. 1. The computations are based on the exact Sommerfeld formula as transformed by Biedenharn (1956) but no details on accuracy, numerical methods, and number of points calculated are given. The results are presented in two graphs, from which g may be read within a few percent as functions of the frequency ($\nu/(\nu_0 Z^2) = 10^{-6}$ to 10^{+8}) and the *final* energy of the electron in the emission process, which is the initial energy in the absorption process ($\eta'^{-2} = 10^{-5}$ to 10^{+8}).

Survey of Results

We wish to compile the results over the entire domain of Fig. 1 in a convenient form. This is done in Figs. 4 and 5, which are based both on the approximation formulas from Sec. 4 in their proper domains and on the numerical data from Sec. 5. The coordinates of Fig. 4 correspond precisely with those of Fig. 1; this figure may be used for $\nu/\nu_g < 0.1$. In Fig. 5 the abscissa is ν/ν_g; it may be used for

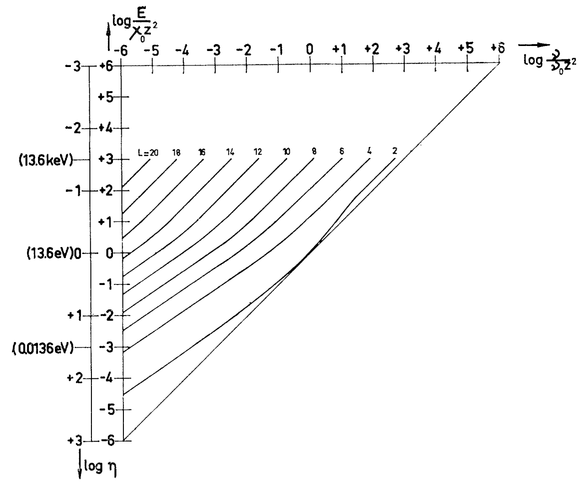

FIG. 4. Survey diagram showing the values of L for relatively long wavelengths.

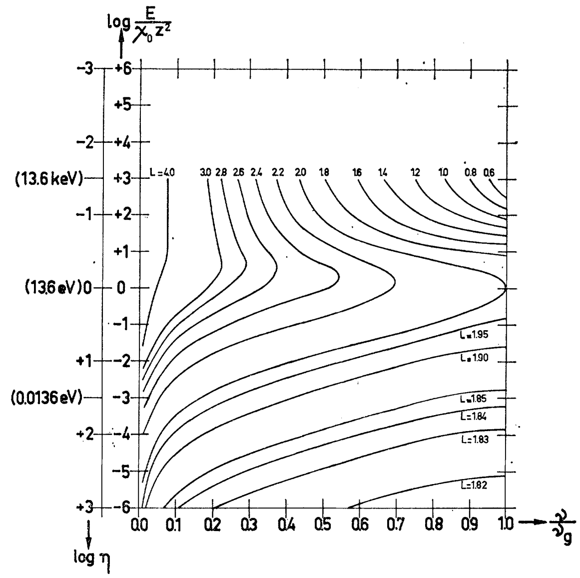

FIG. 5. Survey diagram showing the values of L for relatively short wavelengths.

$\nu/\nu_g > 0.02$. The curves approach the asymptotic direction, $\nu/\nu_g =$ constant, at the top of these figures (small η); at the bottom they virtually coincide with the curves $\xi =$ const.

6. AVERAGING PROCESS FOR THE MAXWELLIAN DISTRIBUTION

Let $E = \frac{1}{2}mv^2$ be the initial energy and $E' = E - h\nu$ be the final energy of the free electron and $g(E,\nu)$ the corresponding Gaunt factor. The free-free emission per unit volume of a thermal gas is obtained by integrating a quantity proportional to g/v over a distribution function proportional to $e^{-E/kT}v^2dv$, where

k denotes the Boltzmann constant.[2] It is thus found that the weighted average of g for a Maxwellian distribution may be expressed in two ways as

$$\bar{g}(T,\nu) = e^{h\nu/kT}\int_{E=h\nu}^{\infty} g(E,\nu)e^{-E/kT}d\,\frac{E}{kT}$$
$$= \int_{E'=0}^{\infty} g(E'+h\nu,\nu)e^{-E'/kT}d\,\frac{E'}{kT}\,. \quad (21)$$

The complete equation for the energy emitted in free-free encounters per cm^3 per $\sec^{-1}$ band width per steradian is

$$\frac{W_{ff}(T,\nu)}{4\pi} = \frac{8Z^2e^6}{3m^2c^3}\left(\frac{2\pi m}{3kT}\right)^{1/2} N_iN_e\bar{g}(T,\nu)e^{-h\nu/kT}\,. \quad (22)$$

These formulas suffice for computing the free-free emission. However, we shall show that it is necessary to consider also the free-bound transitions. The exponential factor in Eq. (22) arises from the fact that free-free emission of frequency ν cannot occur unless the initial electron energy exceeds $h\nu$. However bremsstrahlung in a broader sense can be emitted by electrons of any initial velocity. It includes free-bound radiation, in which the electron is captured by the ion. In any emission experiment these two effects will be measured together.

The free-bound radiation differs from the free-free emission by the quantization of the final energy. The qualitative reasoning given above suggests that the inclusion of the free-bound emission will approximately cancel the exponential factor in Eq. (22). This reasoning simply reverts that of Kramers (1923), who first derived the formula for free-bound radiation by means of the correspondence principle from the classical radiation formula.

We shall first illustrate the situation by a graph of the combined continuum based on Kramers' original equations [(1923), Eq. (56)], i.e., by a computation where $g = 1$. Lumping all factors which do not depend on T or ν into a factor A, we obtain

$$\frac{W_{Kr}(T,\nu)}{4\pi} = Ae^{-h\nu/kT}\left\{1 + 2\theta\sum_{n=m(\nu)}^{\infty} n^{-3}e^{\theta/n^2}\right\}\theta^{1/2}. \quad (23)$$

Here $\theta = h\nu_0Z^2/(kT) = 2\pi^2me^4Z^2/(h^2kT)$ and $m(\nu)$ is the principal quantum number of the lowest bound level to which emission at the frequency ν can occur

$$m(\nu) - 1 < (\nu/\nu_0Z^2)^{1/2} < m(\nu)\,.$$

The quantity $a = W_{Kr}/(4\pi A)$ has been plotted against $\log\,(\nu/\nu_0Z^2)$ in Fig. 6 for four values of the temperature. It is seen that the successive entry of more free-bound continua defines a saw-tooth curve which remains horizontal on the average, so that the exponential drop of the free-free emission (dashed curves) is canceled quite well by the free-bound contributions. The teeth become invisible at low frequencies (the radio domain) or at high temperatures (small θ). In those situations the free-bound contribution is negligible.

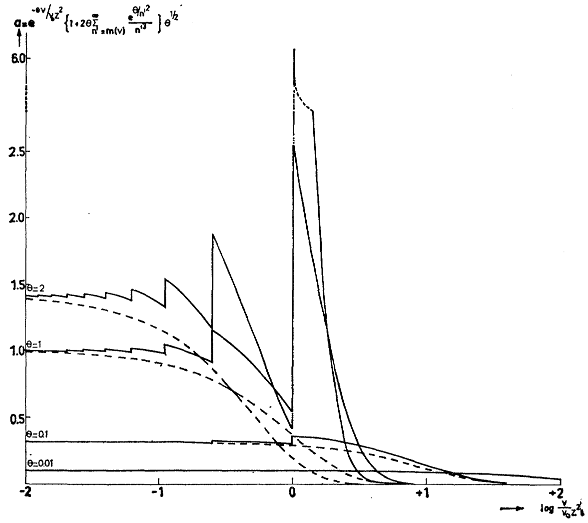

FIG. 6. Emission spectrum from a thermal gas at various temperatures according to Kramers' formulas. Upper curves: combined continuum; lower curves: free-free emission only.

Insertion of the relevant Gaunt factors now gives the rigorous expression

$$[W(T,\nu)/4\pi]d\nu = N_iN_eR\bar{v}e^{-h\nu/kT}$$
$$\times\,\{\bar{g}(T,\nu) + f(T,\nu)\}d(h\nu/kT)\,. \quad (24)$$

Here, $R = 4\pi Z^2e^6/(3^{3/2}mhc^3) = 18.24 \times 10^{-35}Z^2$ erg cm^2; $\bar{v} = (8kT/\pi m)^{1/2}$, $\bar{g}(T,\nu)$ has been defined above and the new term is

$$f(T,\nu) = 2\theta\sum_{n=m(\nu)}^{\infty} n^{-3}e^{\theta/n^2}g_n(\nu)\,, \quad (25)$$

where $g_n(\nu)$ is the Gaunt factor for the transition from the free level $E = -h\nu_0Z^2/n^2 + h\nu$ to the bound level $E' = -h\nu_0Z^2/n^2$.

7. AVERAGE GAUNT FACTORS FOR FREE-FREE EMISSION

(a) Analytical Integration

The expression for the Born approximation (12)

[2] Confusion with the wave number k is not possible from here on.

can be integrated analytically over the Maxwellian distribution. Starting from the form

$$g = \pi^{-1}\sqrt{3}\ln(4\nu_0/\nu)$$

Elwert (1954) finds [cf. Erdélyi (1953), Vol. II, Eq. 9.7(5)]

$$\bar{g}(T,\nu) = \pi^{-1}\sqrt{3}\ln(4kT/(h\nu\zeta)) . \quad (26)$$

This formula has been discussed recently by Oster [(1961), Eq. (152)]. Starting from the form

$$g = \pi^{-1}\sqrt{3}\ln\{(\eta' + \eta)/(\eta' - \eta)\}$$

Greene (1958) finds [cf. Erdélyi (1954), Vol. I, p. 149, Eq. (19)]

$$\bar{g}(T,\nu) = \pi^{-1}\sqrt{3}\, e^{h\nu/2kT} K_0(h\nu/2kT) . \quad (27)$$

Here $\ln \zeta = C = 0.5772 \cdots$ is Euler's constant and $K_0(z)$ is the modified Bessel function (e.g. Watson [(1958), pp. 78, 698]).

It might seem that Greene's expression is more general than Elwert's, because Elwert's expression derives from it in the limit $h\nu \ll kT$. However, the validity domains (within 1%) of the original formulas practically coincide and are limited to the long-wave limit $h\nu/E < 0.01$. Hence, the validity of the average formulas is similarly limited to $h\nu/kT < 0.01$, say, and the more complicated form with the Bessel function has no advantage (see Fig. 7).

Kulsrud (1954) has integrated the Born approximation numerically for $1 \leqslant h\nu/kT \leqslant 7$. For these frequencies, however, the Born approximation is not suitable.

Another approximation formula suitable for analytical integration is the low-energy (long-wavelength) limit of g, coinciding in its domain of validity with "the Kramers formula" [(16) and (19), respectively]. The integration yields [cf. Erdélyi (1953), Vol. II, Eq. 9.7(5)]

$$\bar{g}(T,\nu) = \frac{\sqrt{3}}{\pi}\left\{\frac{3}{2}\ln\frac{kT}{h\nu_0 Z^2} - \ln\frac{\nu}{4\nu_0 Z^2} - \frac{5}{2}C\right\}, \quad (28)$$

where C is again Euler's constant. This result has been discussed by Scheuer [(1960), Eq. (7)] and Oster [(1961), Eq. (134)]. These authors investigated the derivation of this expression from the one-velocity formula.

(b) Numerical Integration

Grant (1958) has suggested a rapid method for integrating over a Maxwellian distribution by a quadrature formula [Chandrasekhar, (1950)], in which the integrand should be computed in only a few points, i.e., at the zeros of Laguerre polynomials in E'/kT. Several examples are given.

Berger (1956) has computed average Gaunt factors both from his numerical evaluation of the exact Sommerfeld formula and from numerical data based on the Born-Elwert approximation (11). Both averages are given for eight temperatures between 0.5 eV and 100 eV and 20 wavelengths between 500 Å and 10 000 Å (for $Z = 1$).

This work has later been extended by Greene (1958), who, as a by-product of a calculation of the total power, computed the ratio and difference between the exact value of $\bar{g}$ and the value $\bar{g}_B$ based on the Born approximation [our Eq. (12)]. The work is based on Berger's tables and on asymptotic formulas for very small and very large $h\nu/kT$. The results are given in the form of graphs of $\bar{g}$, $\bar{g}/\bar{g}_B$ and $\bar{g}$-$\bar{g}_B$ plotted against $h\nu/kT$ as well as separate graphs showing the dependence on T (ranging from 10^4 to 10^9°K) for the three wavelengths: 12.4 Å, 3000 Å, and 1 cm.

Karzas and Latter (1957) have also computed the average Gaunt factor by numerical integration of exact values. Their results, which cover the intervals $h\nu/kT = 10^{-4}$ to 30 and $\chi_0/kT = 10^{-3}$ to 10^3 are presented in the form of graphs in which $\bar{g}$ is plotted against one of these quantities with the other as a parameter.

Kazachevskaya and Ivanov-Kholodny (1959) have computed the average Gaunt factor for 17 values of the wavelength between $\lambda = 1$Å and $\lambda = 10^{10}$Å and temperatures 10^6°K and 2×10^6°K, with a view to application to the continuous solar emission. They give no details on the calculations of the average Gaunt factor, which they consider exact. For comparison results from the Born-Elwert formula and the Born approximation are tabulated.

(c) Domains in the Temperature-Frequency Diagram

In analogy with the energy-frequency diagram for electrons of a given velocity (Fig. 1) we have constructed a temperature-frequency diagram which permits a quick survey of the availability of values or formulas for the average Gaunt factor.

In this diagram (Fig. 7) Berger's (1956) results form a ladder-like structure of horizontal dotted lines B.

The intervals G, covered in Greene's graphs, are rather widely spread over the diagram.

Karzas and Latter's (1957) data cover the largest domain, the circumference of which has been marked KL.

The line BH at $kT = h\nu_0 Z^2$ represents the data of Table II and part of Fig. 9.

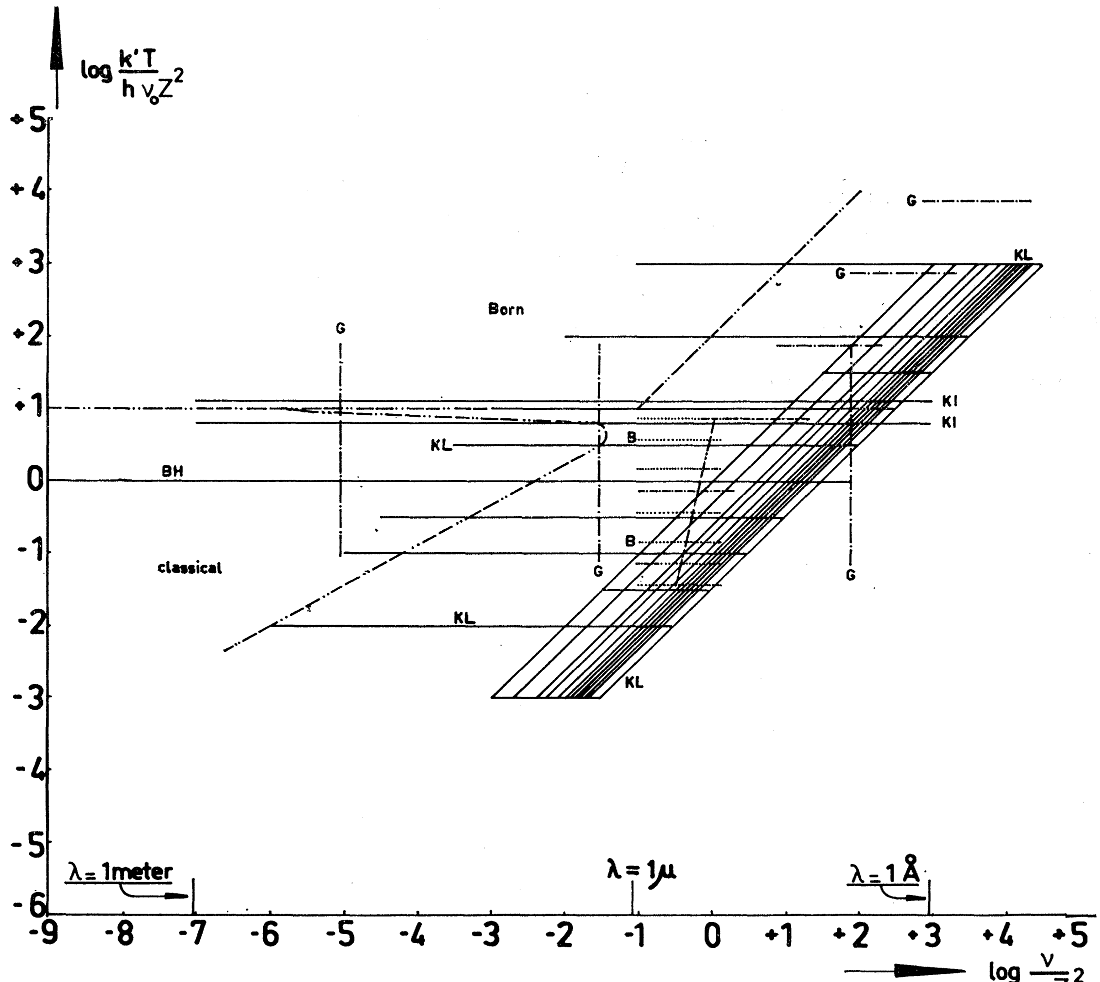

FIG. 7. Numerical results available in the literature: Berger's results are denoted by the dotted lines ......; Greene's by ·—·—·—·; Karzas and Latter's by the solid lines marked KL; Kazachevskaya and Ivanov-Kholodny's by the solid lines marked KI. The line BH represents the data of the present work (Table II and part of Fig. 9). The boundaries of the domains of 2 per cent accuracy of the Born and the Born-Elwert approximations and the classical limit have been indicated by ··—··—·· .

In the same Fig. 7 the boundaries of the domains of validity of some approximations have been indicated by broken lines. The upper left domain belongs to the Born approximation, for which either the expression containing a modified Bessel function or the even simpler logarithmic expression may be used.

The lower left domain is the domain of the classical limit of the Gaunt factor.

From Berger's (1956) results we determined the 2% boundary for the Born-Elwert approximation. This boundary is the sloping line in Fig. 7, crossing Berger's intervals B. This approximation, however, still requires numerical integration.

The conclusions reached in Sec. 2 for the density independence of the one-velocity formulas, may, of course, be taken over into the case of a Maxwellian velocity distribution. Hence in this paper we have not discussed expressions for the average Gaunt factor that are density dependent, like the ones, e.g., given by Smerd and Westfold [(1949), Eq. (3.31)] and Oster [(1959), Eq. (39)] (for these expressions the upper impact parameter is taken to be $p_a = N^{-1/3}$ and the emitted frequency is assumed to obey $\nu < \nu_p$, respectively).

8. FREE-BOUND EMISSION

The Gaunt factor for the transition from the free level, $E = h\nu_0 Z^2/\eta^2$, to the bound level, $E' = -h\nu_0 Z^2/n^2$, has been discussed in detail by Menzel and Pekeris (1935). The exact expression is very similar to that for free-free transitions. However, the available approximation formulas, also given by Menzel and Pekeris, cover only a very restricted domain of validity.

The function $f(T,\nu)$, defined in Eq. (25), contains this Gaunt factor but is not itself an average Gaunt factor. In order to determine this function, the values of $g_n(\nu)$ for a fixed frequency ν and for n ranging from $m(\nu)$ to ∞ are required. Using formula (1.36) by Menzel and Pekeris, we have computed[3] the values of $g_n(\nu)$ for the frequency interval $10^{-4} \leqslant \nu/\nu_0 Z^2 \leqslant 10^4$ and n ranging from 1 to 10. The results are given in Fig. 8.

The summation, necessary to calculate the function $f(T,\nu)$, may be split into two parts, the first part, $\sum_{n=m}^{5}$, to be performed by adding the separate terms. For the second part, $\sum_{n=6}^{\infty}$, an interpolation for $g_n(\nu)$ must be made between $n = 5$ and $n = \infty$. This may be done without great loss of accuracy by

$$g_n = g_\infty + (25/n^2)(g_5 - g_\infty),$$

where g_5 and g_∞ have to be taken for the same ν. We may remark here that for $n = \eta' = \infty$ the free-bound and the free-free Gaunt factors coincide, so that the formulas (5) to (10) can be used. Thus, we get

$$f(T,\nu) = 2\theta \sum_{n=m}^{5} g_n \frac{e^{\theta/n^2}}{n^3} + 2\theta g_\infty \sum_{n=6}^{\infty} \frac{e^{\theta/n^2}}{n^3} + 50\theta(g_5 - g_\infty) \sum_{n=6}^{\infty} \frac{e^{\theta/n^2}}{n^5}. \tag{29}$$

The infinite sums $\sum_{n=6}^{\infty}$ can readily be evaluated, either by replacing them by integrals from 5.5 to ∞ or (more accurately) by developing the exponentials.

Two examples may serve for an illustration:

	$\theta = 1,\ \nu/(\nu_0 Z^2) = 1.26$	$\theta = 1,\ \nu/(\nu_0 Z^2) = 0.316$
g_1	0.855	...
g_2	1.049	0.915
g_3	1.080	1.015
g_4	1.091	1.048
g_5	1.095	1.063
g_∞	1.099	1.090
first term of f	5.128	0.430
second term of f	0.037	0.037
third term of f	0.000	0.000
	$f = 5.165$	$f = 0.467$

At very low frequencies ($\nu < 0.04\nu_0 Z^2$) we cannot compute f by this method, because $m > 5$. Generally, however, the free-bound contribution to the average Gaunt factor is small for these frequencies and we may put $g_n(\nu) = 1$ within one percent error (cf. Fig. 8) and replace the sum by an integral.

[3] The authors are indebted to Dr. T. A Griffy for programming this expression for the IBM-704.

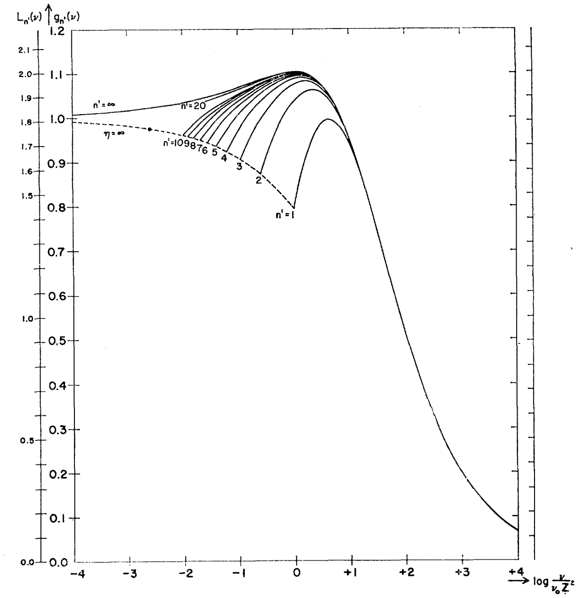

FIG. 8. Gaunt factor $g_n(\nu)$ versus frequency for free-bound transitions ($L_n = \pi 3^{-1/2} g_n$). The intersection of the $n = 20$ curve with the $\eta = \infty$ limit has been indicated by a dot.

9. THE COMBINED FREE-FREE AND FREE-BOUND CONTINUUM

We may now return to formula (24) which gives the intensity of the combined continuum for an ionised gas with a Maxwellian distribution of the electron velocities. The values of $\bar{g} + f$ are given in Table II for $\theta = 1$, which means $T = 158\ 000°K$ in the case of hydrogen. This function has been plotted on a logarithmic scale in Fig. 9, where it is seen that from $\nu/\nu_0 Z^2 = 0.1$ on, i.e., from $\lambda = 1$ micron down to shorter waves, the free-bound continuum becomes increasingly important.

Figure 9 also shows the values of $\bar{g}$ for the temperatures 10^3, 10^4, 10^5, 10^6, 10^7, and 10^8°K in the frequency range in which they can be calculated from the classical limit or Born approximation within 2% error.

The product $b = e^{-h\nu/kT}(\bar{g} + f)\theta^{1/2}$ is proportional to the energy emitted per unit frequency band. It has been plotted, again for $\theta = 1$, in Fig. 10. The function b differs from the function a plotted in Fig. 6 only by the inclusion of the Gaunt factor and the graphs are directly comparable with the same (logarithmic) frequency scale and the same (linear) ordinate scale. For instance, the ratio of the linear magnitudes of the first jump (the Lyman discontinuity) in the two figures is $1.60/2.00 = 0.80 =$ the

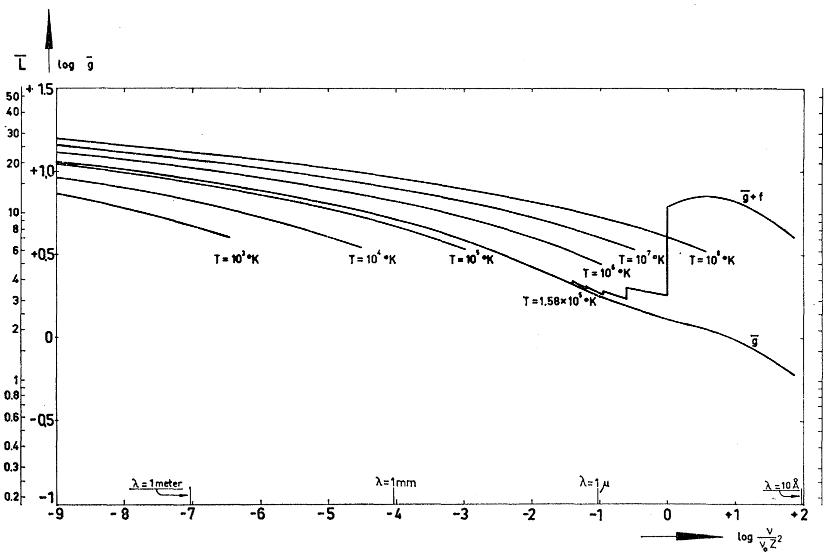

FIG. 9. Average Gaunt factor $\bar{g}\ (T,\nu)$ versus frequency at given temperatures ($L = \pi 3^{-1/2}\bar{g}$). For the temperature $kT = h\nu_0 Z^2$ the free-bound transitions have been taken into account. The temperatures given in °K refer to the case of hydrogen ($Z = 1$).

Gaunt factor at the Lyman limit. The most striking difference between the two graphs is the strong slope in Fig. 10, which arises entirely from the logarithmic increase of $\bar{g}$ towards low frequencies.

The results may be summarized in a few conclusions:

(1) Unless $h\nu \ll kT$, it is not useful to compute $\bar{g}(T,\nu)$ only, for $\bar{g}(T,\nu) + f(T,\nu)$ is the quantity to be compared with experimental emission data.

(2) In the absence of computed values of $f(T,\nu)$, it is best simply to use $\bar{g}(T,\nu)$ and drop the exponential factor.

(3) The same function $\bar{g}(T,\nu) + f(T,\nu)$ occurs in the absorption coefficient of a hydrogen-like gas in thermal equilibrium (Maxwellian distribution and Saha formula referring to the same temperature).

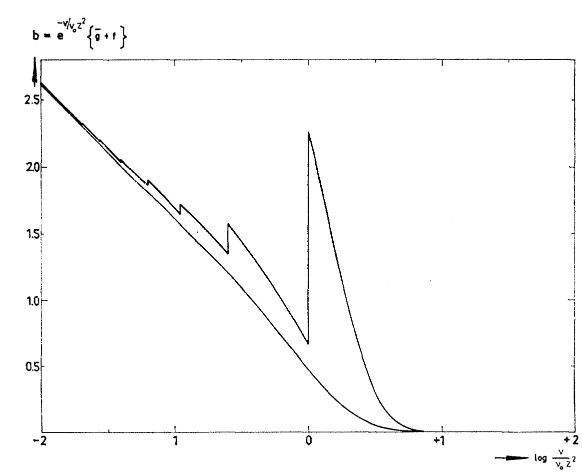

FIG. 10. Emission spectrum from a Maxwellian distribution of electrons at the temperature $kT = h\nu_0 Z^2$ with exact Gaunt factors taken into account. Upper curve: combined continuum; lower curve: free-free emission only.

TABLE II. Average Gaunt factors for $\theta = 1$.

$\log(\nu/\nu_0 Z^2)$	$\bar{g}+f$	$\log(\nu/\nu_0 Z^2)$	$\bar{g}+f$	$\log(\nu/\nu_0 Z^2)$	$\bar{g}+f$
−9	11.47	−1.500	2.21	+0	6.15
−8.5	10.83	−1.398	2.12	+0.05	6.30
−8	10.20	−1.398	2.13	+0.10	6.43
−7.5	9.56	−1.204	1.99	+0.15	6.55
−7	8.93	−1.204	2.02	+0.20	6.65
−6.5	8.29	−1.097	1.94	+0.25	6.75
−6	7.66	−0.954	1.83	+0.30	6.85
−5.5	7.02	−0.954	1.91	+0.45	7.02
−5	6.39	−0.900	1.88	+0.60	7.04
−4.5	5.75	−0.750	1.80	+0.81	6.87
−4	5.12	−0.602	1.72	+1.10	6.31
−3.5	4.48	−0.602	2.00	+1.50	5.14
−3	3.85	−0.500	1.97	+1.84	4.05
−2.5	3.23	−0.330	1.91		
−2	2.66	−0	1.81		

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The Normal Constants of Motion in Quantum Mechanics Treated by Projection Technique*

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1. INTRODUCTION

IN the treatment of both one-particle and many-particle systems, the constants of motion play an important role in studying the time-dependent phenomena and the stationary states. In the classical theory, the constants of motion were physical quantities like the energy, the momentum, the angular momenta, etc. In quantum theory, the list of the constants of motion has been considerably extended and includes, not only the spin and the total angular momentum, but also such operators as the permutations, the translations, the crystal symmetry operators, etc. The physical quantities are represented by Hermitian operators, whereas the latter quantities correspond to another type of operators, the so-called *normal operators*, which may be considered as a generalization of the Hermitian operators. The normal constants of motion are essential for classifying the energy levels, and we will here give a short outline of their theory by means of a projection operator formalism.

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