Relativistic Partial Wave Analysis*

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1. INTRODUCTION

VERY often in current work on relativistic theories of particle reactions of type

$$A+B \to C+D, \tag{1.1}$$

one requires the analysis of the S or T matrix element in terms of partial amplitudes. In such a situation the procedure adopted is to restrict attention to the center-of-mass system and there to use the corresponding nonrelativistic results.[1] Recently, Joos[2] has discussed the reaction (1.1) in a fully relativistic manner, based on the theory[3,4] of irreducible representations of the Poincaré[5] or inhomogeneous Lorentz Group P, and has obtained the partial wave analysis of its matrix element in a form valid for a general frame of reference. Of course, when one specializes his result to the center-of-mass system of (1.1), one obtains a result differing from the corresponding nonrelativistic result only in multiplicative factors due to the use of a different normalization of states in the two contexts.

It is to be expected that, in the future, relativistic theories of reactions

$$A+B \to 1+2+\cdots+n \tag{1.2}$$

will be developed, in which case one will require the partial wave analysis of their matrix elements in relativistic form. The present paper is designed to extend the work of Joos in this direction. The procedures to be followed for the treatment of the general n case of (1.2) are developed and explicit formulas are presented for the case with three outgoing particles. One important conclusion that emerges is that it is essential, for all production reactions, to adopt a fully relativistic approach to partial wave analysis. Otherwise the angular dependence of their matrix elements will not be exhibited correctly. In particular, the method of using corresponding nonrelativistic results in the center-of-mass system of the reaction fails in this manner.

Since one of the possible fields of application of this work is to the dynamical theory of strong interactions, based on the analyticity and unitarity properties of partial amplitudes, it is essential to know not only the partial wave analysis of matrix elements, but also the reciprocal formulas for projecting partial amplitudes out of full matrix elements. This problem is tackled for reactions (1.1) and (1.2), although explicit results for the latter are given only in the $n=3$ case.

Møller's formulas[6] for the invariant cross sections for reactions (1.1) and (1.2) are developed into explicit forms suitable for use in connection with the above partial wave analysis of their matrix elements. In this context, some "optical theorems" are proved.

The particles treated in the subsequent work are relativistic particles of arbitrary (nonzero) masses and arbitrary (integral or half-integral) spins, but questions which relate to the identity of particles are not touched on at all.

Sections 2–4 contain a review of the work of Joos[2] together with some minor additions. The basic ideas of his analysis are as follows. Although each of the particles of the relativistic state of two particles can be described by an irreducible representation of P, their direct product is reducible. The direct integral, which expresses its reduction into irreducible parts, is named the Clebsch-Gordon (C-G) series of P for the direct product. The coefficients which appear in the definition of the basis states of each irreducible representation of P contained in the series in terms of the direct product basis states, are called the C-G coefficients of P for the direct product. These names are used because of analogy with the three-dimensional rotation group.[7–10] The C-G series of P for the case of the direct product of two single particle irreducible representations of P is set up in Sec. 2, and Joos's formula for the corresponding C-G coefficient of P is derived in Sec. 3. In Sec. 4, this work is applied to the derivation of the relativistic partial wave analysis of the S-matrix element for (1.1).

In Sec. 5, the C-G series of the direct product of two not-necessarily-single-particle irreducible representations of P is set up, and is applied in Sec. 6 to the problem of giving the relativistic partial wave analysis

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[1] J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. **14**, 258 (1952).

[2] H. Joos, *Bemerkungen zur Phase-Shift Analysis auf Grund der Darstellungstheorie der inhomogenen Lorentzgruppe* (Oberwolfach, 1959).

[3] E. Wigner, Ann. Math. **40**, 149 (1939).

[4] A. S. Wightman, *Lectures on Invariance in Relativistic Quantum Mechanics* (Les Houches, 1960).

[5] For a general discussion of the theory of P and its representations see reference 4.

[6] C. Møller, Kgl. Danske Videnskab, Selskab, Mat.-fys. Medd. **23**, 1 (1945).

[7] For a discussion of the properties of quantities associated with the three-dimensional rotation group, see references 8–10.

[8] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[9] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[10] I. M. Gelf'and and Z. Šapiro, Am. Math. Soc. Trans. **2**, 207 (1956).

of the S-matrix element for (1.2). The part of the discussion relating to the general n case of (1.2) is of a qualitative nature, particularly with reference to the nonuniqueness of possible results. This nonuniqueness is due to the number of different manners in which final particles can be coupled by means of C-G series. Only for the $n=3$ case of (1.2) are explicit formulas presented and briefly commented on. In Sec. 7 the orthogonality properties of C-G coefficients of P are developed, and in Sec. 8 they are used to derive projection formulas for the partial amplitudes of (1.1) and (1.2). Sections 9 and 10 are occupied with the work on invariant cross sections and optical theorems, respectively.

2. THE SIMPLE C-G SERIES OF P

The simple C-G series of P, which arises in the reduction of the direct product of two single-particle irreducible representations of P, is set up in this section.

Irreducible representations of P are referred to by means of the notation $[w,j]$. The labels w and j, which up to an equivalence uniquely characterize the irreducible representations concerned, are the eigenvalues of the invariants of P, which these representations possess. For example, the irreducible representation $[\kappa,\sigma]$ can be used to describe a single relativistic particle of rest mass κ and spin σ. To obtain such a description one introduces into the Hilbert space of states of the representation $[\kappa,\sigma]$, a canonical system of basis vectors

$$|k\nu[\kappa\sigma]\rangle \tag{2.1}$$

specified by the following properties:

(a) They are eigenstates of the linear momentum operator P_μ with

$$P_\mu|k\nu[\kappa\sigma]\rangle=k_\mu|k\nu[\kappa\sigma]\rangle$$

and

$$k^2=(k_0)^2-\mathbf{k}^2=\kappa^2.$$

(b) They are eigenstates of the z component[11] of the relativistic spin operator $\mathbf{S}(k)$ for the particle,[12,13] with

$$S_z(k)|k\nu[\kappa\sigma]\rangle=\nu|k\nu[\kappa\sigma]\rangle$$
$$S_\pm(k)|k\nu[\kappa\sigma]\rangle=\{(\sigma\mp\nu)(\sigma\pm\nu+1)\}^{\frac{1}{2}}|k\nu\pm1[\kappa\sigma]\rangle$$

where

$$S_\pm(k)=S_x(k)\pm iS_y(k).$$

(c) They have the invariant normalization[14]

$$\langle k'\nu'[\kappa'\sigma']|k\nu[\kappa\sigma]\rangle =2k_0\delta(\mathbf{k}-\mathbf{k}')\delta(\nu\nu')\delta(\kappa\kappa')\delta(\sigma\sigma'). \tag{2.2}$$

[11] Throughout what follows an arbitrarily assigned spatial direction is taken as z axis or axis of quantization.

[12] For an explicit definition of $\mathbf{S}(k)$ in terms of the operators of P, see references 2 and 13: of course, its momentum dependence is a necessary feature of a relativistic particle theory.

[13] A. J. Macfarlane: Ph. D. Thesis, University of London, 1961.

[14] If a composite label x contains n individual labels $a, b\cdots h$, then the notation $\delta(xx')$, implies the product $\delta(aa')\delta(bb')\cdots\delta(hh')$ of n Kronecker symbols.

(d) They have known transformation properties with respect to P. These are quoted and used in the next section.

In order to set up the simple C-G series of P, consider the direct product

$$[\kappa',\sigma']\otimes[\kappa'',\sigma''] \tag{2.3}$$

of two single-particle representations of P, each of whose Hilbert spaces is spanned by a canonical system of basis vectors of the above type. The representation (2.3) of P is not itself irreducible, but it can be written as a direct integral of irreducible representations $[w,j]$ of P. This direct integral is the simple C-G series of P for the direct product (2.3).

One begins by expressing those basis vectors, which provide a canonical basis in the Hilbert space of each $[w,j]$ contained in the C-G series of (2.3), in terms of the direct product basis vectors

$$|k'\nu'[\kappa'\sigma'],k''\nu''[\kappa''\sigma'']\rangle.$$

The former basis vectors are denoted by

$$|pm[wj],\eta\alpha\rangle, \tag{2.4}$$

the first two labels specifying them within the Hilbert space of the $[w,j]$ concerned, just as the labels k and ν do for $[\kappa,\sigma]$ above. Unless $\sigma'=\sigma''=0$ there occurs, for each j-value, a certain multiplicity of equivalent $[w,j]$, so that η is needed as a degeneracy label to distinguish them in some manner. The precise nature of the label η is discussed in the next section. Finally the composite label α, $\alpha\equiv\kappa'$, σ', κ'', σ'', serves to indicate the direct product which has given rise to the family of basis vectors (2.4). In agreement with the normalization (2.2) of single particle basis vectors, the basis vectors (2.4) have the unitary relationship

$$|pm[wj],\eta\alpha\rangle = \sum_{\nu'\nu''}\int\frac{d^3k'}{2k_0'}\int\frac{d^3k''}{2k_0''}|k'\nu'[\kappa'\sigma'],k''\nu''[\kappa''\sigma'']\rangle \times\langle k'\nu'[\kappa'\sigma'],k''\nu''[\kappa''\sigma'']|pm[wj],\eta\alpha\rangle, \tag{2.5}$$

to the direct product basis vectors, whose labels may be conveniently rearranged to give the notation

$$|k'\nu'k''\nu'',\alpha\rangle.$$

The coefficient

$$\langle k'\nu'[\kappa'\sigma'],k''\nu''[\kappa''\sigma'']|pm[wj],\eta\alpha\rangle,$$

whose labels may similarly be rearranged to give

$$\langle k'\nu'k''\nu''\alpha|pm[wj],\eta\alpha\rangle, \tag{2.6}$$

is the C-G coefficient of P for the direct product (2.3). Formally it might seem possible to introduce a more general coefficient by setting

$$|pm[wj],\eta\beta\rangle = \sum_{\alpha}\sum_{\nu'\nu''}\int\frac{d^3k'}{2k_0'}\int\frac{d^3k''}{2k_0''}|k'\nu'k''\nu'',\alpha\rangle \times\langle k'\nu'k''\nu'',\alpha|pm[wj],\eta\beta\rangle,$$

but since one evidently has

$$\langle k'\nu'k''\nu'',\alpha|pm[wj],\eta\beta\rangle = \delta(\alpha\beta)\langle k'\nu'k''\nu'',\alpha|pm[wj],\eta\alpha\rangle, \quad (2.7)$$

the attempt reduces to (2.5). Hence the label α is dropped from (2.6) and (2.5) appears in its final form

$$|pm[wj],\eta\alpha\rangle = \sum_{\nu'\nu''}\int\frac{d^3k'}{2k_0'}\int\frac{d^3k''}{2k_0''}|k'\nu'k''\nu'',\alpha\rangle \times\langle k'\nu'k''\nu''|pm[wj],\eta\rangle. \quad (2.8)$$

As a natural extension of (2.2), one imposes on the basis vectors (2.4) the invariant normalization

$$\langle p'm'[w'j'],\eta'\alpha'|pm[wj],\eta\alpha\rangle = 2p_0\delta(\mathbf{p}-\mathbf{p}')\delta(mm')\delta(w-w')\delta(jj')\delta(\eta\eta')\delta(\alpha\alpha'). \quad (2.9)$$

In agreement with this, one may now give the inverse relationship to (2.8) in the form

$$|k'\nu'k''\nu'',\alpha\rangle = \sum_{jm\eta}\int dw\int\frac{d^3p}{2p_0}|pm[wj],\eta\alpha\rangle \times\langle k'\nu'k''\nu''|pm[wj],\eta\rangle^*. \quad (2.10)$$

Equation (2.10) may be regarded as the C-G series of P for the direct product (2.3).

3. THE SIMPLE C-G COEFFICIENT OF P

The principal aim of this section is to give a heuristic derivation of Joos' formula[2] for the simple C-G coefficient

$$\langle k'\nu'k''\nu''|pm[wj],\eta\rangle \quad (3.1)$$

of P. This derivation follows the intuitive approach of transforming to the center-of-mass system and assumes that the methods of nonrelativistic angular momentum theory apply there.

The transformation of the coefficient (3.1) to the center-of-mass system can be achieved using Wigner's theory[3,4] of the representations of P.

If the element $\{a,L\}$ of P is given explicitly as a transformation of coordinates

$$x_\mu \rightarrow x_\mu' = L_\mu{}^\nu x_\nu + a_\mu,$$

or, in matrix notation by

$$x\rightarrow x' = L\cdot x + a,$$

then one may define the representations of P by the correspondence

$$\{a,L\}\rightarrow U(a,L),$$

with

$$\{a_1,L_1\}\{a_2,L_2\} = \{a_3,L_3\}\rightarrow U(a_1,L_1)U(a_2,L_2) = U(a_3,L_3),$$

where

$$a_3 = a_1 + L_1a_2,\quad L_3 = L_1L_2,$$

and with $U(a,L)$ unitary. In terms of the unitary operators $U(a,L)$, one may give the law of transformation under the general element $\{a,L\}$ of P of the basis vectors (2.1) of the irreducible representation $[\kappa,\sigma]$ of P. It is[15]

$$\langle k\nu[\kappa\sigma]|U(a,L) = \exp(ik\cdot a)\textstyle\sum_\mu D_{\nu\mu}{}^\sigma(R(k,L)) \times\langle L^{-1}\cdot k\mu[\kappa\sigma]|. \quad (3.2)$$

Herein $R(k,L)$ is a purely spatial[16] rotation, dependent on the four-momentum k and the Lorentz transformation L, and $D^\sigma(R(k,L))$ is the usual[17] $(2\sigma+1)\times(2\sigma+1)$ matrix representative of it with unitary property

$$\textstyle\sum_\mu D_{\nu\mu}{}^\sigma(R(k,L))^* D_{\nu'\mu}{}^\sigma(R(k,L)) = \delta(\nu\nu'). \quad (3.3)$$

A 4×4 matrix representation of $R(k,L)$ is given by

$$R(k,L) = L(k)\cdot L\cdot L(L^{-1}\cdot k)^{-1}, \quad (3.4)$$

where $L(k)$ and $L(L^{-1}k)$ are examples of a notation strictly reserved for the pure rotation-free Lorentz transformation that carries the indicated four vector into its rest system. Explicitly, if $\tilde{k} = (\kappa,\mathbf{0})$, then

$$\tilde{k} = L(k)\cdot k$$

or

$$\tilde{k}_\mu = L(k)_\mu{}^\nu k_\nu,$$

with

$$L(k)_0{}^\nu = L(k)_\nu{}^0 = k^\nu/\kappa,\quad k^\nu = (k_0, -\mathbf{k}),$$
$$L(k)_i{}^j = \delta_{ij} + k_ik_j/[\kappa(\kappa+k_0)],\quad i,\ j\neq 0. \quad (3.5)$$

Similarly, since, for each set of values η, α, the basis vectors (2.4) form a canonical basis for the irreducible representation $[w,j]$ of P indicated, one has their transformation law

$$\langle pm[wj],\eta\alpha|U(a,L) = \exp(ip\cdot a)\textstyle\sum_n D_{mn}{}^j(R(p,L)) \times\langle L^{-1}\cdot pn[wj],\eta\alpha|. \quad (3.6)$$

Also, for the direct product states, one has[18]

$$\langle k'\nu'k''\nu'',\alpha|U(a,L) = \exp[i(k'+k'')\cdot a]\sum_{\mu'\mu''}D_{\nu'\mu'}{}^{\sigma'}(R(k',L)) \times D_{\nu''\mu''}{}^{\sigma''}(R(k'',L))\langle L^{-1}\cdot k'\mu'L^{-1}\cdot k''\mu'',\alpha|. \quad (3.7)$$

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[15] cf. reference 4, Eq. (3.24).

[16] One can indeed prove, that for a general Lorentz transformation L, R as defined by (3.4) and (3.5) is a purely spatial rotation. See reference 2.

[17] See references 8–10.

[18] cf. reference 4, Eq. (4.7).

Applying $U(a,1)$ to both sides of (2.8), and using Eqs. (3.6) and (3.7), one may deduce that the coefficient (3.1) vanishes unless $p=k'+k''$. Hence one may factor out of it a four-momentum conservation δ-function in the covariant form

$$2p_0\delta(\mathbf{p}-\mathbf{r})\delta(w-\epsilon) \tag{3.8}$$

where

$$r=k'+k'', \quad r^2=\epsilon^2.$$

Since $L(p)\cdot p=\tilde{p}=(w,\mathbf{0})$, one may apply $U(0,L(p))$ to both sides of (2.8), and use Eqs. (3.6) and (3.7) to transform the coefficient (3.1) to the center-of-mass system, which is uniquely defined because the coefficient (3.1) vanishes unless $p=r$. One obtains the result

$$\begin{aligned}\langle k'\nu'k''\nu''|pm[wj],\eta\rangle &= \sum_{\mu'\mu''} D_{\nu'\mu'}{}^{\sigma'}(R(k',p))D_{\nu''\mu''}{}^{\sigma''}(R(k'',p)) \\ &\quad\times\langle L(p)\cdot k'\mu' L(p)\cdot k''\mu''|\tilde{p}m[wj],\eta\rangle, \end{aligned}\tag{3.9}$$

with[19]

$$\begin{aligned}R(k',p)&=R(k',L(p)),\\ R(k'',p)&=R(k'',L(p)).\end{aligned}\tag{3.10}$$

The next stage is the evaluation of the center-of-mass coefficient:

$$\langle v'\mu'v''\mu''|\tilde{p}m[wj],\eta\rangle, \tag{3.11}$$

with

$$v'=L(p)\cdot k', \quad v''=L(p)\cdot k''.$$

It will be convenient to use a temporary notation

$$\begin{aligned}\langle v'\mu'v''\mu''|\tilde{p}m[wj],\eta\rangle &= 2w\delta(\mathbf{v}'+\mathbf{v}'')\delta(v_0'+v_0''-w) \\ &\quad\times\langle \mathbf{v}'\mu'\mathbf{v}''\mu''|\mathbf{0}m[wj],\eta\rangle,\end{aligned}\tag{3.12}$$

where the δ-function part of the right side has arisen by specializing (3.8) to the rest system of p. The coefficient (3.11) appears naturally [cf. Eq. (2.8)], according to

$$\begin{aligned}|\tilde{p}m[wj],\eta\alpha\rangle &= \sum_{\mu'\mu''}\int\frac{d^3v'}{2v_0'}\int\frac{d^3v''}{2v_0''}2w\delta(\mathbf{v}'+\mathbf{v}'')\delta(v_0'+v_0''-w) \\ &\quad\times|v'\mu'v''\mu'',\alpha\rangle\langle\mathbf{v}'\mu'\mathbf{v}''\mu''|\mathbf{0}m[wj],\eta\rangle,\end{aligned}\tag{3.13}$$

when one seeks to build total angular momentum eigenfunctions in the space of center-of-mass system states of two particles of spins σ' and σ''. Now in this system one may define the relative orbital angular momentum of the two particles, just as in nonrelativistic theory, and describe it by a spherical harmonic

$$Y_{ll_z}(\mathbf{v}/|\mathbf{v}|), \tag{3.14}$$

where $\mathbf{v}=\mathbf{v}'-\mathbf{v}''$. One now sees that the l value of the harmonic will be contained in the degeneracy label η, for, if one couples l, σ', σ'' according to the scheme

$$l+(\sigma'+\sigma'')\rightarrow l+s\rightarrow j, \tag{3.15}$$

then specification of both l and s is necessary to distinguish the multiplicity of different ways of reaching any j value, given only σ' and σ'' initially. Thus, with

$$\eta\equiv l,\, s$$

one expects that the coefficient

$$\langle\mathbf{v}'\mu'\mathbf{v}''\mu''|\mathbf{0}m[wj],\eta\rangle$$

will be proportional to[20]

$$\sum_{l_zs_z} C(\sigma'\sigma''s\mu'\mu''s_z)C(lsjl_zs_zm)Y_{ll_z}(\mathbf{v}/|\mathbf{v}|). \tag{3.16}$$

At this point, one can say that the essential structure of the coefficient (3.1) is determined by Eqs. (3.8), (3.9), and (3.16), and that the basis vectors (2.4) are now explicitly defined in terms of the direct product states by Eq. (2.8) at least up to a normalization factor. To obtain this factor, one must use the normalization (2.9) of the basis vectors (2.4). More precisely, one has to establish the consistency of the results so far obtained with (2.9), and hence obtain the normalization factor necessary to complete Joos's formula[2] for the coefficient (3.1). It will be convenient to present the details of this calculation in Sec. 7 which deals with the orthogonality properties of the C-G coefficients of P. Thus, anticipating the result (7.12), one may use also Eqs. (3.8), (3.9), and (3.16) to give Joos's formula:

$$\begin{aligned}\langle k'\nu'k''\nu''|pm[wj],\eta\rangle &= 2w^{\frac{1}{2}}[\lambda(w^2,\kappa'^2,\kappa''^2)]^{-\frac{1}{4}}2p_0\delta(\mathbf{p}-\mathbf{r})\delta(w-\epsilon) \\ &\quad\times\sum_{\mu'\mu''}D_{\nu'\mu'}{}^{\sigma'}(R(k',r))D_{\nu''\mu''}{}^{\sigma''}(R(k'',r)) \\ &\quad\times\sum_{l_zs_z}C(\sigma'\sigma''s\mu'\mu''s_z)C(lsjl_zs_zm)Y_{ll_z}(\mathbf{e}),\end{aligned}\tag{3.17}$$

where

$$\eta\equiv l,\, s, \quad r=k'+k'', \quad r^2=\epsilon^2.$$

In the normalization factor, the notation used in an instance of the general abbreviation

$$\lambda(a,b,c)=a^2+b^2+c^2-2(bc+ca+ab). \tag{3.18}$$

Thus, here, one has

$$\lambda(w^2,\kappa'^2,\kappa''^2)=[w^2-(\kappa'+\kappa'')^2][w^2-(\kappa'-\kappa'')^2],$$

which relates the concise to the familiar form of the expression. For the unit spatial vector $\mathbf{e}$, one has [cf. Eqs. (3.11) and (3.14)] the formula

$$e_i=\epsilon[\lambda(\epsilon^2,\kappa'^2,\kappa''^2)]^{-\frac{1}{2}}L(r)_i{}^{\mu}(k'-k'')_{\mu}, \tag{3.19}$$

with λ in the sense of (3.18) and $L(r)$ in the sense of (3.5).

Before passing to the use of the formula (3.17), it is

[19] Subsequent use of the notation $R(u,v)$, where u and v are four-momentum arguments, is always in the sense of (3.10).

[20] The summations over l_z, s_z are dummies, but it is very often convenient to carry them along.

necessary to discuss several points regarding it. Firstly, a more concise expression for $\mathbf{e}$ is sought. To obtain this, it is necessary to introduce a certain four vector, dependent on k' and k'', which will be named the relative four momentum of k' and k'' and denoted by the letter q. This vector, which has previously appeared in the work of Gårding and Wightman, and Michel,[21] is constructed from k' and k'' so as to satisfy

$$r\cdot q=0, \tag{3.20}$$

$$q^2+1=0. \tag{3.21}$$

These equations determine that q has the form

$$q=\epsilon[\lambda(\epsilon^2,\kappa'^2,\kappa''^2)]^{-\frac{1}{2}} \times\{k'-k''-[(\kappa'^2-\kappa''^2)/\epsilon^2](k'+k'')\}. \tag{3.22}$$

Since, by definition $\bar{r}_i=L(r)_i{}^\mu r_\mu=0$, it can be seen, from (3.19) and (3.22) that

$$e_i=L(r)_i{}^\mu q_\mu, \tag{3.23}$$

which is the required expression for $\mathbf{e}$. Also, from (3.5) and (3.20), one obtains

$$e_0=L(r)_0{}^\mu q_\mu=0.$$

Of course, the set $e_\mu=(e_0,\mathbf{e})$ of components is simply the four-vector q transformed to rest frame of r, in which the scalar Eqs. (3.20) and (3.21) become

$$e_0=0,\quad \mathbf{e}^2=1. \tag{3.24}$$

Further, one may use (3.5), (3.20), and (3.21) to give the expression for $\mathbf{e}$:

$$\mathbf{e}=\mathbf{q}-[q_0/(\epsilon+r_0)]\mathbf{r}. \tag{3.25}$$

Secondly, some abbreviation of (3.17) is required. It will be convenient to write

$$\langle k'\nu'k''\nu''|pm[wj],\eta\rangle =2w^{\frac{1}{2}}[\lambda(w^2,\kappa'^2,\kappa''^2)]^{-\frac{1}{2}}2p_0\delta(\mathbf{p}-\mathbf{r})\delta(w-\epsilon) \times P(k'k'',\nu'\nu'',jm\eta), \tag{3.26}$$

where $P(k'k'',\nu'\nu'',jm\eta)$ is defined to be the rest of the right side of (3.17). This is an adequate notation: The spherical harmonic contained in $P(k'k''\cdots)$ has arguments given by (3.23) with r and q the total and relative four momenta of its two four-momentum arguments. It will also be convenient for the statement of the integral properties of the C-G coefficients of P. A further useful notation is[22]

$$P(k'k'',\nu'\nu'',jm\eta)=\textstyle\sum_{s_z}\Gamma(\sigma'\sigma''s\nu'\nu''s_z)Y_{jm\eta}(\mathbf{e},s_z). \tag{3.27}$$

The two parts of (3.27) are defined by

$$\Gamma(\sigma'\sigma''s\nu'\nu''s_z)=\sum_{\mu'\mu''}D_{\nu'\mu'}{}^{\sigma'}(R(k',r))D_{\nu''\mu''}{}^{\sigma''}(R(k'',r)) \times C(\sigma'\sigma''s\mu'\mu''s_z), \tag{3.28}$$

and

$$Y_{jm\eta}(\mathbf{e},s_z)=\textstyle\sum_{l_z}C(lsjl_zs_zm)Y_{ll_z}(\mathbf{e}). \tag{3.29}$$

[21] See reference 4, Chap. 4.

[22] As before, the summations in (3.27) and (3.29) are dummies.

The notation (3.28) is adequate if one bears in mind that the supressed momentum arguments are those which correspond uniquely to σ' and σ'', respectively. Its utility stems from the fact that one has

$$\sum_{\nu'\nu''}\Gamma(\sigma'\sigma''s\nu'\nu''s_z)^*\Gamma(\sigma'\sigma''t\nu'\nu''t_z)=\delta(st)\delta(s_zt_z), \tag{3.30}$$

as a result of the unitary properties of D matrices and of C-G coefficients of the rotation group.

Thirdly, by use of a coupling scheme other than that of (3.15), one obtains an alternative C-G coefficient to Joos' coefficient, as given by (3.17). For, if one couples l, σ', and σ'' according to

$$(l+\sigma')+\sigma''\rightarrow h+\sigma''\rightarrow j, \tag{3.31}$$

then specification of both l and h is necessary to distinguish the multiplicity of different ways of reaching any j value, given only σ' and σ'' initially. To obtain a formula for the coefficient

$$\langle k'\nu'k''\nu''|pm[wj],\zeta\rangle, \tag{3.32}$$

where

$$\zeta\equiv l,h,$$

one readily sees, from the derivation of (3.17) given above, that the only change needed on the right side of (3.17) is the replacement of

$$\textstyle\sum_{s_z}C(\sigma'\sigma''s\mu'\mu''s_z)C(lsjl_zs_zm)$$

by

$$\textstyle\sum_{h_z}C(l\sigma'hl_z\mu'h_z)C(h\sigma''jh_z\mu''m). \tag{3.33}$$

To abbreviate the formula so obtained for the coefficient (3.32), one may write

$$\langle k'\nu'k''\nu''|pm[wj],\zeta\rangle =2w^{\frac{1}{2}}[\lambda(w^2,\kappa'^2,\kappa''^2)]^{-\frac{1}{2}}2p_0\delta(\mathbf{p}-\mathbf{r})\delta(w-\epsilon) \times P(k'k'',\nu'\nu'',jm\zeta\rangle \tag{3.34}$$

Hereafter, the coefficients (3.1) and (3.32) will be referred to as η-type and ζ-type C-G coefficients of P, and the labels η and ζ will be used strictly in the senses of Eqs. (3.15) and (3.31).

4. PARTIAL WAVE ANALYSIS OF THE S-MATRIX ELEMENT FOR REACTION (1.1)

The partial wave analysis of the matrix element

$$\langle CD|S|AB\rangle,$$

or, more precisely, of the matrix element

$$\langle k_C\nu_Ck_D\nu_D|S|k_A\nu_Ak_B\nu_B\rangle, \tag{4.1}$$

will be given and discussed here. The character labels of the initial and final particles are implied by the use of the particles labels A, B, C, and D as subscripts.

The state $|k_A\nu_Ak_B\nu_B\rangle$ is a direct product state for the direct product $[\kappa_A,\sigma_A]\otimes[\kappa_B,\sigma_B]$ of the initial state particles, and from (2.8) one gets its C-G series

$$|k_A\nu_A k_B\nu_B\rangle = \sum_{jm\eta_{AB}} \int dw \int \frac{d^3p}{(2p_0)} |pm[wj],\eta_{AB}\rangle \times \langle k_A\nu_A k_B\nu_B | pm[wj],\eta_{AB}\rangle^*.$$

Using (3.26) this becomes

$$|k_A\nu_A k_B\nu_B\rangle = 2E^{\frac{1}{2}}[\lambda(E^2,\kappa_A^2,\kappa_B^2)]^{-\frac{1}{4}} \times \sum_{jm\eta_{AB}} P(k_Ak_B,\nu_A\nu_B,jm\eta_{AB})^* \times |Km[Ej],\eta_{AB}\rangle, \quad (4.2)$$

where $k_A+k_B=K$, $K^2=E^2$. One has also a like result for the final state. Since the S matrix in a relativistic theory commutes with the operators P_μ (translations) and $M_{\mu\nu}$ (space-time rotations) of P, one has a result of the type

$$\langle K'm'[E'j'],\eta_{CD}|S|Km[Ej],\eta_{AB}\rangle = 2K_0\delta(\mathbf{K}-\mathbf{K}')\delta(mm')\delta(E-E')\delta(jj') \times \langle\eta_{CD}|S_j(E)|\eta_{AB}\rangle, \quad (4.3)$$

with

$$K'=k_C+k_D, \quad K'^2=E'^2.$$

The quantities

$$\langle\eta_{CD}|S_j(E)|\eta_{AB}\rangle \quad (4.4)$$

are the partial amplitudes of the S-matrix element (4.1). From (4.2), its final state counterpart and (4.3), one obtains directly the partial wave analysis for (4.1)

$$\langle k_C\nu_C k_D\nu_D|S|k_A\nu_A k_B\nu_B\rangle = 4E[\lambda(E^2,\kappa_A^2,\kappa_B^2)\lambda(E^2,\kappa_C^2,\kappa_D^2)]^{-\frac{1}{4}} \times 2K_0\delta(\mathbf{K}-\mathbf{K}')\delta(E-E') \times \sum_{JM\eta_{AB}\eta_{CD}} P(k_Ck_D,\nu_C\nu_D,JM\eta_{CD}) \times P(k_Ak_B,\nu_A\nu_B,JM\eta_{AB})^*\langle\eta_{CD}|S_J(E)|\eta_{AB}\rangle. \quad (4.5)$$

The arguments of the initial and final spherical harmonics are given by

$$(\mathbf{e}_{AB})_i = L(K)_i{}^\mu(q_{AB})_\mu, \quad (\mathbf{e}_{CD})_i = L(K)_i{}^\mu(q_{CD})_\mu, \quad (4.6)$$

where q_{AB} and q_{CD} are the relative four-momenta of k_A and k_B, and of k_C and k_D, respectively, in the sense of Eqs. (3.20) to (3.22), and where

$$K=k_A+k_B=k_C+k_D.$$

Equation (4.5) is quite a complicated result, when one bears in mind the meaning [Eqs. (3.17) and (3.26)] of the coefficients $P(\cdots)$ involved.

First, consider (4.5) in the case $\sigma_A=\sigma_B=\sigma_C=\sigma_D=0$, in which case the summation on the right becomes

$$\sum_{JM} Y_{JM}(\mathbf{e}_{CD})S_J(E)Y_{JM}^*(\mathbf{e}_{AB}) = \sum_J [(2J+1)/4\pi]P_J(\mathbf{e}_{CD}\cdot\mathbf{e}_{AB})S_J(E), \quad (4.7)$$

so that $\mathbf{e}_{CD}\cdot\mathbf{e}_{AB}$ plays the role of the scattering angle. Since $(e_{AB})_0=(e_{CD})_0=0$, one has

$$e_{AB}\cdot e_{CD} = -\mathbf{e}_{AB}\cdot\mathbf{e}_{CD} = q_{AB}\cdot q_{CD} = -x, \quad (4.8)$$

from the definition of e_{AB}, e_{CD}. Thus x is the scalar variable which provides a generalization of the "barycentric scattering angle" to a reference system wherein $|\mathbf{K}|\neq 0$. This indicates the importance of describing the kinematics of the reaction (1.1) by the set K, q_{AB}, q_{CD} of three independent four vectors. Since the latter pair were constructed so as to satisfy.

$$K\cdot q_{AB}=q_{AB}^2+1=K\cdot q_{CD}=q_{CD}^2+1=0, \quad (4.9)$$

it follows that the only pair of scalar variables one can use are the physically important pair E^2 and x.

Second, consider the case of $K=(E,\mathbf{0})$, so that

$$\mathbf{k}_A=-\mathbf{k}_B=k_i\mathbf{n}_i, \quad \mathbf{k}_C=-\mathbf{k}_D=k_f\mathbf{n}_f$$

where $\mathbf{n}_i^2=\mathbf{n}_f^2=1$. The magnitudes k_i and k_f are related to E by the familiar equations

$$4E^2k_i^2=\lambda(E^2,\kappa_A^2,\kappa_B^2), \quad 4E^2k_f^2=\lambda(E^2,\kappa_C^2,\kappa_D^2), \quad (4.10)$$

so that (4.5) becomes[23]

$$\langle k_C\nu_C k_D\nu_D|S|k_A\nu_A k_B\nu_B\rangle = 2(k_fk_i)^{-\frac{1}{2}}2E\delta(\mathbf{k}_f)\delta(k_{C0}+k_{D0}-E) \times \sum_{JM\eta_{AB}\eta_{CD}} C(\sigma_A\sigma_Bs_{AB}\nu_A\nu_Bs_{ABz}) \times Y_{JM\eta_{AB}}(\mathbf{n}_i,s_{ABz})^*C(\sigma_C\sigma_Ds_{CD}\nu_C\nu_Ds_{CDz}) \times Y_{JM\eta_{CD}}(\mathbf{n}_f,s_{CDz})\langle\eta_{CD}|S_J(E)|\eta_{AB}\rangle. \quad (4.11)$$

When (4.11) is compared with the corresponding nonrelativistic results,[1] it is seen to differ from them only in multiplicative factors, due to different normalizations of states.

5. THE GENERAL C-G SERIES OF P

In order to generalize previous work to the reaction (1.2) with n outgoing particles, some further theory is required. This relates to the study of the direct product of two not necessarily single particle representations of P.

Consider, first, the direct product

$$[w',j']\otimes[w'',j''] \quad (5.1)$$

where the two representations involved are irreducible, with bases of the type discussed in Sec. 2, i.e.,

$$|p'm'[w'j'],\eta'\alpha'\rangle \text{ and } |p''m''[w''j''],\eta''\alpha''\rangle \quad (5.2)$$

where α', α'' indicate the single-particle representations of P, which have led to $[w',j']$ and $[w'',j'']$, respectively.

[23] The notation of (3.29) is here being used.

The direct integral which expresses the reduction of (5.1) into its irreducible constituents is the C-G series of P for (5.1). The aim, as before, is to express the basis vectors, which provide a canonical basis in the Hilbert space of each $[W,J]$ that occurs in this series, in terms of the direct products of basic vectors (5.2). The required new basis vectors can be written as

$$|PM[WJ],\eta\alpha\gamma\rangle, \tag{5.3}$$

where P and M specify the states within the Hilbert space of $[W,J]$ and η is the degeneracy label distinguishing the equivalent $[W,J]$ occurring in the C-G series of P for (5.1). The composite labels α and γ are

$$\alpha\equiv w',j',w'',j''; \quad \gamma\equiv\eta',\alpha',\eta'',\alpha''. \tag{5.4}$$

They are evidently necessary to indicate how the $[W,J]$ have been built up from single particle representations of P. The explicit definition of the states (5.3) is

$$|PM[WJ],\eta\alpha\gamma\rangle=\sum_{m'm''}\int\frac{d^3p'}{2p_0'}\int\frac{d^3p''}{2p_0''}|p'm'p''m'',\alpha\gamma\rangle \times\langle p'm'p''m''|PM[WJ],\eta\rangle, \tag{5.5}$$

where the right-hand states are direct-product states with labels rearranged in accordance with (5.4). The coefficient

$$\langle p'm'p''m''|PM[WJ],\eta\rangle \tag{5.6}$$

is the C-G coefficient of P for (5.1). All the labels contained in α, γ are unaffected by the transformation (5.5), and the attempt to introduce a more general coefficient reduces to (5.5) by means of

$$\langle p'm'p''m'',\beta\delta|PM[WJ],\eta\alpha\gamma\rangle =\delta(\alpha\beta)\delta(\gamma\delta)\langle p'm'p''m''|PM[WJ],\eta\rangle. \tag{5.7}$$

Herein, the notation is somewhat loose, since α contains the continuous variables w' and w''. It may be used without causing confusion, if it is remarked that delta functions are implied by Kronecker symbols for continuous variables, whenever these occur in composite labels. In terms of the same conventional notation, one may give the normalization of the states (5.3)

$$\langle P'M'[W'J'],\bar{\eta}\bar{\alpha}\bar{\gamma}|PM[WJ],\eta\alpha\gamma\rangle =2P_0\delta(\mathbf{P}-\mathbf{P}')\delta(MM')\delta(W-W')\delta(JJ') \times\delta(\eta\bar{\eta})\delta(\alpha\bar{\alpha})\delta(\gamma\bar{\gamma}). \tag{5.8}$$

The above discussion may be completely generalized to the case

$$[w',j']\otimes[w'',j''], \tag{5.9}$$

where these representations have bases

$$|p'm'[w'j'],\beta'\rangle, \quad |p''m''[w''j''],\beta''\rangle, \tag{5.10}$$

where β', β'' are now those sets of variables, some of which may be continuous, which indicate exactly how the indicated representations have been built up from single particle representations. One may denote the basis vectors of any $[W,J]$ that occurs in the C-G series of (5.9), by

$$|PM[WJ],\eta\alpha\beta\rangle \tag{5.11}$$

with α as given by (5.4) and $\beta\equiv\beta'$, β''; and the C-G coefficient of P that arises in their definition is

$$\langle p'm'p''m''|PM[WJ],\eta\rangle,$$

which is the same coefficient as (5.6) regardless of β. The states (5.11) have normalization containing a factor $\delta(\beta\bar{\beta})$ in the above conventional sense.

Thus the most general C-G coefficient of P appears in

$$|PM[WJ],\eta\alpha\beta\rangle=\sum_{m'm''}\int\frac{d^3p'}{2p_0'}\int\frac{d^3p''}{2p_0''}|p'm'p''m'',\alpha\beta\rangle \times\langle p'm'p''m''|PM[WJ],\eta\rangle, \tag{5.12}$$

and the inverse equation

$$|p'm'p''m'',\alpha\beta\rangle=\sum_{JM\eta}\int dW\int\frac{d^3P}{2P_0}|PM[WJ],\eta\alpha\beta\rangle \times\langle p'm'p''m''|PM[WJ],\eta\rangle^* \tag{5.13}$$

is the most general C-G series of P.

From inspection of the derivation of (3.17), it is immediate to conclude that the formula for the coefficient (5.6) is obtained from (3.17) simply by a change of notation.

In the above work, η defined by the scheme

$$l+(j'+j'')\rightarrow l+s\rightarrow J \tag{5.14}$$

has always been written. It need hardly be mentioned that the entire discussion can be written for ζ-type coupling, with ζ defined by the scheme

$$(l+j')+j''\rightarrow h+j''\rightarrow J. \tag{5.15}$$

6. PARTIAL WAVE ANALYSIS OF THE S-MATRIX ELEMENT FOR REACTION (1.2)

In this section, the topic to be discussed is the partial wave analysis of the matrix element

$$\langle 12\cdots n|S|AB\rangle,$$

or, more precisely, of the matrix element:

$$\langle k_1\nu_1k_2\nu_2\cdots k_n\nu_n|S|k_A\nu_Ak_B\nu_B\rangle. \tag{6.1}$$

The bulk of the discussion is of the final state, since the C-G series of the initial state has already been given, Eq. (4.2).

The final state is the continued direct product of the states of type

$$|k_\alpha\nu_\alpha\rangle,\ \alpha=1,\ 2\cdots n, \tag{6.2}$$

and familiar questions about the order of coupling them are immediately raised. For a start, consider a particularly simple mode of procedure.

One first combines $[\kappa_1,\sigma_1]$ and $[\kappa_2,\sigma_2]$, obtaining, as in Sec. 4

$$|k_1\nu_1k_2\nu_2\rangle=2\epsilon_2^{\frac{1}{2}}[\lambda(\epsilon_2^2,\kappa_1^2,\kappa_2^2)]^{-\frac{1}{2}}\times\sum_{j_2m_2\eta_2}P(k_1k_2,\nu_1\nu_2,j_2m_2\eta_2)^*\times|r_2m_2[\epsilon_2j_2],\eta_2\rangle,\quad(6.3)$$

where $r_2=k_1+k_2$, $r_2^2=\epsilon_2^2$, and $\eta_2\equiv l_2$, s_2, according to the scheme

$$l_2+(\sigma_1+\sigma_2)\rightarrow l_2+s_2\rightarrow j_2.\quad(6.4)$$

The arguments of the spherical harmonic contained in $P(k_1k_2\nu_1\nu_2,j_2m_2\eta_2)$ are given by

$$(\mathbf{e}_2)_i=L(r_2)_i{}^\mu(q_2)_\mu,\quad(6.5)$$

where r_2 and q_2 are the total and relative four momenta of k_1 and k_2, the latter term being in the sense of Eqs. (3.20) to (3.22). Next one combines $[\epsilon_2,j_2]$ and $[\kappa_3,\sigma_3]$. It is desirable here to use ζ-type coupling, corresponding to the scheme

$$(l_3+\sigma_3)+j_2\rightarrow h_3+j_2\rightarrow j_3,\quad(6.6)$$

in which case from the work of Sec. 5 one has

$$|r_2m_2,k_3\nu_3,\alpha_3\rangle=2\epsilon_3^{\frac{1}{2}}[\lambda(\epsilon_3^2,\epsilon_2^2,\kappa_3^2)]^{-\frac{1}{2}}\times\sum_{j_3m_3\zeta_3}P(r_2k_3,m_2\nu_3,j_3m_3\zeta_3)^*\times|r_3m_3[\epsilon_3j_3],\zeta_3\alpha_3\rangle,\quad(6.7)$$

where $r_3=r_2+k_3$, $r_3^2=\epsilon_3^2$, and $\alpha_3\equiv\eta_2$, ϵ_2, j_2. The arguments of the spherical harmonic contained in $P(r_2k_3,m_2\nu_3,j_3m_3\zeta_3)$ are given by

$$(\mathbf{e}_3)_i=L(r_3)_i{}^\mu(q_3)_\mu,\quad(6.8)$$

where r_3 and q_3 are the total and relative four momenta of r_2 and k_3. One now goes on to combine $[\epsilon_3,j_3]$ and $[\kappa_4,\sigma_4]$ according to the same procedure, and so on until all $[\kappa_\alpha,\sigma_\alpha]$ have been annexed one at a time. Successively then one forms the states

$$|r_4m_4[\epsilon_4j_4],\zeta_4\alpha_4\rangle,\quad\vdots\quad|r_nm_n[\epsilon_nj_n],\zeta_n\alpha_n\rangle,$$

with

$$\alpha_4\equiv\zeta_3,\epsilon_3,j_3,\alpha_3,\quad\vdots\quad\alpha_n\equiv\zeta_{n-1},\epsilon_{n-1},j_{n-1},\alpha_{n-1},\quad(6.9)$$

and otherwise obvious notations. This corresponds to the coupling scheme

$$\begin{aligned}&l_2+(\sigma_1+\sigma_2)\rightarrow l_2+s_2\rightarrow j_2;\quad\eta_2\equiv l_2,s_2,\\&(l_3+\sigma_3)+j_2\rightarrow h_3+j_2\rightarrow j_3;\quad\zeta_3\equiv l_3,h_3,\\&\quad\vdots\\&(l_n+\sigma_n)+j_{n-1}\rightarrow h_n+j_{n-1}\rightarrow j_n;\quad\zeta_n\equiv l_n,h_n.\end{aligned}\quad(6.10)$$

Partial amplitudes of (6.1) appropriate to the above procedure are defined by

$$\begin{aligned}\langle r_nm_n[\epsilon_nj_n],\zeta_n\alpha_n|S|Km[Ej],\eta_{AB}\rangle&=2K_0\delta(\mathbf{K}-\mathbf{r}_n)\delta(mm_n)\delta(E-\epsilon_n)\delta(jj_n)\\&\times\langle\zeta_n,\zeta_{n-1}j_{n-1},\cdots,\zeta_3j_3,\eta_2j_2\\&\times|S_j(\epsilon_2,\cdots,\epsilon_{n-1},E)|\eta_{AB}\rangle,\end{aligned}\quad(6.11)$$

with the discrete and continuous labels involved in ζ_n, α_n separated in accordance with usual practice. Sufficient discussion has now been given to allow the partial wave analysis of (6.1) corresponding to the scheme (6.10) to be written down. To describe the mode of procedure used above in a pictorial sense,[24–26] one may say that the final-state particles have been combined into a single increasing cluster by adding particle 3 to the (1+2) system, then adding 4 to the [(1+2)+3] system, . . . until all n particles are used up. There is however an abundant choice of alternative procedures available. Present methods are able to handle them, as a single further example will show.

Supposing one forms the first p particles into a single increasing cluster, then one ultimately reaches, in the manner indicated above, final states for the cluster

$$|r_pm_p[\epsilon_pj_p],\zeta_p\alpha_p\rangle.$$

Suppose then that one decides to form the remaining $(n-p)$ particles into a second increasing cluster. It will be convenient to rename the particles $(m+1)$, $(m+2)\cdots n$ as $1'$, $2'\cdots p'$ where $p'=n-p$. Then for the second cluster, one reaches final states

$$|r_{p'}m_{p'}[\epsilon_{p'}j_{p'}],\zeta_{p'}\alpha_{p'}\rangle,$$

in obvious notation. It only remains to combine the clusters, i.e., to form $[\epsilon_p,j_p]\otimes[\epsilon_{p'},j_{p'}]$, which can be handled by the methods of Sec. 5. The only remark to be made regarding this final step of combining clusters is that it seems desirable to use the η type of coupling.

One is left with the conclusion that the only difficulties in giving the partial wave analysis of (6.1), for any clustering of the final particles whatever, are notational ones.

Explicit formulas will now be given for $n=3$ case of reaction (1.2), i.e.

$$A+B\rightarrow1+2+3.$$

The partial wave analysis of the S-matrix element for this reaction is obtained directly by combining Eqs.

[24] The language is taken from the corresponding nonrelativistic theory, references 25, 26.

[25] L. M. Delves, Nuclear Phys. 20, 275 (1960).

[26] D. Jepsen and J. O. Hirschfelder, Proc. Nat. Acad. Sci. U. S. 45, 249 (1959).

(4.2), (6.3), (6.7), and the $n=3$ case of (6.11):

$$\langle k_1\nu_1k_2\nu_2k_3\nu_3|S|k_A\nu_Ak_B\nu_B\rangle$$
$$=2\epsilon_2^{\frac{1}{2}}[\lambda(\epsilon_2^2,\kappa_1^2,\kappa_2^2)]^{-\frac{1}{4}}2\epsilon_3^{\frac{1}{2}}[\lambda(\epsilon_3^2,\epsilon_2^2,\kappa_3^2)]^{-\frac{1}{4}}$$
$$\times 2E^{\frac{1}{2}}[\lambda(E^2,\kappa_A^2,\kappa_B^2)]^{-\frac{1}{4}}2K_0\delta(\mathbf{K}-\mathbf{r}_3)\delta(E-\epsilon_3)$$
$$\times\sum P(k_1k_2,\nu_1\nu_2,j_2m_2\eta_2)P(r_2k_3,m_2\nu_3,JM\zeta_3)$$
$$\times\langle\zeta_3\eta_2j_2|S_J(\epsilon_2,E)|\eta_{AB}\rangle$$
$$\times P(k_Ak_B,\nu_A\nu_B,JM\eta_{AB})^*, \quad (6.12)$$

with the summation over j_2, m_2, η_2, ζ_3, J, M, η_{AB}. Also,

$$r_2=k_1+k_2,\quad r_2^2=\epsilon_2^2,$$
$$r_3=r_2+k_3,\quad r_3^2=\epsilon_3^2,$$
$$K=k_A+k_B,\quad K^2=E^2,$$

and the arguments of the spherical harmonics contained in the successive coefficients on the right are given by (6.5), (6.8), and (4.6).

Equation (6.12) will now be specialized to the case[27] when all particles are spinless, and $K=(E,\mathbf{0})$. Then the summation on the right becomes

$$\sum_{JMl_2l_3}Y_{JMl_2l_3}(\mathbf{e}_2,\mathbf{e}_3)\langle l_2l_3|S_J(\epsilon_2,E)|AB\rangle \times Y_{JM}^*(\mathbf{e}_{AB}), \quad (6.13)$$

with

$$Y_{JMl_2l_3}(\mathbf{e}_2,\mathbf{e}_3)=\sum_{m_2m_3}C(l_2l_3Jm_2m_3M) \times Y_{l_2m_2}(\mathbf{e}_2)Y_{l_3m_3}(\mathbf{e}_3). \quad (6.14)$$

Herein $\mathbf{e}_3$ and $\mathbf{e}_{AB}$ are unit vectors parallel to $\mathbf{r}_2(=-\mathbf{k}_3)$ and $\mathbf{k}_A(=-\mathbf{k}_B)$ respectively. Formula (6.5) must still be used for $\mathbf{e}_2$. Thus (6.13) exhibits one feature, the angular dependence of $Y_{l_2m_2}(\mathbf{e}_2)$, wherein it differs radically from the corresponding nonrelativistic result,[28] even though the center-of-mass system is being referred to.

The conclusion is a perfectly general one for production reactions. If one tries to describe an angular momentum problem in relativistic theory by using nonrelativistic results in the center-of-mass system of the situation concerned, then one is certainly following an erroneous procedure unless only two particle states are involved. For aside from normalization factors and possible omission of essential D matrices, one thereby fails[29] to find the true angular dependence of the results sought.

7. ORTHOGONALITY PROPERTIES

The aims of the present section are: firstly, to exhibit the consistency of Eqs. (2.8) and (3.17) with (2.9); secondly, to obtain the normalization factor in (3.17); and thirdly to derive the orthogonality properties of the functions

$$P(k'k'',\nu'\nu'',jm\eta),\quad P(k'k'',\nu'\nu'',jm\zeta).$$

These aims will be approached simultaneously.

Consider first the coefficient

$$\langle k'k''|pm[wj]\rangle \quad (7.1)$$

which occurs in the reduction of $[\kappa',0]\otimes[\kappa'',0]$. In Sec. 3, it was shown that, as a result of the known structure of irreducible representations of P, the formula for the coefficient (7.1) must contain the factors

$$2p_0\delta(\mathbf{p}-\mathbf{r})\delta(w-\epsilon)Y_{jm}(\mathbf{e}), \quad (7.2)$$

where $r=k'+k''$, $r^2=\epsilon^2$, $e_i=L(r)_i{}^\mu q_\mu$, and q is the relative four momentum of k' and k''. It is now necessary to exhibit the consistency of the formula

$$\langle k'k''|pm[wj]\rangle=\alpha(w)2p_0\delta(\mathbf{p}-\mathbf{r})\delta(w-\epsilon)Y_{jm}(\mathbf{e}) \quad (7.3)$$

where $\alpha(w)$ is a normalization factor, with the normalization of the states $|pm[wj],\kappa'\kappa''\rangle$, obtained from (2.9) in the form

$$\langle p'm'[w'j'],\kappa'\kappa''|pm[wj],\kappa'\kappa''\rangle$$
$$=2p_0\delta(\mathbf{p}-\mathbf{p}')\delta(mm')\delta(w-w')\delta(jj')$$
$$=\int d^3k'/(2k_0')\int d^3k''/(2k_0'')\langle k'k''|pm[wj]\rangle \times\langle k'k''|p'm'[w'j']\rangle^*, \quad (7.4)$$

and hence obtain $\alpha(w)$. Inserting (7.3) into (7.4) leads to

$$|\alpha(w)|^2\int d^3k'/(2k_0')\int d^3k''/(2k_0'')2p_0\delta(\mathbf{p}-\mathbf{r})\delta(w-\epsilon) \times Y_{jm}(\mathbf{e})Y_{j'm'}^*(\mathbf{e})=\delta(mm')\delta(jj'), \quad (7.5)$$

and, if the integral on the left can be converted into the orthogonality integral of spherical harmonics, consistency is indeed established. In order to do this, it will be necessary to introduce into (7.5) a certain change of variables, similar to that used by Wightman,[21] when confronted by an integral like (7.5). It consists of the replacement of k' and k'' by their total and relative four momenta

$$r=k'+k'',\quad r^2=\epsilon^2,$$
$$q=\epsilon[\lambda(\epsilon^2,\kappa'^2\kappa''^2)]^{-\frac{1}{2}} \times\{k'-k''-[(\kappa'^2-\kappa''^2)/\epsilon^2](k'+k'')\}, \quad (7.6)$$

the latter having been constructed [cf. Eqs. (3.20) to (3.22)] so as to satisfy $r\cdot q=q^2+1=0$. The solutions of (7.6), namely

$$k'=\{(\epsilon^2+\kappa'^2-\kappa''^2)r+\epsilon[\lambda(\epsilon^2,\kappa'^2,\kappa''^2)]^{\frac{1}{2}}q\}/(2\epsilon^2),$$
$$k''=\{(\epsilon^2-\kappa'^2+\kappa''^2)r-\epsilon[\lambda(\epsilon^2,\kappa'^2,\kappa''^2)]^{\frac{1}{2}}q\}/(2\epsilon^2),$$

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[27] This topic will be treated in more detail in a forthcoming paper by the author.
[28] R. G. Newton and L. Fonda, Phys. Rev. **120**, 394 (1960).
[29] V. Ritus, J. Exptl. Theoret. Phys. (U.S.S.R.) **10**, 152 (1960).

automatically satisfy $k'^2=\kappa'^2$, $k''^2=\kappa''^2$, if r and q satisfy $r^2=\epsilon^2$, $r\cdot q=q^2+1=0$. The result of the change (7.6) of variables is[4]

$$\int\frac{d^3k'}{2k_0'}\int\frac{d^3k''}{2k_0''}\rightarrow\frac{1}{2}\int d\epsilon\ [\lambda(\epsilon^2,\kappa'^2,\kappa''^2)]^{\frac{1}{2}}\times\int\frac{d^3r}{2r_0}\int d^4q\delta(r\cdot q)\delta(q^2+1),\quad(7.7)$$

so that (7.5) becomes

$$\tfrac{1}{2}|\alpha(w)|^2[\lambda(w^2,\kappa'^2,\kappa''^2)]^{\frac{1}{2}}\int d^4q\delta(p\cdot q)\delta(q^2+1)\times Y_{jm}(\mathbf{e})Y_{j'm'}{}^*(\mathbf{e})=\delta(mm')\delta(jj'),\quad(7.8)$$

with $\mathbf{e}$ given by $e_i=L(p)_i{}^\mu q_\mu$. A further change of variables

$$q\rightarrow e:\quad e_\mu=L(p)_\mu{}^\nu q_\nu,\quad(7.9)$$

with

$$\int d^4q\delta(p\cdot q)\delta(q^2+1)\rightarrow\int d^4e\delta(we_0)\delta(e^2+1),=1/(2w)\int d\Omega(\mathbf{e}),\quad(7.10)$$

converts (7.8) into

$$|\alpha(w)|^2\int d\Omega(\mathbf{e})Y_{jm}(\mathbf{e})Y_{j'm'}{}^*(\mathbf{e})=4w[\lambda(w^2,\kappa'^2,\kappa''^2)]^{-\frac{1}{2}}\delta(mm')\delta(jj').\quad(7.11)$$

Now, by virtue of the orthogonality property of spherical harmonics, the required consistency proof for the case of the coefficient (7.1) is complete, and one may set

$$\alpha(w)=2w^{\frac{1}{2}}[\lambda(w^2,\kappa'^2,\kappa''^2)]^{-\frac{1}{4}},\quad(7.12)$$

in agreement with (3.17).

In the analogous treatment of the coefficient

$$\langle k'\nu'k''\nu''|pm[wj],\eta\rangle\quad(7.13)$$

the result

$$\int\frac{d^3k'}{2k_0'}\int\frac{d^3k''}{2k_0''}2p_0\delta(\mathbf{p}-\mathbf{r})\delta(w-\epsilon)\rightarrow\frac{[\lambda(w^2,\kappa'^2,\kappa''^2)]^{\frac{1}{2}}}{4w}\int d\Omega(\mathbf{e})\quad(7.14)$$

still holds good, so that if the functions

$$P(k'k'',\nu'\nu'',jm\eta)$$

have the integral property

$$\sum_{\nu'\nu''}\int d\Omega(\mathbf{e})P(k'k'',\nu'\nu'',jm\eta)^*P(k'k'',\nu'\nu'',j'm'\eta')=\delta(jj')\delta(mm')\delta(\eta\eta'),\quad(7.15)$$

then the $\alpha(w)$ of the general coefficient is still given by (7.12), and the general consistency proof is complete. To establish (7.15), it is convenient to use the notations of (3.27) to (3.29). One easily verifies the result

$$\sum_{s_z}\int d\Omega(\mathbf{e})Y_{jmls}{}^*(\mathbf{e},s_z)Y_{j'm'l's}(\mathbf{e},s_z)=\delta(jj')\delta(mm')\delta(ll'),$$

by using the orthogonality property of spherical harmonics and those of C-G coefficients of the rotation group. The use of this last result and (3.30) now leads directly to (7.15), as required.

The ζ analog of (7.15), namely

$$\sum_{\nu'\nu''}\int d\Omega(\mathbf{e})P(k'k'',\nu'\nu'',jm\zeta)^*P(k'k'',\nu'\nu'',j'm'\zeta')=\delta(jj')\delta(mm')\delta(\zeta\zeta')\quad(7.16)$$

can likewise be proved directly.

8. PROJECTION FORMULAS FOR PARTIAL AMPLITUDES

It is the aim of the present section to derive formulas for projecting the partial amplitudes

$$\langle\eta_{CD}|S_J(E)|\eta_{AB}\rangle,\quad(8.1)$$

$$\langle\zeta_3\eta_2j_2|S_J(\epsilon_2,E)|\eta_{AB}\rangle,\quad(8.2)$$

respectively, out of the energy-shell matrix elements

$$\langle k_C\nu_Ck_D\nu_D|S(E)|k_A\nu_Ak_B\nu_B\rangle,$$
$$\langle k_1\nu_1k_2\nu_2k_3\nu_3|S(E)|k_A\nu_Ak_B\nu_B\rangle.$$

Energy-shell matrix elements are defined in relation to those of (4.1) and (6.1) by[30]

$$\langle k_C\nu_Ck_D\nu_D|S|k_A\nu_Ak_B\nu_B\rangle=\delta(K-K')\langle k_C\nu_Ck_D\nu_D|S(E)|k_A\nu_Ak_B\nu_B\rangle\quad(8.3)$$

and an analogous equation extracting the factor

$$\delta(K-r_n)$$

out of (6.1). As before

$$K=k_A+k_B,\quad K'=k_C+k_D,\qquad r_n=k_1+k_2+\cdots+k_n.\quad(8.4)$$

Before treating the general case of (8.1), it is helpful to consider the case with $\sigma_A=\sigma_B=\sigma_C=\sigma_D=0$. In this case, from

$$\langle k_Ck_D|S(E)|k_Ak_B\rangle=8E^2[\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)\lambda(E^2,\kappa_C{}^2,\kappa_D{}^2)]^{-\frac{1}{4}}\times\textstyle\sum_{JM}Y_{JM}(\mathbf{e}_{CD})S_J(E)Y_{JM}{}^*(\mathbf{e}_{AB}),\quad(8.5)$$

one obtains, using the orthogonality property of

[30] One can write $\delta(K-K')=(K_0/E)\delta(\mathbf{K}-\mathbf{K}')\delta(E-E')$.

spherical harmonics

$$Y_{JM}^*(\mathbf{e}_{AB})S_J(E) = [\lambda(E^2,\kappa_A^2,\kappa_B^2)\lambda(E^2,\kappa_C^2,\kappa_D^2)]^{\frac{1}{2}}/(8E^2) \times \int d\Omega(\mathbf{e}_{CD})Y_{JM}^*(\mathbf{e}_{CD})\langle k_Ck_D|S(E)|k_Ak_B\rangle. \quad (8.6)$$

Hence, using the addition theorem of spherical harmonics

$$S_J(E) = [\lambda(E^2,\kappa_A^2,\kappa_B^2)\lambda(E^2,\kappa_C^2,\kappa_D^2)]^{\frac{1}{2}}/(8E^2) \times \int d\Omega(\mathbf{e}_{CD})P_J(\mathbf{e}_{CD}\cdot\mathbf{e}_{AB})\langle k_Ck_D|S(E)|k_Ak_B\rangle$$
$$= [\lambda(E^2,\kappa_A^2,\kappa_B^2)\lambda(E^2,\kappa_C^2,\kappa_D^2)]^{\frac{1}{2}}/(8E^2) \times \int d^4q_{CD}\delta(K\cdot q_{CD})\delta(q_{CD}^2+1)P_J(x) \times \langle k_Ck_D|S(E)|k_Ak_B\rangle \quad (8.7)$$

where $x=-q_{CD}\cdot q_{AB}$ is the scalar variable introduced in Sec. 4. The projection formula (8.7) is manifestly covariant and reduces to a familiar form in the center-of-mass, $|\mathbf{K}|=0$.

In the general spin case, a closely parallel procedure to the above is followed. From the partial wave analysis (4.5), one obtains, using (7.15) and (8.3), the result

$$\sum_{\eta_{AB}} P(k_Ak_B,\nu_A\nu_B,JM\eta_{AB})^*\langle\eta_{CD}|S_J(E)|\eta_{AB}\rangle = [\lambda(E^2,\kappa_A^2,\kappa_B^2)\lambda(E^2,\kappa_C^2,\kappa_D^2)]^{\frac{1}{2}}/(8E^2) \times \sum_{\nu_C\nu_D}\int d\Omega(\mathbf{e}_{CD})P(k_Ck_D,\nu_C\nu_D,JM\eta_{CD})^* \times \langle k_C\nu_Ck_D\nu_D|S(E)|k_A\nu_Ak_B\nu_B\rangle. \quad (8.8)$$

Although one could now use (7.15) again to do the initial state part of the projection, a method which avoids integration over initial state variables is sought. To this end, the important result:

$$4\pi\sum_{\nu'\nu''m} P(k'k'',\nu'\nu'',jm\eta)^*P(k'k'',\nu'\nu'',jm\eta') = (2j+1)\delta(\eta\eta'), \quad (8.9)$$

will be proved. Notations (3.27) to (3.29) are again useful.

By use of symmetry properties of C-G coefficients of the rotation group, one obtains the result[31]

$$\sum_{m_2m} C(j_1j_2jm_1m_2m)C(j_1'j_2jm_1'm_2m) = (2j+1)/(2j_1+1)\delta(j_1j_1')\delta(m_1m_1')$$

[31] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 791.

and, hence, using the addition theorem of spherical harmonics, the result

$$4\pi\sum_{ms_z} Y_{jmls}^*(\mathbf{e},s_z)Y_{jml's}(\mathbf{e},s_z) = (2j+1)\delta(ll').$$

Combining this equation with (3.30) leads to direct proof of (8.9). By applying (8.9) to (8.8), one now gets the desired projection formula

$$(2J+1)\langle\eta_{CD}|S_J(E)|\eta_{AB}\rangle = 4\pi[\lambda(E^2,\kappa_A^2,\kappa_B^2)\lambda(E^2,\kappa_C^2,\kappa_D^2)]^{\frac{1}{2}}/(8E^2) \times\sum P(k_Ak_B,\nu_A\nu_B,JM\eta_{AB}) \times\int d\Omega(\mathbf{e}_{CD})P(k_Ck_D,\nu_C\nu_D,JM\eta_{CD})^* \times\langle k_C\nu_Ck_D\nu_D|S(E)|k_A\nu_Ak_B\nu_B\rangle, \quad (8.10)$$

with the summation over ν_A, ν_B, ν_C, ν_D, and M.

Similarly for the case of (8.2), by applying (7.15) and (7.16) to the final state, and (8.9) to the initial state, one may prove

$$(2J+1)\langle\zeta_3\eta_2j_2|S_J(\epsilon_2,E)|\eta_{AB}\rangle = 4\pi[\lambda(\epsilon_2^2,\kappa_1^2,\kappa_2^2)\lambda(E^2,\epsilon_3^2,\kappa_3^2)\lambda(E^2,\kappa_A^2,\kappa_B^2)]^{\frac{1}{2}}/(16\epsilon_2^{\frac{1}{2}}E^2) \times\sum P(k_Ak_B,\nu_A\nu_B,JM\eta_{AB})\int d\Omega(\mathbf{e}_3)d\Omega(\mathbf{e}_2) \times P(r_2k_3,m_2\nu_3,JM\zeta_3)^*P(k_1k_2,\nu_1\nu_2,j_2m_2\eta_2)^* \times\langle k_1\nu_1k_2\nu_2k_3\nu_3|S(E)|k_A\nu_Ak_B\nu_B\rangle \quad (8.11)$$

with the summation over ν_A, ν_B, ν_1, ν_2, m_2, ν_3, and M. One may easily obtain a result like (8.11) for a matrix element with an (arbitrary) n-particle final state, for any possible coupling sheme.

9. CROSS SECTIONS

Invariant cross-section formulas for the reactions (1.1) and (1.2) are developed in this section.

Present work has so far dealt only with S-matrix elements but, of course, it applies also to T-matrix elements where

$$S=1+iT. \quad (9.1)$$

In analogy with (8.3) and (8.4), one defines "energy shell" elements of T by extraction of factors $\delta(K-K')$ and $\delta(K-r_n)$. The invariant cross-section formula for (1.1) may now be given in terms of the matrix element

$$\langle k_C\nu_Ck_D\nu_D|T(E)|k_A\nu_Ak_B\nu_B\rangle,$$

by[6,32]

$$\sigma(AB\to CD) = (2\pi)^{-4}\sum(\phi_{CD}/\rho_{AB}) \times|\langle k_C\nu_Ck_D\nu_D|T(E)|k_A\nu_Ak_B\nu_B\rangle|^2, \quad (9.2)$$

[32] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 163.

where $\sum$ implies a summation over final spin states and an average over initial ones. The factors ϕ_{CD} and ρ_{AB} are, respectively, the density of final states factor for the CD system, and incident-flux factor for the AB system. In consequence of the normalization (2.2) of one particle states, ρ_{AB} is given by the expression

$$(2\pi)^6\rho_{AB}=2[\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)]^{\frac{1}{2}}. \tag{9.3}$$

The expression for ϕ_{CD} is

$$\phi_{CD}=\int\frac{d^3k_C}{2k_{C0}}\int\frac{d^3k_D}{2k_{D0}}\delta(K-K'). \tag{9.4}$$

A change of variables may now be introduced into (9.4) as in Sec. 8. The new variables are K' and q_{CD}, the total and relative four-momenta of k_C and k_D. Using a result like (7.7) and then a result like (7.10) one is lead successively to

$$\phi_{CD}=\frac{[\lambda(E^2,\kappa_C{}^2,\kappa_D{}^2)]^{\frac{1}{2}}}{4E}\int d^4q_{CD}\delta(K\cdot q_{CD})\delta(q_{CD}{}^2+1)$$
$$=\frac{[\lambda(E^2,\kappa_C{}^2,\kappa_D{}^2)]^{\frac{1}{2}}}{8E^2}\int d\Omega(\mathbf{e}_{CD}), \tag{9.5}$$

with $\mathbf{e}_{CD}$ as previously defined. From (9.2), (9.5), one obtains the following expression for the differential cross section

$$\frac{d\sigma(AB\to CD)}{d\Omega(\mathbf{e}_{CD})}=\left(\frac{2\pi}{4E}\right)^2\left[\frac{\lambda(E^2,\kappa_C{}^2,\kappa_D{}^2)}{\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)}\right]^{\frac{1}{2}}$$
$$\times\sum|\langle k_C\nu_Ck_D\nu_D|T(E)|k_A\nu_Ak_B\nu_B\rangle|^2, \tag{9.6}$$

with $\sum$ as in (9.2). In agreement with the remarks of Chew,[33] one may introduce the physical amplitude for reaction (1.1) by setting

$$\langle k_C\nu_Ck_D\nu_D|F(E)|k_A\nu_Ak_B\nu_B\rangle$$
$$=(2\pi/4E)\langle k_C\nu_Ck_D\nu_D|T(E)|k_A\nu_Ak_B\nu_B\rangle, \tag{9.7}$$

the definition of F being unique up to a phase. Into (9.6), one may insert the partial wave analysis:

$$\langle k_C\nu_Ck_D\nu_D|T(E)|k_A\nu_Ak_B\nu_B\rangle$$
$$=8E^2[\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)\lambda(E^2,\kappa_C{}^2,\kappa_D{}^2)]^{-\frac{1}{4}}$$
$$\times\sum P(k_Ck_D,\nu_C\nu_D,JM\eta_{CD})\langle\eta_{CD}|T_J(E)|\eta_{AB}\rangle$$
$$\times P(k_Ak_B,\nu_A\nu_B,JM\eta_{AB})^*, \tag{9.8}$$

with summation over J, M, η_{CD}, and η_{AB}. Then, on using the results (7.15) and (8.9), the result

$$\sigma(AB\to CD)$$
$$=\frac{1}{(2\sigma_A+1)(2\sigma_B+1)}\frac{(4\pi E)^2}{\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)}$$
$$\times\sum_{J\eta_{CD}\eta_{AB}}\frac{2J+1}{4\pi}|\langle\eta_{CD}|T_J(E)|\eta_{AB}\rangle|^2 \tag{9.9}$$

follows. Various partial cross sections may be defined, e.g.,

$$\sigma(AB\to CD)=\sum_{J\eta_{CD}\eta_{AB}}\sigma_J(\eta_{AB}\to\eta_{CD}). \tag{9.10}$$

The cross section for (1.2) is defined by a formula identical in structure to (9.2), with the density of final states factor given by

$$\phi_{12\cdots n}=\left[\prod_{\alpha=1}^{n}\int\frac{d^3k_\alpha}{2k_{\alpha0}}\right]\delta(K-r_n) \tag{9.11}$$

with $r_n=k_1+k_2+\cdots+k_n$. A more convenient expression for (9.11) is sought. Suppose the partial wave analysis of the T-matrix element for (1.2) to be used corresponds to the formation of final particles into a single cluster. Then (9.11) ought to be transformed by means of the following successive changes of variable:

$$\begin{array}{lll} k_1,k_2\to r_2,q_2; & q_2\to e_2: & (e_2)_\mu=L(r_2)_\mu{}^\nu(q_2)_\nu, \\ r_2,k_3\to r_3,q_3; & q_3\to e_3: & (e_3)_\mu=L(r_3)_\mu{}^\nu(q_3)_\nu, \\ & \vdots & \\ r_{n-1},k_n\to r_n,q_n; & q_n\to e_n: & (e_n)_\mu=L(r_n)_\mu{}^\nu(q_n)_\nu, \end{array} \tag{9.12}$$

with each line governed by equations like (7.7) and (7.10). In the $(\alpha-1)$th line, the pair r_α and q_α are the total and relative four momenta of the pair of four vectors $r_{\alpha-1}$ and k_α. One has

$$r_\alpha=r_{\alpha-1}+k_\alpha=k_1+k_2+\cdots+k_\alpha, \tag{9.13}$$

and q_α is constructed out of $r_{\alpha-1}$ and k_α so that

$$r_\alpha\cdot q_\alpha=(r_{\alpha-1}+k_\alpha)\cdot q_\alpha=q_\alpha{}^2+1=0. \tag{9.14}$$

Hence, for $\alpha=2,\cdots n$, one has

$$q_\alpha=\frac{\epsilon_\alpha}{[\lambda(\epsilon_\alpha{}^2,\epsilon_{\alpha-1}{}^2,\kappa_\alpha{}^2)]^{\frac{1}{2}}}$$
$$\times\left\{r_{\alpha-1}-k_\alpha-\frac{(\epsilon_{\alpha-1}{}^2-\kappa_\alpha{}^2)}{\epsilon_\alpha{}^2}(r_{\alpha-1}+k_\alpha)\right\}, \tag{9.15}$$

with $r_\alpha{}^2=\epsilon_\alpha{}^2$, provided that r_1 and ϵ_1 imply k_1 and κ_1 where necessary. In the notation of Sec. 6, the lines of (9.12) correspond to the lines of the angular momentum scheme (6.10), or

$$\begin{array}{ll} [\kappa_1,\sigma_1]\otimes[\kappa_2,\sigma_2]\to\sum[\epsilon_2,j_2], & \text{with } P(k_1k_2,\nu_1\nu_2,j_2m_2\eta_2), \\ [\epsilon_2,j_2]\otimes[\kappa_3,\sigma_3]\to\sum[\epsilon_3,j_3], & \text{with } P(r_2k_3,m_2\nu_3,j_3m_3\zeta_3), \\ \vdots & \vdots \\ [\epsilon_{n-1},j_{n-1}]\otimes[\kappa_n,\sigma_n]\to\sum[\epsilon_n,j_n], & \text{with } P(r_{n-1}k_n,m_{n-1}\nu_n,j_nm_n\zeta_n), \end{array}$$

[33] G. F. Chew, *Lectures on Dynamical Theory of Strong Interactions* (Les Houches, 1960).

which contain spherical harmonics whose arguments are given by

$$\mathbf{e}_2:\quad e_{2\mu}=L(r_2)_\mu{}^\nu(q_2)_\nu,$$
$$\mathbf{e}_3:\quad e_{3\mu}=L(r_3)_\mu{}^\nu(q_3)_\nu,$$
$$\vdots$$
$$\mathbf{e}_n:\quad e_{n\mu}=L(r_n)_\mu{}^\nu(q_n)_\nu.$$

As in (7.7) and (7.10), one finds that the successive lines of (9.12) are governed by

$$\int\frac{d^3k_1}{2k_{10}}\int\frac{d^3k_2}{2k_{20}}\rightarrow\int d\epsilon_2\frac{[\lambda(\epsilon_2{}^2,\kappa_1{}^2,\kappa_2{}^2)]^{\frac{1}{2}}}{4\epsilon_2}\int\frac{d^3r_2}{2r_{20}}\int d\Omega(\mathbf{e}_2)$$
$$\int\frac{d^3r_2}{2r_{20}}\int\frac{d^3k_3}{2k_{30}}\rightarrow\int d\epsilon_3\frac{[\lambda(\epsilon_3{}^2,\epsilon_2{}^2,\kappa_3{}^2)]^{\frac{1}{2}}}{4\epsilon_3}\int\frac{d^3r_3}{2r_{30}}\int d\Omega(\mathbf{e}_3)$$

and so on. Hence one obtains

$$\phi_{12\cdots n}=\prod_{\alpha=2}^{n}\left\{\int d\epsilon_\alpha\frac{[\lambda(\epsilon_\alpha{}^2,\epsilon_{\alpha-1}{}^2,\kappa_\alpha{}^2)]^{\frac{1}{2}}}{4\epsilon_\alpha}\times\int d\Omega(\mathbf{e}_\alpha)\right\}\frac{\delta(E-\epsilon_n)}{2E},\quad(9.16)$$

the energy conservation delta-function being left for the sake of conciseness of notation. The transformation of (9.11) to be used along with different partial wave analysis of the T-matrix element of (1.2) can be achieved by quite similar methods.

For the case of $n=3$, with final particles coupled according to $(1+2)+3$, one has

$$\frac{d\sigma(AB\rightarrow 123)}{d\epsilon_2}=\left(\frac{2\pi}{4E}\right)^2\left[\frac{\lambda(\epsilon_2{}^2,\kappa_1{}^2,\kappa_2{}^2)\lambda(E^2,\epsilon_2{}^2,\kappa_3{}^2)}{16\epsilon_2{}^2\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)}\right]^{\frac{1}{2}}\times\sum\int d\Omega(\mathbf{e}_2)\int d\Omega(\mathbf{e}_3)\times|\langle k_1\nu_1k_2\nu_2k_3\nu_3|T(E)|k_A\nu_Ak_B\nu_B\rangle|^2,\quad(9.17)$$

with $\sum$ used in the same sense as in (9.2). Inserting into (9.17) the partial wave analysis of

$$\langle k_1\nu_1k_2\nu_2k_3\nu_3|T(E)|k_A\nu_Ak_B\nu_B\rangle$$

as obtained from (6.12), (9.1), and (8.3), one can use (7.15), (7.16), and (8.9) to verify that

$$\frac{d\sigma(AB\rightarrow 123)}{d\epsilon_2}=\frac{1}{(2\sigma_A+1)(2\sigma_B+1)}\frac{(4\pi E)^2}{\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)}\times\sum[(2J+1)/4\pi]\times|\langle\zeta_3j_2\eta_2|T_J(\epsilon_2,E)|\eta_{AB}\rangle|^2.\quad(9.18)$$

Partial cross sections are defined by

$$d\sigma(AB\rightarrow 123)/d\epsilon_2=\sum d\sigma_J(\eta_{AB}\rightarrow\zeta_3j_2\eta_2)/d\epsilon_2,\quad(9.19)$$

with the summation, as in (9.18), over J, η_{AB}, ζ_3, j_2, and η_2.

10. OPTICAL THEOREMS

The object of this final section is to derive optical theorems and partial optical theorems, which are simply concise ways of expressing the requirements of unitarity on certain matrix elements and partial amplitudes, respectively.

In terms of the T matrix, the unitarity equations $SS^\dagger=S^\dagger S=1$ of the S matrix become

$$2\,\mathrm{Im}T=TT^\dagger.\quad(10.1)$$

For a value of the total energy above the physical threshold of (1.1), the element of (10.1) for this reaction can be written in the form

$$\langle k_C\nu_Ck_D\nu_D|2\,\mathrm{Im}T(E)|k_A\nu_Ak_B\nu_B\rangle=\sum_N\delta(K-K_N)\langle k_C\nu_Ck_D\nu_D|T(E)|N\rangle\times\langle N|T(E)^\dagger|k_A\nu_Ak_B\nu_B\rangle,\quad(10.2)$$

with $K=k_A+k_B=k_C+k_D$. Here $\sum_N$ involves a sum over a complete set of real particle states. Of course, only states with threshold energies below the value of E in question in (10.2) actually contribute to the equation.

First consider an E value such that the only type of state which contributes to the right side of (10.2) is the two particle state FG. Of course, if the states AB, CD, and FG are not the same, AB and CD themselves will also contribute, by assumption, but this will be disregarded.

One can write the FG contribution to the right of (10.2) as

$$C_{FG}=\sum_{\nu_F\nu_G}\phi_{FG}\langle k_C\nu_Ck_D\nu_D|T(E)|k_F\nu_Fk_G\nu_G\rangle\times\langle k_F\nu_Fk_G\nu_G|T(E)^\dagger|k_A\nu_Ak_B\nu_B\rangle\quad(10.3)$$

with ϕ_{FG} given by (9.4) or (9.5). One may insert into (10.3) results of type (9.8) and use (7.15) to give

$$C_{FG}=8E^2[\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)\lambda(E^2,\kappa_C{}^2,\kappa_D{}^2)]^{-\frac{1}{4}}\times\sum_{JM\eta_{AB}\eta_{CD}}P(k_Ck_D,\nu_C\nu_D,JM\eta_{CD})\times P(k_Ak_B,\nu_A\nu_B,JM\eta_{AB})^*\{\sum_{\eta_{FG}}\langle\eta_{CD}|T_J(E)|\eta_{FG}\rangle\times\langle\eta_{FG}|T_J(E)^\dagger|\eta_{AB}\rangle\}.\quad(10.4)$$

For the E value in question, C_{FG} provides the entire right side of (10.2), whose left side, from (9.8) is

$$8E^2[\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)\lambda(E^2,\kappa_C{}^2,\kappa_D{}^2)]^{-\frac{1}{4}}\times\sum_{JM\eta_{AB}\eta_{CD}}P(k_Ck_D,\nu_C\nu_D,JM\eta_{CD})\times P(k_Ak_B,\nu_A\nu_B,JM\eta_{AB})^*\times\langle\eta_{CD}|\mathrm{Im}T_J(E)|\eta_{AB}\rangle,\quad(10.5)$$

so that orthogonality, expressed explicitly by formulas (7.15) and (8.9), gives

$$\langle\eta_{CD}|2\mathrm{Im}T_J(E)|\eta_{AB}\rangle = \sum_{\eta_{FG}} \langle\eta_{CD}|T_J(E)|\eta_{FG}\rangle\langle\eta_{FG}|T_J(E)^\dagger|\eta_{AB}\rangle. \quad (10.6)$$

Consider next the case of (10.3), when the value of E is such that in addition to states of type FG, states of type 123 also contribute. Then

$$C_{123}=\sum_{\nu_1\nu_2\nu_3} \phi_{123}\langle k_C\nu_C k_D\nu_D|T(E)|k_1\nu_1k_2\nu_2k_3\nu_3\rangle \times\langle k_1\nu_1k_2\nu_2k_3\nu_3|T(E)^\dagger|k_A\nu_Ak_B\nu_B\rangle, \quad (10.7)$$

with ϕ_{123} given by (9.11) or (9.16). One may insert the appropriate partial wave analyses and then use (7.15) and (7.16) to caste (10.7) into the same form as (10.4). In the case of C_{123}, the factor contained within curly brackets is

$$\sum_{\zeta_3\eta_2j_2}\int d\epsilon_2\langle\eta_{CD}|T_J(\epsilon_2,E)|\zeta_3\eta_2j_2\rangle \times\langle\zeta_3\eta_2j_2|T_J(\epsilon_2,E)^\dagger|\eta_{AB}\rangle. \quad (10.8)$$

Orthogonality, applied to (10.2), now gives a result like (10.6) with a term (10.8) added to the right side.

It is obvious that such results can be obtained for any E value whatever. The above discussion of (10.2) will now be applied to the writing of optical theorems. Such theorems exist for elastic scattering processes

$$A+B\rightarrow A+B, \quad (10.9)$$

and it will suffice to illustrate the method for the case of reaction (10.9) at an E value above its threshold such that the only competing reaction is

$$A+B\rightarrow 1+2+3. \quad (10.10)$$

The desired optical theorem will be a relation between the imaginary part of the forward physical amplitudes for (10.9) and the total cross sections

$$\sigma(AB\rightarrow AB),\quad \sigma(AB\rightarrow 123).$$

From (9.8), one gets

$$\sum_{\nu_A\nu_B}\langle k_A\nu_Ak_B\nu_B|T(E)|k_A\nu_Ak_B\nu_B\rangle =\frac{8E^2}{[\lambda(E^2,\kappa_A^2,\kappa_B^2)]^{\frac{1}{2}}}\sum_{J\eta_{AB}}\frac{2J+1}{4\pi} \times\langle\eta_{AB}|T_J(E)|\eta_{AB}\rangle, \quad (10.11)$$

where the fact that only partial amplitudes with the same initial and final labellings contribute is a consequence of (8.9). For the imaginary part of such amplitudes above work gives

$$\langle\eta_{AB}|2\,\mathrm{Im}T_J(E)|\eta_{AB}\rangle = \sum_{\eta_{AB}'}|\langle\eta_{AB}|T_J(E)|\eta_{AB}'\rangle|^2 +\int d\epsilon_2\sum_{\zeta_3\eta_2j_2}|\langle\eta_{AB}|T_J(\epsilon_2,E)|\zeta_3\eta_2j_2\rangle|^2, \quad (10.12)$$

so that, on using definitions (9.10) and (9.19) of partial cross sections, one has the partial optical theorem

$$\langle\eta_{AB}|2\,\mathrm{Im}T_J(E)|\eta_{AB}\rangle =\beta_J(E)\left\{\sum_{\eta_{AB}'}\sigma_J(\eta_{AB}\rightarrow\eta_{AB}') +\sum_{\zeta_3\eta_2j_2}\int d\epsilon_2 d\sigma_J(\eta_{AB}\rightarrow\zeta_3\eta_2j_2)/d\epsilon_2\right\}, \quad (10.13)$$

with $\beta_J(E)$ given by

$$4\pi E^2\beta_J(E)=[(2\sigma_A+1)(2\sigma_B+1)/(2J+1)]\lambda(E^2,\kappa_A^2,\kappa_B^2).$$

By summing over J, η_{AB} and multiplying by a suitable factor, one converts (10.13) into the optical theorem

$$[(2\sigma_A+1)(2\sigma_B+1)]^{-1} \times\sum_{\nu_A\nu_B}\langle k_A\nu_Ak_B\nu_B|\mathrm{Im}T(E)|k_A\nu_Ak_B\nu_B\rangle =[\lambda(E^2,\kappa_A^2,\kappa_B^2)]^{\frac{1}{2}}/(4\pi^2) \times\{\sigma(AB\rightarrow AB)+\sigma(AB\rightarrow 123)\}, \quad (10.14)$$

or, on using (9.7)

$$4\pi[(2\sigma_A+1)(2\sigma_B+1)]^{-1} \times\sum_{\nu_A\nu_B}\langle k_A\nu_Ak_B\nu_B|\mathrm{Im}F(E)|k_A\nu_Ak_B\nu_B\rangle =([\lambda(E^2,\kappa_A^2,\kappa_B^2)]^{\frac{1}{2}}/2E) \times\{\sigma(AB\rightarrow AB)+\sigma(AB\rightarrow 123)\}. \quad (10.15)$$

Since in the center-of-mass system, the barycentric three-momentum of the AB system has magnitude given by

$$[\lambda(E^2,\kappa_A^2,\kappa_B^2)]^{\frac{1}{2}}/(2E), \quad (10.16)$$

the result (10.15) will there assume a familiar form.[34]

Similarly one can obtain optical theorems for an E value at which many reactions compete with the elastic channel. At any given E value, the optical theorem for (10.9) assumes the form as (10.15), the term within the curly brackets on the right now being the sum of the total cross sections of all reactions possible at the E value.

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[34] N. N. Bogoliubov and V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), p. 560.

Finally, result (10.6) will be used to make contact with the results of Møller,[6] for the cross section, for the elastic scattering of spinless particles, in terms of phase shifts. Since only one channel, the elastic scattering channel is being considered, one may write (10.6) in the form

$$2\,\mathrm{Im}T_J(E)=|T_J(E)|^2, \qquad (10.17)$$

so that $T_J(E)$ can be written as

$$T_J(E)=2\sin\delta_J(E)\exp[i\delta_J(E)], \qquad (10.18)$$

where $\delta_J(E)$ is a real phase shift. Using (9.9) and (10.18) one then obtains

$$\sigma(AB\rightarrow AB)=16\pi E^2[\lambda(E^2,\kappa_A{}^2,\kappa_B{}^2)]^{-1}\times\sum_J(2J+1)\sin^2\delta_J(E), \qquad (10.19)$$

which, when one remembers (10.16), can be seen to be identical with result (212) of Møller's paper.

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GLOSSARY OF NOTATION

κ: rest mass of particle.
σ: spin of particle.
k: momentum of particle, $k^2=\kappa^2$.
μ, ν: z component of spin of particle.
p, P, K, r: momentum of system of particles, $p^2=w^2$, $P^2=W^2$, $K^2=E^2$, $r^2=\epsilon^2$.
q: relative four-momentum.
j, J: total angular momentum.
m, n, M: z component thereof.
l: orbital angular momentum
s, h: intermediate angular momenta in $j=l+\sigma'+\sigma''$ etc.
$\eta\equiv l, s$.
$\zeta\equiv l, h$.
$\alpha\equiv\kappa', \sigma', \kappa'', \sigma''$ or w', j', w'', j''.
$\gamma\equiv\eta', \alpha', \eta'', \alpha''$.
$\lambda(a,b,c)=a^2+b^2+c^2-2(bc+ca+ab)$.
$L(p)$: 4×4 matrix describing the pure Lorentz transformation that carries p into its rest system.
$\tilde{p}$: $\tilde{p}=L(p)\cdot p=(w,\mathbf{0})$.
R: a spatial rotation; $R(k,L)$, $R(u,v)$ defined by Eqs. (3.4) and (3.10).
$D^j(R)$: its $(2j+1)\times(2j+1)$ matrix representative.
$\delta(ab)$: Kronecker delta, $\delta(ab)=0$ or 1 according as $a\neq b$ or $a=b$.
δ: phase shift, $\delta_J(E)$.
$\sigma(AB\rightarrow CD)$ etc. $\cdots$ cross sections.
ρ_Σ: incident flux factor for the system Σ of particles.
ϕ_Σ: density of final states factor for the system Σ of particles.
$\mathbf{e}$: $e=(e_0,\mathbf{e})=(0,\mathbf{e})=L(r)\cdot q$, where r, q are the total and relative four-momenta of the same pair of momenta.
$\int d\Omega(\mathbf{e})$: integration over the polar angles of $\mathbf{e}$.