13. G. Cortellessa and A. Reale, *Cinematica della fotoproduzione in idrogeno e decadimento in due mesoni π di un "mesone ω^0" di massa 700 MeV*, Report No. ISS 61/36. [Tables are given for the reactions: $\gamma + p \rightarrow \omega^0 + p$ from the threshold up to 1500 MeV, and $\omega^0 \rightarrow \pi^+ + \pi^-$, (0–800 MeV). Here, the "ω^0 meson" is a neutral particle having rest energy equal to 700 MeV.]
14. G. Cortellessa and A. Reale, *Cinematica della fotoproduzione in idrogeno del mesone ω^0 con massa di 787 MeV*, Report No. ISS 61/41. (Range of bombarding energies from threshold up to 1500 MeV.)
15. G. Cortellessa, *Cinematica della disintegrazione del mesone ω^0 di massa 310 MeV in $\pi^0 + \gamma$*, Report No. ISS 61/49.

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Conformal Invariance in Physics*

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CONTENTS

A. BACKGROUND

1. Preliminaries

WHEREAS conformal transformations and conformal invariance are well understood in mathematics, this is not the case in physics. Most physicists believe that conformal invariance has no physical meaning and consequently should not play any role in physics; but very few seem to have studied the matter.

The present paper exhibits the conformal covariance of certain basic equations of present-day physics. This covariance is valid generally only when one relaxes the requirement that rest masses are constants; rather, it is necessary that all rest masses transform in a certain conformally covariant way. The fundamental charge e, as well as Planck's constant h, remains constant.

The conformal covariance of these equations leads to the question of the physical significance of conformal invariance and of the concomitant mass

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transformations; the possible experimental detection of the consequences of conformal invariance becomes essential.

The present paper is mainly devoted to a discussion of conformal transformations and to a study of the covariance of various equations. The questions just raised will be answered only briefly (Sec. 10); a more complete answer is contained in other papers on this subject (see references 22 and 23 below).

2. Brief History of Conformal Transformations in Physics

Since the study of conformal transformations of various basic equations of physics has had a rather varied history, it is appropriate to note here some of the important developments in this connection during the last fifty years.

Conformal invariance was first introduced into physics in 1909 when Cunningham and Bateman[1] showed that Maxwell's equations are covariant not only under the 10-parameter Lorentz group, but under the larger 15-parameter conformal group C_0. It was soon observed that the equations of motion of charged particles (masses $m \neq 0$) are not conform-invariant.

In 1936 Page[2] developed a "new relativity" which was later recognized to be "special relativity" based on the conformal group C_0 instead of the Lorentz group.[3] This meant among other things that not only all coordinate systems with constant relative velocity are equivalent, but in fact all those with constant relative acceleration. To make this physically meaningful, one has to restrict oneself to *local* measurements, i.e., determine distances by reflecting light signals from distant points, etc. In Page's electrodynamics rest masses are no longer constants.

After Weyl's theory of gravitation and electrodynamics (1918) was proposed, interest arose in extending general relativity by removing the restriction of the invariance of proper differential line elements $ds^2 \neq 0$. An examination of the ideas underlying Weyl's theory suggested that light signals determine only *ratios* of the elements of the metric tensor $g_{\mu\nu}$. The determination of $g_{\mu\nu}$ itself requires an object like a clock that is much less fundamental than a light ray. If one attempts to base general relativity exclusively on light signals, one is led to "conformal general relativity," corresponding to a transformation group C which no longer refers only to flat space. As was first pointed out by Einstein,[4] in such a theory it is no longer possible to transfer yardsticks from one space-time point to another and the rate of clocks will depend on their history. Only local measurements (i.e., local comparisons of space and time intervals) are meaningful. So far, however, this idea has not been particularly fruitful in general relativity. As Pauli[5] puts it, "This point of view was soon abandoned by him and others, as it does not seem to have any physical meaning." Page's relativity theory, which was mentioned above, is the special relativistic analog of these ideas (C_0 is a subgroup of C), and kinematic cosmology associated with the names of Schrödinger, Infeld and Schild, and others, is closely connected with conformal invariance.

During the last 25 years various equations of physics were repeatedly proven to be covariant. However, it was primarily Schouten and Haantjes[6] who pointed out the necessity of transforming masses in a conformally covariant way (viz., as invariants of weight $-1/2$). Also, conform-invariant wave equations have been constructed[7] and a conform-invariant quantum field theory has been proposed.[8] We shall return to these attempts in Sec. 9.

B. CONFORMAL TRANSFORMATIONS AND WEYL SPACE

3. Definition of Conformal Transformations

Conformal transformations can be formulated in different ways. It is very important to distinguish clearly these different formulations, because they have different physical interpretation in terms of observer and observed, and because equations belonging to different formulations may easily be confused. Such a confusion can lead to mathematical inconsistencies even when two formulations are equivalent.

(a) *Conformal Point Transformations*

In the following it will be essential to distinguish between points in a space and the coordinate systems used to specify them. We shall denote different points by x, $\bar{x}$, $\bar{\bar{x}}$, etc., whereas the coordinate systems will

[1] E. Cunningham, Proc. London Math. Soc. **8**, 77 (1909); H. Bateman, *ibid.* **8**, 223 (1910).
[2] L. Page, Phys. Rev. **49**, 254, 946 (1936); L. Page and N. I. Adams, *ibid.* **49**, 466 (1936).
[3] H. T. Engstrom and M. Zorn, Phys. Rev. **49**, 701 (1936); H. P. Robertson, *ibid.* **49**, 755 (1936).
[4] A. Einstein, Sitzber. preuss. Akad. Wiss., Physik.-math. Kl. 1921, 261.
[5] W. Pauli, *Theory of Relativity* (Pergamon Press, New York, 1958), p. 224.
[6] J. A. Schouten and J. Haantjes, Koninkl. Ned. Akad. Wetenschap. Proc. **43**, 1288 (1940).
[7] F. Gürsey, Nuovo cimento **3**, 988 (1956); J. A. McLennan, *ibid.* **5**, 640 (1957); H. A. Buchdahl, *ibid.* **11**, 496 (1959). References to earlier work can be found in these papers.
[8] R. L. Ingraham, Nuovo cimento **12**, 825 (1954); J. Wess, *ibid.* **18**, 1086 (1960).

be denoted by the indices characterizing the covariant or contravariant components of various quantities. Thus, x^μ, $x^{\mu'}$, etc., denote components of x measured in the coordinate systems S, S', etc.[9] We shall sometimes write x or x' to indicate that the point is measured by S or S'.

Consider the point transformation (sometimes called "active transformation")

$$\bar{x} = f^\mu(x) , \tag{3.1}$$

which determines the components of the point $\bar{x}$ in S when the components of the point x are known in the same coordinate system S. By definition, this transformation is one to one and analytic in a given domain D.

Although the following discussion is obviously considerably more general, we shall assume that we are concerned with points in a Riemann space of four dimensions and with indefinite metric tensor $g_{\mu\nu}(x)$. This metric tensor is characterized by the requirement that in a local geodesic coordinate system it can be brought into the Minkowski form $\eta_{\mu\nu}$ with signature $+2$.

The line element of a time-like curve is then given by

$$d\tau^2(x) = -\, g_{\mu\nu}(x)dx^\mu dx^\nu . \tag{3.2}$$

The point transformation (3.1) yields

$$d\bar{x}^\mu = \partial_\alpha \bar{x}^\mu dx^\alpha , \tag{3.3}$$

which determines the difference of the components of the two points $\bar{x}$ and $\bar{x} + d\bar{x}$ into which the two infinitesimally nearby points x and $x + dx$ are mapped by (3.1).

A conformal point transformation is characterized by the property that the line element $d\tau(\bar{x})$ at the point $\bar{x}$ is related to the line element at x by a scalar function $\sigma(x)$,

$$d\tau(\bar{x}) = \surd(\sigma) \cdot d\tau(x) . \tag{3.4}$$

This property implies

$$g_{\mu\nu}(\bar{x})d\bar{x}^\mu d\bar{x}^\nu = \sigma(x)g_{\alpha\beta}(x)dx^\alpha dx^\beta , \tag{3.5}$$

with the restriction

$$\sigma(x) > 0 . \tag{3.6}$$

This relation can be written

$$g_{\mu\nu}(\bar{x})\partial_\alpha \bar{x}^\mu \partial_\beta \bar{x}^\nu = \sigma(x)g_{\alpha\beta}(x) . \tag{3.7}$$

It can be seen easily that Eq. (3.7) is equivalent to (3.4) and that it can therefore be used as a defining equation for conformal point transformations.

We note that all quantities in all equations above refer to the *same* coordinate system. If we identify a coordinate system S with an observer, we can say that we have here a mapping of the domain D of points into the domain $\bar{D}$ of points, both domains being in the space of the observer S.

We further note that the structure on the left-hand side of (3.7) should be clearly distinguished from a coordinate transformation $S \to S'$ in which the *same* point P occurs with components x^μ and $x^{\mu'}$ corresponding to the two coordinate systems,

$$g_{\mu'\nu'}(x')\partial_\alpha x^{\mu'}\partial_\beta x^{\nu'} = g_{\alpha\beta}(x) . \tag{3.8}$$

(b) Conformal Coordinate Transformations

A coordinate transformation $S \to S'$ (sometimes called "passive transformation") is a one-to-one relation

$$x^{\mu'} = h^{\mu'}(x) \tag{3.9}$$

of the components of the points x of a domain D as seen by two different observers (coordinate systems). We again assume it to be analytic. Infinitesimally close points transform according to

$$dx^{\mu'} = \partial_\alpha x^{\mu'} dx^\alpha , \tag{3.10}$$

and the metric tensor according to

$$g_{\mu'\nu'}(x') = \partial_{\mu'}x^\alpha \partial_{\nu'}x^\beta g_{\alpha\beta}(x) . \tag{3.11}$$

These relations imply

$$d\tau^2(x') = d\tau^2(x) . \tag{3.12}$$

in contradistinction to (3.4), unless $\sigma = 1$.

Equation (3.11) is typical of the transformation of any tensor field $T_{\alpha\beta\ldots}{}^{\mu\nu\cdots}$, which proceeds according to

$$T_{\alpha'\beta'\ldots}{}^{\mu'\nu'\cdots}(x') = \partial_\mu x^{\mu'}\partial_\nu x^{\nu'} \cdots \partial_{\alpha'}x^\alpha \partial_{\beta'}x^\beta \cdots T_{\alpha\beta\ldots}{}^{\mu\nu\cdots}(x) . \tag{3.13}$$

In order to determine the definition of a conformal coordinate transformation corresponding to the conformal point transformation (3.7), we need to establish first the general relationship between coordinate transformations and point transformations.

A point transformation can be associated with each coordinate transformation by requiring the relationship

$$\bar{x}^{\mu'} \doteq x^\mu , \quad \text{or} \quad \bar{x}' \doteq x . \tag{3.14}$$

This means that for a given relation of the components of the point x in two coordinate systems, a

[9] This notation agrees with that by J. A. Schouten, *Ricci-Calculus* (Springer-Verlag, Berlin, Germany, 1954), 2nd ed.

point $\bar{x}$ is associated with x in such a way that the components of $\bar{x}$ with respect to S' are the same as the components of x with respect to S. The "dot equal" sign in (3.14) indicates that this equality is valid only in the coordinate systems indicated in the equation. A definite correlation of the labelings μ and μ' is thus established.

By means of (3.14) the relation between the point transformation (3.1) and the coordinate transformation (3.9) is obtained:

$$\bar{x}^{\mu'} = h^{\mu'}(\bar{x}) = h^{\mu'}(f(x)) \doteq x^\mu$$

implies that the function h^μ in (3.9) is the inverse transformation to (3.1); if (3.1) implies

$$x^\mu = F^\mu(\bar{x}) \; , \tag{3.1$'$}$$

then (3.9) is

$$x^{\mu'} = F^{\mu'}(\bar{x}) = h^{\mu'}(x) \; . \tag{3.15}$$

These relations can also be written

$$\bar{x}^\mu = f^\mu(x) \doteq f^\mu(\bar{x}')$$

from which follows, using (3.14),

$$\partial\bar{x}^\mu/\partial x^\alpha \doteq \partial\bar{x}^\mu/\partial\bar{x}^{\alpha'} \; . \tag{3.16}$$

This establishes the relationship between point transformations and coordinate transformations.

The *conformal* coordinate transformation may now be inferred from the corresponding point transformation (3.7). It is given by

$$x^\mu = f^\mu(x') \tag{3.15$'$}$$

where f^μ is the same function as in (3.1) and is such that it implies (3.7).

Finally, we note that (3.4) and (3.12) are consistent equations. Substituting (3.16) into (3.7) we find

$$g_{\mu\nu}(\bar{x}) \; \frac{\partial\bar{x}^\mu}{\partial\bar{x}^{\alpha'}} \; \frac{\partial\bar{x}^\nu}{\sigma\bar{x}^{\beta'}} \doteq \sigma(x)g_{\alpha\beta}(x) \; .$$

Using (3.11) for $\bar{x}$, this gives

$$g_{\alpha'\beta'}(\bar{x}') \doteq \sigma(x)g_{\alpha\beta}(x) \; . \tag{3.17}$$

Therefore, with (3.7),

$$d\tau^2(\bar{x}') \doteq \sigma(x)d\tau^2(x) \; . \tag{3.18}$$

which shows the consistency of (3.4) and (3.12):

$$d\tau^2(\bar{x}') = d\tau^2(\bar{x}) = \sigma(x)d\tau^2(x) \; .$$

(c) *Conformal Transformations of Tensor Fields*

Consider a vector field $A_\mu(x)$. A coordinate transformation of its covariant components is given by (3.13) as

$$A_{\alpha'}(\bar{x}') = \bar{\partial}_{\alpha'}\bar{x}^\mu A_\mu(\bar{x}) \tag{3.19}$$

which, by means of (3.16), is

$$A_{\alpha'}(\bar{x}') \doteq \partial_\alpha\bar{x}^\mu A_\mu(\bar{x})$$

We can therefore define a new field $\bar{A}_\alpha(x)$ such that

$$A_{\alpha'}(\bar{x}') \doteq \bar{A}_\alpha(x) \; . \tag{3.20}$$

For the contravariant components a similar relation holds.

However, it now follows from (3.7) and (3.20) that one *cannot* identify $\bar{A}^\alpha(x)$ with $A^\alpha(x)$ and $\bar{A}_\alpha(x)$ with $A_\alpha(x)$, but only either

$$\begin{cases} \bar{A}^\alpha(x) = A^\alpha(x) \\ \bar{A}_\alpha(x) = \sigma A_\alpha(x) \end{cases} \tag{3.21}$$

or

$$\begin{cases} \bar{A}_\alpha(x) = A_\alpha(x) \\ \bar{A}^\alpha(x) = (1/\sigma)\, A^\alpha(x) \; . \end{cases} \tag{3.22}$$

Identities (3.21) can be proven by substitution of the first Eq. (3.21) into (3.7); similarly for Eq. (3.22).

The result (3.21) and (3.22) can be stated as follows: If the components of a field $A(x)$ transform as a covariant vector under a conformal point transformation, then the contravariant components transform like an affine contravariant vector with weight factor σ^{-1},

$$A^\mu(\bar{x}) = (1/\sigma)\partial_\alpha\bar{x}^\mu A^\alpha(x) \; . \tag{3.23}$$

Conversely, if under a conformal point transformation we have a contravariant vector, then the corresponding covariant components transform like

$$A_\mu(\bar{x}) = \sigma\bar{\partial}_\mu x^\alpha A_\alpha(x) \; . \tag{3.24}$$

As a consequence, the length of a vector $A(x)$ transforms under a conformal point transformation (3.1) and (3.7) as

$$A_\mu(\bar{x})A^\mu(\bar{x}) = [1/\sigma(x)]A_\nu(x)A^\nu(x) \; , \tag{3.25}$$

whereas the length of a contravariant vector transforms as

$$A_\mu(\bar{x})A^\mu(\bar{x}) = \sigma(x)A_\nu(x)A^\nu(x) \; . \tag{3.25$'$}$$

These considerations can obviously be generalized to tensors of arbitrary degree.

(d) *Conformal Transformations of the Metric Tensor*

Consider the conformal point transformations defined by (3.7). If we use the symbol

$$g^c_{\mu\nu}(x) = \sigma(x)g_{\mu\nu}(x) \tag{3.26}$$

Eq. (3.7) becomes

$$g^c_{\mu\nu}(x) = \partial_\mu\bar{x}^\alpha\partial_\nu\bar{x}^\beta g_{\alpha\beta}(\bar{x}) \; . \tag{3.27}$$

This looks like a coordinate transformation, but x and $\bar{x}$ refer to two different points in the same coordinate system rather than to different coordinates of the same point.

The conformal coordinate transformation characterized by (3.17) can be written, using (3.26),

$$g^{c}_{\mu\nu}(x) \doteq g_{\mu'\nu'}(\bar{x}') \tag{3.28}$$

for

$$x \doteq \bar{x}' . \tag{3.29}$$

This indicates that one can give a definition of conformal transformations which does not refer to either point or coordinate transformations, but is consistent with both of these:

Given a metric manifold characterized by the metric tensor $g_{\mu\nu}$ we define

$$g^{c}_{\mu\nu}(x) = \sigma(x)g_{\mu\nu}(x) , \quad g^{\mu\nu}_{c}(x) = [1/\sigma(x)]g^{\mu\nu}(x) , \tag{3.30}$$

where σ is an arbitrary positive differentiable function of x. We refer to (3.30) as the "conformal transformation of the metric tensor." These transformations form a group C_g.

The totality of all manifolds differing from each other only by elements of C_g is called a conformal space. Elements of length $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ in such a space clearly have no absolute meaning because a comparison of lengths at two different points involves the arbitrary function σ [see also (3.25)], but the ratio of two infinitesimal lengths is well defined when both lengths refer to the same point. Also, angles are well defined at each point:

$$\cos \alpha = \frac{g_{\mu\nu}dx^\mu \delta x^\nu}{(g_{\alpha\beta}dx^\alpha dx^\beta)^{1/2}(g_{\rho\sigma}\delta x^\rho \delta x^\sigma)^{1/2}} \tag{3.31}$$

is invariant under C_g. This is the reason for the name "conformal."

The conformal point and coordinate transformations are now seen to be combinations of the conformal transformations of the metric (3.30), with equations of the type (3.27) or (3.11) characterizing the tensor nature of $g_{\mu\nu}$. We shall denote the corresponding group of transformations by C. It has the group C_g and the group of all coordinate transformations as proper subgroups. We call C the "extended conformal group."

From these considerations we conclude that when we deal with equations that are form-invariant (covariant) under coordinate transformations, it will be necessary only to check covariance under C_g in order to assure covariance under C.

In the following we shall therefore be concerned only with C_g and coordinate transformations.

4. Riemann Space and Weyl Space

(a) *Riemann Space and the Groups C_g and C_0*

The Riemann space $\mathfrak{R}$ underlying the theory of relativity is characterized by a symmetric metric (Riemann) tensor $g_{\mu\nu}$, which is restricted by the conditions

$$g_{00} < 0 , \quad \sum\nolimits_{i,k} g_{ik}\xi^i\xi^k > 0 , \quad (i,k = 1,2,3) \tag{4.1}$$

for all vectors ξ^μ. The element of proper time $d\tau$, defined by time-like dx^μ,

$$d\tau^2 = - g_{\mu\nu}(x)dx^\mu dx^\nu \tag{4.2}$$

is invariant under all coordinate transformations. The latter are defined by[10]

$$x'^\mu = h^\mu(x) , \tag{4.3}$$

with h^μ being a real differentiable functions with a nonvanishing Jacobian. We are not concerned with space or time reversal transformations and we shall admit only such coordinate transformations (4.3) that leave the properties (4.1) invariant, i.e.,

$$g'_{\mu\nu}(x') = \partial'_\mu x^\alpha \partial'_\nu x^\beta g_{\alpha\beta}(x)$$

must also satisfy (4.1).

The invariance of proper time under coordinate transformations must be contrasted to its lack of invariance under the conformal transformations of the metric tensor C_g. Clearly, $g^c_{\mu\nu}$ also satisfies (4.1), but

$$d\tau_c^2 = - g^c_{\mu\nu}dx^\mu dx^\nu = \sigma(x)d\tau^2 \tag{4.4}$$

is not invariant under C_g. Nevertheless, the space $\mathfrak{R}^c$ based on the metric tensor $g^c_{\mu\nu}$ is as good a Riemannian space as $\mathfrak{R}$. The group C_g gives a one-to-one mapping of $\mathfrak{R}$ onto $\mathfrak{R}^c$.

The Christoffel symbol in $\mathfrak{R}$,

$$\left\{ {\mu \atop \alpha\beta} \right\} = \tfrac{1}{2} g^{\mu\lambda}(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\beta} - \partial_\lambda g_{\alpha\beta}) \tag{4.5}$$

transforms under C_g as

$$\left\{ {\mu \atop \alpha\beta} \right\}^c = \left\{ {\mu \atop \alpha\beta} \right\} + \tfrac{1}{2} (\delta^\mu_\alpha s_\beta + \delta^\mu_\beta s_\alpha - s^\mu g_{\alpha\beta}) \tag{4.6}$$

where

$$s_\mu = \partial_\mu \ln \sigma . \tag{4.7}$$

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[10] The customary notation used in physics is employed here. In the notation of Sec. 3 which distinguishes between active and passive transformations Eq. (4.3) would be written $x^{\mu'} = h^{\mu'}(x)$. In what follows x^μ and x'^μ refer to the same point seen by different coordinates (observers) S and S'.

Correspondingly, the curvature tensor in $\mathfrak{R}^c$ differs from that in $\mathfrak{R}$ by terms depending on s_μ.

The group C_0 of *special* or restricted conformal transformations is defined as the set of those transformations in C that transforms flat space into flat space. This means that the functions $\sigma(x)$ are no longer arbitrary, but are now restricted by the condition that

$$R^{\mu}{}_{\nu\rho\sigma} = 0 \quad \text{implies} \quad R^{c\mu}{}_{\nu\rho\sigma} = 0\,, \qquad (4.8)$$

which requires as a necessary and sufficient condition that[11]

$$2(\partial_\mu s_\nu - s_\alpha\{{}^{\alpha}_{\mu\nu}\}) = s_\mu s_\nu - \tfrac{1}{2}\, g_{\mu\nu} s_\lambda s^\lambda\,. \qquad (4.9)$$

These equations can be solved for s_μ; they determine σ within a multiplicative constant. The corresponding transformations from Minkowski space $[g_{\mu\nu} = \eta_{\mu\nu};\ \eta_{\mu\nu} = 0(\mu \neq \nu);\ \eta_{ii} = 1 = -\eta_{00}]$ to a flat space with metric $g^c_{\mu\nu} = \sigma\eta_{\mu\nu}$ were classified by Haantjes.[11] They constitute the transformation group C_0, the restricted conformal group.

The group C_0 is the 15-parameter Lie group often referred to as "conformal transformations." It is the group considered by Bateman and Cunningham[1] and by most physicists writing on conformal invariance. It consists of the space-time translations

$$x'^{\mu} = x^\mu + \alpha^\mu\,, \qquad \text{(4 parameters)} \quad (4.10\text{a})$$

the proper homogeneous Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^\nu\,, \qquad \text{(6 parameters)} \quad (4.10\text{b})$$

the dilatation (or scale) transformation

$$x'^{\mu} = s x^\mu\,, \qquad \text{(1 parameter)} \quad (4.10\text{c})$$

and the acceleration transformation

$$x'^{\mu} = (1 + 2a^\alpha x_\alpha + x^2 a^2)^{-1}(x^\mu + a^\mu x^2) \qquad \text{(4 parameters)}. \quad (4.10\text{d})$$

In the latter $x^2 = x_\alpha x^\alpha$ and $a^2 = a_\alpha a^\alpha$ refer to Minkowski space. The 15 parameters are α_μ, $\Lambda^{\mu}{}_{\nu}$, s, and a^μ. By an extension to Riemann spaces with nondefinite metric of a well-known theorem due to Liouville, Haantjes[11] showed that every element of C_0 can be composed from motions and inversions only.

The group C_0 is obviously a subgroup of C; whereas most of what follows will be valid for C, we shall have occasion to limit our considerations to the important special case C_0.

(b) *Invariance of Derivatives*

Since we are concerned with the group C, and in particular with the group C_g, it is desirable to define quantities whose transformation properties under C_g are explicit. The following concepts will therefore be useful.

A *Weyl tensor* of weight n and k indices is a Riemann tensor of k indices which transforms under (3.30) as

$$T_c = \sigma^n T\,. \qquad (4.11)$$

Examples are the metric tensor $g_{\mu\nu}$ which is a Weyl tensor of weight $+1$, dx^μ (Weyl vector of weight zero), and $d\tau$ (Weyl scalar of weight 1/2).

For the formulation of a conform-invariant differential equation we need a conform-invariant covariant derivative. A Riemann space [defined by (4.1) and (4.2)] has an affine connection which is the Christoffel symbol (4.5) constructed from the metric tensor $g_{\mu\nu}$. The covariant derivative of a vector V^μ is therefore defined as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + V^\alpha \{{}^{\mu}_{\alpha\nu}\} \qquad (4.12)$$

The left-hand side is a Riemann tensor because the Christoffel symbol is a solution of the transformation equation of linear connections

$$L'^{\mu}_{\alpha\beta} = \partial_\lambda x'^{\mu}(\partial'_\alpha x^\rho \partial'_\beta x^\sigma L^\lambda_{\rho\sigma} + \partial'_\alpha \partial'_\beta x^\lambda)\,. \qquad (4.13)$$

The Christoffel symbol satisfies (4.13) under the coordinate transformations (4.3). Obviously, any other solution of (4.13) when used in (4.12) would also produce a ∇_ν such that $\nabla_\nu V^\mu$ is a Riemann tensor. Any such choice will make ∇_ν a covariant vector.

As is seen from (4.6), the transformations C_g produce

$$\begin{aligned}\nabla^c_\nu V^\mu_c &= \partial_\nu V^\mu_c + V^\alpha_c \{{}^{\mu}_{\alpha\nu}\}^c \\ &= \nabla_\nu V^\mu + \tfrac{1}{2}\,(V^\mu s_\nu + \delta^\mu_\nu V^\alpha s_\alpha - s^\mu V_\nu) \qquad (4.14)\end{aligned}$$

where we assumed that V^μ is a Weyl vector of weight zero. Thus, ∇_ν is not a Weyl vector, because $\nabla_\nu V^\mu$ is not a Weyl Tensor. If the last term were absent, ∇_ν would be a Weyl vector of weight zero.

Let us attempt to obtain ∇_ν as a zero-weight Weyl vector by use of a connection different from the Christoffel symbols. We define the symmetric connection

$$\Gamma^\mu_{\alpha\beta} = \{{}^{\mu}_{\alpha\beta}\} - \tfrac{1}{2}\,(\delta^\mu_\alpha \kappa_\beta + \delta^\mu_\beta \kappa_\alpha - \kappa^\mu g_{\alpha\beta})\,. \qquad (4.15)$$

Then we have

$$\overset{c}{\Gamma}{}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} \qquad (4.16)$$

provided κ_μ transforms according to

$$\overset{c}{\kappa}_\mu = \kappa_\mu + s_\mu \qquad (4.17)$$

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[11] J. Haantjes, Koninkl. Ned. Akad. Wetenschap. Proc. **40**, 700 (1937).

under C_g. The expression

$$\nabla_\nu V^\mu \equiv \partial_\nu V^\mu + V^\alpha \Gamma^\mu_{\alpha\nu}$$

will then be a Weyl tensor of weight zero provided $\Gamma^\mu_{\alpha\beta}$ satisfies (4.13).

Now $\Gamma^\mu_{\alpha\beta}$ should be a characteristic feature of the space. Since we have a metric space, $\Gamma^\mu_{\alpha\beta}$ should depend only on the metric and its derivatives. To this end we note that if κ_μ is of the form

$$\kappa_\mu = \partial_\mu \ln \phi \tag{4.18}$$

then (4.15) can be written as

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} G^{\mu\lambda}(\partial_\alpha G_{\lambda\beta} + \partial_\beta G_{\lambda\alpha} - \partial_\lambda G_{\alpha\beta}) \tag{4.19}$$

where

$$G^{\mu\lambda}G_{\lambda\nu} = \delta^\mu_\nu\,, \quad G_{\mu\nu} = g_{\mu\nu}/\phi \tag{4.20}$$

Since κ_μ must transform according to (4.17), the requirement that ϕ depend on $g_{\mu\nu}$ yields uniquely (except for a constant factor)

$$\phi = |g|^{1/4} \quad \text{where} \quad g \equiv \det g_{\mu\nu} \tag{4.21}$$

or

$$G_{\mu\nu} = g_{\mu\nu}/|g|^{1/4} \tag{4.22}$$

The fundamental conformal-affine tensor $G_{\mu\nu}$ is form-invariant under C_g so that the form-invariance under C_g of $\Gamma^\mu_{\alpha\beta}$, Eq. (4.16), is obvious.

However, the choice (4.21) for ϕ is not satisfactory for the invariance of the covariant derivative under the general group C, because if $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbol constructed from $G_{\mu\nu}$, it does not satisfy (4.13). This follows immediately from the fact that g is *not* a scalar. The space based on

$$ds^2 = \pm\, G_{\mu\nu}dx^\mu dx^\nu \tag{4.23}$$

is therefore not a Riemann space, since $G_{\mu\nu}$ is not a Riemann tensor. Such a space is called a conformal-affine space, since ds^2 is conform-invariant and since the tensors one encounters are in general no longer Riemann tensors, but affine tensors of which the Riemann tensors are a special case.

We shall not work explicitly in conformal-affine space, but shall follow an equivalent approach in terms of tensor densities. These are more familiar to the physicist than the affine tensors. At the same time, invariance under coordinate transformations will appear in the usual well-known form of covariance of equations.[11a]

[11a] A conformal-affine Riemann space is used for a discussion of Weyl's theory by P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1946).

Therefore, we shall not adopt (4.18) and (4.21) *a priori*, but work with (4.15) and derive a dependence of κ_μ on the metric which assures the vector properties of κ_μ, and therefore satisfies (4.13).

(*c*) *Weyl Space*

A Weyl space is defined as a space with a real symmetric metric tensor and a symmetric connection given by (4.15). In our case we have to add that the space is 4 dimensional, and that the conditions (4.1) on the metric will always be assumed to hold. We shall denote this space by $\mathcal{W}$.

A Weyl space reduces to a Riemann space, $\mathcal{R}$, if and only if $\kappa_\mu = 0$. However, when κ_μ is a gradient it is equivalent to a Riemann space, as we shall see later. The most important difference between these two spaces is that $\nabla_\mu g_{\alpha\beta} = 0$ in $\mathcal{R}$, while in $\mathcal{W}$

$$\nabla_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \left\{{\lambda \atop \alpha\mu}\right\} g_{\lambda\beta} - \left\{{\lambda \atop \beta\mu}\right\} g_{\lambda\alpha} + \kappa_\mu g_{\alpha\beta}\,.$$

According to a well-known identity, the first three terms on the right-hand side vanish, so that

$$\nabla_\mu g_{\alpha\beta} = \kappa_\mu g_{\alpha\beta}\,. \tag{4.24}$$

A space with this property is said to have a semi-metric connection.

The geometrical meaning of (4.24) is the following: Let a vector have the contravariant components V^μ and the covariant components V_μ; then its length is defined by

$$l_V^2 \equiv V_\mu V^\mu = g_{\mu\nu}V^\mu V^\nu\,. \tag{4.25}$$

Let δ be the covariant differential. Then a parallel displacement of the contravariant vector $\delta V^\mu = 0$, does *not* imply a parallel displacement of the covariant vector. This is of course related to the situation characterized by (3.21) and (3.22). We have from $\delta V^\mu = 0$,

$$\delta V_\mu = \delta(g_{\mu\nu}V^\nu) = V^\nu \delta g_{\mu\nu} = V^\nu dx^\lambda \nabla_\lambda g_{\mu\nu} = V_\mu \kappa_\alpha dx^\alpha \tag{4.26}$$

and

$$\delta l_V^2 = V^\mu \delta V_\mu = l_V^2 \kappa_\alpha dx^\alpha \tag{4.27}$$

or

$$\delta \ln l_V^2 = \kappa_\alpha dx^\alpha\,. \tag{4.27'}$$

Thus, parallel displacement of a vector changes its length. This is not the case in $\mathcal{R}$, where $\kappa_\lambda = 0$. We note, however, that null vectors ($l_V^2 = 0$) remain null vectors, as is evident from (4.27).

This is of course exactly the same situation as in

Weyl's theory[12] of gravitation and electricity of 1918. That theory failed, because from (4.27) it follows that the length of an object depends on its history, which is physically untenable. To what extent this difficulty also exists in a conform-invariant theory will be discussed elsewhere. We note here only the following: The conformal invariance of the theory implies as was mentioned earlier [see the remarks preceding (3.31)], that comparison of lengths at *different* space-time points is *a priori* meaningless, whereas (infinitesimal) lengths at the same point can be compared. The only comparison of measurements at two points a finite distance apart must therefore be made by means of light signals; their invariant nature was noted above.

An important quantity in Riemann geometry is the contracted Christoffel symbol

$$\{{}^{\alpha}_{\mu\alpha}\} = \partial_\mu \ln \sqrt{|g|} \,. \tag{4.28}$$

For a Weyl connection

$$\Gamma_\mu \equiv \Gamma^{\alpha}_{\mu\alpha} = \{{}^{\alpha}_{\mu\alpha}\} - 2\kappa_\mu = \partial_\mu \ln \sqrt{|g|} - 2\kappa_\mu \,. \tag{4.29}$$

In order to express κ_μ in terms of the metric we recall the notion of tensor density. A tensor density of weight w, $T(w)$, is related to the corresponding tensor T of weight zero by

$$T(w) = (\sqrt{|g|})^w T \,. \tag{4.30}$$

The covariant derivative of a tensor density of weight w is so defined that it is again a tensor density of weight w, but with one more index. Thus, for a vector density $\mathfrak{v}(w)$,

$$\nabla_\mu \mathfrak{v} = \partial_\mu \mathfrak{v} + \mathfrak{v}^\alpha \Gamma^{\nu}_{\alpha\mu} - w\Gamma_\mu \mathfrak{v}^\nu \tag{4.31a}$$

$$\nabla_\mu \mathfrak{v}_\nu = \partial_\mu \mathfrak{v}_\nu - \mathfrak{v}_\alpha \Gamma^{\alpha}_{\mu\nu} - w\Gamma_\mu \mathfrak{v}_\nu \,. \tag{4.31b}$$

From these equations the covariant derivative of tensors of any index number follows. In particular, for a scalar density of weight w

$$\nabla_\mu \mathfrak{s} = \partial_\mu \mathfrak{s} - w\Gamma_\mu \mathfrak{s} \,. \tag{4.32}$$

The determinant g is itself a scalar density of weight 2,

$$\nabla_\mu g = \partial_\mu g - 2\Gamma_\mu g$$

or

$$\nabla_\mu \ln \sqrt{|g|} = \partial_\mu \ln \sqrt{|g|} - \Gamma_\mu \,. \tag{4.33}$$

We note that while we adopt the usual convention of using gothic letters to designate densities, this notation is not used for the determinant of the metric tensor. Using (4.29),

$$\kappa_\mu = \tfrac{1}{4} \nabla_\mu \ln |g| \,, \tag{4.34}$$

which is the desired relation. This equation replaces (4.18) and (4.21). The important point is that (4.34) establishes κ_μ as a vector, contrary to (4.18).

Parenthetically, it may be remarked that the connection between tensor densities and affine tensors is obtained by noting the relation

$$g' = J^2 g \,, \quad J = \det \partial'_\mu x^\nu > 0 \,, \tag{4.35}$$

which follows from the tensor character of $g_{\mu\nu}$. Expressing the transformation law of tensor densities in terms of J rather than g one establishes that, when a metric is defined, affine tensors of weight w and tensor densities of weight w are identical concepts.

In order to avoid confusion between tensor densities of weight w and Weyl tensors of weight n, we shall denote the latter as W_n tensors. Thus, we might have a W_n tensor density of weight w; obviously, the tensor density transformation properties of a W_n tensor density are unchanged by a conformal transformation, since σ is a scalar.

Turning now to conformal transformation, we see that (4.34) implies the transformation property (4.17) that assures the invariance of the Weyl connection under C_σ.

We are now ready to study the invariance under C of various equations. This invariance can be expressed in a Weyl space as follows: The laws of physics are invariant under C if they are invariant

(a) under all coordinate transformations,

(b) under all metric gauge transformations, being the simultaneous transformations of the metric C_σ,

$$g_{\mu\nu} \to \overset{c}{g}_{\mu\nu} = \sigma g_{\mu\nu} \,, \tag{3.30}$$

and the gauge transformations on

$$\kappa_\mu \to \overset{c}{\kappa}_\mu = \kappa_\mu + \partial_\mu \ln \sigma \,. \tag{4.17}$$

We shall adopt the following procedure for the study of C invariance of a given equation:

(a) An equation (or set of equations) is assumed to be given covariantly in $\mathfrak{M}$, the Minkowski space, defined as the Euclidean flat space of four dimensions whose symmetric metric tensor satisfies (4.1), and in which it is consequently always possible to transform to the metric $\eta_{\mu\nu}$ with vanishing connection.

(b) Generalization to an equation covariant in any space $\mathfrak{A}$ with symmetric linear connection and symmetric (real) metric tensor satisfying (4.1). This will be accomplished by replacing the coordinate deriva-

[12] H. Weyl, Sitzber. preuss. Akad. Wiss., Physik-math. Kl. 1918, 465; Math. Z. 2, 384 (1918); Ann. Phys. 59, 101 (1919); *Space, Time and Matter* (Dover Publications, Inc., New York).

tives by covariant derivatives and the derivatives with respect to parameters (proper time) by covariant derivatives with respect to these parameters

$$\partial_\mu \rightarrow \nabla_\mu, \quad \frac{d}{d\tau} \rightarrow \frac{\delta}{\delta\tau} = \frac{dx^\mu}{d\tau}\nabla_\mu \tag{4.36}$$

(c) The invariance of the resulting equation under C_g is studied. This is greatly simplified by specializing from the space $\mathfrak{A}$ to a Weyl space $\mathfrak{W}$. The Weyl connection assures the C_g invariance of the covariant derivatives. If C_g invariance can be obtained for a *covariant* equation in $\mathfrak{W}$, C invariance is assured.

(d) The C-invariant equation is expressed in terms of the derivatives covariant in $\mathfrak{R}$ with additional terms depending explicitly on κ_μ.

In concluding this section we would like to add the following remarks. First, it is clear that the use of a Weyl space to study conformal invariance adds an essential feature, viz., the vector κ_μ. The conformal space has no connection specified, whereas the knowledge of κ_μ determines the conform invariant Weyl connection. Secondly, we have a more general space underlying our descriptions than the Riemann space. However, we can always return to $\mathfrak{R}$ by specifying κ_μ as the gradient of a scalar function. In that case the Weyl space is equivalent to a Riemann space, as follows from (4.6) and from (4.18) to (4.20). Specifically, if we identify $\kappa_\mu = s_\mu$, then

$$\Gamma^\mu_{\alpha\beta} = \{{}^{\mu}_{\alpha\beta}\}^c, \tag{4.37}$$

i.e., the Weyl connection constructed with $g_{\mu\nu}$ and $\kappa_\mu = \partial_\mu \ln\sigma$ is the Christoffel symbol for the metric tensor $g^c_{\mu\nu} = \sigma g_{\mu\nu}$. However, in the following discussion it will be irrelevant whether or not we assume $\kappa_\mu = s_\mu$.

C. CONFORMAL INVARIANCE OF FUNDAMENTAL EQUATIONS

5. Maxwell's Equations[13]

In $\mathfrak{M}$ with $g_{\mu\nu} = \eta_{\mu\nu}$ the electromagnetic field is given by an antisymmetric tensor $F_{\mu\nu}$ which satisfies the equations

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{5.1}$$

$$\partial_\mu F^{\mu\nu} = -j^\nu \tag{5.2}$$

where the current density is

$$j^\mu(x) = e\int_{-\infty}^{\infty} \delta(x-z)\frac{dz^\mu}{d\tau}d\tau \tag{5.3}$$

for a point charge on the world-line $z^\mu(\tau)$, and is

$$j^\mu(x) = \rho(x)u^\mu(x) \tag{5.4}$$

for finite charge distributions; both are vectors. ρ is the invariant charge density and $u^\mu(x)$ the four-velocity vector at a space-time point x.

For the generalization to $\mathfrak{A}$ we assume first that $F_{\mu\nu}$ is an antisymmetric tensor in general. Then

$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0 \tag{5.5}$$

and

$$g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\nabla_\mu \ln\sqrt{|g|} + \nabla_\mu(g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}) = -j^\nu \tag{5.6}$$

are the covariant generalizations of (5.1) and (5.2), provided j^μ is a vector in $\mathfrak{A}$. Since the connection in $\mathfrak{A}$ is assumed to be symmetric, (5.5) reduces to the form (5.1) and is therefore independent of the connection. For the same reason,

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma_\alpha F^{\alpha\nu} \tag{5.7}$$

and, therefore, from (4.33)

$$\begin{gathered}\nabla_\mu F^{\mu\nu} + (\nabla_\mu \ln\sqrt{|g|})F^{\mu\nu} = \partial_\mu F^{\mu\nu} + (\partial_\mu \ln\sqrt{|g|})F^{\mu\nu} \\ (1/\sqrt{|g|})\nabla_\mu(F^{\mu\nu}\sqrt{|g|}) = (1/\sqrt{|g|})\partial_\mu(F^{\mu\nu}\sqrt{|g|}).\end{gathered} \tag{5.8}$$

Note that differentiations always act only on the immediately following function unless otherwise indicated.

Equation (5.6) can thus be written in a form very similar to (5.2), making use of the tensor density of weight 1

$$\mathfrak{F}^{\mu\nu} = \sqrt{(|g|)}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} \tag{5.9}$$

and the vector density of weight 1,

$$\mathfrak{j}^\mu = j^\mu\sqrt{|g|} \tag{5.10}$$

(5.6) becomes

$$\partial_\mu \mathfrak{F}^{\mu\nu} = -\mathfrak{j}^\nu \tag{5.11}$$

valid in $\mathfrak{A}$. This equation is obviously also independent of the connection.

Equations (5.1) and (5.11) are identical with the well-known form of Maxwell's equations in general relativity ($\mathfrak{R}$). However, it is not trivial to point out that they hold not only in $\mathfrak{R}$, but even in $\mathfrak{A}$.

C_g invariance is now very simple. We assume that $F_{\mu\nu}$ is a W_0 tensor in the sense of (3.22). Since dx^μ is a W_0 vector and (5.1) holds in $\mathfrak{A}$ and therefore in particular in $\mathfrak{W}$, this equation is C_g invariant. Similarly, $\mathfrak{F}_{\mu\nu}$ is a W_0-tensor density with $w = 1$, and (5.11) holds in W. Hence, it is also C_g invariant, pro-

[13] The results of Secs. 5 and 6 were first obtained by Schouten and Haantjes (references 6 and 14).

vided $\mathfrak{j}^\mu$ is a W_0-vector density with $w = 1$. To verify this we define a generalized δ function.

Let $f(x)$ be a scalar function; then

$$f(x) = \int \delta(x - y) f(y) d^4y\,, \quad (d^4y = dy^0 dy^1 dy^2 dy^3) \tag{5.12}$$

defines the four-dimensional δ function. It is a scalar density ($w = 1$), since d^4y is a scalar density ($w = -1$). This is true in $\mathcal{W}$ as well as in $\mathcal{R}$. Therefore,

$$\mathfrak{j}^\mu(x) = e\int_{-\infty}^{\infty} \delta(x - z)\frac{dz^\mu}{d\tau}\, d\tau \tag{5.13}$$

is a vector density ($w = 1$). In $\mathfrak{M}$ this equation of course reduces to (5.3) since vector densities are indistinguishable from vectors ($|g| = 1$). Now the δ function is W_0, as is evident from (5.12). Since e is a number and dz^μ is W_0, it follows that $\mathfrak{j}^\mu$ is indeed W_0.

Analogous to the generalization of (5.3) to (5.13) is the generalization of (5.4) to

$$\mathfrak{j}^\mu(x) = \varrho(x) u^\mu(x) \tag{5.14}$$

where ϱ is a scalar density ($w = 1$), since $u^\mu(x) = dx^\mu/d\tau$ is a vector. Under C_g, ϱ must be $W_{1/2}$, since u^μ is $W_{-1/2}$ and the resulting $\mathfrak{j}^\mu$ is W_0. The total charge

$$Q = \int \mathfrak{j}^\mu(x) d\sigma_\mu \qquad \sigma = \text{space-like surface} \tag{5.15}$$

is a W_0 scalar. Here, $d\sigma_\mu$ is a W_0-vector density ($w = -1$), as follows from Gauss' theorem ($\mathfrak{j}_\mu = 0$ for large space-like distances),

$$\begin{aligned} Q_2 &= \int_{\sigma_2} \mathfrak{j}^\mu d\sigma_\mu = \int_{\sigma_1} \mathfrak{j}^\mu d\sigma_\mu + \left(\int_{\sigma_2} \mathfrak{j}^\mu d\sigma_\mu - \int_{\sigma_1} \mathfrak{j}^\mu d\sigma_\mu\right) \\ &= Q_1 + \int \partial_\mu \mathfrak{j}^\mu d^4x\,. \end{aligned}$$

The last integrand is a W_0-scalar density ($w = 1$) and vanishes identically, because

$$\nabla_\nu \mathfrak{j}^\nu = \partial_\nu \mathfrak{j}^\nu = \partial_\nu \partial_\mu \mathfrak{F}^{\mu\nu} = 0 \tag{5.16}$$

establishing charge conservation.

The electromagnetic energy-momentum tensor, defined by

$$T^\mu_\nu = F^{\mu\lambda} F_{\lambda\nu} + \tfrac{1}{4}\,\delta^\mu_\nu F^{\alpha\beta} F_{\alpha\beta} \tag{5.17}$$

has the divergence

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + T^\alpha_\nu \Gamma^\mu_{\alpha\mu} - T^\mu_\alpha \Gamma^\alpha_{\nu\mu}\,.$$

This yields by means of (4.33)

$$\nabla_\mu \mathfrak{T}^\mu_\nu \equiv \nabla_\mu (T^\mu_\nu \sqrt{|g|}) = \partial_\mu \mathfrak{T}^\mu_\nu - \mathfrak{T}^\mu_\alpha \Gamma^\alpha_{\nu\mu}\,. \tag{5.18}$$

This divergence has the same value whether computed with a Weyl connection or a Riemann connection, because

$$\begin{aligned} \mathfrak{T}^\mu_\alpha \Gamma^\alpha_{\nu\mu} &= \mathfrak{T}^\mu_\alpha \left\{{\alpha \atop \nu\mu}\right\} - \tfrac{1}{2}\,(\mathfrak{T}^\mu_\nu \kappa_\mu + \mathfrak{T}^\mu_\mu \kappa_\nu - \mathfrak{T}_{\nu\alpha}\kappa^\alpha) \\ &= \mathfrak{T}^\mu_\alpha \left\{{\alpha \atop \nu\mu}\right\}\,. \end{aligned} \tag{5.19}$$

The last equation follows from $\mathfrak{T}^{\mu\nu} = \mathfrak{T}^{\nu\mu}$ and

$$\mathfrak{T}^\mu_\mu = 0\,. \tag{5.20}$$

Thus, $\nabla_\mu \mathfrak{T}^\mu_\nu$ does not depend on κ_μ. It is a W_0-tensor density ($w = 1$). The well-known conservation law which follows from (5.1), viz.,

$$\nabla_\mu \mathfrak{T}^\mu_\nu = F_{\nu\alpha} \mathfrak{j}^\alpha\,, \tag{5.21}$$

therefore does not depend on κ_μ either. Furthermore, both sides of (5.21) are W_0 vectors of weight 1.

In summary, then, we see that Maxwell's equations in the form (5.1) and (5.11), as well as the conservation laws (5.16) and (5.21) are C invariant. In addition they are independent of κ_μ. Physically, this means that the validity of these equations does not depend on whether or not the length of a non-null vector changes under parallel displacement. Consequently, such a change, if it existed, would never be observed by means of phenomena described by Maxwell's equations alone.

The C_0 invariance of Maxwell's equations, discovered by Bateman and Cunningham,[1] is a special case of our results.

6. The Lorentz Equation[13,14]

In $\mathfrak{M}$ the Lorentz equation is

$$dv^\mu/d\tau = (e/m) F^{\mu\nu} v_\nu \tag{6.1}$$

where the constants e and m are the charge and (rest) mass of a particle whose world-line $x^\mu(\tau)$ has tangent

$$v^\mu = dx^\mu/d\tau \tag{6.2}$$

(velocity four-vector). The fields are the retarded fields due to all the other charges.

The simplest generalization to $\mathfrak{A}$ seems to be

$$\delta v^\mu/d\tau = (e/m) g^{\mu\alpha} F_{\alpha\beta} v^\beta\,, \tag{6.3}$$

where we expressed the right-hand side explicitly in terms of the W_0 tensor $F_{\mu\nu}$ and the vector v^μ. The left-hand side involves the covariant derivative (4.36)

$$\delta v^\mu/d\tau = dv^\mu/d\tau + \Gamma^\mu_{\alpha\beta} v^\alpha v^\beta\,. \tag{6.4}$$

[14] J. A. Schouten and J. Haantjes, Physica, 1, 869 (1934).

When we apply a conformal transformation on the metric tensor, we see that v^μ is a $W_{-1/2}$ vector,

$$v^\mu \rightarrow v_c^\mu = (1/\sqrt{\sigma})v^\mu \,, \tag{6.5}$$

and one obtains, since σ is a scalar,

$$\frac{\delta v_c^\mu}{d\tau} = \frac{1}{\sqrt{\sigma}}\frac{\delta v^\mu}{d\tau} + v^\mu \frac{d}{d\tau}\frac{1}{\sqrt{\sigma}} \,.$$

The equation of motion (6.3) then becomes

$$\frac{\delta v^\mu}{d\tau} - v^\mu \frac{d}{d\tau} \ln \sqrt{\sigma} = \frac{1}{\sigma}\frac{e}{m} g^{\mu\alpha} F_{\alpha\beta} v^\beta \tag{6.6}$$

which is obviously *not* form-invariant.

Form-invariance under C_g can be obtained, however, by (a) making the fundamental quantity of the equation not v^μ but the W_0-vector density of weight $w = 1/4$,

$$\mathfrak{v}^\mu \equiv |g|^{1/8} v^\mu \,, \tag{6.7}$$

and by (b) assuming m to be a $W_{-1/2}$ scalar,

$$m_c = m/\sqrt{\sigma} \,. \tag{6.8}$$

Specifically, we propose instead of (6.3) the equation in W

$$\delta \mathfrak{v}^\mu / d\tau = (e/m) g^{\mu\alpha} F_{\alpha\beta} \mathfrak{v}^\beta \,. \tag{6.9}$$

The covariant derivative here is, using (4.29) and (4.31),

$$\begin{aligned}\delta\mathfrak{v}^\mu/d\tau &= v^\alpha \nabla_\alpha \mathfrak{v}^\mu = d\mathfrak{v}^\mu/d\tau + v^\alpha \mathfrak{v}^\beta \Gamma^\mu_{\beta\alpha} - \tfrac{1}{4} v^\alpha \Gamma_\alpha \mathfrak{v}^\mu \\ &= d\mathfrak{v}^\mu/d\tau + v^\alpha \mathfrak{v}^\beta \{{}^{\,\mu}_{\alpha\beta}\} - v^\alpha \mathfrak{v}^\beta (\delta^\mu_\alpha \kappa_\beta - \tfrac{1}{2} \kappa^\mu g_{\alpha\beta}) \\ &\quad - \tfrac{1}{4} v^\alpha \mathfrak{v}^\mu (\partial_\alpha \ln \sqrt{|g|} - 2\kappa_\alpha)\end{aligned}$$

A comparison between the covariant derivative in $\mathcal{W}$ and in $\mathcal{R}$ is now easily made. From (4.28),

$$\left(\frac{\delta \mathfrak{v}^\mu}{d\tau}\right)_{\mathcal{W}} = \left(\frac{\delta \mathfrak{v}^\mu}{d\tau}\right)_{\mathcal{R}} + v^\alpha \mathfrak{v}^\beta K^\mu_{\alpha\beta} \tag{6.10}$$

where the tensor

$$K^\mu_{\alpha\beta} \equiv \tfrac{1}{2} (\kappa^\mu g_{\alpha\beta} - \kappa_\beta \delta^\mu_\alpha) \,. \tag{6.11}$$

Equation (6.9) is manifestly covariant, since both sides are vector densities of weight 1/4. Under a transformation of the group C_g, however, the left-hand side is $W_{-1/2}$ because of $d\tau$, while the right-hand side is W_{-1} because of $g^{\mu\alpha}$, as long as m is considered a number. The assumption (6.8) removes this difficulty and makes (6.9) C invariant. The $W_{-1/2}$ character of m was first proposed by Schouten and Haantjes.[6]

At this point we interrupt the formal development in order to interject a brief remark on the physical meaning of the mass transformation (6.8). For a long time such a transformation was *a priori* excluded by many physicists, because rest masses are assumed to be universal constants. If one excludes the mass transformation, the Lorentz equation is not conformally covariant and neither are any of the other equations which we shall discuss for particles of mass $m \neq 0$. However, a closer examination of the physical meaning of conformal transformations (see Sec. 10 for further discussion) reveals that—at least for the group C_0—these transformations correspond to the introduction of static force fields and, in particular, of a homogeneous gravitational field. When m is looked at as a rest energy (rather than a mass) it must contain the potential energy associated with the position of the particle in this field. Since σ measures the strength of the field, m must naturally depend on σ. There is consequently good reason for the mass transformation (6.8). At the same time, this interpretation clarifies the origin dependence of the conformal transformations.

Returning to the conformally covariant Lorentz equation (6.9), we can see its relation to (6.3) by writing it in the form

$$\frac{1}{|g|^{1/8}} \frac{\delta}{d\tau}(|g|^{1/8} v^\mu) = \frac{e}{m} g^{\mu\alpha} F_{\alpha\beta} v^\beta \,. \tag{6.9$'$}$$

In $\mathcal{R}$ $\delta g = 0$ and (6.9′) becomes identical with (6.3), but in $\mathcal{W}$ we have because of (6.10)

$$(\delta v^\mu/d\tau)_{\mathcal{R}} + K^\mu_{\alpha\beta} v^\alpha v^\beta = (e/m) g^{\mu\alpha} F_{\alpha\beta} v^\beta \,. \tag{6.9$''$}$$

When $F_{\mu\nu} = 0$, this equation is the geodesic equation in $\mathcal{W}$ for $d\tau^2 > 0$ (i.e., for particles with nonzero rest mass)

$$\frac{dv^\mu}{d\tau} + (\{{}^{\,\mu}_{\alpha\beta}\} + K^\mu_{\alpha\beta}) v^\alpha v^\beta = 0 \,. \tag{6.12}$$

The complete dependence on κ_μ is contained in $K^\mu_{\alpha\beta}$, (6.11), which can also be expressed by[15]

$$\begin{aligned}K_{\alpha\beta\lambda} &= - K_{\beta\alpha\lambda} \equiv g_{\alpha\mu} K^\mu_{\beta\lambda} \\ &= \tfrac{1}{8} (g_{\beta\lambda} \nabla_\alpha \ln |g| - g_{\alpha\lambda} \nabla_\beta \ln |g|) \,.\end{aligned} \tag{6.13}$$

Obviously, these terms will become important only when the gravitational field is sufficiently inhomogeneous ($\nabla_\mu \ln|g|$ is large enough). As a matter of principle, however, Eq. (6.9) does depend on κ_μ, and therefore depends on the change that the length of a vector experiences under parallel displacement.

7. The Lorentz-Dirac Equation

In special relativity the equation of motion of a

[15] We remind the reader that our differentiation always acts only on the function immediately following the operator unless otherwise indicated.

particle of charge e which can be approximated by a point charge satisfies the equation

$$\frac{dv^\mu}{d\tau} = \frac{e}{m} F^{\mu\nu} v_\nu + \frac{2}{3}\frac{e^2}{m}\left(\frac{d^2v^\mu}{d\tau^2} - \frac{dv_\lambda}{d\tau}\frac{dv^\lambda}{d\tau} v^\mu\right). \tag{7.1}$$

This equation was derived by Dirac[16] from Maxwell's equations and the conservation laws. The term proportional to e^2/m describes the reaction effects on the motion due to the particle's own electromagnetic field. It is essential for a consistent treatment of radiation effects.[17] If radiation is negligible, the Lorentz-Dirac equation reduces to the Lorentz equation (preceding section).

As in (6.3) we can formally generalize (7.1) from $\mathfrak{M}$ to a covariant equation in $\mathfrak{A}$.

$$\frac{\delta v^\mu}{d\tau} = \frac{e}{m} g^{\mu\alpha} F_{\alpha\beta} v^\beta + \frac{2}{3}\frac{e^2}{m}\left(\frac{\delta^2 v^\mu}{d\tau^2} - \frac{\delta v^\alpha}{d\tau}\frac{\delta v^\beta}{d\tau} g_{\alpha\beta} v^\mu\right) \tag{7.2}$$

Such a generalization, however, has little physical meaning. It is neither conformally invariant, nor does it in general have the same relationship to the conservation laws as in $\mathfrak{M}$. For $\mathfrak{R}$ it was shown by DeWitt and Brehme,[18] that an additional term on the right-hand side of the form

$$\frac{e^2}{m} v^\alpha \int_{-\infty}^{\tau} f^\mu_{\alpha\beta} v^\beta(\tau') d\tau' \tag{7.3}$$

is necessary to restore this relationship. However, this term differs from zero only when the curvature tensor does not vanish. In flat space no such term arises, so that it does not enter into C_0 invariance; in that case the equations are related to the conservation laws in the sense of Dirac's 1938 paper.

In order to obtain a Lorentz-Dirac type equation in a curved space which is invariant under the extended conformal group C, it would be necessary to repeat the DeWitt-Brehme analysis in terms of conformal general relativity. Such an analysis exceeds the purpose of the present paper. However, it seems plausible that it could be carried out with the use of W_0 quantities throughout.

It is obviously suggestive to generalize (7.2) into a conformally invariant equation by means of the same procedure which was successful for the Lorentz equation.

From (6.3) and (6.9′) we see that in that equation the conformally covariant generalization of the covariant derivative of v_μ was

$$\frac{\delta v^\mu}{d\tau} \to \frac{1}{|g|^{1/8}}\frac{\delta}{d\tau}(|g|^{1/8} v^\mu) = \frac{1}{|g|^{1/8}}\frac{\delta}{d\tau}\mathfrak{v}^\mu .$$

Since v_μ is $W_{-1/2}$ we might expect that the covariant derivative of $d\mathfrak{v}_\mu/d\tau$ which is also $W_{-1/2}$ will be generalized in the same way. Thus, we try

$$\frac{\delta}{d\tau}\left(\frac{\delta\mathfrak{v}^\mu}{d\tau}\right) \to \frac{1}{|g|^{1/8}}\frac{\delta}{d\tau}\left(|g|^{1/8}\frac{\delta\mathfrak{v}^\mu}{d\tau}\right).$$

If we now use the W_0-vector density ($w = 1/4$) $\mathfrak{v}^\mu$ as the basic quantity in (7.2) instead of v^μ, we see that every term in

$$\frac{\delta\mathfrak{v}^\mu}{d\tau} = \frac{e}{m} g^{\mu\alpha} F_{\alpha\beta}\mathfrak{v}^\beta + \frac{2}{3}\frac{e^2}{m}\left[\frac{1}{|g|^{1/8}}\frac{\delta}{d\tau}\left(|g|^{1/8}\frac{\delta\mathfrak{v}^\mu}{d\tau}\right) - \frac{g_{\alpha\beta}}{|g|^{1/4}}\frac{\delta\mathfrak{v}^\beta}{d\tau}\frac{\delta\mathfrak{v}^\alpha}{d\tau}\mathfrak{v}^\mu\right] \tag{7.4}$$

has the same transformation property, i.e., it is a $W_{-1/2}$ vector density of weight 1/4, provided we again adopt the mass transformation (6.8). The equation is therefore manifestly C invariant. For reasons explained previously, however, only C_0 invariance is meaningful in this case.

The same result can also be obtained by introducing only W_0 quantities, i.e., by using in addition to $\mathfrak{w}^\mu$,

$$d\mathfrak{s} = d\tau/|g|^{1/8}, \quad \mathfrak{m} = |g|^{1/8} m, \tag{7.5}$$

and the conform invariant metric tensor densities $G_{\mu\nu}$ and $G^{\mu\nu}$ of (4.20). The covariant Lorentz-Dirac equation (7.2) can now be written in a form containing only W_0 quantities,

$$\frac{\delta\mathfrak{v}^\mu}{d\mathfrak{s}} = \frac{e}{\mathfrak{m}} G^{\mu\alpha} F_{\alpha\beta}\mathfrak{v}^\beta + \frac{2}{3}\frac{e^2}{\mathfrak{m}}\left(\frac{\delta^2\mathfrak{v}^\mu}{d\mathfrak{s}^2} - G_{\alpha\beta}\frac{\delta\mathfrak{v}^\alpha}{d\mathfrak{s}}\frac{\delta\mathfrak{v}^\beta}{d\mathfrak{s}}\mathfrak{v}^\mu\right), \tag{7.6}$$

which, upon substitution leads again to (7.4).

In analogy with (6.9′), Eq. (7.4) can also be written

$$\frac{1}{|g|^{1/8}}\frac{\delta}{d\tau}(|g|^{1/8} v^\mu) = \frac{e}{m} g^{\mu\alpha} F_{\alpha\beta} v^\beta + \frac{2}{3}\frac{e^2}{m}\left[\frac{1}{|g|^{1/4}}\frac{\delta}{d\tau}\left(|g|^{1/8}\frac{\delta(|g|^{1/8} v^\mu)}{d\tau}\right) - \frac{g_{\alpha\beta}}{|g|^{1/4}}\frac{\delta(|g|^{1/8} v^\alpha)}{d\tau}\frac{\delta(|g|^{1/8} v^\beta)}{d\tau} v^\mu\right]. \tag{7.4′}$$

It reduces to (7.2) for $\delta g = 0$.

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[16] P. A. M. Dirac, Proc. Roy. Soc. (London) **A 167**, 148 (1938).

[17] F. Rohrlich, *Lectures in Theoretical Physics*, Boulder Summer Institute, (Interscience Publishers, Inc., New York, 1960) Vol. II, p. 240; Ann. Phys., **13**, 93 (1961); Nuovo cimento **21**, 811 (1961).

[18] B. S. DeWitt and R. W. Brehme, Ann. Phys., **9**, 220 (1960).

8. Variational Principle

To the extent that the equations considered in Secs. 5–7 result from a variational principle, it is clear that the C invariance of these equations could also be established by starting with a C-invariant variational principle and maintaining C invariance throughout the subsequent calculations.

Thus, the principle

$$\delta \int \mathfrak{m} d\mathfrak{s} - \delta \int eA_\mu dx^\mu + \delta \int \tfrac{1}{4} \mathfrak{F}^{\mu\nu} F_{\mu\nu} d^4x = 0 \quad (8.1)$$

produces Maxwell's equations upon variation of A_μ, and it produces the Lorentz equation upon variation of $x_\mu(\mathfrak{s})$. The latter variation was used also by Infeld and Schild.[19]

All quantities in (8.1) are of W_0 type. The fields are related in a W_0-invariant way to the potentials which play the basic role in (8.1)

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (8.2)$$

independent of the connection, in any $\mathfrak{C}$.

The Lorentz-Dirac equation apparently cannot be obtained directly from a variational principle.

Let us next consider briefly the conservation laws which arise from C-invariant variational principles. According to Noether's theorem, to every continuous transformation of coordinates which makes the variation of the action zero, there corresponds an invariant, the generator of this transformation, which is conserved. For example, the momentum and angular momentum are the generators of infinitesimal translations and rotations, respectively, for Lorentz invariant Lagrangians.

Formally, we get the same expressions in a C-invariant theory as in other theories. For example, an invariant particle Lagrangian of the form $L(x,\mathfrak{v};\mathfrak{s})$, with the variational principle

$$\delta \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} L(x,\mathfrak{v};\mathfrak{s}) d\mathfrak{s} = 0 \quad (8.3)$$

will lead to the Euler-Lagrange equations

$$(d/d\mathfrak{s})(\partial L/\partial \mathfrak{v}^\mu) - (\partial L/\partial x^\mu) = 0\,, \quad (8.4)$$

and the conservation laws

$$d/d\mathfrak{s}[(\partial L/\partial \mathfrak{v}^\mu)\delta x^\mu] = 0\,. \quad (8.5)$$

The form of the generator for particular transformations can be read off from (8.5).

One could easily derive the conservation laws which arise from (8.1). We won't do this, but will rather obtain the form of the generators for various transformations for the particle part of the Lagrangian in (8.1). We thus consider the variational principle for free particle motion:

$$\delta \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{m} d\mathfrak{s} = \delta \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{m}(-\mathfrak{v}^\mu G_{\mu\nu} \mathfrak{v}^\nu)^{1/2} d\mathfrak{s}\,. \quad (8.6)$$

This variational principle of course generates the correct free particle geodesic, Eq. (6.12).

To illustrate the application of Noether's theorem to conform-invariant theories, we shall consider two transformations of the group C_0, given in (4.10): translations and transformations to uniform acceleration.

The generator of infinitesimal translations

$$\delta x^\mu = \epsilon^\mu \quad (8.7)$$

is the generalization of the momentum:

$$P_\mu = \mathfrak{m}\mathfrak{v}_\mu = mv_\mu\,. \quad (8.8)$$

It is a W_0 vector.

Next, consider the infinitesimal conformal transformation corresponding to uniform acceleration:

$$\delta x^\mu = (2x^\mu x_\lambda - x^\nu x_\nu \delta_\lambda^\mu)\alpha^\lambda \quad (8.9)$$

where α_μ is an infinitesimal four-vector. The corresponding generator, which participates in a conservation law, is

$$\mathfrak{A}_\mu = x^\alpha x^\beta (G_{\alpha\beta} P_\mu - 2G_{\mu\beta} P_\alpha) \quad (8.10)$$

It is a W_0 vector density, $w = -1/2$. The associated conservation law does not seem to have any simple physical meaning.

9. Conformal Invariance in Quantum Mechanics

The extension of the study of conformal invariance into quantum mechanics is strongly suggested by the conformal transformation of mass, obtained in classical mechanics (Sec. 6), viz., that the mass is a $W_{-1/2}$ scalar. This means that m transforms under C_g like a reciprocal covariant length [e.g., like $(d\tau)^{-1}$]; it forecasts the $W_{1/2}$-scalar properties of Compton wave lengths with Planck's constant and the velocity of light remaining constants of type W_0.

Since conformal invariance is intimately connected with relativity, we have in mind here relativistic quantum mechanics, and possibly relativistic quantum field theory. The basic equation of interest is therefore Dirac's equation, and the first theory to be considered would be relativistic quantum electrodynamics, first in its semiclassical form, and then in a quantum field theoretical formulation.

The conformal invariance of the Dirac equation

[19] P. A. M. Dirac, Ann. Math. **37**, 429 (1936); O. Veblen, Proc. Natl. Acad. Sci. U. S. **21**, 484 (1935); L. Infeld and A. Schild, Phys. Rev. **70**, 410 (1946).

has been studied by many investigators,[14,19] most of them concerned with C_0 invariance; but Pauli[20] considered C invariance.

In order to apply the mathematical techniques used in the preceding sections to the Dirac equation, it is first of all necessary to write it in $\mathfrak{R}$. This was done by Schrödinger[21] and by Bargmann.[21] The formalism obtained must then be generalized to a Weyl space. Having achieved covariance in this space we can proceed exactly as in Sec. 5 to prove C invariance.

However, this invariance proof is greatly simplified if one restricts oneself to C_0 invariance, i.e., if one restricts the proof outlined above to flat spaces. On physical grounds this restriction is completely acceptable as long as one is not concerned with the unification of quantum mechanics with general relativity. We shall therefore consider only C_0 invariance.

We start with the free Dirac equation. In $\mathfrak{M}$ with $g_{\mu\nu} = \eta_{\mu\nu}$ we have

$$(\gamma^\mu \partial_\mu + m)\psi = 0 \tag{9.1}$$

where

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} . \tag{9.2}$$

In curvilinear coordinates (flat Riemann space) the last equation must be written

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \quad \text{or} \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} , \tag{9.3}$$

so that the Dirac matrices are now functions of space and time. Taking the covariant derivative one finds

$$\nabla_\lambda \gamma_\mu = 0 . \tag{9.4}$$

Thus, the Dirac matrices are still "constants" in the sense that their covariant derivatives vanish. It should be noted, however, that (9.4) is no longer valid when the space has nonvanishing curvature.[21]

The Dirac equation (9.1) is now generalized to

$$(\gamma^\mu \nabla_\mu + m)\psi = 0 . \tag{9.5}$$

This is a covariant equation. The covariance proof proceeds in analogy to the usual proof of Lorentz covariance. In particular, the invariance of the anticommutation relation (9.3) under similarity transformations is essential.

Since we are restricting ourselves to flat space the equation (9.5) can be further simplified. We note that in Cartesian coordinates

$$\nabla_\mu \psi = \partial_\mu \psi \tag{9.6}$$

and we return to the form (9.1).

[20] W. Pauli, Helv. Physica, **13**, 204 (1940).

[21] E. Schrödinger, Sitzber. preuss. Akad. Wiss, Physik-math. 105 (1932); V. Bargman, *ibid.* 346 (1932).

The form (9.5) of the free Dirac equation is also valid in a flat Weyl space, the covariant derivative now containing the Weyl connection. The spinor ψ must then be of type W_0. The conformal transformations C_g will leave (9.3) unchanged if the Dirac matrices transforms as

$$\gamma_c^\mu = (1/\sqrt{\sigma})\gamma^\mu , \quad \gamma_\mu^c = (\sqrt{\sigma})\gamma_\mu . \tag{9.7}$$

The Dirac equation (9.3) is clearly form-invariant under this transformation provided the mass transforms according to (6.8). This establishes the invariance of the free Dirac equation under the restricted conformal transformations (4.10).

In interaction with the electromagnetic field the Dirac particle satisfies

$$[\gamma^\mu \nabla(_\mu + ieA_\mu) + m]\psi = 0 . \tag{9.8}$$

Since A_μ is a W_0 vector this equation is clearly also C_0 invariant.

For the construction of the bilinear covariants we note the existence of a Hermitian matrix A such that

$$\gamma_\mu^+ = -A\gamma_\mu A^{-1} .$$

Because of (9.3) A will also be a function of x. In a special representation $A = i\gamma_0$ so that A will be $W_{1/2}$ like all γ_μ according to (9.7). We then define $\bar{\psi} = \psi^* A$.

The current density is now defined by

$$\mathfrak{j}^\mu = -ie\bar{\psi}\gamma^\mu\psi . \tag{9.9}$$

It is of type W_0 since $\bar{\psi}$ is W_0 and A^μ is W_0. It satisfies

$$\nabla_\mu \mathfrak{j}^\mu = 0 , \tag{9.10}$$

because of (9.8), its conjugate equation, and (9.4).

These results together with Maxwell's equations establish a C_0-invariant relativistic quantum electrodynamics.

These considerations can be extended to quantum field theories of particles of spin zero and of spin one. In particular, the Klein-Gordon equation for a scalar field ϕ,

$$(\partial_\mu \partial^\mu - m^2)\phi = 0 , \tag{9.11}$$

can be generalized to a conformally invariant equation, contrary to some claims.[20] To this end we first generalize to $\mathfrak{R}$:

$$(1/\sqrt{|g|})\partial_\mu[(\sqrt{|g|})g^{\mu\nu}\partial_\nu\phi] = m^2\phi$$

or

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = m^2\phi . \tag{9.12}$$

This follows from $\nabla_\mu\phi = \partial_\mu\phi$ and $\nabla_\lambda g_{\mu\nu} = 0$. Equation (9.12) can be adopted without formal change when one uses a Weyl connection rather than a Christoffel symbol. Furthermore, this equation is also invariant under conformal transformations (extended group C) because ∇_μ is a W_0 vector and $g^{\mu\nu}$ and m^2 transform in exactly the same way under C_g.

If one identifies the momentum, angular momentum, and other operators with the infinitesimal generator of translation, rotation, etc., in the usual way, one obtains the corresponding commutation relations from the structure equations of the group C_0 which is a Lie group. Of special interest is the result emerging from these relations, that the square of the length of the momentum four-vector, $P_\mu P^\mu$, has no discrete eigenvalues other than zero, as was shown by Wess.[8] We need not follow his proof, because with the usual coordinate representation of the operator P_μ,

$$P_\mu \leftrightarrow (\hbar/i)\nabla_\mu \qquad (9.13)$$

his result is obvious from (9.12), which can be written

$$g^{\mu\nu}P_\mu P_\nu \phi = -m^2\phi \,. \qquad (9.14)$$

Since m is *not* a W_0 scalar, but a $W_{-1/2}$ scalar, a general conformal transformation does not leave m^2 invariant, but permits m^2 to take on a continuum of values according to (6.8). Only $m = 0$ remains invariant under (6.8).

10. The Physical Meaning of Conformal Invariance

The result obtained in the last section, viz., that $P_\mu P^\mu$ has no nonvanishing discrete eigenvalues, seems very disturbing. It appears to exclude particles of finite mass from a conformal quantum field theory. However, this conclusion is not correct. As was pointed out in connection with (6.8), conformal transformations of C_0 which are not Lorentz transformations can be interpreted as apparent gravitational fields ("apparent" because C_0 refers to flat space). The mass m includes the potential energy in such a field. The rest mass of a particle is a relative concept in this framework. Nevertheless, a special value can be singled out in much the same way as in general relativity by measuring the mass of a particle at rest in a local geodesic coordinate system with Minkowski metric. We can identify this value with "the" rest mass (i.e., rest mass in the usual sense).

The interpretation of the mass transformation given above derives from a detailed study of the acceleration transformation.[22] These transformations are exhibited in (4.10). Their physical interpretation is as follows. When applied to an inertial frame of reference they transform a particle from rest to uniform acceleration (hyperbolic motion). Such a motion is equivalent to the presence of a constant homogeneous gravitational field according to the equivalence principle. Thus, an acceleration transformation is equivalent to switching on such a gravitational field. Consequently, the rest energy of a particle changes from mc^2 to $mc^2 + mgh$ (in first order of g) and its mass becomes

$$m \to m_c = m(1 + gh/c^2) \,. \qquad (10.1)$$

It can be shown that this factor is exactly what the mass transformation (6.8) specifies in much greater generality. Thus, a conformal transformation corresponds to a change of the (apparent) force field acting on the particle and the mass transformation represents the corresponding change in rest energy which takes account of the change in potential energy.

A fundamental difference between restricted conformal relativity (based on the group C_0) and special relativity (based on the Lorentz group) appears in the measurement of lengths and time intervals. These quantities no longer have an absolute meaning. The only meaningful comparison of lengths and time intervals is by means of light rays. As was mentioned in Sec. 3(d), only a local comparison of infinitesimal lengths and times is possible, in general. This is not surprising in view of the presence of accelerating fields (apparent gravitational fields) which depend on position and time and which are changed with every conformal transformation which is not a Lorentz transformation.

A detailed study of this physical picture[22] reveals that conformal transformations are a special way of describing certain phenomena which in general relativity are accounted for by a restricted class of coordinate transformations. All results obtained by study of the group C_0 are in fact equivalent to certain special cases of general relativity and no new physical results are predicted. In some cases no simple physical meaning can be attached to these transformations.

The question of the physical meaning of conformal covariance of the field equations under C, i.e., in the presence of true gravitational fields (curved space), will be dealt with in a future publication.[23]

In summary, we can state the physical meaning of the covariance of equations under conformal transfor-

[22] T. Fulton, F. Rohrlich, and L. Witten, "Physical Consequences of a Coordinate Transformation to a Uniformly Accelerated Frame", Nuovo cimento (to be published).

[23] See also F. Rohrlich, T. Fulton and L. Witten, Bull. Am. Phys. Soc. 6, 346 (1961).

mations restricted to flat space (group C_0) as follows: Any transformation of this type can be regarded by an inertial observer (Minkowski space) as a change from one apparent gravitational field to another. Field free space, of course, is a special case of such a field. In a Weyl space $\mathcal{W}$ the form of a C_0-invariant equation is not affected by such a change of fields. All particles freely falling in such a field follow geodesics in $\mathcal{W}$. However, the basic physical quantities are different from the usual ones: Instead of the rest mass m one has here the conformal mass $\mathfrak{m}$ which is the sum of potential and rest energy in the language of the inertial observer; instead of the proper time $d\tau$ one has the conformal proper time $d\mathfrak{s}$ which is associated with $\mathfrak{m}$ such that $\mathfrak{m}d\mathfrak{s} = md\tau$ [Eq. (7.5)]. In terms of these quantities and corresponding derived quantities such as $\mathfrak{v}^\mu = dx_\mu/d\mathfrak{s}$ instead of $v^\mu = dx^\mu/d\tau$, the physical theories based on C_0-invariant equations (classical electrodynamics, relativistic quantum field theories) are characterized geometrically by the geodesics in $\mathcal{W}$ and the angles of their intersections. Length comparisons have only local significance, since the metric tensor is known only within a factor σ which is a function of position.

For each coordinate system in the flat $\mathcal{W}$, an equivalent flat Riemann space can be found when the vector κ_μ is chosen to be a gradient $\kappa_\mu = \partial_\mu \ln\sigma$. (See the end of Sec. 4.) Two Riemann spaces constructed in this way with two different values of σ will have the same linear connections, but different metric tensors. However, a "conformal Riemann space" can be defined which is not based on the tensor $g_{\mu\nu}$ and the corresponding Christoffel symbol, but is defined by the tensor density $G_{\mu\nu} = g_{\mu\nu}/|g|^{1/4}$ and the corresponding Christoffel symbol. This conformal Riemann space is invariant under general conformal transformations (group C) because G is invariant. But it is not a metric space, because $d\mathfrak{s}^2 = G_{\mu\nu}dx^\mu dx^\nu$ is not invariant even under coordinate transformations.

Geodesics of particles with finite mass are invariant under C in a Weyl space or in a conformal Riemann space $\mathfrak{R}$. This is not so for null geodesics (paths of particles with zero mass, photons and neutrinos); these are invariant also in $\mathfrak{R}$. The interpretation of this result lies in the fact that for an observer in a Weyl space the apparent gravitational fields are geometrized and thereby eliminated from explicit appearance, while for an inertial observer each such field presents a very different physical situation. This difference, however, can only be ascertained by observations on particles with finite rest masses. No such difference exists when observations are restricted to measurements by means of light rays only.