Had we known of hyperfine structure in the early days of atomic physics, however, it would have been a mistake to insist that any theory should explain the effect. Historically, *all* dynamical theories in physics have had limitations on their domain of validity, no matter how general they seemed when they were proposed. We must not be too greedy.

Note added: After preparation of this manuscript I became aware of an article by V. N. Gribov, J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 667 (1961), which discusses the importance of the Pomeranchuk trajectory in high energy scattering.

REVIEWS OF MODERN PHYSICS VOLUME 34, NUMBER 3 JULY, 1962

Field Theories with Persistent One Particle States. I. General Formalism*

M. DRESDEN AND PETER B. KAHN†
State University of Iowa, Iowa City, Iowa

CONTENTS

1. MOTIVATION AND PURPOSE

In the last few years considerable attention has been devoted to the study of the general aspects of field theory. Currently there are actually several rather different approaches. One approach is based on the notion of local field operators; general principles such as relativistic invariance and local commutativity are assumed. On this basis an extensive, mathematically precise theory may be developed. Initiated by Wightman [1],[1] this approach through the work of Jost [2], Haag [3], Lehman, Symanzik, and Zimmerman [4], and many others (see the extensive bibliography in Schweber [5], has led to a considerable deepening of the mathematical basis of field theory and to specific results of physical interest, such as proofs of the *TCP* theorem (Jost [6]) and the connection between spin and statistics (Burgoyne [7]). Another approach described most explicitly by Chew [8], also utilizes the principle of relativistic invariance. The notions of the "local field" and field operators, so essential in the abstract approach, are eliminated as far as possible in this approach. In fact the ultimate hope of this general philosophy is that the relevant physical principles may be expressed exclusively in terms of the analytic properties of S-matrix elements. If, according to this attitude, a particular assumption about the analytic character of the S matrix, such as the Mandelstam representation, cannot be proven, using the general principles of field theory—this is yet another indication that field theory is incorrect, inconsistent, and irrelevant. Thus using the *assumed* analytic properties of S-matrix elements as given by the Mandelstam representation (for two incoming and two outgoing particles), together with the requirements of unitarity and relativistic invariance, a detailed theoretical framework has been constructed, which has been successful

* Work supported in part by the National Science Foundation.

† Now at the State University of New York, Long Island Center, Oyster Bay, Long Island, New York.

[1] The references are to be found at the end of this paper.

in correlating a large amount of empirical material (Chew [9]; Chew and Mandelstam [10]).

Many studies of course, do not adopt either one of these extreme viewpoints. Very often results obtained by abstract methods are used, although the postulates in which these results were based are not necessarily accepted.

One of the difficulties in using the abstract form of field theory is the often mentioned circumstance that in spite of the general and broad character of the basic postulates, no nontrivial examples of such field theories appear to exist[2] (Lehman, Symanzik, and Zimmerman [11]). There are no (nontrivial) finite Lagrangian theories, with canonical commutation rules, satisfying all the usually accepted principles of quantum field theory. In dispersion theory the Lagrangian formulation of field theory is used as a heuristic tool to obtain, for example, expressions for the residues of the poles. Just as in quantum (or classical) electrodynamics, specific models of the electron were at one time employed in order to obtain the "model independent" features of the theory; in dispersion theory one hopes to obtain from a study of certain Lagrangians at least some of the features that are in fact independent of the Lagrangian. Symmetry principles and the substitution law are cases in point.

Whereas the mathematical formulation of the abstract field theory is in principle completed once the postulates are written down, the mathematical framework of dispersion theory requires, in principle, statements about the analytic character of *all* the S-matrix elements. The analytic properties of the S-matrix elements as conjectured by Mandelstam refer to reactions where one has two incoming and two outgoing particles. Hence, one needs nontrivial extensions of the Mandelstam representation. The guide of perturbation theory and conventional field theory in this context is of doubtful value. The extension of the Mandelstam representation to more complicated S-matrix elements appears extremely difficult. In fact it was shown by Landshoff and Treiman [15] that for the simplest production process ($N + \pi \rightarrow N + \pi + \pi$) the amplitude as a function of only one variable (with the other four variables needed to describe the process, fixed at physical values) exhibits a complicated analytic behavior. (This result depends on the choice of variables; the ones picked were the scalar products of four-momenta.) This illustrates that the analyticity properties of the more complicated S-matrix elements cannot be conjectured as extensions of the results obtained in perturbation theory. In principle the abstract form of field theory can make contributions to the study of these questions, for the S-matrix elements in question can be expressed in terms of the Wightman functions. Through study of the Wightman functions, and investigations of the domains of analyticity of these functions (Kallen and Wightman [16]), some properties of the S-matrix elements can be obtained. So far, however, the properties actually proven are not sufficient to deduce or suggest the Mandelstam representation, let alone possible extensions. Instead of deriving the general analytic character of the S-matrix elements from such general principles, one can attempt to obtain information about the location of the singularities of certain amplitudes by studying the Feynman graphs corresponding to these amplitudes (Landau [17]). The rules so derived (Cutkosky [18]) may be considered as the basis of a theory. The validity of this theory of course depends ultimately on its power to predict experimental results. In addition, it is desirable that such a theory—perhaps in an approximate sense only—should elucidate the reliability as well as the limitation of such notions as Lagrangians, field equations, and field operators.

The general motivation for the present study stems from the desire to gain a more detailed understanding of the relationships between the three forms of field theory: the abstract version, utilizing general principles only; dispersion theory which postulates the analytic properties of the S-matrix elements; and the conventional or Lagrangian approach which operates with specific Lagrangians, definite equations of motion, and definite commutation rules. It would be unreasonable to expect that one type of approach could rigorously be deduced from another, but they certainly should have something to do with one another. One might hope that such a study would throw some light on the reasons for the apparent validity (in a formal sense in any case) of perturbation theory in situations where the very notion of a local field operator would be of doubtful significance. In addition, if the interrelation were better understood, the various forms of field theory might possibly be combined to obtain a more satisfactory theory which would combine the mathematical precision of the abstract theories with the predictive power of the dispersion relations. Such a combined theory would be of a somewhat hybrid character; it would un-

[2] Several examples of especially simple nontrivial field theories have been studied. A particular one, a modification of Thirring's model [12] is described in the Northwestern thesis of one of us (PBK) (Dresden, McGlinn, and Kahn [13]). The only nontrivial theory obtained was one where the charge superselection rule was violated. If one demands the customary symmetries, the S matrix turns out to be unity.

doubtedly have features of its own. Although this is the general idea behind the present study, this work unfortunately makes practically no contribution to the specific problem outlined so far. Actually the problem studied *is* the relationship between these three approaches to field theory, but a significant alteration (and simplification) has been made; the theories considered are to be invariant with respect to just space and time translations, and space rotations. Thus invariance is assumed with respect to the *Euclidean* group, not the Lorentz group. There are of course several advantages in considering nonrelativistic theories:

(a) Within nonrelativistic quantum field theory it is a fairly easy matter to construct examples, this is in striking contrast with the relativistic situation. Some of these examples can be discussed in detail and so yield a possibility of testing general principles and specific conjectures.

(b) The significance of bound states in a nonrelativistic theory is reasonably well understood. In relativistic theory this is a rather tenuous point, although there is in principle no difficulty in introducing stable bound states within the abstract framework (Zimmerman [19]). In nonrelativistic theories the situation is certainly much better understood (Barut and Ruei [20]).

(c) In relativistic and nonrelativistic theories the unitarity of the S matrix is expressed in much the same manner. Actually certain simplifications occur in the nonrelativistic situation (Sec. 3) which makes the unitarity conditions much easier to handle and thus more useful.

(d) There are now several proofs of the validity of the Mandelstam representation in potential scattering, provided that the potentials satisfy certain regularity conditions (Klein [21]; Blankenbecler, Goldberger, Khuri, and Treiman [22]; and Bowcock and Martin [23]). Although these proofs are of great interest, they refer to potential scattering only. Some typical field theoretic features such as the creation and annihilation of particles, are not included in the description. Part of the purpose of the theoretical ideas, developed here, was to provide theories which will be flexible enough to allow the creation or annihilation of particles, yet simple enough to be handled mathematically.

(e) It is sometimes possible within nonrelativistic quantum mechanics to establish relations between various type descriptions. Recently Martin [24] showed how the discontinuity across an unphysical cut in the complex energy plane, which is a notion typical of the dispersion theoretic approach, is directly related to the inverse Laplace transform of the potential (which obviously belongs to the Lagrangian description).

If—as seems rather likely now—the physical description of strong interaction processes can be more effectively carried out in a dispersion theoretic framework than in the framework of abstract field theory, many questions of field theoretic character remain. Many of the field theoretic notions have some, perhaps only approximate or limiting, significance. Such concepts as states represented by vectors in a Hilbert space or that of field operators probably will keep some such meaning. To guess at those interrelations can sometimes be done, more simply within the context of rather contrived theories. It is with these kinds of theories, that the present paper is concerned.

There are, of course, obvious disadvantages in using nonrelativistic considerations exclusively. Most serious perhaps is the fact that in general one can conclude very little about a relativistic theory from the behavior of a corresponding nonrelativistic one. Indeed several examples will be presented in this work, where the nonrelativistic theory leads to strikingly different results from the relativistic one. Physically one loses the important particle, antiparticle relationship in a nonrelativistic framework. This relation is of extreme importance in dispersion theory—it has no obvious counterpart nonrelativistically, again diminishing the importance of the discussion. In many nonrelativistic theories one has no vacuum polarization effects,[3] so this feature too is lost from the relativistic theory. It is perhaps well to explain why the invariance group considered in most of this paper is the Euclidean group and *not* the Galilean group. The basic reason for this choice stems from the following circumstance: in most of this paper, the interactions scattering of just one kind of scalar neutral particle is considered. If one were to demand invariance with respect to the Galilean group, one knows from the work of Bargmann [25], that in that case the number of particles is necessarily conserved. Inasmuch as one of the purposes of this paper is the development of field theoretic models for production reactions and other field theoretic processes, it would be pointless to exclude such processes by requiring Galilean invariance. If just Euclidean invariance is demanded, one retains the possibility of constructing particle nonconserving theories. On the other hand if

[3] It is possible to construct Euclidean invariant theories, which do exhibit vacuum polarization effects. An example will be given in Sec. 5, but these theories will not be discussed extensively in this paper.

a variety of particles of different masses can interact, one may still require Galilean invariance, without implication of the conservation of the total number of particles.[4] Nonrelativistic field theory has been studied in a paper by Redmond and Uretzky [26]. The methods used as well as the motivation for that study are similar to those of the present paper. But it would be well to state the basic difference in approach. These authors *assume* the Schrödinger equation (in a second quantized form) and deduce from it the asymptotic properties of certain Heisenberg field operators, which are then shown to agree with the requirements usually made in abstract field theory. In this paper, by contrast, the asymptotic properties are *assumed*; the Schrödinger equation is not. But the purpose of this paper is precisely to investigate how far these (and other) general properties specify the theory in question. Thus, the starting point for the present considerations is similar to that in the abstract theories. Additional assumptions are made as they are needed, but an attempt is made to keep the discussion as general as possible, consistent with the goal of yet obtaining specific results. In this connection it is probably worth observing that one of the mathematical complications of field theory, the existence of nonequivalent representations of the commutation relations, is common to both relativistic and nonrelativistic theories. The assumptions made about the kind of representations allowed have actually profound dynamical consequences. One of the purposes of this paper is to exhibit some of the dynamical consequences of such general (nondynamical) appearing principles.

The general philosophy underlying this paper is, therefore, to see how far one can come—just what specific results may be obtained from an appropriate adaptation of the known principles of field theory. In the course of the discussion, it became evident that there was another general assumption which could, in this nonrelativistic framework, be made in a rather natural manner. This assumption is the *persistence* of one-particle states. Physically this means that the state vector describing a state of just a single particle is independent of the time—this state persists for all times. One describes *stable* particles. The physical and *bare* particles are always the same. Even though in such theories there can be *no* vacuum polarization, no self energy effects for a single particle, if states involving more than one particle are considered, virtual (reactive) effects can occur and they do have observable consequences. As one might anticipate, certain of the mathematical features of these persistent theories are a good deal simpler than those of other theories. Still, a large class of quite complicated physical systems is included among these persistent systems, and their mathematical description is far from trivial. The number of particles need not, and in general is not, conserved in such theories. All of ordinary quantum mechanics (also the theory of potential scattering) is contained in this persistent framework.

One of the important and interesting questions in current field theory is the extent to which the analytic properties of the S matrix are determined by, or in turn determine, the dynamical character of a theory. For instance, one might attempt—within the framework of such persistent theories—to prove the Mandelstam representation. This should be easier than in general field theories—but more difficult than in the potential scattering case. Conversely, one could attempt to just add the analytic properties as independent assumptions and investigate to what extent the character of the theory is fixed by these assumptions. The main purpose of this paper is to provide a setting in which these questions may be conveniently studied. This paper is devoted to the development of a formalism for such persistent theories. One limitation should be mentioned. One of the important features of the Mandelstam representation is the fact that the amplitudes of different reactions in which a number of particles may participate are *all* described by the *same* analytic function. This becomes a trivial statement in the case of nonrelativistic potential scattering, and almost as trivial in the persistent particle theories, when the participating particles are all identical. When different kinds of particles—with the possibility of the formation of bound states between them—are involved the restrictions imposed by the Mandelstam representation are much more severe. Thus the proper type of theory to study in this connection is a many channel theory. Most of the discussion of these theories is given in a succeeding paper.

Section 2 contains a fairly leisurely discussion of the assumptions made. In Sec. 2(a) the usual assumptions are noted; Sec. 2(b) contains an extensive discussion of the persistence assumptions together with some immediate consequences.

In Sec. 3 the various elementary consequences of these assumptions are described. In 3(a) it is demonstrated that such theories do exist and some examples are given. In 3(b) the character of the two-particle states is investigated. Section 3(c) contains a number

[4] This point was clarified in an interesting discussion with Dr. M. Peshkin.

of remarks and some speculations about the general character of persistent theories.

In Sec. 4 the formal structure of these persistent theories is described. In 4(a) the reduction formulas, which are very similar to the relativistic ones, are derived. Section 4(b) contains an extensive discussion of the restrictions imposed on the τ functions (the vacuum expectations values of time ordered products) by the persistence requirement. Section 4(c) contains a derivation of additional asymptotic identities. In 4(d) the information these identities yield about the τ functions is analyzed.

The final section contains a number of somewhat disconnected comments and remarks. The fact that Lagrangian, Galilean invariant theories (for *one* particle type) necessarily conserve the number of particles is demonstrated in 5(a). The extensions of the theory to many particle types is indicated in 5(b). An example of a nonpersistent theory is described in 5(c). Some comments about unsettled or partially settled questions are made at the end.

It is perhaps of interest to note, that even though the formalism set up here was primarily intended for discussion of problems in field theory, there might well be other in fact more realistic physical situations to which the formalism is applicable. The only physical requirement is really that the entities participating in the processes can *asymptotically* be described as nonrelativistic stable entities with *well* defined masses. This would include a certain part of nuclear reaction theory as well as portions of the theory of chemical reaction.

There are other problems, especially in superconductivity, to which this formalism (by chance) might be applicable. There one deals with a Galilean invariant theory of persistent particles (electrons and phonons); the number of individual particles is not conserved. Unfortunately it does not appear that the persistence notion, which is certainly intuitively appealing, can be incorporated into a relativistic framework. In such a relativistic theory, the requirement of persistence (as formulated here) causes the theory to be free (Jost's theorem).

In nonrelativistic theories, however, the notion appears to be useful; perhaps some appropriate relativistic modification will in time be found.

2. ASSUMPTIONS

a. Usual Assumptions [5]

The following formal description is to be applicable to systems consisting of uncharged particles of mass m. These particles can interact with one another; they can scatter; in the interaction processes, particles of the same kind may be created or annihilated. In this present section it is assumed that only particles of *one* kind are involved; certain, not altogether trivial, modifications need to be made when different kinds of particles are described. (See Sec. 4.) The description given is in terms of a quantized field theory. In harmony with the program outlined in Sec. 1, the specific dynamical assumptions will be kept to a minimum. Several of the more general appearing assumptions have profound dynamical consequences —these will be noted in the sequel. The statement that the theory describes particles of mass m, is to be understood in the sense that whenever the particles are separated by large spatial distances, the particles behave as nonrelativistic particles of mass m (described by a nonrelativistic Schrödinger equation). Measurements of the momenta and the energy of these particles can be made in this asymptotic limit.

The first part of the discussion is an adaptation of the treatment of Lehman, Symanzik, and Zimmerman [4] to this nonrelativistic situation. The notation and method of exposition follows a previous discussion rather closely (Dresden [14]). In this section the assumptions usually made will be briefly stated, with a minimum of discussion and motivation. In Sec. 2(b) the special assumptions characterizing the present study will be considered in more detail. The assumptions are as follows.

I. The field in the sense of quantum field theory is described by field operators $\psi(\mathbf{x},t)$ and $\psi^\dagger(\mathbf{x},t)$. Operators will generally be written in the Heisenberg picture. These operators can act on states, written as Ψ or $|\Psi\rangle$. These states are elements (rays) in some appropriate Hilbert space. The domains of the operators are presumably the complete Hilbert space; in any case ψ and $\psi^\dagger$ shall possess domains which are everywhere dense in the Hilbert space.

II. The theory is invariant with respect to the operations of the Euclidean group,[6] i.e., with respect to space rotations, space translations, and time translations.

The invariance requirement may be expressed in

---

[5] One reason for discussing these well-known axioms once again is to give partial answers to the criticisms of Stapp [27] leveled against these axioms. Whereas the axioms are to some extent arbitrary (as all axioms are bound to be), reasonable justifications exist; they are not altogether capricious.

[6] When dealing with situations in which different kinds of particles with different masses are considered, it is possible to require invariance with respect to the Galilean group instead of the Euclidean group (Hamermesh [26]). In that case one can have particle nonconserving theories as well as invariance with respect to the Galilean group. If only one kind of particle is considered, as in the present section, one cannot have both such invariance and particle nonconservation. (See Sec. 5.)

the following manner: For every transformation G of the Euclidean group there exists a unitary transformation of the field operators such that

$$\psi(G^{-1}(\mathbf{x},t)) = U(G)\psi(\mathbf{x},t)U^{-1}(G) \tag{1a}$$

$$\psi^{\dagger}(G^{-1}(\mathbf{x},t)) = U(G)\psi^{\dagger}(\mathbf{x},t)U^{-1}(G)\ . \tag{1b}$$

When (1) is applied to the infinitesimal time and space translations, the existence of the time and space displacement operators H and P_k ($k = 1,2,3$) follows in the usual fashion. H and P_k satisfy the commutation rules

$$[P_k,\psi] = i\partial\psi/\partial x_k \tag{2a}$$

$$[H,\psi] = -i\partial\psi/\partial t\ , \tag{2b}$$

where P_k and H are Hermitian operators. Their Hermitian character follows from the unitary character of U. (From the existence of an operator U, as given by (2), no signs can be inferred. The signs in (2) are picked in such a manner, that in theories described by a Lagrangian, where specific forms for P_k and H in terms of ψ and $\psi^{\dagger}$ are available, H as defined by (2b), coincides with the energy.) It is a typical feature of the abstract field theories, that the *existence* of the operators P_k and H is guaranteed by the invariance requirement. However, no functional dependence of P_k and H on ψ and $\psi^{\dagger}$ is given. As always (2a) and (2b) may be written as[7]

$$\psi(\mathbf{x},t) = \exp\{-i[\mathbf{P}(\mathbf{x}-\mathbf{x}_0) - H(t-t_0)]\} \times \psi(\mathbf{x}_0 t_0)\exp\{i\mathbf{P}(\mathbf{x}-\mathbf{x}_0) - iH(t-t_0)\}\ . \tag{2c}$$

III. In any field theory (any quantum theory for that matter) one has to impose commutation rules between the field operators. In relativistic field theories one customarily demands local commutativity (Wightman [1], Schweber [5]); operators at points separated by a space-like interval $(\mathbf{x}\text{-}\mathbf{y})^2 - (x_0\text{-}y_0)^2 > 0$ commute. A straight forward transcription of this requirement to the present nonrelativistic situation would yield the vanishing of the equal time commutators:

$$[\psi(\mathbf{x},t),\psi(\mathbf{y},t)] = 0 \quad (\mathbf{x}-\mathbf{y})^2 > 0\ , \tag{3a}$$

$$[\psi(\mathbf{x},t),\psi^{\dagger}(\mathbf{y},t)] = 0 \quad (\mathbf{x}-\mathbf{y})^2 > 0\ . \tag{3b}$$

In the relativistic considerations nothing is said about the commutator at the apex of the light cone where $(\mathbf{x}\text{-}\mathbf{y})^2 - (x_0\text{-}y_0)^2 = 0$. This same fact is expressed in (3) by the inequality $(\mathbf{x}\text{-}\mathbf{y})^2 > 0$.

[7] Vectors in three dimensions are denoted by $\mathbf{x}$, the time is written as x_0 or t. When x is used it stands for $\mathbf{x},t$. AB stands for $\mathbf{AB} - A_0B_0$. Entities such as $(\mathbf{A},A_0)$ or AB do not possess any special transformation properties under the Euclidean group.

In this paper however, a more stringent assumption is made:

$$[\psi(\mathbf{x},t),\psi(\mathbf{y},t)] = 0\ , \tag{4a}$$

$$[\psi(\mathbf{x},t),\psi^{\dagger}(\mathbf{y},t)] = \delta(\mathbf{x}-\mathbf{y})\ . \tag{4b}$$

It is clear that (4) is different from (3) only in the value assigned to the commutator at zero separation. This modification has—as will be seen presently—profound consequences. The rationale for the assumption (4) will be discussed in Sec. 2(b). For the present it may suffice to note that all Lagrangian type field theories (such as quantum electrodynamics) have commutation rules for the field operators of the type given by (4).

IV. To formulate the asymptotic condition it is, as always, necessary to introduce a (complete) set of c number functions $f_\alpha(\mathbf{x},t)$. These functions satisfy the free particle Schrödinger equation[8]:

$$\frac{1}{i}\frac{\partial f_\alpha}{\partial t} = \frac{\Delta}{2m}f_\alpha \quad S_x f_\alpha = 0 \quad S_x \equiv \frac{1}{i}\frac{\partial}{\partial t} - \frac{\Delta}{2m}\ , \tag{5a}$$

$$-\frac{1}{i}\frac{\partial f_\alpha^*}{\partial t} = \frac{\Delta}{2m}f_\alpha^* \quad S_x^{\dagger}f_\alpha^* = 0 \times S_x^{\dagger} \equiv -\frac{1}{i}\frac{\partial}{\partial t} - \frac{\Delta}{2m}\ . \tag{5b}$$

In addition the f's satisfy the orthogonality and completeness relations:

$$\int d^3x f_\alpha(\mathbf{x},t)f_\beta^*(\mathbf{x},t) = \delta_{\alpha\beta} \tag{6a}$$

$$\sum_\alpha f_\alpha(\mathbf{x},t)f_\alpha^*(\mathbf{y},t) = \delta(\mathbf{x}-\mathbf{y})\ . \tag{6b}$$

The function G defined by

$$G(\mathbf{x}-\mathbf{y},t-t') \equiv \textstyle\sum_\alpha f_\alpha(\mathbf{x},t)f_\alpha^*(\mathbf{y},t') \tag{6c}$$

will be frequently used. There are of course many ways in which such a set of functions can be picked. It is convenient (and for strict mathematical arguments essential) that these functions shall be test functions in the sense of distribution theory (Schwartz [28]), i.e., they should be differentiable infinitely many times and vanish outside a compact set. (A special choice of such function is given by Redmond and Uretzky [26]). Using these smooth functions, one now defines an average of the field operators by[9]

$$\psi_\alpha(t) = \int d^3x f_\alpha^*(\mathbf{x},t)\psi(\mathbf{x},t)\ , \tag{7a}$$

$$\psi_\alpha^{\dagger}(t) = \int d^3x f_\alpha(\mathbf{x},t)\psi^{\dagger}(\mathbf{x},t)\ . \tag{7b}$$

[8] $\hbar$ has been put equal to unity.

[9] The location of α on ψ_α or ψ^α is determined by convenience only; it has nothing to do with co- or contravariant indices. One could define a smoothed-out operator $(\psi^{\dagger})_\alpha = \int d^3x f_\alpha^*(\mathbf{x}_1t)\psi^{\dagger}(\mathbf{x}_1t)$, which is distinct from (7b), but such an operator will never be used in this paper.

From (7a) it follows that

$$\psi(\mathbf{x},t) = \sum_\alpha \psi_\alpha(t) f_\alpha(\mathbf{x},t) \, . \tag{7c}$$

The operators $\psi_\alpha(t)$ and $\psi_\alpha^\dagger(t)$ will be referred to as "*smoothed-out*" operators. The incoming and outgoing particles, are (asymptotically in any case) free particles. Consequently, they can be described in the language of field theory in terms of the operators $\psi_{\text{in}}(\mathbf{x},t)$ and $\psi_{\text{out}}(\mathbf{x},t)$ which satisfy[10]

$$S_x \psi_{\text{in}}(\mathbf{x},t) = S_x \psi_{\text{out}}(\mathbf{x},t) = 0 \, , \tag{8a}$$

$$S_x^\dagger \psi_{\text{in}}^\dagger(\mathbf{x},t) = S_x^\dagger \psi_{\text{out}}^\dagger(\mathbf{x},t) = 0 \, , \tag{8b}$$

$$[\psi_{\text{in}}(\mathbf{x},t), \psi_{\text{in}}^\dagger(\mathbf{y},t)] = [\psi_{\text{out}}(\mathbf{x},t), \psi_{\text{out}}^\dagger(\mathbf{y},t)] = \delta(\mathbf{x} - \mathbf{y}) \, . \tag{8c}$$

Smoothed-out operators similar to (7) may also be introduced for the *in* and *out* fields. From (8) and (5) one shows immediately that

$$\partial \psi_{\text{in}}^\alpha / \partial t = \partial \psi_{\text{out}}^\alpha / \partial t = 0 \, . \tag{9}$$

The smoothed-out *in* and *out* operators are *time independent; $\psi_\alpha(t)$* as defined by (7a) is, of course, *time dependent*. The asymptotic condition (Lehman, Symanzik, and Zimmerman [4]) can now be phrased as

$$\lim_{t\to+\infty} \langle \Phi | \psi_\alpha(t) | \Psi \rangle = \langle \Phi | \psi_{\alpha\,\text{out}} | \Psi \rangle \tag{10a}$$

$$\lim_{t\to-\infty} \langle \Phi | \psi_\alpha(t) | \Psi \rangle = \langle \Phi | \psi_{\alpha,\text{in}} | \Psi \rangle \, . \tag{10b}$$

This formulation expresses in a more precise manner the qualitative physical idea mentioned previously: at large spatial separation the particles behave as *free* particles. Actually (10) refers to time limits $t \to +\infty$ and $t \to -\infty$, rather than to spatial separations. If, however, no stable new entities can be formed, the limit at infinite times (positive and negative) refers to a similar limit at infinite spatial separations. The theory is therefore in essence a scattering theory.

The postulates I–IV, all have some more or less direct connection with experimentation. They express the *assumption* that certain attributes of the field are at least in principle measurable. In this connection it is important to note that the introduction and use of the averaged (smoothed-out) operators corresponds (in some sense) to the fact that only space time averages over finite regions can be experimentally measured. Thus the use of test functions is here considered to be an essential physical feature, rather than a device which merely allows a more rigorous mathematical formulation. In quantum electrodynamics such space time averages are known to be measurable in principle (Bohr and Rosenfeld [30]). In that theory one therefore definitely needs quantities of the type $\int d^4 x \, f \, \langle \Psi | F_{\mu\nu} | \Psi \rangle$; F is an electromagnetic field operator. It is not certain that in all field theories, for instance those involving strong interactions, one can indeed measure these or similar space time averages. Hence there, the need for a description in terms of field operators is perhaps not as compelling.

The asymptotic condition IV, expresses (or is intended to express) the possibility of making measurements of momentum and energy on the free (asymptotic) particles. These particles are *physical* particles, which must be distinguished from the *bare* particles of a Lagrangian field theory. Thus the field operators are already renormalized. (It was in fact one of the original purposes of the "abstract" version of field theory to set up a formalism, which would deal exclusively with "already renormalized" field operators, physical masses, and physical charges so that no divergent quantities would enter at any stage.)

The commutation relations, or more precisely the local commutativity which they express are consistent with the physical notion of causality, but they certainly do not follow from that requirement. Consequently, one may well have grave doubts about the physical validity of the specific form of assumption III. Still if the theory has to be expressed in terms of field operators at all, some assumption of this general character appears necessary. There is now a vigorous school (Chew and Stapp [31]) which contends that the very notions of field operators, states, and the like are more of a hindrance than a help in the understanding of strong interaction phenomena. If it were indeed so that, all that in *principle* (not in practice) were measurable would be the properties of free particles—it would make sense to attempt to eliminate the "in principle" unobservable quantities; so to say the theory and experiment would only make contact in these "asymptotic" free particle observations. Just properties so obtained would be the appropriate ones to enter the theory; field operators and vacuum expectation values would at best be auxiliary entities—and certainly not essential. In electrodynamics, in any case, one can measure more than just these free particle properties—thus the field theoretic concepts there will likely play a role for sometime to come. Inasmuch as particles which interact strongly can interact electromagnetically as well, one may well need field theoretic ideas to describe *some* of the possible interactions of strongly inter-

[10] It is demonstrated in many places (Schiff [29]) that the formulation as given by (8) is precisely equivalent to a description of free particles by means of an ordinary Schrödinger equation—utilizing symmetric wave functions.

acting particles, even if the strong interaction themselves could indeed be described without recourse to these notions.

b. Persistence Assumptions

From the commutation rules (4) for the operators and the definition of the smoothed-out operators $\psi_\alpha(t)$ in (7) follow the commutation rules for the averaged operators:

$$[\psi_\alpha(t),\psi_\beta^\dagger(t)] = \delta_{\alpha\beta}\,, \tag{11a}$$

$$[\psi_\alpha(t),\psi_\beta(t)] = [\psi_\alpha^\dagger(t),\psi_\beta^\dagger(t)] = 0\,. \tag{11b}$$

It is important to note, that nothing can be said about the commutation rule of the ψ_α operators at different times $[\psi_\alpha(t),\, \psi_\beta(t')]$. The averaged operators of the *in* and *out* fields $\psi_{\alpha,\mathrm{in}}\ \psi_{\beta,\mathrm{out}}$, satisfy the same commutation rules

$$[\psi_{\alpha,\mathrm{in}},\psi_{\beta\,\mathrm{in}}^\dagger] = [\psi_{\alpha,\mathrm{out}},\psi_{\beta\,\mathrm{out}}^\dagger] = \delta_{\alpha\beta}\,. \tag{12}$$

(This follows directly from [8].) Since the $\psi_{\alpha,\mathrm{in}}$ and $\psi_{\beta,\mathrm{in}}$ are independent of the time [9], their commutation rules are known for all time in contrast, with those for $\psi_\alpha(t)$ and $\psi_\beta(t'.)$ These commutation rules suggest the definition of two operators:

$$N_{\mathrm{in}} = \sum_\alpha N_{\alpha,\mathrm{in}} \equiv \sum_\alpha \psi_{\alpha,\mathrm{in}}^\dagger \psi_{\alpha,\mathrm{in}} = \int d^3x\, \psi_{\mathrm{in}}^\dagger(\mathbf{x},t)\psi_{\mathrm{in}}(\mathbf{x},t), \tag{13}$$

$$N(t) \equiv \sum_\alpha N_\alpha(t) \equiv \sum_\alpha \psi_\alpha^\dagger(t)\psi_\alpha(t) = \int d^3x\, \psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t). \tag{14}$$

(One can obviously define N_{out} in a manner similar to (13). It is known from the work of Gårding and Wightman [32] that the commutation rules (11), still allow infinitely many inequivalent representations. One can fix the representations of the *in* and *out* operators by demanding the existence of vectors in the Hilbert space, with the property

$$\psi_{\alpha,\mathrm{in}}|0_{\mathrm{in}}\rangle = 0 \quad \text{all } \alpha\,, \tag{15a}$$

$$\psi_{\alpha,\mathrm{out}}|0_{\mathrm{out}}\rangle = 0 \quad \text{all } \alpha\,. \tag{15b}$$

Moreover, the assumption expressed by (15) also guarantees the existence of N_{in} and N_{out} defined formally by (13) as well-defined operators in the Hilbert space. (15a) and (15b) thus determine the representations of the *in* and *out* operators. It is customary to assume in addition to (15a) and (15b) that $|0_{\mathrm{in}}\rangle = |0_{\mathrm{out}}\rangle$. The in and out vacuum states are identical. This in turn guarantees that the in and out operators belong to equivalent representations—as such there exists a unitary transformation which takes the operators ψ_{in} into ψ_{out}. (This transformation is in fact accomplished by the S matrix.) In this paper a much more restrictive assumption in addition to (15) is made. It is clear that to fix the representations of the $\psi_\alpha(t)$ operators, defined by (11), one can again appeal to the analysis of Gårding and Wightman [32] or Wightman and Schweber [33]. In this case one would, in order to fix the representation of the operators $\psi_\alpha(t)$, assume the existence of a state $|0_t\rangle$, which has the property that

$$\psi_\alpha(t)|0_t\rangle = 0 \quad \text{all } \alpha\,. \tag{16}$$

(1b) together with (11) provides a unique specification of the representations of the operators $\psi_\alpha(t)$; a representation by the way, which guarantees the existence of $N(t)$ as given by (14). The assumption (16) appears physically quite reasonable; it guarantees that at any t the system possesses a well-defined vacuum state $|0_t\rangle$, as well as an operator for the total number of particles $N(t)$. In spite of this reasonable and general appearance, (16) is a very restrictive assumption which has specific dynamical consequences. It may well be that the assumption of (16) for all time is actually in conflict with the equations of motion. To recognize the dynamical implications of (16), consider a physical system, where the dynamics is given in terms of the time evolution of the operators $\psi_\alpha(t)$ (through an equation of motion, for example). This allows one to relate the operator $\psi_\alpha(t)$ to the operator $\psi_\alpha(t_0)$ at a previous time. Note that (16) guarantees the existence of a state $|0_{t_0}\rangle$, such that $\psi_\alpha(t_0)|0_{t_0}\rangle = 0$. Suppose that for the system in question[11] $\psi_\alpha(t) = \psi_\alpha(t_0) + c_\alpha(t,t_0)$ where c_α is a given c number function of t. In order for a vacuum state $|0_t\rangle$ to exist [as required by (1b)], there must exist a state such that $\psi_\alpha(t_0)|0_t\rangle = -c_\alpha(t,t_0)|0_t\rangle$, i.e., $\psi_\alpha(t_0)$ must possess an eigenstate with c_α as an eigenvalue. In order for $|0\rangle$ to be a proper vector in the Hilbert space, one must have in addition, that $\sum_\alpha |c_\alpha(t,t_0)|^2 < \infty$. (See Sec. 5, or Wightman and Schweber [33].) Inasmuch as c depends on the dynamics (in the special case discussed in Sec. 5, c_α depends on the character of the source), it is clear that in general the requirement $\sum|c_\alpha|^2 < \infty$ can not be met. Hence, the requirement (1b) excludes particular kinds of systems and interactions. This simple example is intended to illustrate the circumstance that a general requirement such as (1b) has direct and profound dynamical consequences.

Actually in this paper a further, even more restrictive assumption is made, namely, that one can

[11] An example of a Lagrangian system where this precise relation holds (fixed source scalar meson theory) is given in Sec. 5. Further details about this example are worked out there as well.

define a vacuum state which is independent of time or that $|0_t\rangle$ in (1b) is such that $(d/dt)|0_t\rangle = 0$. It is clear that this represents a further limitation of the systems describable by this theory. Vacuum polarization effects, for example, cannot be described by this scheme.

These considerations may be summarized in the fifth assumption:

V. It is assumed that there exists unique states $|0_{\rm in}\rangle$, $|0_{\rm out}\rangle$, $||0\rangle$ in the Hilbert space such that

$$\begin{aligned} \psi_{\alpha,\rm in}|0_{\rm in}\rangle &= 0 \quad \text{all } \alpha\,, \\ \psi_{\alpha,\rm out}|0_{\rm out}\rangle &= 0 \quad \text{all } \alpha\,, \\ \psi_\alpha(t)|0\rangle &= 0 \quad \text{all } \alpha \text{ and } t\,. \end{aligned} \tag{17}$$

(17) is the assumption which will be used in the sequel. It is perhaps of some interest to show that if one assumes (16) rather than (17), the existence of a time-independent vacuum state is a consequence of a particular form of the equations of motion. For if the dynamics leads to the result that

$$(\partial\psi_\alpha/\partial t)|0_t\rangle = 0 \tag{18}$$

then, one can always define a time-independent vacuum state. To show this, note that from (16) one sees that

$$(\partial\psi_\alpha/\partial t)|0_t\rangle + \psi_\alpha(\partial/\partial t)|0_t\rangle = 0\,, \tag{19a}$$

hence, using the assumption (18), it follows that

$$\psi_\alpha(t)\,(\partial/\partial t)|0_t\rangle = 0\,. \tag{19b}$$

Therefore, $\psi_\alpha(t)$ annihilates the state $\partial/\partial t|0_t\rangle$. Since one had assumed in (16) the existence of a *unique* state $|0_t\rangle$ annihilated by the $\psi_\alpha(t)$ (19b) implies that

$$(\partial/\partial t)|0_t\rangle = b_t|0_t\rangle\,. \tag{19c}$$

Here b_t is some number (not a vector, not an operator) which could depend on t. Actually b_t is purely imaginary for $b_t = \langle 0_t|(\partial/\partial t)0_t\rangle$. From the fact $\langle 0_t|0_t\rangle = 1$ one deduces that the real part of $\langle 0_t|\partial 0_t/\partial t\rangle$ equals zero. Therefore b_t may be written as $i\omega_t$. Now (19c) yields the result that for any t

$$|0_t\rangle = \exp\,\{i\int^t dt'\omega(t')\}\,|0\rangle\,.$$

Here $|0\rangle$ is a *time-independent* vector, hence the fact that $\psi_\alpha(t)|0_t\rangle = 0$, for all t, implies that $\psi_\alpha(t)|0\rangle = 0$, for all t.

Equation (18) is a useful relation inasmuch as it allows the possibility of checking the more stringent requirement (17) once the dynamics of a system is given. It is perhaps worth observing that if one *assumes* that a vacuum state $|0_t\rangle$ is invariant under a time translation, the validity of (16) at one time t implies its validity at all times. For under a time translation $t' = t + \tau$, the field operators transform according to (2c):

$$\psi(\mathbf{x},t+\tau) = U^{-1}(\tau)\psi(\mathbf{x},t)U(\tau)\,, \tag{20a}$$

while

$$U(\tau)|0_t\rangle = |0_t\rangle\,. \tag{20b}$$

(20b) is the assumed invariance of $|0_t\rangle$. Now

$$\psi(\mathbf{x},t+\tau)|0_t\rangle = U^{-1}\psi(\mathbf{x},t)|0_t\rangle = 0\,. \tag{20c}$$

Thus there exists a state $|0_t\rangle$ which is annihilated by $\psi(\mathbf{x},t+\tau)$ for all τ, hence the same vacuum state exists for all time.

It would seem reasonable that the formulation of the requirements to be imposed on the vacuum state as given by (17) and (18) is preferable over that given by (20a) and (20b), the dynamical restrictions imposed by the requirement of the existence of a vacuum can be seen more explicitly from (18).

Finally, it follows from (17) in conjunction with the asymptotic condition (10) that $|0_{\rm in}\rangle = |0\rangle = |0_{\rm out}\rangle$. Not only are the *in* and *out* vacuum states identical (this is also the case in relativistic theories) but these vacuum states are identical with the "physical vacuum." To see this observe that for any $|\Phi\rangle$

$$\langle\Phi|\psi_\alpha(t)|0\rangle = 0 = \lim_{t\to-\infty}\langle\Phi|\psi_\alpha(t)|0\rangle = \langle\Phi|\psi_{\alpha,\rm in}|0\rangle\,. \tag{21a}$$

[The first equality follows from (17), the second is obvious, the third is the asymptotic condition.] Since (21) holds for any state vector Φ, one has $\psi_{\alpha,\rm in}|0\rangle = 0$ (for all α), (21b) but from (17) $\psi_{\alpha,\rm in}|0_{\rm in}\rangle = 0$, for all α. Combining this with (21b) it follows that $|0_{\rm in}\rangle$ and $|0\rangle$ can be different by a phase at most. The same argument can be used to show the identity up to a phase of $|0\rangle$ and $|0_{\rm out}\rangle$. The identity of these vacuum states, represents of course a tremendous simplification as compared to relativistic field theories. From this identity of the three states $|0\rangle = |0_{\rm in}\rangle = |0_{\rm out},\rangle$ it also becomes evident that no vacuum polarization effects can possibly be described by this present formalism. (The vacuum state will from now on be denoted by $|0\rangle$.)

It is easy to check from (11) and (14) that the state $\psi_\alpha^\dagger(t)|0\rangle \equiv |\alpha(t)\rangle$ is an eigenstate of $N(t)$ with eigenvalue 1:

$$\begin{aligned} N(t)|\psi_\alpha^\dagger(t)|0\rangle \equiv N(t)|\alpha(t)\rangle &= |\alpha(t)\rangle \\ &= \psi_\alpha^\dagger(t)|0\rangle\,. \end{aligned} \tag{22}$$

The final perhaps most restrictive postulate to be made is now that the *one* particle states as defined by

$\psi_\alpha^\dagger(t)|0\rangle \equiv |\alpha(t)\rangle$ shall be *independent* of the time. Thus the *one* particle states are assumed to persist unchanged in the course of time. Stated more formally—the assumption made is as follows:

VI. The persistence assumption. It is explicitly assumed that the states $\psi_\alpha^\dagger(t)|0\rangle \equiv |\alpha(t)\rangle$ are independent of the time.

$$(\partial/\partial t)(\psi_\alpha^\dagger(t)|0\,\rangle = (\partial/\partial t)|\alpha(t)\,\rangle = 0\,. \qquad (23)$$

Thus in the present theory there is neither vacuum polarization nor are there self-energy effects—a vacuum stays a vacuum—and a one-particle state remains a one-particle state. It is no doubt obvious that this is just the purpose of the persistence assumptions. As a final remark it is important to note that VI and the asymptotic condition imply the identity of the one-particle in-states, the one-particle out-states, and the general one-particle states. The one-particle in-states are obtained as $\psi_{\alpha,\mathrm{in}}^\dagger|0\rangle \equiv |\alpha_{\mathrm{in}}\rangle$. $|\alpha_{\mathrm{in}}\rangle$ is obviously (9) independent of time. If Φ is an arbitrary state, one has from the asymptotic conditions

$$\lim_{t\to-\infty}\langle\Phi|\psi_\alpha^\dagger(t)|0\rangle = \langle\Phi|\psi_{\alpha,\mathrm{in}}|0\rangle = \langle\Phi|\alpha_{\mathrm{in}}\rangle\,. \qquad (24a)$$

On the other hand, from the persistence condition

$$\lim_{t\to-\infty}\langle\Phi|\psi_\alpha^\dagger(t)|0\rangle = \lim_{t\to-\infty}\langle\Phi|\alpha(t)\rangle = \langle\Phi|\alpha\rangle\,. \qquad (24b)$$

Hence one has that for all Φ,

$$\langle\Phi|\alpha_{\mathrm{in}} - \alpha\rangle = 0\,. \qquad (24c)$$

This implies the equality of $|\alpha\rangle$ and $|\alpha_{\mathrm{in}}\rangle$. The identical argument shows the identity of $|\alpha\rangle$ and $|\alpha_{\mathrm{out}}\rangle$, hence the identity of the one particle *in* and *out* states follows. Physically the identity of these *one*-particle states follows immediately from the persistence condition: if *one*-particle states remain unchanged in the course of time it is clear that *one*-particle incoming, outgoing, and intermediate states must all be the same.

The further discussion will be concerned with systems which satisfy the conditions I–VI.

3. ELEMENTARY CONSEQUENCES: MISCELLANEOUS COMMENTS

a. Examples

Before embarking upon a detailed investigation of the consequences of the axioms I–VI, it is well to check whether worthwhile physical systems can be expected to satisfy the postulates. Indeed it might appear that the assumptions made are so stringent that only very trivial systems can be described by them. This is actually not the case, quite complicated systems are included among these "persistent systems." One can show that a *sufficient* condition for the persistence axioms (V and VI) to be satisfied is that the Heisenberg field operators $\psi(\mathbf{x},t)$ satisfy an equation of motion of the form[12]

$$S\psi = \psi^\dagger F(\psi^\dagger,\psi)\psi\,. \qquad (25a)$$

It should be noted that all that is essential in (25a) is the $S\psi$ operator on the left side and the *location* of $\psi^\dagger$ and ψ on the right hand side. $F(\psi,\psi^\dagger)$ can be an arbitrary functional of ψ and $\psi^\dagger$. Actually expressions for $S\psi$ of the type

$$S_x\psi(\mathbf{x},t) = \iint d\xi d\xi'\psi^\dagger(\xi)F(\psi^\dagger,\psi)\psi(\xi')K(x-\xi-\xi'), \qquad (25b)$$

where K is a c number, possess the general character of (25a). (25a) indicates the general structure of the equation of motion. The result, that if (25a) holds, the system is persistent, is correct for all such general expressions no matter what F is. What is to be shown, is just that (25a) implies the two persistence conditions $(\partial\psi_\alpha/\partial t)|0\rangle = 0$, $(\partial/\partial t)\psi_\alpha^\dagger|0\rangle = 0$. To show this, start from the observation that $\psi_\alpha(t)|0\rangle = 0$, hence by (7c)

$$\psi(\mathbf{x},t)|0\,\rangle = \textstyle\sum_\alpha f_\alpha(\mathbf{x},t)\psi_\alpha(t)|0\,\rangle = 0\,. \qquad (26)$$

[It is important to again note the difference between the relatistic and the nonrelativistic situations. In the relativistic case the field operator $A(x)$ possesses *both* positive and negative frequencies, hence $A(x)|\Omega\rangle \neq 0$ (Ω is the vacuum)—this is different from (26).] From (7a) one sees by straight differentiation that

$$-\frac{1}{i}\frac{\partial\psi_\alpha}{\partial t} = \int d^3x f_\alpha^*\Big(-\frac{1}{i}\frac{\partial}{\partial t}+\frac{\Delta}{2m}\Big)\psi$$
$$= -\int d^3x f_\alpha^*(S_x'\psi)\,. \qquad (27a)$$

Here (5) has been used, two spatial partial integrations[13] were carried out. Similarly,

$$\frac{\partial\psi_\alpha^\dagger}{\partial t} = -i\int d^3x f_\alpha(S_x\psi)^\dagger\,. \qquad (27b)$$

It is now very straight forward to show that if the field operators ψ satisfy an equation of the general character (25), the persistence conditions (18) and (23) follow.

---

[12] S is always defined by (5a) $S_x \equiv \dfrac{1}{i}\dfrac{\partial}{\partial t} - \dfrac{\Delta}{2m}$.

[13] The f functions are supposed to be obtained in such a manner, that the out integrated parts, in the partial integration may be put equal to zero.

Indeed from (27a) combined with (25) one sees

$$\frac{\partial\psi_\alpha}{\partial t}|0\rangle = i\int d^3x f_\alpha^*(S_x\psi)|0\rangle$$
$$= i\int d^3x f_\alpha^*\psi^\dagger F(\psi^\dagger,\psi)\psi|0\rangle = 0\,.$$

The last equality follows from (26); this is the "vacuum" persistence condition (18). To show the "one particle" persistence condition (25), it is only necessary to observe that the structure of $(S\psi)^\dagger$ via (25) is identical with that of (25) $(S\psi)^\dagger = \psi^\dagger F^\dagger\psi$. Again $(S\psi)^\dagger$ has an annihilation operator ψ at the right, a creation operator $\psi^\dagger$ at the left. Thus one obtains, combining this observation with (27b),

$$\frac{\partial\psi_\alpha^\dagger}{\partial t}|0\rangle = -i\int d^3x f_\alpha(S_x\psi)^\dagger|0\rangle$$
$$= -i\int d^3x f_\alpha\psi^\dagger F^\dagger\psi|0\rangle = 0\,.$$

This indeed is the persistence condition (23). Thus it has been shown that systems possessing equations of motion of the type given by (25) always satisfy the persistence conditions. The character of the functional F is completely arbitrary. As long as the $\psi^\dagger$ operator is at the left and the ψ operator is at the right, the proof proceeds as given. Thus F could contain $\partial\psi/\partial x$ or $\partial\psi/\partial t$ terms; however F cannot contain differentiation operators which are to act on the ψ outside the F. From the fact that theories satisfying (25) are included among the persistent theories, a class of Lagrangian theories satisfying the persistence conditions may be obtained. The Lagrangian only need to be so chosen that field equations of the form (25) result. As examples one can consider first

$$L = L_0 - \tfrac{1}{2}\iint d^3x d^3y \psi^\dagger(x)\psi^\dagger(y)V(x-y)\psi(x)\psi(y)\,. \tag{28}$$

L_0 is the Lagrangian of the free Schrödinger field, the Hamiltonian is in this case given by

$$H = -\frac{1}{2m}\int\psi^\dagger\Delta\psi d^3x + \tfrac{1}{2}\iint d^3x d^3y\psi^\dagger(x)\psi^\dagger(y)V(x-y)\psi(x)\psi(y)\,. \tag{28a}$$

L and H are recognized as the well-known Lagrangian and Hamiltonian in a quantized theory for a system of particles interacting with two body forces derived from a potential V. It is easy[14] to verify that the equations of motion of the Heisenberg operators are just

$$S_x\psi = -\int d^3y\psi^\dagger(y)V(x-y)\psi(x)\psi(y)\,. \tag{29a}$$

This indeed has the form (25), hence ordinary nonrelativistic quantum mechanics is indeed a "persistent" theory.

It is clear that if three-body interactions are included, say, by using a term of the type

$$\iiint d^3x d^3y d^3z\psi^\dagger(x)\psi^\dagger(y)\psi^\dagger(z)V(xyz)\psi(x)\psi(y)\psi(z)$$

in the Lagrangian, the resulting field equations again have the general character of (25). Just a number of terms of the type

$$\iint d^3y d^3z\psi^\dagger(y)\psi^\dagger(z)V(xyz)\psi(x)\psi(y)\psi(z)$$

will be added on to the equation of motion (29a), thus retaining the required persistent character. Actually the example (28) is not a particularly interesting one. The number operator N, commutes with H. This system consists of a fixed number of interacting particles. Whereas it is legitimate to use field theoretic methods, the system could equally well be described by an N body Schrödinger equation. This is no longer the case in the next example of a persistent theory,

$$L = L_0 - \iiint d^3x d^3y d^3z \times [\psi^\dagger(x)\psi^\dagger(y)\psi^\dagger(z)V(xyz)\psi(x)\psi(y) + \psi^\dagger(x)\psi^\dagger(y)V(xyz)\psi(x)\psi(y)\psi(z)]\,. \tag{30}$$

In this case the equation of motion becomes[15]

$$\begin{aligned} S_x\psi = -\iint d^3y d^3z\{&\psi^\dagger(y)\psi^\dagger(z)V(xyz)\psi(x)\psi(y) \\ &+ \psi^\dagger(y)\psi^\dagger(z)V(yxz)\psi(y)\psi(x) \\ &+ \psi^\dagger(y)\psi^\dagger(z)V(zyx)\psi(x)\psi(y) \\ &+ \psi^\dagger(y)(V(xyz) \\ &+ V(yxz))\psi(x)\psi(y)\psi(z)\}\,. \end{aligned} \tag{30a}$$

It is quite clear that the equation of motion again is of the type given by (25) so that the physical system described by (30) is again a persistent one. It is important to note that (30) describes processes in which 3 particles may be created and 2 destroyed. Thus the N operator, will not in general commute with H [constructed from (30)]. This, therefore, is an example of a situation where there are no vacuum fluctuations, the one particle states persist (the "physical" and bare particles are the same), but the number of particles is *not* conserved.

The term added to L_0 in (30), (the identical term, will be added with a plus sign to the Hamiltonian) had to be symmetrized in order that the total Hamil-

[14] This could be done either using (2b), by just calculating $[H,\psi]$ with H given by (28b), or by writing out the Lagrange equations, which follow from (28).

[15] The easiest way to obtain these equations is to use functional differentiation.

tonian is a Hermitian operator. This is the reason that (30) is actually the simplest particle nonconserving persistent theory. A term in H of the type $\int\int \psi^\dagger\psi^\dagger V\psi$ is a particle nonconserving term alright; it would produce a term $\int \psi^\dagger V(xy)\psi(y)$ in the equation of motion, which is persistent. But the Hermitian character of the Hamiltonian requires the presence of an additional term $\int\int dxdy\psi^\dagger V\psi\psi$ in the Hamiltonian. This term in turn produces an additional term $\int dyV(xy)\psi\psi$ in the expression for $(S_x\psi)$ in the equation of motion. This clearly does not agree with the requirement (25), hence such a theory is not in general a persistent one. (It is well to note that the condition (25) is a sufficient condition, it is probably not a necessary condition although this so far is unproven.) These examples should suffice to demonstrate that Lagrangian persistent systems do exist. All nonrelativistic, many particle systems interacting via two, three, or n body potentials are included. In addition the persistent systems include nonlocal potentials, as well as systems interacting in a manner in which the number of particles in individual events is not necessarily conserved [see (30)]. Thus there exists an extensive class of persistent systems. Some of them might be of considerable physical interest.

b. Simple Properties, Anti-Jost Theorem

By assumption (VI) $\psi_\alpha^\dagger(t)|0\rangle \equiv |\alpha\rangle$ is time independent. It is important to note that $\psi^\dagger|\mathbf{x}(t)|0\rangle$ is an eigenstate of $N(t)$, but it, itself is a time dependent state. For

$$\psi\ (\mathbf{x},t)|0\rangle = \textstyle\sum_\alpha f_\alpha^*(\mathbf{x},t)\psi_\alpha^\dagger(t)|0\rangle \equiv \sum_\alpha f_\alpha^*(\mathbf{x},t)|\alpha\rangle. \quad (31)$$

Thus, $\psi^\dagger(\mathbf{x},t)|0\rangle$ is actually a superposition of one-particle states $N|\alpha\rangle = |\alpha\rangle$, with time dependent coefficients. Whereas $\psi_\alpha^\dagger|(t)|0\rangle$ is by assumption independent of the time, the state $\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)|0\rangle$ is a two-particle state, an eigenstate of the number operator $N(t)$, which depends on t_1, but not on t_2. To check this it is only necessary to observe that $\psi_\beta^\dagger(t_2)|0\rangle$ is independent of t_2, therefore, one can pick for t_2 any time value convenient in $\psi_\beta^\dagger(t_2)|0\rangle$. Therefore,

$$\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)|0\rangle = \psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_1)|0\rangle\,. \qquad (32)$$

Consequently, $N(t_1)$ $(\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)|0)$ may be directly obtained, indeed $\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)|0\rangle$ is an eigenstate of $N(t_1)$ with eigenvalue 2. It is a two-particle state. Thus $\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)|0\rangle$ depends on *one* time only; it is occasionally written as $|\alpha\beta t_1\rangle$. In a completely analogous fashion one sees that $\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)\psi_\gamma^\dagger(t_3)|0\rangle = \psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)\psi_\gamma^\dagger(t_2)|0\rangle$. No further simplification is possible; $\psi_\gamma^\dagger(t_2)|0\rangle = |\gamma\rangle$ is a one-particle state, $\psi_\beta^\dagger(t_2)\psi_\gamma^\dagger(t_2)|0\rangle = |\beta\gamma t_2\rangle$ a two-particle state at t_2. Nothing special is known about the action of $\psi_\alpha^\dagger(t_1)$ on a two-particle state. Hence, $\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)\psi_\gamma^\dagger(t_3)|0\rangle$ is not in general an eigenstate of the number operator, neither at time t_1, nor at time t_2. A state such as $\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)\psi_\gamma^\dagger(t_3)|0\rangle$ or any more complicated state, in general, is not a state with a definite number of particles. (In general means $t_1 \neq t_2 \neq t_3$.) The only general state where the number of particles is definite is a two-particle state $\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)|0\rangle$. In this connection just as $\psi^\dagger(\mathbf{x},t)|0\rangle$ is a superposition of one-particle states (31), one has

$$\begin{aligned}\psi^\dagger(\mathbf{x}_1t_1)\psi^\dagger(\mathbf{x}_2t_2)|0\rangle &= \sum_{\alpha,\beta} f_\alpha^*(\mathbf{x}_1t_1)f_\beta^*(\mathbf{x}_2t_2)\psi_\alpha^\dagger(t_1)\psi_\beta^\dagger(t_2)|0\rangle \\ &= \sum_{\alpha,\beta} f_\alpha^*(\mathbf{x}_1t_1)f_\beta^*(\mathbf{x}_2t_2)|\alpha\beta t_1\ \rangle. \qquad (33)\end{aligned}$$

Thus, $\psi^\dagger(\mathbf{x}_1t_1)\psi^\dagger(\mathbf{x}_2t_2)|0\rangle$ does depend on two times t_1,t_2; $|\alpha\beta t\rangle$, of course, depends on t_1 only. It is easy to check that $\psi^\dagger(x_1t_1)\psi^\dagger(x_2t_2)|0\rangle$ is still an eigenstate of $N(t_1)$ with eigenvalue 2.

The simple properties of the one-particle states are of course reflected in simple expressions for some of the vacuum expectation values of ordered or unordered products of field operators. It is of interest to recall the result of *Jost* proven for relativistic fields. Within the framework of relativistic field theory Jost [6] proved that if $\langle 0|A(x)A(y)|0\rangle$ equals the free field value,[16] then the theory is free. Here A is the (relativistic) field operator. The phrase, "the theory is free," means that the field operators $A(x)$ satisfy a Gorden Klein equation: $(\Box - m^2)A(x) = 0$ *and* the commutation relations between the field operators are again those of a free field of mass m. This remarkable result therefore shows that effects of interaction in a relativistic theory are necessarily present in the second Wightman function $\langle 0|A(x)A(y)|0\rangle$ (or the two point Green's function). This result is in striking contrast with the situation under discussion in this paper. Instead of the result of Jost in which the second Wightman function must show some effects of the interaction, in the *persistent* theories the second Wightman function is always the one for free fields no matter how complicated the actual theory is. This result, which might well be called the "anti-Jost theorem" is an immediate consequence of the persistence condition. First write the free Wightman function:

$$\begin{aligned}W_2^{\text{free}} &\equiv \langle 0|\psi_{\text{in}}(x)\psi_{\text{in}}^\dagger(y)|0\rangle \\ &= \textstyle\sum_\alpha f_\alpha(\mathbf{x}_1t_1)f_\beta^*(\mathbf{y}t_2)\langle 0|\psi_{\alpha,\text{in}}\psi_{\beta,\text{in}}^\dagger|0\rangle \\ &= \textstyle\sum_\alpha f_\alpha(\mathbf{x}_1t_1)f_\alpha^*(\mathbf{y}_1t_2) = G(\mathbf{x}-\mathbf{y},t_1-t_2)\,. \qquad (34)\end{aligned}$$

[16] Thus the second Wightman function is the same as the second Wightman function for a free field with mass m.

The definition (6c) has been used for the last equality. The commutation rules of the $\psi_{\alpha,\mathrm{in}}$ coupled with the definition of the vacuum (V) lead directly to (34). The calculation of the actual Wightman function proceeds in the identical manner: using the development (7c) one has

$$W_2(x,y) = \langle 0|\psi(\mathbf{x}t_1)\psi^\dagger(\mathbf{y}t_2)|0\rangle = \sum\nolimits_{\alpha,\beta} f_\alpha(\mathbf{x}_1t_1)f_\beta^*(\mathbf{y}_1t_2)\langle 0|\psi_\alpha(t_1)\psi_\beta^\dagger(t_2)|0\rangle\,. \tag{35}$$

Now $\psi_\beta^\dagger(t_2)|0\rangle$ is a time-independent state, in fact a *one*-particle *in* state (24c). Similarly $\langle 0|\psi_\alpha(t_1)| \equiv \langle\alpha_1|$. Hence the scalar product $\langle 0|\psi_\alpha(t_1)\psi_\beta^\dagger(t_2)|0\rangle = \delta_{\alpha\beta}$. This reduces the actual Wightman function $W_2(x,y)$ to the free Wightman function, as a glance at (34) will show.

There are apart from $W(1,2^\dagger) = \langle 0|1,2^\dagger|0\rangle$ actually three more twofold Wightman functions[17] $W(1,2)$, $W(1^\dagger,2)$, $W(1^\dagger,2^\dagger)$. These all vanish by virtue of the vanishing of $\psi(\mathbf{x}_1t)|0\rangle$, or $\langle 0|\psi^\dagger(\mathbf{x}_1t)|$.

There are eight threefold Wightman functions. All vanish. Some vanish by virtue of the relations just quoted. There are only two which do not vanish in this obvious manner. Consider, for example,

$$W(1,2,3^\dagger) = \langle 0|\psi(\mathbf{x}_1t_1)\psi(\mathbf{x}_2t_2)\psi^\dagger(\mathbf{x}_3t_3)|0\rangle = \sum\nolimits_{\alpha,\beta} f_\alpha(\mathbf{x}_2t_2)f_\beta^*(x_3t_3)\langle 0|\psi(\mathbf{x}_1t_1)\psi_\alpha(t_2)\psi_\beta^\dagger(t_3)|0\rangle\,. \tag{36}$$

Since $\psi_\beta^\dagger(t_3)|0\rangle$ is independent of the time, one may write the scalar product in (3b) as

$$\langle 0|\psi(\mathbf{x}_1t_1)\psi_\alpha(t_2)\psi_\beta^\dagger(t_3)|0\rangle = \langle 0|\psi(\mathbf{x}_1t_1)\psi_\alpha(t_2)\psi_\beta^\dagger(t_2)|0\rangle = \langle 0|\psi(\mathbf{x}_1t_1)|0\rangle\delta_{\alpha\beta} + \langle 0|\psi(\mathbf{x}_1t_1)\psi_\beta^\dagger(t_2)\psi_\alpha(t_2)|0\rangle = 0\,.$$

The commutation relations (11) have been used. Thus $W(1,2,3^\dagger)$ vanishes. In the same manner one shows that $W(1,2^\dagger3^\dagger)$ vanishes.

There are (2^4) or 16 fourfold Wightman functions. Some of those are again obviously equal to zero (by virtue of $\psi(\mathbf{x}_1t)|0\rangle = 0$, or $\langle 0|\psi^\dagger(\mathbf{x}_1t) = 0)$. The remaining ones are $W(1,2,3,4^\dagger)$, $W(1,2,3^\dagger,4^\dagger)$, $W(12^\dagger,34^\dagger)$, and $W(12^\dagger,3^\dagger,4^\dagger)$. Of these $W(1,2^\dagger,3^\dagger,4^\dagger)$ and $W(1,2,3,4^\dagger)$ are zero again. As proof, this last function may be written as

$$W(1,2,3,4^\dagger) = \sum\nolimits_{\alpha\beta} f_\alpha(\mathbf{x}_3t_3)f_\beta^*(\mathbf{x}_4t_4)\langle 0|\psi(\mathbf{x}_1t_1)\psi(\mathbf{x}_2t_2)\psi_\alpha(t_3)\psi_\beta^\dagger(t_4)|0\rangle\,. \tag{37}$$

[17] As stated before x, often stands for $\mathbf{x}_1x_0$. It is convenient to write functions such as $\langle 0|\psi(\mathbf{x}_1t_1)\psi^\dagger(y_2t_2)|0\rangle$ as $\langle 0|1,2^\dagger|0\rangle$. This is often done in the sequel. They are also often denoted by $W(1,2^\dagger)$. This is an obvious and convenient notation. $W(x,y^\dagger)$ is also used.

The scalar product $\langle\ \ \rangle$ in (37) may be written (using the persistence condition) as

$$\begin{aligned}\langle 0|\psi(\mathbf{x}_1t_1)\psi(\mathbf{x}_2t_2)\psi_\alpha(t_3)\psi_\beta^\dagger(t_4)|0\rangle &= \langle 0|\psi(\mathbf{x}_1t_1)\psi(\mathbf{x}_2t_2)\psi_\alpha(t_3)\psi_\beta^\dagger(t_3)|0\rangle \\ &= \langle 0|\psi(\mathbf{x}_1t_1)\psi(\mathbf{x}_2t_2)\psi_\beta^\dagger(t_3)\psi_\alpha(t_3)|0\rangle \\ &\quad + \delta_{\alpha\beta}\langle 0|\psi(\mathbf{x}_1t_1)\psi(\mathbf{x}_2t_2)|0\rangle = 0\,.\end{aligned}$$

This derivation in fact shows that any Wightman function of the form $W(\cdots lmn^\dagger) = \langle 0|\cdots lmn^\dagger|0\rangle = 0$. On the other hand a Wightman function of the type $W(|\cdots lm^\dagger n^\dagger) = \langle 0|\cdots lm^\dagger n^\dagger|0\rangle$ is in general not zero, irrespective of the number of annihilation operators which precede the $m^\dagger n^\dagger$. This originates from the fact that the application of $m^\dagger n^\dagger|0\rangle$ creates a state of the type already indicated in (33)—a time dependent mixture of two-particle states. The general character of such a W function is $\langle 0|\cdots\psi_\mu(t_\mu)\psi_\lambda^\dagger(t_\lambda)\psi_2^\dagger(t_2^\dagger)|0\rangle$. Since the commutation rules for unequal times of $\psi_\mu(t_\mu)$ and $\psi_\lambda^\dagger(t_\lambda)$ are not given, this expression can in general not be reduced any further.

The only nonzero fourfold Wightman functions are $W(1,2,3^\dagger\,4^\dagger)$ and $W(1,2^\dagger\,3\,4^\dagger)$. Using the persistence conditions in the same way as before; the following results are easily obtained:

$$W(1,2^\dagger34^\dagger) = G(\mathbf{x}_1-\mathbf{x}_2,t_1-t_2)G(\mathbf{x}_3-\mathbf{x}_4,t_3-t_4) \tag{38a}$$

$$W(1,2,3^\dagger4^\dagger) = \sum_{\alpha\beta\mu\nu} f_\alpha(\mathbf{x}_1t_1)f_\beta(\mathbf{x}_2t_2)f_\mu^*(\mathbf{x}_3t_3)f_\nu^*(\mathbf{x}_4t_4)\,\langle 0|\psi_\alpha(t_2)\psi_\beta(t_2)\psi_\mu^\dagger(t_3)\psi_\nu^\dagger(t_3)|0\rangle\,. \tag{38b}$$

This last Wightman function is the first one, which in any way reflects the character of the system. W_2 was shown to be always equal to the free-field Wightman function. All threefold vacuum expectation values vanish, $W(1,2^\dagger,3,4^\dagger)$ is a product of free-field functions. The vacuum expectation value occuring in (38b) depends only on two times t_2 and t_3. In a physical situation where a number of particles are interacting via potentials, the $W(1,2,3^\dagger\,4^\dagger)$ function would explicitly depend on those potentials.

It was pointed out on several occasions, that nothing is known about the commutation rules of the operators $\psi_\alpha(t)$ and $\psi_\beta(t^1)$, or $\psi_\beta^\dagger(t^1)$. It was this fact which precluded a further reduction or simplification of the fourfold Wightman function, for example. Actually something is known about these commutators—but it is of a rather negative character: If the commutators of $\psi_\alpha(t)$, $\psi_\beta(t^1)$, $\psi_\beta^\dagger(t^1)$ are c numbers, the theory described by such operators is necessarily a theory of noninteracting particles, a free theory. Thus the inability to commute the destruction operator

$\psi_\beta(t_2)$ in (38b) with $\psi_\mu^\dagger(t_3)$ and $\psi_\nu^\dagger(t_3)$ is not merely a lack of knowledge. If equations of motion were given, one could in principle perform such exchanges. However, the theorem just stated, asserts that in this process new operators would appear; only if the theory were a free theory would just c numbers occur in this commutation process. This circumstance makes an explicit evaluation of Wightman functions easy in that case. In the cases of interest however, this simplifiction can not possibly occur. The proof[18] of the result just announced is conveniently split into two (simple) parts: (1) If the operators $\psi_\alpha(t)$, $\psi_\beta(t')$, $\psi_\beta^\dagger(t')$ satisfy c number commutation rules, then these c numbers are the same as those giving the commutators of free fields. Formally, if

$$[\psi_\alpha(t),\psi_\beta(t')] = C_{\alpha\beta}(t,t') \,, \tag{39a}$$

$$[\psi_\alpha(t),\psi_\beta^\dagger(t')] = C'_{\alpha\beta}(t,t') \tag{39b}$$

with C and C' c numbers, then

$$C_{\alpha\beta}(t_1t') = 0 \,,\, C'_{\alpha\beta}(t_1t') = \delta_{\alpha\beta} \,. \tag{39c}$$

Proof. It follows directly from (39a) and from the fact that $C_{\alpha\beta}$ is a c number that

$$\langle 0|[\psi_\alpha(t),\psi_\beta(t')]|0\rangle = C_{\alpha\beta}(t_1t') \,.$$

But it is clear that

$$\langle 0|[\psi_\alpha(t),\psi_\beta(t')]|0\rangle = \langle 0|(\psi_\alpha(t)\psi_\beta(t') - \psi_\beta(t')\psi_\alpha(t)|0\rangle = 0 \,.$$

(17) has been used. This shows that $C_{\alpha\beta} = 0$. From (39b) and the fact that $C'_{\alpha\beta}(t,t')$ is a c number one sees that

$$\langle 0|[\psi_\alpha(t),\psi_\beta^\dagger(t')]|0\rangle = C'_{\alpha\beta}(t,t') = \langle 0|\psi_\alpha(t)\psi_\beta^\dagger(t')|0\rangle \,. \tag{39d}$$

One can now again apply the persistence condition; $\psi_\beta^\dagger(t')|0\rangle$ in (39d) is a *one*-particle state independent of t'. Thus one may pick $t' = t$ in (39d). An appeal to (11a) shows immediately that $C'_{\alpha\beta}(t,t') = \delta_{\alpha\beta}$. Hence if the commutation rules are c numbers one has necessarily that

$$[\psi_\alpha(t),\psi_\beta(t')] = 0 \quad \text{for all t and t}' \,, \tag{40a}$$

$$[\psi_\alpha(t),\psi_\beta^\dagger(t')] = \delta_{\alpha\beta} \quad \text{for all t and t}' \,. \tag{40b}$$

Next one shows that (40) implies that $\partial\psi_\alpha/\partial t = 0$. To show this it is best to invoke the asymptotic condition. From (40a) one sees that

$$\lim_{t'\to-\infty} \langle \Phi[\psi_\alpha(t),\psi_\beta(t')]\Psi\rangle = 0 \text{ for all } \Phi \text{ and } \Psi \,. \tag{41a}$$

[18] It probably goes without saying, that the basic postulated I–VI of a persistent theory, are all *made* in this case.

Applying the asymptotic condition (10b) one obtains

$$[\psi_\alpha(t),\psi_{\beta,\text{in}}] = 0 \,. \tag{42a}$$

A similar relation is obtained from (40b)

$$[\psi_\alpha(t),\psi_{\beta,\text{in}}^\dagger] = \delta_{\alpha\beta} \,. \tag{42b}$$

Now differentiate both (42a) and (42b) with respect to t, this yields

$$[\partial\psi_\alpha/\partial t,\psi_{\beta,\text{in}}] = 0 \quad \text{for all } \beta \,,$$
$$[\partial\psi_\alpha/\partial t,\psi_{\beta,\text{in}}^\dagger] = 0 \quad \text{for all } \beta \,.$$

Thus $\partial\psi_\alpha/\partial t$ commutes with *all* operators $\psi_{\beta,\text{in}}$ and $\psi_{\beta,\text{in}}^\dagger$. Therefore $\partial\psi_\alpha/\partial t$ is a c number, call it C''. Since $\partial\psi_\alpha/\partial t$ is a c number: $\langle 0|\partial\psi_\alpha/\partial t|0\rangle = C'' = \partial/\partial t \langle 0|\psi_\alpha|0\rangle = 0$. Hence it has been demonstrated that if the commutation rules of $\psi_\alpha(t)$, $\psi_\beta(t')$, $\psi_\beta^\dagger(t')$ are c numbers, $\partial\psi_\alpha/\partial t$ and also $\partial\psi_\alpha^\dagger/\partial t$ (the proof follows the identical pattern) vanish; both ψ_α and $\psi_\alpha^\dagger$ are independent of the time. To complete the proof that the ψ's indeed do describe a free field, write the expansion $\psi(\mathbf{x},t) = \sum_\alpha \psi_\alpha f_\alpha(\mathbf{x},t)$. Since ψ_α is independent of t, one has $S_x\psi(\mathbf{x},t) = 0$. [(5a) has been used.] Thus, ψ satisfies the field free equation; the free particle commutation rules are guaranteed by (40). This completes the proof of the q number character of the commutators, for any nontrivial (i.e., nonfree) theory.

c. Remarks and Speculations

The differences between a relativistic theory and a nonrelativistic persistent theory are of course immense. This difference starts in the very beginning of the development. It was possible in the nonrelativistic theory to define a time dependent number operator $N(t) = \sum_\alpha \psi_\alpha^\dagger(t)\psi_\alpha(t)$ (14). The eigenvalues of this operator are 0, 1, $\cdots$, etc. In the relativistic theory such operators can be defined for the asymptotic fields (the in and out fields) but not for the interpolating fields. The possibility of defining and using the $N(t)$ operator, in turn depended on the *assumed* equal time commutation rule (4): $[\psi(\mathbf{x},t),\psi^\dagger(\mathbf{y},t)] = \delta(\mathbf{x}-\mathbf{y})$. This is distinct from the less restrictive commutation rule (3) which just states that

$$[\psi(\mathbf{x}_1t),\psi^\dagger(\mathbf{y}_1t)] = 0 \quad \text{for } \mathbf{x} \neq \mathbf{y} \,. \tag{43}$$

This commutation rule is actually the proper transcription to the nonrelativistic situation of the vanishing of the commutator for (nonzero) space-like separations in the relativistic case. In principle nothing is stated in these relativistic situations, about the behavior of these commutators at the point $\mathbf{x} = \mathbf{y}$. It is known from the examples in Lagrangian field theories, that singularities occur at those points. Even though the statement $[\psi(\mathbf{x},t),\psi^\dagger(\mathbf{y},t)] = 0$ for $\mathbf{x} \neq \mathbf{y}$, it does not rigorously imply the commutation rule $[\psi(\mathbf{x},t),\psi^\dagger(\mathbf{y},t)] = \delta(\mathbf{x}-\mathbf{y})$ [it was already mentioned previously that (4) is a more stringent asymptotic than (31)]; still within the general framework of field theory assumed here, this is "practically" the case. If one assumes that the commutator $[\psi(\mathbf{x},t),\psi^\dagger(\mathbf{y},t)]$ is a distribution (this is clearly a restrictive assumption), which is zero for $\mathbf{x} \neq \mathbf{y}$, a theorem of Schwartz [28] may be invoked. This theorem reference [28], [Vol. I, p. 99, Theorem XXXV] asserts that if a distribution is nonzero at $x = 0$ *only*, it may uniquely be written as a *finite* sum of δ distributions and derivatives thereof. Using this theorem it follows that

$$[\psi(\mathbf{x}_1 t),\, \psi^\dagger(\mathbf{y}_1 t)] = \sum_{n=0}^{N} c_n(t)\delta^n(\mathbf{x}-\mathbf{y})\ . \quad (44)$$

Here $\delta^{(n)}$ stands for the nth spatial derivative. Written more explicitly (44) reads[19]:

$$[\psi(\mathbf{x},t),\psi^\dagger(\mathbf{y},t)] = c_o\delta(\mathbf{x}-\mathbf{y}) + c_i(\partial/\partial x_i)\delta(\mathbf{x}-\mathbf{y}) + c_{ij}(\partial^2/\partial x_i\partial x_j)\delta(\mathbf{x}-\mathbf{y}) + \cdots\ . \quad (44a)$$

In (44a) the c's can depend on t, they could in principle be operators. If it is assumed that the equal time commutators are c numbers, the coefficients in (44a) have to be c numbers as well. On this basis, one can again calculate the commutation relations of the averaged operators ψ_α and $\psi_\beta^\dagger$. Instead of the previously obtained result (11a) one now obtains[20]:

$$[\psi_\alpha(t),\psi_\beta^\dagger(t)] \equiv \Gamma_{\alpha\beta} = \delta_{\alpha\beta} + c_i\Gamma^i_{\alpha\beta} + c_{ij}\Gamma^{ij}_{\alpha\beta} + \cdots\ . \quad (44b)$$

Here the Γ's are ordinary numbers, which depend on time and on the choice of the set f.

$$\Gamma^i_{\alpha\beta} = \int d^3x f^*_\alpha(\mathbf{x}_1 t)\frac{\partial f_\beta}{\partial x_i}\ , \quad (44c)$$

$$\Gamma^{ij}_{\alpha\beta} = \int d^3x f^*_\alpha(\mathbf{x}_1 t)\frac{\partial^2 f_\beta}{\partial x_i \partial x_j}\ . \quad (44d)$$

It is clear that $\Gamma_{\alpha\beta}$ defined in (44b) is again a c number if the coefficients are c numbers.

It is important to observe that the commutation rule (44b) for the averaged field operators $\psi_\alpha(t)$, again allows the construction of a number operator $N_\alpha(t) = \psi^\dagger_\alpha(t)\psi_\alpha(t)$.

In the special case in which one would pick plane waves for the f_α (a choice which mathematically is not really allowed), the Γ's would all be diagonal and one would obtain

$$[\psi_\alpha(t),\psi_\beta^\dagger(t)] = \delta_{\alpha\beta}\Gamma^o_\alpha \equiv \delta_{\alpha\beta}|1 + K_{\alpha,i}C_i - K_{\alpha,i}K_{\alpha,j}C_{ij} + \cdots\ . \quad (44e)$$

Here $K_{\alpha,i}$ is the ith component of the α wave vector

$$f_\alpha(x,t) = e^{i(K_{i\alpha}x_i - k_{o,\alpha}t)}\ . \quad (44f)$$

In this special case the eigenvalues of $N_\alpha(t)$ are integer multiples of $\Gamma^{(o)}_\alpha$. Since N_α and N_β commute in this same special case, the eigenvalues of N are given by $\sum n_\alpha\Gamma^o_\alpha$. This expression depends explicitly on the coefficients c, which characterize the commutator. In the more general situation where the Γ's are not diagonal, but still c numbers, one can make a similar argument. Presumably the $\Gamma_{\alpha\beta}$ as a matrix can be diagonalized by a unitary matrix, so that $R\Gamma R^{-1} = D$ diagonal. In that case, $\psi' = R\psi$ and $\psi^{\dagger\prime} = \psi^\dagger R^{-1}$ satisfy the commutation rules:

$$[\psi'_\alpha(t),\psi^{\dagger\prime}_\beta(t)] = \delta_{\alpha\beta}D_{\alpha\alpha}\ . \quad (45)$$

Hence, this more general situation can be described in exactly the same manner as the previous one. The question of real physical interest is, of course, whether theories built on the commutation relations (44e) have any different content from those based on the usual commutation relations (4). It appears that the constants c would play an important role in such a theory. It is tempting to speculate on the possibility of introducing interactions in a theory through a commutation rule of the type (44a) or (44b). The constants c would then play the role of coupling constants. To investigate whether or not interactions could be so introduced, it is imperative to keep the free field equations for the Heisenberg operators. Otherwise one would mix the effect of the presence of interactions in the equations of motion and the alteration of the commutation rules. Actually it would be interesting if one could introduce or simulate interactions in this manner, while keeping the free field equations. Whether or not it is possible depends of course on what other assumptions are made.[21] If one, for example, assumes that the ψ operator satisfies a free field equation, as well as an asymptotic condition, where the asymptotic fields ψ_{in} and ψ_{out} satisfy the *usual* commutation rules; then if the field ψ itself satisfies c number equal time commutation rules, these commutation rules must indeed be the usual ones. This is easily shown by noting that $S_\alpha\psi = 0$ implies immediately that $\psi_\alpha(t)$ the averaged field is independent of time. Then using the asymp-

[19] The usual summation convention is used. ($i = 1,2,3.$)

[20] If one assumes that the coefficients are c numbers, one may put $c_o = 1$ in (44a). This has been done in (44b).

[21] The authors are grateful to Professor Fritz Coester for interesting discussions regarding this point.

totic condition in the manner indicated on several previous occasions one sees that $\Gamma_{\alpha\beta} = \delta_{\alpha\beta}$. Thus the free field equations, the asymptotic condition, and (44) are not compatible. In a sense what one would like is a time dependence in the c's in (44a) such that $C_o(t) \to 1$, as $t \to \pm\infty$, while $C_n(t) \to 0$ as $t \to \pm\infty$. This might appear to be a way to express the fact that these coefficients in some sense describe the interaction. Actually this too is impossible as long as one insists on c number coefficients c. For one has, using the invariance condition (2c), that

$$[\psi(\mathbf{x}_1 t), \psi^\dagger(\mathbf{y}_1 t)] = e^{iH(t-t0)}[\psi(\mathbf{x}_1 t_0)\psi^\dagger(\mathbf{y}_1 t_0)]e^{-iH(t-t_0)}$$
$$= \sum_n C_n(t_0)\delta^n(\mathbf{x}-\mathbf{y}) \qquad (46)$$

$$\sum C_n(t)\delta^n(\mathbf{x}-\mathbf{y}) = \sum C_n(t_0)\delta^n(\mathbf{x}-\mathbf{y})\,. \qquad (46a)$$

(46a) is to be valid for all t and t_0. Hence $C_n(t)$ must be independent of t. Thus one cannot arbitrarily prescribe a time dependence of the coefficients $C_n(t)$. These considerations show that whether or not theories based on (44) can be constructed depends crucially on the other assumptions made. The fact that in such theories with altered commutation rules, one still can define number operators, points to the possibility of introducing some kind of persistence condition. This would in a way be a curious kind of condition. Referring back to (45) one would expect that $(\psi_\alpha')^\dagger|0\rangle$ is a time independent state. $\psi_\alpha'^\dagger$ is actually an infinite combination of the operators $\psi_\beta^\dagger R_{\beta\alpha}^\dagger$ corresponding to the degrees of freedom β of the field. It might be amusing to investigate such theories, although it is not evident that any one of these notions and ideas can be readily generalized to relativistic situations.

4. FORMAL RESULTS

a. Reduction Formulas

It is reasonable to anticipate that the special assumptions made about the vacuum and one-particle states will lead to simplifications in the formalism. Some special instances of this general situation, the vanishing of the threefold Wightman functions; the anti-Jost theorem, were noted before. The results to be obtained in this section are mathematical consequences of the assumptions I–VI; the *states* which are employed in this theory are many particle *in* and *out* states

$$\Phi_{\text{in}}^{(\alpha)} \equiv \Phi_{\text{in}}^{\alpha_1\cdots\alpha_k} \equiv |\alpha_1 \cdots \alpha_k \text{ in}\rangle$$
$$= (m_1!\cdots)^{-\frac{1}{2}}(\psi_{\text{in}}^{\alpha 1})^\dagger \cdots (\psi_{\text{in}}^{\alpha_k})^\dagger|0\rangle\,, \qquad (47a)$$

$$\Phi_{\text{out}}^{(\beta)} \equiv \Phi_{\text{out}}^{\beta_1\cdots\beta_l} \equiv |\beta_1 \cdots \beta_l \text{ out}\rangle$$
$$= (m_1!\cdots)^{-\frac{1}{2}}(\psi_{\text{out}}^{\beta_1})^\dagger \cdots (\psi_{\text{out}}^{\beta_l})^\dagger|0\rangle\,. \qquad (47b)$$

These states are time independent. They are eigenstates of the number operators N_{in} and N_{out}. The m_1 in (47a) indicates the number of indices among the $\alpha_1 \cdots \alpha_k$ equal to α_1. This factor guarantees the normalization of $\Phi_{\text{in}}^{(\alpha)}$ if the vacuum states are assumed to be normalized. One could in principle deal with states of the type

$$\Phi^{\alpha_1}(t) \equiv (m_1!\cdots)^{-1/2}[\psi^{\alpha_1}(t)]^\dagger \cdots [\psi^{\alpha_k}(t)]^\dagger|0\rangle\,. \qquad (47c)$$

Such a state would be dependent on the time, at time t it is an eigenstate of $N(t)$, in general it would *not* be an eigenstate of N at any other time. For the further discussion the *reduction formulas* are of paramount importance. The derivation of such formulas for the present theory is practically identical with the derivation in the relativistic situation. (Actually it is simpler here.) Thus, just the results are given here.[22]

$$\langle 0|T(1\cdots n)|\Phi_{\text{in}}^{\alpha}\rangle = i\int d^4\xi f_\alpha(\xi)S_\xi^\dagger\tau(1\cdots n,\xi^\dagger)\,, \qquad (48a)$$

$$\langle 0|T(1\cdots n)|\Phi_{\text{in}}^{\alpha_1\cdot\alpha_k}\rangle$$
$$= i^k\int\cdots\int d^4\xi_1\cdots d^4\xi_k f_{\alpha_1}(\xi_1)\cdots f_{\alpha_k}(\xi_k)$$
$$\times S_{\xi_1}^\dagger \cdots S_{\xi_k}^\dagger\tau(1\cdots n,\xi_1^\dagger\cdots\xi_k^\dagger) \qquad (48b)$$

$$\langle\Phi_{\text{out}}^{\beta_1\cdots\beta_l}|T(1\cdots n)|0\rangle$$
$$= i^l\int\cdots\int d^4\eta_1\cdots d^4\eta_l f_{\beta_1}^*(\eta_1)\cdots f_{\beta_l}^*(\eta_l)$$
$$\times S\eta_1\cdots S\eta_l\tau(\eta_1\cdots\eta_l,1\cdots n)\,, \qquad (48c)$$

$$\langle\Phi_{\text{out}}^{\beta_1-\beta_l}|T(1\cdots n)|\Phi_{\text{in}}^{\alpha_1-\alpha_k}\rangle$$
$$= i^{k+l}\int\cdots\int d^4\eta_1 d^4\eta_l d\xi_1\cdots d^4\xi_k$$
$$\times f_{\alpha_1}(\xi_1)\cdots f_{\alpha_k}(\xi_k)f_{\beta_1}^*(\eta_1)\cdots f_{\beta_1}^*(\eta_l)$$
$$\times S\eta_1\cdots S\eta_l\cdot S_{\xi_1}^\dagger\cdots S_{\xi_k}^\dagger\tau$$
$$\times(\eta_1\cdots\eta_l,1\cdots n,\xi_1^\dagger\cdots\xi_k^\dagger). \qquad (48d)$$

In these reduction formulas the meaning of the symbols is the same as it was before.[23] $T(l\cdots n)$ is an abbreviation for the time ordered product of n operators. (The earliest one in time stands farthest to the right in the product.) The $l\cdots n$ stands for the space time coordinates of the Heisenberg field operators. $\tau(l\cdots n)$ is the vacuum expectation value of the time ordered product:

$$\tau(1\cdots n) \equiv \langle 0|T(1\cdots n)|0\rangle$$
$$\equiv \langle 0|T(\psi(\mathbf{x}_1 t_1)\cdots\psi(\mathbf{x}_n t_n)|0\rangle\,. \qquad (49a)$$

---

[22] In the identities derived later, some of the detailed mechanics, used in deriving the reduction formulas are presented. These same methods applied to the present formulas would directly yield the quoted results.

[23] It should be noted that the integrals are taken over space time regions; $d^4x = d^3x dx_0$ in spite of the nonrelativistic character of the theory.

It is convenient to indicate the creation or destruction character of an operator in the vacuum expectation value of a product by a ξ or $\xi^\dagger$, appended to the variable which locates the space time point. This is done in (48). Thus in (48d) all the $\xi_1 \cdots \xi_k$ in the τ function refer to creation operators—the $\eta_1 \cdots \eta_l$ all refer to annihilation operators. Nothing (in general) need be said about the operators $l \cdots n$, occurring in this same τ. (48d) is valid no matter what kind operators these $l \cdots n$ are. But the character of the ξ and η has to be as indicated by the notation. The S matrix which is (as always) the overlap of the *in* and *out* states is given by

$$\begin{aligned} S_{(\beta)(\alpha)} &= \langle \Phi_{\text{out}}^{\beta_1 \cdots \beta_l} | \Phi_{\text{in}}^{\alpha_1 \cdots \alpha_k} \rangle \\ &= i^{k+l} \int \cdots \int d^4\xi_1 \cdots d^4\xi_k d^4\eta_1 \cdots d^4\eta_l f_{\alpha_1}(\xi_1) \\ &\quad \cdots f_{\alpha_k}(\xi_k) f_{\beta_1}^*(\eta_1) \cdots f_{\beta_l}^*(\eta_l). \\ &\quad S_{\eta_1} \cdots S_{\eta_l} S_{\xi_1}^\dagger \cdots S_{\xi_k}^\dagger \tau(\eta_1 \cdots \eta_l, \xi_1^\dagger \cdots \xi_k^\dagger) . \end{aligned} \tag{49b}$$

From these results one can observe that the expression of the S matrix in terms of τ functions is apart from trivial modifications, the same as that in the relativistic theory. Hence as far as these formal expressions are concerned the persistence assumptions do not produce any simplifications.

b. τ System

In the relativistic field theories, one can utilize the unitarity of the S matrix (or the assumed completeness of the set of *in* and *out* states) to obtain a set of nonlinear integro-differential equations for the τ functions. Similar procedures can be used here. It will be seen that in this case the persistence conditions give rise to substantial simplifications. To derive this system of equations, start from the operator identity[24]

$$\begin{aligned} T(1 \cdots n, 1^\dagger \cdots m^\dagger) &= \sum_{i^\dagger=1}^{m} \theta(1 - i^\dagger) \cdots \theta(m^\dagger - i^\dagger) \\ &\quad \times T(1 \cdots n, 1^\dagger \cdots i^{\dagger\star} m^\dagger) \psi^\dagger(i^\dagger) \\ &\quad + \sum_{i=1}^{n} \theta(1 - i) \cdots \theta(m^\dagger - i) \\ &\quad \times T(1 \cdots i^\star \cdots n, 1^\dagger \cdots m^\dagger) \psi(i) . \end{aligned} \tag{50}$$

The notation $T(l \cdots n, l^\dagger \cdots i^{\dagger *} \cdots m^\dagger)$ means that the operator $\psi^\dagger(i)$ is to be omitted from the product. Now take the vacuum expectation value of (50). Since $\psi(i)|0\rangle = 0$, the second term does not contribute, and one obtains

$$\begin{aligned} \tau(1 \cdots n, 1^\dagger \cdots m^\dagger) &= \sum_{i^\dagger=1}^{m} \theta(1 - i^\dagger) \cdots \theta(m^\dagger - i^\dagger) \\ &\quad \times \langle 0 | T(1 \cdots n, 1^\dagger \cdots i^{\dagger\star} \cdots m^\dagger) \psi^\dagger/(i^\dagger) | 0 \rangle \\ &= \sum_{i^\dagger=1}^{m} \theta(1 - i^\dagger) \cdots \theta(m^\dagger - i^\dagger) \\ &\quad \times \sum_{\alpha_{\text{in}}} \langle 0 | T(1 \cdots n, 1^\dagger \cdots i^{\dagger\star} m^\dagger | \alpha_{\text{in}} \rangle \langle \alpha_{\text{in}} | \psi^\dagger(i^\dagger) | 0 \rangle . \end{aligned} \tag{51}$$

The second equality in (51) follows in the usual manner by inserting a complete set of states in the matrix element. The complete set picked is the set of *in* states. Thus $|\alpha_{\text{in}}\rangle$ stands for the *one-*, *two-*, and n-particle *in* states. The sum in (51) is over all those states. It is known however, that because of the persistence conditions $\psi^\dagger(i^\dagger)|0\rangle$ is a linear combination of *one*-particle states [see (31) and (24c)]. The one-particle states are one-particle *in* states.

$$\psi^\dagger(i^\dagger)|0\rangle = \textstyle\sum_\beta f_\beta^*(\mathbf{x}_1 t_1)|\beta\rangle = \sum_\beta f_\beta^*(\mathbf{x}_i t_i)|\beta_{\text{in}}\rangle . \tag{52}$$

In (51) the expression $\langle \alpha_{\text{in}} \psi^\dagger | (i)^\dagger | 0 \rangle$ occurred. Since the *one*-particle *in* states are orthogonal to *in* states having a different number of particles; it follows using (52) that this matrix element is zero, unless $|\alpha_{\text{in}}\rangle$ refers to a *one*-particle *in* state. The summation $\sum_{\alpha\text{in}}$ which in fact stands for $\sum_k (\sum_{\alpha_1} \cdots \sum_{\alpha_k})$ therefore, reduces to a summation over the *one*-particle states. Thus (51) becomes [using (52)]

$$\begin{aligned} \tau(1 \cdots n, 1^\dagger \cdots m^\dagger) &= \sum_{i^\dagger=1}^{m} \theta(1 - i^\dagger) \cdots \theta(m^\dagger - i^\dagger) \\ &\quad \times \sum_{\alpha} \langle 0 | T(1 - n, 1^\dagger \cdots i^{\dagger\star} m^\dagger) | \alpha \rangle f_\alpha^*(x_i) . \end{aligned}$$

Now one obtains using the reduction formula (48a) and the definition of the function G:

$$\begin{aligned} \tau(1 \cdots n, 1^\dagger \cdots m^\dagger) &= i \sum_{i^\dagger=1}^{m} \sum_{\alpha} \theta(1 - i^\dagger) \cdots \theta(m^\dagger - i^\dagger) \\ &\quad \int d^4\xi G(\xi - x_i) \\ &\quad \times S_\xi^\dagger \tau(1 \cdots n, 1^\dagger \cdots i^{\dagger\star} \cdots m^\dagger, \xi^\dagger) . \end{aligned} \tag{53}$$

It is clear that (53) is a linear relation. In addition, (53) is a relation which involves only τ functions of a *given* order, on the left side $\tau(1 \cdots n, 1^\dagger \cdots m^\dagger)$; it appears that on the right side, the operator $\psi^\dagger(x_i)$ is to be omitted, but $\psi^\dagger(\xi)$ is to be included in the τ function, so that indeed the right side involves the same type τ function again. Thus (53) plays the role of an identity which has to be satisfied in a persistent theory. The structure of (53) indicates that the

---

[24] $\theta(t) = 0$ if $t \langle 0$; $\theta(t) = 1$ if $t \rangle 0$; $\theta(t) = 1/2$ if $t = 0$.

variables $1 \cdots n$ play no role in this particular identity, they just enter as parameters. On the other hand, integrals and sums have to be carried out over the ξ variables. There exists an identity in which the roles of the creation and destruction operators are reversed. This identity is obtained in a completely analogous fashion. Instead of starting from (50) one starts from the identity:

$$T(1 \cdots n,1^\dagger \cdots m^\dagger) = \sum \theta(i-1) \cdots \theta(i-m^\dagger)$$
$$\psi(i)T(1 \cdots i^\star \cdots n,1^\dagger \cdots m^\dagger)$$
$$+\sum \theta(i^\dagger - 1) \cdots \theta(i^\dagger - m^\dagger)$$
$$\times \psi^\dagger(i)T(1 \cdots n,1^\dagger \cdots i^{\dagger\star} \cdots m^\dagger) . \qquad (54)$$

The procedure already described in deriving (53) is now imitated. When inserting a complete set of states as in (51) a set of *out* states is used. This will require the use of the reduction formulas, in the form (48c). Putting this together the counterpart of (53) is obtained:

$$\tau(1 \cdots n,1^\dagger \cdots m^\dagger) = i\sum \theta(i-1) \cdots \theta(i-m^\dagger)$$
$$\times \int d\eta G(x_i - \eta) S_\eta \tau(\eta,1 \cdots i^\star - n,1^\dagger \cdots m^\dagger) . \qquad (55)$$

Equations (53) and (55) are direct consequences of the persistence conditions. They are integral relations, valid for all the functions in a persistent theory.

In the simplest case ($n = l$, $m = l$) these identities become[25]

$$\tau(x,\xi) = i\theta(x-\xi)\int d^4\eta G(\eta - \xi) S_\eta^\dagger \tau(x,\eta) , \qquad (56a)$$

$$\tau(x,\xi) = i\theta(x-\xi)\int d^4y G(x-y) S_y \tau(x,\xi) . \qquad (56b)$$

It is already known that in any persistent theory, the twofold Wightman functions are those of a free field (35a). In a persistent theory

$$\tau(x,\xi) = \langle 0 | T(\psi(x)\psi^\dagger(\xi))|0\rangle ,$$
$$= \theta(x-\xi)G(x-\xi) , \qquad (57)$$

(35a) has been used. Since the identities (56) have to be satisfied in a persistent theory it follows that $\tau(x,\xi)$ as given by (57) should satisfy the relations (56) identically. It is easy to check that this indeed is the case. A more interesting and as yet unanswered question is just what conditions have to be added to the equations (50a) and (50b) so as to make the solution of these equations unique.

The actual content of the relations (53) and (55) will be studied in the next section in conjunction with another set of identities the "asymptotic identities."

Note added in proof. The authors have profited from discussions with Dr. Edwards. The application of Zimmerman's method to this nonrelativistic situation was given independently by Dr. Edwards.

c. Asymptotic Identities

It is clear from the way in which they were derived, that the identities (53) and (55) depend in an essential manner on the persistence assumptions. In the relativistic theory, we would therefore not expect identities of this type to occur. It is for that reason, extremely interesting to observe, that Zimmerman [34] within the context of the relativistic theory has obtained relations, which although not identical with the relations (53) and (55), have a remarkably similar structure. This is somewhat surprising, since the identity of the states $\psi_\alpha^\dagger(t)|0\rangle$ and $\psi_{\alpha,\mathrm{in}}^\dagger|0\rangle$ (which itself is a direct consequence of the persistence condition and which is *not* true in relativistic theories) was the crucial step which allowed the simplification of the nonlinear coupled τ system, to a linear relation involving only one τ function at a time. The derivation of the new relations given here, follows Zimmerman's procedure in a general way. Since his proofs were somewhat abbreviated, and the present situation is genuinely different, the necessary formulas will be derived here. It is important to state explicitly that the derivation of these identities *does not* in any way depend on the persistence conditions. The results obtained here (with obvious modifications) apply to relativistic theories as well. In order to distinguish between these two types of identities, (53) and (55) will be referred to as "persistent" identities, while the ones under discussion now will be called the "asymptotic" or the "Zimmerman" identities.

It is convenient to define in addition to the function G, given by (6c), two auxiliary functions[26]:

$$G_R(x) = \theta(x)G(x) \qquad G_R(x) = 0 \quad x_0 < 0 , \qquad (58a)$$

$$G_A(x) = -\theta(-x)G(x) \quad G_A(x) = 0 \quad x_0 > 0 . \qquad (58b)$$

Using the definition of S, one immediately established that

$$iS_xG_R(x) = \delta(x) ,$$
$$iS_xG_A(x) = \delta(x) ,$$
$$iS_x^\dagger G_R(\xi - x) = \delta(\xi - x) ,$$
$$iS_x^\dagger G_A(\xi - x) = \delta(\xi - x) . \qquad (59)$$

[25] The notation in (56) has been changed from that of (55) and (53). It is frequently convenient to indicate the space time points referring to annihilation operators $\psi(x)$ by x in general, by Latin letters; those referring to creation operators $\psi^\dagger(\xi)$ by Greek letters.

[26] In harmony with the convention adopted in the beginning; x stands for $\mathbf{x},x_0$, $\delta(x)$ is a four-dimensional δ function. $\theta(x)$ refers only to the time variable, strictly speaking this should be written $\theta(x_0)$.

[The completeness relation (6b) was used in obtaining (59).]

The identities in question are:

$$T(x_2 \cdots x_n)\psi_{\text{in},\alpha} = \psi_{\alpha,\text{out}}T(x_2 \cdots x_n) - i\int d^4x f^*_\alpha(x) S_x T(x_2 \cdots x_n x)\ , \tag{60a}$$

$$T(x_2 \cdots x_n)\psi^\dagger{}_{\text{in},\alpha} = \psi^\dagger_{\alpha,\text{out}}T(x_2 \cdots x_n) + i\int d^4\xi f_\alpha(\xi) S^\dagger_\xi T(x_2 \cdots x_n,\xi^\dagger)\ , \tag{60b}$$

$$T(x_2 \cdots x_n)\psi_{\text{in}}(x_1) = T(x_1 \cdots x_n) - i\int d^4\xi G_R(x_1 - \xi) S_\xi T(\xi,x_2 \cdots x_n)\ , \tag{61a}$$

$$\psi_{\text{out}}(x_1)T(x_2 \cdots x_n) = T(x_1 \cdots x_n) - i\int d^4\xi G_A(x_1 - \xi) S_\xi T(\xi,x_2 \cdots x_n)\ , \tag{61b}$$

$$T(x_2 \cdots x_n)\psi^\dagger_{\text{in}}(x) = T(x_2 \cdots x_n,x^\dagger) - i\int d^4\xi G_A(\xi - x) S^\dagger_\xi T(x_2 \cdots x_n,\xi^\dagger)\ , \tag{62a}$$

$$\psi^\dagger_{\text{out}}(x)T(x_2 \cdots x_n) = T(x_2 \cdots x_n,x^\dagger) - i\int d^4\xi G_R(\xi - x) S^\dagger_\xi T(x_2 \cdots x_n,\xi^\dagger)\ . \tag{62b}$$

It is clear that the identities so far written down are *operator* identities. From these operator relations follow relations for the vacuum expectation values. Of special interest are the following:

$$\langle 0|\psi_{\text{out}}(x)T(x_2 \cdots x_n)|0\rangle = i\int d^4x' G(x - x') S_{x'}\langle 0|T(x',x_2 \cdots x_n)|0\rangle\ , \tag{63a}$$

$$\langle 0|T(x_2 \cdots x_n)\psi^\dagger_{\text{in}}(x)|0\rangle = i\int d^4x' G(x' - x) S^\dagger_{x'}\langle 0|T(x_2 \cdots x_n,x'^\dagger)|0\rangle\ . \tag{63b}$$

The identities which show the already mentioned remarkable similarity to the persistent identities are:

$$\tau(x_1 \cdots x_n) = i\int d^4\xi G_R(x_1 - \xi) S_\xi \tau(\xi_1 x_2 \cdots x_n)\ , \tag{64a}$$

$$\tau(x_2 \cdots x_n,x^\dagger) = i\int d^4\xi G_R(\xi - x) S^\dagger_\xi \tau(x_2 \cdots x_n,\xi^\dagger)\ . \tag{64b}$$

Proofs. The proof of (60a) follows the, by now traditional, use of the asymptotic condition. Let Φ and Ψ be arbitrary states; then the asymptotic condition, (10) leads immediately to the relation

$$\begin{aligned}\langle\Phi|T(x_2 \cdots x_n)\psi_{\text{in},\alpha}|\Psi\rangle &= \lim_{t\to-\infty}\langle\Phi|T(x_2 \cdots x_n)\psi_\alpha(t)|\Psi\rangle \\ &= \lim_{t\to-\infty}\langle\Phi|\int d^3x f^*_\alpha(\mathbf{x}_1 t)T(x_2 \cdots x_n,x)|\Psi\rangle\ .\end{aligned} \tag{65a}$$

The last equality in (65a) follows from the expression for $\psi_\alpha(t)$ in terms of $\psi(\mathbf{x},t)$; since the limit, as $t\to -\infty$, has to be taken of the whole expression, the operator $\psi(\mathbf{x},t)$ may be taken inside the T product. This is what is written in (65a). One next utilized the obvious relation that

$$\begin{aligned}\int d^4x \frac{\partial}{\partial t}\left(f^*_\alpha(\mathbf{x},t)T(x_1x_2 \cdots x_n)\right) &= \lim_{t\to+\infty}\int d^3x f^*_\alpha(\mathbf{x},t)T(x_1x_2 \cdots x_n) \\ &\quad - \lim_{t\to-\infty}\int d^3x f^*_\alpha(\mathbf{x},t)T(x_1x_2 \cdots x_n)\ .\end{aligned} \tag{65b}$$

This gives the result

$$\begin{aligned}\langle\Phi|T(x_2 \cdots x_n)\psi_{\text{in},\alpha}|\Psi\rangle &= \lim_{t\to+\infty}\langle\Phi|\int d^3x f^*_\alpha(\mathbf{x}_1 t)\psi(\mathbf{x}_1 t)T(x_2 \cdots x_n)|\Psi\rangle \\ &\quad - \langle\Phi|\int d^4x\left(\frac{\partial f^*_\alpha}{\partial t}T(\cdots) + f^*_\alpha\frac{\partial T(\cdots)}{\partial t}|\Psi\rangle\right.\ .\end{aligned} \tag{65c}$$

Partial integrations in the second term, use of the asymptotic conditions in the first, (one already did use in the first term of (65c), that $\lim t\to+\infty$ $T(x_2 \cdots x_n,x) = \lim t\to+\infty\psi(\mathbf{x},t)T(x_2 \cdots x_n)$ lead immediately to

$$\begin{aligned}\langle\Phi|T(x_2 \cdots x_n)\psi_{\text{in},\alpha}|\Psi\rangle &= \langle\Phi|\psi_{\alpha,\text{out}}T(x_2 \cdots x_n)|\Psi\rangle \\ &\quad - i\langle\Phi|\int d^4x f^*_\alpha(x) S_x T(x_2 \cdots x_n,x)|\Psi\rangle\ .\end{aligned} \tag{65d}$$

Since (65d) is valid for *any* states Φ and Ψ, the operator relation (60a) follows. An identical type procedure yields (60b). The identities (60a) and (60b) which involve the smoothed-out *in* and *out* operators may be transcribed in terms of results using the field operators in $\psi_{\text{in}}(\mathbf{x},t)$ and $\psi_{\text{out}}(\mathbf{x},t)$. Multiply (60a) by $f_\alpha(x)$ and sum over α; this gives immediately

$$T(x_2 \cdots x_n)\psi^\dagger_{\text{in}}(x) = \psi^\dagger_{\text{out}}(x)T(x_2 \cdots x_n) + i\int d^4\xi G(\xi - x) S^\dagger_\xi T(x_2 \cdots x_n,\xi^\dagger)\ , \tag{66a}$$

$$T(x_2 \cdots x_n)\psi_{\text{in}}(x) = \psi_{\text{out}}(x)T(x_2 \cdots x_n) - i\int d^4\xi G(x - \xi) S_\xi T(\xi,x_2 \cdots x_n)\ . \tag{66b}$$

By taking the vacuum expectation values of these operator relations, noting that both $\psi^\dagger_{\text{in}}(x)|0\rangle = 0$ $\langle 0|\psi_{\text{in}} = 0$, one obtains the relations (63).

The proof of identities of the type (61) proceeds along slightly different lines. To show (61a) and (61b), consider two operators Q_R and Q_A *defined* by

$$\begin{aligned}Q_R(x_1 \cdots x_n) &\equiv T(x_1 \cdots x_n) - i\int d^4\xi G_R(x_1 - \xi) S_\xi T(\xi,x_2 \cdots x_n)\ , \\ Q_A(x_1 \cdots x_n) &\equiv T(x_1 \cdots x_n) - i\int d^4\xi G_A(x_1 - \xi) S_\xi T(\xi,x_2 \cdots x_n)\ .\end{aligned} \tag{67}$$

From (67) and (59) it follows immediately that

$$S_{x_1}Q_R = S_{x_1}Q_A = 0 \,. \tag{68a}$$

Furthermore, using the property that $G_R(x) = 0$ as $x_0\langle 0$, one sees that

$$\lim_{x_{1,0}\to-\infty} Q_R(x_1\cdots x_n) = \lim_{x_{1,0}\to-\infty} T(x_1\cdots x_n) = \lim_{x_{1,0}\to-\infty} T(x_2\cdots x_n)\psi_{\text{in}}(x_1)\,, \tag{68b}$$

$$\lim_{x_{1,0}\to+\infty} Q_A(x_1\cdots x_n) = \lim_{x_{1,0}\to+\infty} T(x_1\cdots x_n) = \lim_{x_{1,0}\to+\infty} \psi_{\text{out}}(x_1)T(x_2\cdots x_n)\,. \tag{68c}$$

Strictly speaking, the limits in (68) should be written in terms of matrix elements.

Finally, one sees from (67) that

$$Q_R - Q_A = i\int d\xi G(x_1-\xi)S_\xi T(\xi,x_2\cdots x_n) = T(x_2\cdots x_n)\psi_{\text{in}}(x) - \psi_{\text{out}}(x)T(x_2\cdots x_n)\,. \tag{68d}$$

(66b) has been used in obtaining (68d).

The conditions (68a) and (68d) determine Q_R (and Q_A) uniquely as functions of x_1. [The Eq. (68a) for Q_A and Q_R are linear in time—boundary conditions in time are provided by (68b) or (68c)—these conditions suffice to determine Q_R and Q_A uniquely; (68d) is in a way a consistency condition, or additional check.) It is easy to check that Q_R and Q_A defined by

$$Q_R(x_1\cdots x_n) = T(x_2\cdots x_n)\psi_{\text{in}}(x_1)\,, \tag{69a}$$

$$Q_A(x_1\cdots x_n) = \psi_{\text{out}}(x_1)T(x_2\cdots x_n)\,, \tag{69b}$$

satisfy the conditions (68). Since these conditions have a unique solution, the identities (61) follow via (67) and (69). This proves (61); the proof of (62) is identical, it can safely be omitted.

From the identities (61) and (62), the relations for the vacuum expectation values follow directly. Take, for instance, the vacuum expectation value of (61a). From the fact that $\psi_{\text{in}}(x_1)|0\rangle = 0$, one sees, by inspection, that the identity (64a) follows. In an identical manner (62b) produces (64b).

Thus all the identities given, have now been proven. The similarity in structure of the persistence identities and the asymptotic identities (64) is indeed striking. If one recalls that $G_R(x-\xi) = \theta(x-\xi)G(x-\xi)$, one sees by inspection that (64a) and (53) are different only with respect to integrations over the θ functions. This of course, causes considerable differences in the significance of these relations. This will be studied in the next section.

d. Significance of the Persistent Identities

In a sense the designation "identities" for (53), (55), and (64) is unfortunate; these relations are certainly not satisfied for *arbitrary* functions $\tau(x_1\cdots x_n)$. On the other hand these relations are certainly not to be considered as *equations*, from which the τ functions can be actually computed. (From the general character of the theories comprised in the "persistence category," this could hardly be expected.) Actually, the persistence identities, especially when combined with the asymptotic identities, give general requirements which the τ functions must satisfy. These general requirements have the character of boundary or limiting conditions. To clarify this situation consider for simplicity the identities which must be satisfied by the function $\tau(x,\xi)$.[27] They are

$$\tau(x,\xi) = i\theta(x_0-\xi_0)\int d^4\eta G(\eta-\xi)S_\eta^\dagger\tau(x,\eta)\,, \tag{70a}$$

$$\tau(x,\xi) = i\theta(x_0-\xi_0)\int d^4y G(x-y)S_y\tau(y,\xi)\,, \tag{70b}$$

$$\tau(x,\xi) = i\int d^4y G_R(x-y)S_y\tau(y,\xi) = i\int d^4y\theta(x_0-y_0)G(x-y)S_y\tau(y,\xi)\,, \tag{71a}$$

$$\tau(x,\xi) = i\int d^4\eta G_R(\eta-\xi)S_\eta^\dagger\tau(x,\eta) = i\int d^4\eta\theta(\eta_0-\xi_0)G(\eta-\xi)S_\eta^\dagger\tau(x,\eta)\,. \tag{71b}$$

It is easy to check (and reassuring) that $\tau(x_1\xi) = \theta\,(x_0-\xi_0)G(x-\xi)$ indeed does satisfy all four relations (70) and (71). To study the *general* nature of these identities; consider (70b) and (71a) together. One can make partial integrations in (70a), with respect to all the variables which are integrated and which occur in S_y. Thus one can make two partial integrations with respect to the space variables $\mathbf{y}$, one with respect to the time variable y_0. In the partial integration, one will obtain an out-integrated part. It appears reasonable from the character of the f functions to assume that the out integrated parts vanish at very large spatial distances. (This will not be the case at either positive or negative large *times*.) Combining this with the fact that $\theta(t) = 0$ for $t<0$ (70b) reduces to a limiting statement:

$$\tau(x,\xi) = \theta(x_0-\xi_o)\left[\lim_{y_0\to+\infty}\int d^3yG(x-y)\tau(y,\xi) - \lim_{y_0\to-\infty}\int d^3yG(x-y)\tau(y,\xi)\right]. \tag{72a}$$

[27] This function must of course be the same as $\theta(x_0-\xi_0)$ $G(x-\xi)$; as was proven before. The discussion which follows is only intended as an illustration of the situation encountered in general; not as a clumsy way to discuss this particular function.

The integrations in (72a) are over all three dimensional space, at infinitely remote (future or past) times.

The same partial integration process can, of course, be applied to (71a). The only difference between the two cases, stems from the fact that $S_y^\dagger G_R(x-y) = \delta(x-y)$. This causes a cancellation of $\tau(x,\xi)$ in (71a). The remarks made previously, concerning the vanishing (or nonvanishing) of the out-integrated parts, apply here as well. In this manner (71a) becomes:

$$0 = \lim_{y_0\to+\infty}\int d^3y\theta(x_\circ - y_0)G(x-y)\tau(y,\xi) - \lim_{y_0\to-\infty}\int d^3y\theta(x_\circ - y_0)\tau(y,\xi)G(x-y)\,. \quad (72b)$$

Since, in this expression, $x \equiv (\mathbf{x},x_0)$ is presumably finite it follows (if one may exchange the limiting process involving y_0 and the integral involving $\mathbf{y}$) that, the first term always vanishes since $x_0 - y_0$ is negative; one obtains

$$\lim_{y_0\to-\infty}\int d^3y\tau(y,\xi)G(x-y) = 0\,. \quad (72c)$$

It appears therefore, that the limiting statement (72c) is another form of the asymptotic identity. Actually, (72c) is a very mild restriction. For if $\tau(y,\xi)$ just possesses the form $\theta(y_0-\xi_0)F$, where F is arbitrary, the limiting demand (72c) is already satisfied. (As $y_0 \to -\infty$ for finite ξ_0 the argument of θ becomes negative, hence θ and thus the integral in (72c) vanish in this limit.) Combination of (72c) and (72a) yields the apparently nontrivial condition that:

$$\tau(x,\xi) = \theta(x_0-\xi_0)\lim_{y_0\to+\infty}\int d^3yG(x-y)\tau(y,\xi)\,. \quad (73)$$

Arguments identical in all details, [using (70a) and (71b)] yield

$$\tau(x,\xi) = \theta(x_0-\xi_0)\lim_{\eta_0\to-\infty}\int d^3\eta G(\eta-\xi)\tau(x,\eta)\,. \quad (74)$$

From the form of these relations, one sees clearly that they play the role of auxiliary conditions; they have the general character of a boundary condition. Another way in which the difference between the persistent identities and the asymptotic identities can be seen, is by taking the Fourier transforms of (70b) and (71a). It is straightforward to take the Fourier transform of (71a).

One only needs to define the Fourier transforms of τ and G in the ordinary manner[28]:

$$\tau(x,\xi) \equiv \iint dpdk \exp\{ipx - ik\xi\}\tilde{\tau}(p,k) \quad (75a)$$

$$G_R(x) \equiv \int dp \exp\{ipx\}\tilde{G}_R(p)\,. \quad (75b)$$

Substitutions of (75) into (71a), lead at once to the result that

$$\tilde{\tau}(p,k) = i\tilde{G}_R(p)(\mathbf{p}^2/2m - p_\circ)\tilde{\tau}(p,k)\,. \quad (76)$$

If one works with the general asymptotic identity (64) rather then with (70), one obtains instead of (76)

$$\tilde{\tau}(\cdots p_i \cdots) = i\tilde{G}_R(p_i)(\mathbf{p}_i^2/2m - p_{i,o})\tilde{\tau}(\cdots p_i \cdots)\,. \quad (76a)$$

The expression written here, also the later ones in this section, have only a formal significance. Many of the Fourier transforms used are actually singular. Their precise treatment takes considerable care. The expression (76a) is valid (in this formal sense) for *any* p variable p_i; in the functions $\tilde{\tau}$. (A p variable in $\tilde{\tau}$ is one which corresponds to an x variable in τ, which in turn corresponds to an operator $\psi(x)$ in the time ordered product. A k variable in $\tilde{\tau}$, corresponds to an operator $\psi^\dagger$ in the T product.)

If one calls the Fourier transform[29] of $\theta(x)$

$$\theta(x_0) = \int dt \exp\{ix_0t\}\varphi(t)\,, \quad (77)$$

one checks immediately from (77) and (75b) that

$$\tilde{G}_R(p) = \int dt\varphi(t)\tilde{G}(\mathbf{p},p_0 - t)\,. \quad (77a)$$

This combined with (76a) gives

$$\tilde{\tau}(p,k) = i(\mathbf{p}^2/2m - p_0)\tilde{\tau}(p,k)\int dt\varphi(t)\tilde{G}(\mathbf{p},p_0-t)\,. \quad (77b)$$

To obtain the Fourier transform of the persistence identities, one proceeds as before; substitute (75) and (77) into (70b), then using the fact that θ depends just on x_0, one finds after a short calculation

$$\tilde{\tau}(p,k) = i\int dt\varphi(t)\tilde{G}(\mathbf{p},p_0-t)(\mathbf{p}^2/2m - p_0 - t) \times \tilde{\tau}(\mathbf{p},p_0-t,\mathbf{k},k_0-t)\,. \quad (78)$$

A glance at (77b) and (78) shows again that they express different limiting properties of the Fourier transforms of the τ functions. To study this in more detail would obviously require a more careful handling of the limiting processes which are implicitly involved in (78) and (77). This will be discussed in a later paper. The persistent identities are perhaps the most typical feature of persistent theories.

---

[28] In the succeeding discussion all factors 2π, i, are omitted. The purpose of the calculation is only to show the general form of the relations obtained. The detailed analysis of these relations (which is in fact pretty tricky) will be given on a later occasion.

[29] Note again that numerical factors are omitted. It is important to remember that θ depends on *one* variable only.

5. EXTENSIONS, CLARIFICATIONS AND FURTHER DISCUSSION

a. Unitarity Condition

The persistent identities for the τ functions are linear. This, as has been mentioned a number of times, is in striking contrast with the relativistic situation, where the corresponding relations are *nonlinear*. In fact the nonlinear relations between the τ functions in the relativistic theory are often referred to as the *unitarity* conditions. It is clear, that since the S matrix can be expressed in terms of τ function, the unitarity of the S matrix $SS^\dagger = 1$, will give rise to nonlinear relations between the τ functions; again in the relativistic case, it can be shown [5] that the relations so obtained are equivalent to the ones obtained by starting from the operator identity (50). In the present case, the S matrix is still unitary. (This fact follows at once from the *assumed* completeness of the *in* and *out* states.) Hence, via the reduction formulas, the unitarity must result in a nonlinear relation between the τ functions in the persistent theories, as well. The use of the operator identity (50) leads to perfectly valid identities for the persistent theories, but the unitarity is not expressed thereby. To obtain this set of nonlinear τ equations, one could just substitute the explicit expression for the S matrix in terms of the τ functions into $SS^\dagger = 1$. Here a slight variation of this procedure will be used—an adaptation of a method due to Lehman, Symanzik, and Zimmerman is most convenient. Start from the identity,

$$\textstyle\sum_\beta \langle\alpha|\beta\rangle\langle\beta|Q|\gamma\rangle = \langle\alpha|Q|\gamma\rangle\,. \tag{79}$$

(79) is valid for any states $|\alpha\rangle$ and $|\gamma\rangle$; the sum β must be over a complete set. Q is any operator; pick for Q the time ordered product $T[\psi(x_1)\psi^\dagger(x_2)]$. Then (79) becomes, upon expansion of the right-hand side and the insertion of a complete set

$$\begin{aligned}\textstyle\sum_\beta \langle\alpha|\beta\rangle\langle\beta|T|\psi(x_1)\psi^\dagger(x_2)|\gamma\rangle &\\ = \theta(1-2^\dagger)\textstyle\sum_\beta \langle\alpha|\psi(x_1)|\beta\rangle\langle\beta|\psi^\dagger(x_2)|\gamma\rangle &\\ + \theta(2^\dagger-1)\textstyle\sum_\beta \langle\alpha|\psi^\dagger(x_2)|\beta\rangle\langle\beta|\psi(x_1)|\gamma\rangle\,. &\end{aligned} \tag{80}$$

Next, pick for $|\alpha\rangle$ a k particle *in* state $|\alpha_1\cdots\alpha_k \text{ in}\rangle$; pick for $|\gamma\rangle$ an m particle *in* state $|\gamma_1\cdots\gamma_m \text{ in}\rangle$. The complete set $|\beta\rangle$ is chosen as the complete set of *all* out states $|\beta_1\cdots\beta_l \text{ out}\rangle$ with $l = 0,\cdots\infty$. The summation over $|\beta\rangle$ in (80) is therefore $\sum_l(\sum_{\beta 1}\cdots\sum_{\beta l})$. With this choice of $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$, one sees that the typical form of the term in (80), $\langle\alpha|\beta\rangle = \langle\alpha_1\cdots\alpha_k \text{ in}|\beta_1\cdots\beta_l \text{ out}\rangle$; this is just an S matrix element; it can be directly expressed in terms of τ functions by means of the reduction formulas. The other terms in (80) may be expressed in the same manner. The structure of (80) is such that the summations over β_l may be carried out. After some manipulations one obtains:

$$\begin{aligned}&\sum_l \int\cdots\int dy_1\cdots dy_k d\xi_1\cdots d\xi_l d\xi_1'\cdots d\xi_l' dy_1'\cdots dy_m' f_{\alpha 1}^*(y_1)\cdots f_{\alpha k}^*(y_k) f_{\gamma 1}(y_1')\cdots f_{\gamma m}(y_m')\\ &\quad G(\xi_1-\xi_1')\cdots G(\xi_l-\xi_l')S_{y1}\cdots S_{yk}S'^\dagger_{\xi_1}\cdots S'^\dagger_{\xi l}S'^\dagger_{y_1}\cdots S'^\dagger_{ym}S_{\xi_1}\cdots S_{\xi_0}\\ &\quad[\tau^*(\xi_1\cdots\xi_l,y_1^\dagger\cdots y_k^\dagger)\tau(\xi_1'\cdots\xi_l'x_1x_2^\dagger y_1'\cdots y_m') - \theta(1-2^\dagger)\tau^*(\xi_1\cdots\xi_l x_1 y_1^\dagger\cdots y_k^\dagger)\\ &\quad\tau(\xi_1'\cdots\xi_l'x_2,y_1\cdots y_m) - \theta(2-1^\dagger)\tau^*(\xi_1\cdots\xi_l x_2^\dagger y_1^\dagger\cdots y_k^\dagger)\tau(\xi_1'\cdots\xi_l'x,y_1'^\dagger\cdots y_m'^\dagger)] = 0\,.\end{aligned} \tag{81}$$

(81) is true for *all* $\alpha_1\cdots\alpha_k$ and $\gamma_1\cdots\gamma_m$; since the f's form a complete set it follows that one ends up with the following system

$$\begin{aligned}&\sum_l \int\cdots\int d\xi_1\cdots d\xi_l d\xi_1'\cdots d\xi_l' G(\xi_1-\xi_1')\cdots G(\xi_l-\xi_l')S^\dagger_{\xi_1}\cdots S^\dagger_{\xi l}S_{y_1}\cdots S_{yk}S^\dagger_{y_1}\cdots S^\dagger_{ym}S_{\xi_1}\cdots S_{\xi l}\\ &\quad[\tau^*(\xi_1\cdots\xi_l y_1^\dagger\cdots y_k^\dagger)\tau(\xi_1'\cdots\xi_l'1,2^\dagger y_1'^\dagger\cdots y_m'^\dagger) - \theta(1-2^\dagger)\tau^*(\xi_1\cdots\xi_l x y_1^\dagger\cdots \overset{\dagger}{y}_k)\\ &\quad\tau(\xi_1'\cdots \varsigma_0' x_2^\dagger y_1'^\dagger\cdots y_m'^\dagger) - \theta(2-1^\dagger)\tau^*(\xi_1\cdots\xi_l,x_2^\dagger y_1^\dagger\cdots y_k^\dagger)\tau(\xi_1'\cdots\xi_l'x_1y_1'^\dagger\cdots y_m'^\dagger)] = 0\,.\end{aligned} \tag{82}$$

This set of equations (82) expresses the unitarity condition. Its nonlinear character is no doubt apparent. The time independence of the one-particle states, causes a certain amount of simplification in this system (82), but only for the small l terms in the series. The anticipated nonlinear character remains. It would appear that an effective use of (82) in any manner *different* from a perturbation or iteration method is as difficult here, as it is in the relativistic situation.

b. Euclidean and Galilean Invariance

It was mentioned in the introduction, that the particle nonconserving theories considered in this paper, can only be invariant with respect to the Euclidean group; not with respect to the Galilean group. Invariance, with respect to the latter group, implies that the number of particles is necessarily conserved. Even though this fact is a direct consequence of rather deep group theoretical results, Bargmann [25], more or less heuristic and elementary arguments may be presented to demonstrate this connection. The first point to note, is the behavior of the ordinary nonrelativistic Schrödinger equation under Galilean transformations. Let G be a special Galilean transformation $\mathbf{x} \to \mathbf{x} = \mathbf{x} - \mathbf{u}t$. Suppose $X(\mathbf{x}_1 t)$ and $X'(\mathbf{x}_1' t')$ both satisfy the Schrödinger equation as functions of their respective variables. Then the c number functions $X(\mathbf{x}_1 t)$, $X'(\mathbf{x}_1' t')$ are related by the transformation:

$$X(\mathbf{x}'t') = \exp\{-im[\mathbf{u}\mathbf{x}^1 + (1/2)\mathbf{u}^2 t^1]\}X(\mathbf{x},t) \ . \quad (83)$$

The presence of this phase factor, is essential; without it, X' and X would not both satisfy the same equation. Note further that this phase factor depends on $\mathbf{x}$ and t. Since clearly $X'(\mathbf{x}_1' t') = X(\mathbf{x}_1 t)$ one can say that the Schrödinger wave function *is not* a scalar under Galilean transformations.

To study the effect of Galilean transformations in a quantized field theory it is convenient to develop the field operators $\psi(\mathbf{x}_1 t)$ in the usual fashion; however, the smoothing functions $f_\alpha(\mathbf{x}_1 t)$ are taken to be plane waves[30]:

$$\begin{aligned} f_\alpha(\mathbf{x},t) &= \exp[i(\mathbf{p}_\alpha \mathbf{x} - E_\alpha t)] \\ &\equiv \exp\{mi[\mathbf{v}_\alpha \mathbf{x} - (1/2)\mathbf{v}_\alpha{}^2 t]\} \ . \end{aligned} \quad (84)$$

Here $\mathbf{p}_\alpha, E_\alpha, \mathbf{v}_\alpha$ are, respectively, the momentum, energy, and velocity of a free particle of mass m, having f_α as a wave function. If one develops the free field operator $\psi_{\text{in}}(\mathbf{x}_1 t)$ in terms of these functions $\psi_{in}(\mathbf{x}_1 t) = \sum_\alpha \psi_{\alpha,\text{in}} f_\alpha(\mathbf{x}_1 t)$, the operators $\psi_{\alpha,\text{in}}$ are (as always) time independent. $\psi_{\alpha,\text{in}}|0\rangle$ is a *one*-particle state, of a free particle having a velocity $\mathbf{v}_\alpha$. The simplest way to introduce Galilean invariance, is by requiring that for every Galilean transformation G, there exists a unitary operator U, such that

$$U|0\rangle = |0\rangle \ , \quad (85\text{a})$$

$$U\psi^\dagger_{\alpha,\text{in}}U^{-1} = \psi^\dagger_{(G\alpha),\text{in}} \ . \quad (85\text{b})$$

The first requirement merely states the invariance of the vacuum. $(G\alpha)$ in (85b) specifies the free-particle state which results from the state α, from the application of the Galilean transformation G. Thus, in the case that G is the transformation $\mathbf{x}' = \mathbf{x} + \mathbf{u}t$, $(G\alpha)$ refers to a state where the particle has a velocity $\mathbf{v}_\alpha + \mathbf{u}$. The requirements (85a and b) express the transformation properties, which one can on physical grounds expect from the vacuum and one-particle states under Galilean transformations. Using (85a) and (85b) and the development of $\psi_{\text{in}}(\mathbf{x}_1 t)$, one observes that

$$\begin{aligned} &U\psi_{\text{in}}(\mathbf{x},t)U^{-1} \\ &\quad = \textstyle\sum_\alpha \psi_{(G\alpha),\text{in}} \exp\{mi[\mathbf{v}_\alpha \mathbf{x} - (1/2)\mathbf{v}_\alpha^2 t]\} \ . \end{aligned} \quad (86)$$

Consider the transformation G:

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} + \mathbf{u}t \\ t' &= t \\ \mathbf{v}_\alpha' &= \mathbf{v}_\alpha + \mathbf{u} \ . \end{aligned} \quad (87)$$

Call the state $G\alpha = \alpha'$ (all the characterizations of the state are denoted by a prime). Then straight substitution in (86) yields

$$\begin{aligned} U\psi_{\text{in}}(\mathbf{x},t)U^{-1} &= \textstyle\sum_{\alpha'} \psi_{\alpha'\text{in}} \\ &\quad \exp\{mi[\mathbf{v}_{\alpha'}\mathbf{x}' - (1/2)(\mathbf{v}_{\alpha'})^2 t]\} \\ &\quad \exp\{-im[\mathbf{u}\mathbf{x} + (1/2)\mathbf{u}^2 t]\} \\ &= \exp\{-im[\mathbf{u}\mathbf{x} + (1/2)\mathbf{u}^2 t]\}\psi_{\text{in}}(\mathbf{x}'t') \ . \end{aligned} \quad (88)$$

Thus one observes the important fact that in a Galilean invariant theory, the field operators contain a nontrivial phase factor in their transformation law. (The phase factor is in fact the same as that occurring in the c-number theory.)

The group theoretical basis of the essential character of the phases in the Galilean group can be understood on the basis of a result of Bargmann: If one has a representation $U(L)$ of the Lorentz group, such that

$$U(L_1)U(L_2) = \omega(L_1 L_2)U(L_1 L_2) \ , \quad (89)$$

where $|\omega(L_1 L_2)| = 1$, one can by permissible phase changes, change this into a representation, up to a factor plus or minus one. A similar result holds for the Euclidean group; however, for the Galilean group a corresponding result does not hold. In the Galilean group the phases ω appear; they cannot be removed. If G_1 and G_2 are *general* Galilean transformation ($\mathbf{a}$ represents the space translation, $\mathbf{u}$ an acceleration transformation, τ the time translation, R a rotation) then $\omega(G_1 G_2) = \exp(im[\mathbf{u}R\mathbf{a} + (1/2)\mathbf{u}^2 t])$ (cf. Wightman).

For the present purposes (88) can serve to suggest the relation between particle nonconservation and Galilean invariance. Suppose one has a Lagrangian theory where the Lagrangian L is constructed from the field operators $\psi(\mathbf{x}_1 t)$ and $\psi^\dagger(\mathbf{x}_1 t)$. If the theory is invariant with respect to Galilean transformation, one must require that for every G, there exists a U

[30] No normalization constants are written.

[as in (85)] such that $L^1 = ULU^{-1} = L$. It is clear from (88) that *in general* this will only be possible if the field operators occur in the combination $\psi^\dagger(\mathbf{x}_1t)\psi(x_1t)$. For a ψ occurring singly in L would, upon transformation with the U operator, produce a time and position-time phase which would alter the Lagrangian. Thus one sees that a particle *nonconserving* theory, in which the number of ψ's and $\psi^\dagger$'s in a given term is unequal, can, in general, *not* be Galilean invariant. More explicitly, assume that the Lagrangian is made up of a *sum* of terms having the general structure

$$Q = \int \cdots \int \psi^\dagger(x,t) \cdots \psi^\dagger(x_n,t) \times F(\mathbf{x}_1 \cdot \mathbf{x}_n, \mathbf{y}_1 \cdots \mathbf{y}_m)\psi(\mathbf{y}_1t) \cdots \psi(\mathbf{y}_m,t)\,. \quad (90)$$

(Note that it has been assumed here that all operators occur at the same time.) In the Lagrangian, F is an arbitrary function. The equal-time commutators may be used to rearrange the order of the operators. One can using the equal-times commutation rules always write a general term containing $n\psi^\dagger s$ and $m\psi's$ as a *sum* of terms of the type (90). If one now considers a special Galilean transformation $G:\mathbf{x}' = \mathbf{x} + \mathbf{u}t$, one can directly calculate the transformed expression $Q' = U_G Q U_G^{-1}$. The action of the operator U_G on the field operators is given by (88). Since by assumption all the operators in (90) are taken at the *same* time point, one obtains

$$\begin{aligned} Q' = UQU^{-1} = \int \cdots \int d^3x_1 \cdots d^3y_m \\ \times \exp\{-im_0\mathbf{u}(\mathbf{y}_1 + \cdots \mathbf{y}_m - \mathbf{x}_1 \cdots \mathbf{x}_n) \\ - (1/2)im_0u^2t(m-n)\} \\ \times \psi^\dagger(\mathbf{x}_1,t) \cdots \psi(\mathbf{y}_m,t)F(\mathbf{x}_1 \cdots \mathbf{x}_n, \mathbf{y}, \cdots \mathbf{y}_m)\,. \end{aligned} \quad (90a)$$

The requirement of *form* invariance of the equations of motion under Galilean transformations, can now be stated as the requirement of the *equality* of Q^1 and Q for *all* $\mathbf{u}$. It is clear enough that if in (90) one starts out with as many annihilation as creation operators ($n = m$), and if in *addition* the creation and annihilation operators act pairwise at the same point, then the phases factor in (90a) cancels and indeed $Q^1 = Q$, the theory is Galilean invariant. In that case the total number of particles in the theory is also conserved, for the term Q in the Lagrangian has the form $Q = Q_s$[31]:

$$Q_s = \int \cdots \int d^3x_1 \cdots d^3x_n(\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t) \cdots \psi^\dagger(\mathbf{x}_n,t)\psi(\mathbf{x}_n,t)\,. \quad (91)$$

[31] The m_0 occurring in (90a) is the rest mass of the particles, previously called m. The m in (90a) and (90) refers to the number of annihilation operators in the general term Q.

Define $N(t) = \int d_3\, x\, \psi^\dagger(\mathbf{x}_1t)\psi(\mathbf{x}_1t)$, the number operator. From (91) and the equal-time commutation rule one checks directly that Q_s and N commute. Hence, the Lagrangian and N commute, and the total number of particles is conserved. Of course, this remark only shows that one can have Galilean invariant, particle—conserving theories—it does not show that one implies the other. To sketch the kind of argument involved in showing this connection, it is simplest to consider a special example.

Let Q be a term in the Lagrangian of the form

$$Q = \iiint d^3x d^3y d^3z \psi^\dagger(\mathbf{x}_1t)\psi^\dagger(\mathbf{y}_1t) \times \psi(\mathbf{z}_1t)F(\mathbf{x}-\mathbf{z},\mathbf{y}-\mathbf{z})\,. \quad (92)$$

F in (92) is of the indicated form, but arbitrary otherwise. It will be shown that the existence of such a term is inconsistent with Galilean invariance.

First, make a transformation $G_1:\mathbf{x}' = \mathbf{x} + \mathbf{u}t$, then by the argument just given [see (90a)]

$$\begin{aligned} Q' = U_{G_1}QU_{G_1}^{-1} = \iiint d^3xd^3yd^3z \\ \times \exp[-im\mathbf{u}(\mathbf{x}+\mathbf{y}-\mathbf{z}) - (1/2)im\mathbf{u}^2t] \\ \times \psi^\dagger(\mathbf{x}_1t)\psi^\dagger(\mathbf{y}_1t)\psi(\mathbf{z}_1t)F \\ = Q = \iiint d^3xd^3yd^3z\psi^\dagger(\mathbf{x}_1t)\psi^\dagger(\mathbf{y}_1t)\psi(\mathbf{z}_1t) \\ F(\mathbf{x}-\mathbf{z},\mathbf{y}-\mathbf{z})\,, \end{aligned} \quad (93)$$

(93) must hold for all u.

Next, make a translation $T:\mathbf{x}'' = \mathbf{x}' + \mathbf{a}$. For every translation there exists a transformation of the field operators U_T which satisfies

$$U_T\psi(\mathbf{x}_1t)U_T^{-1} = \psi(\mathbf{x} + \mathbf{a}_1t)\,. \quad (94)$$

The theory is also invariant under translations, therefore,

$$Q'' = U_TQ'U_T^{-1} = U_TQU_T^{-1} = Q\,. \quad (94a)$$

(Q is also unchanged with respect to translations.) Applications of this relation and (94) yields

$$\begin{aligned} Q'' = Q' = Q = \iiint d^3xd^3yd^3z \\ \times \exp\{im\mathbf{ua} - im\mathbf{u}(\mathbf{x}+\mathbf{y}-\mathbf{z}) - (1/2)imu^2t\} \\ \times \psi^\dagger(\mathbf{x}_1t)\psi^\dagger(y_1t)\psi F\,. \end{aligned} \quad (95)$$

[To obtain (95), a change of variables has been made in the integral after (94) has been applied.] (95) must hold for all $\mathbf{u}$ and all $\mathbf{a}$, (93) must hold for all $\mathbf{u}$. Since (95) holds for all a, one has

$$\frac{dQ''}{da_i} = \frac{dQ}{da_i} = 0 = u_i \iiint d^3xd^3yd^3z \times \exp[im\mathbf{ua} - im\mathbf{u}(\mathbf{x}+\mathbf{y}-\mathbf{z}) - (1/2)imu^2t]\psi^\dagger\psi^\dagger\psi F\,. \quad (96a)$$

Since (96a) must hold for *all* **a** and u_i, it follows that the integral must be zero for all a if $u_i \neq 0$. In particular, the integral must vanish if $a = o$, therefore,

$$\iiint d^3x d^3y d^3z \exp\{-im\mathbf{u}(\mathbf{x}+\mathbf{y}-\mathbf{z}) - (1/2)imu^2t\} \times \psi^\dagger\psi^\dagger\psi F = 0 \quad \text{if } u \neq 0 \,. \tag{96b}$$

But, one recognizes (96b) as Q', which by (93) is the same as Q. Thus, (96b) states that $Q = 0$, if $u \neq 0$; but Q is independent of u, hence, Q vanishes identically. This shows that a term of the type (92) is indeed incompatible with the invariance requirements. This type of term is one which would give rise to nonparticle conserving types of interactions. This kind of argument could be given in general. It is clear that this is not a very compelling procedure, especially since the argument which needs to be given, depends to some extent on the structure of the terms in question. A more general argument eliminating the need to investigate terms of specific character in a Lagrangian would be desirable.[32]

There is one feature of these heuristic considerations which is of special interest. Consider the same type of term (92); however, consider the case where $\psi^\dagger(\mathbf{x}_1t)$, $\psi^\dagger(\mathbf{y}_1t)$, $\psi(\mathbf{z}_1t)$ refer to *different* fields with different masses, m_1, m_2, m_3. In that case the phase introduced by the Galilean transformation, for the case of *local* interactions (all field operators are taken at the *same space-time* point) is given by

$$\exp[-i\mathbf{ux}(m_1+m_2-m_3)] \times -\exp[-(1/2)i\mathbf{u}^2(m_1+m_2-m_3)] \,. \tag{97}$$

(The corresponding expression for the case of equal masses m_o is $\exp(-i\mathbf{ux}m_o - \frac{1}{2}i\mathbf{u}^2m_o)$. It now follows by inspection from (97) that the phases will be identically zero if only $m_1 + m_2 = m_3$. In that case the theory is indeed Galilean invariant. This example already shows that it is quite possible to have Galilean invariant theories in which the number of particles is not conserved, but where in the individual interaction processes (characterized by such terms as $\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})\psi(\mathbf{z})$ the creation of two particles and the destruction of one) the total mass is conserved. From this viewpoint, Galilean invariance actually requires the conservation of mass. In a theory with only one kind of particle of a fixed mass present, this necessarily implies conservation of the number of particles. It is however, physically more illuminating to stress the mass conservation aspect of the Galilean invariance. In the group theoretic treatment this same result is expressed in terms of the superselection rule of the mass for the Galilean group. The existence of a state which is a superposition of two states with different masses, is in conflict with Galilean invariance.

The use of the Euclidean group in this paper is thus directly tied to the model described. If one describes a number of particles of different masses, such that the *mass* is conserved in the various processes, one can require Galilean invariance and still have non conservation of particle number.

[32] Dr. Hans Ekstein has informed us that he has constructed such a proof.

c. Many Particle Theory

It is straightforward to generalize the formalism given here to one applicable to the description of several kinds of particles. Since the applications are primarily concerned with systems consisting of several kinds of particles, some of the pertinent formulas are collected here. The system considered contains three kinds of stable particles, masses m_1, m_2, m_3. They are described by fields; the field operators are written as A, B, C. Each field operator is averaged by appropriate functions, $f_{\alpha,A}$, $f_{\alpha B}$, $f_{\alpha C}$ which satisfy the free-particle equations for particles of mass m_1, m_2, m_3, respectively. The fields, A, B, C each have asymptotic fields associated with them called, $A_{\text{in,out}}$, $B_{\text{in,out}}$, $C_{\text{in,out}}$. These *in* field operators satisfy the field free equations, as well as the field free commutation rules.

The asymptotic condition is now

$$\begin{aligned}
\lim_{t\to-\infty}\langle\Phi|A_\alpha(t)|\Psi\rangle &= \langle\Phi|A_{\alpha,\text{in}}|\Psi\rangle \\
\lim_{t\to-\infty}\langle\Phi|B_\alpha(t)|\Psi\rangle &= \langle\Phi|B_{\alpha,\text{in}}|\Psi\rangle \\
\lim_{t\to-\infty}\langle\Phi|C_\alpha(t)|\Psi\rangle &= \langle\Phi|C_{\alpha,\text{in}}|\Psi\rangle \,.
\end{aligned} \tag{98}$$

The equal time commutation rules are the usual ones;

$$\begin{aligned}
[A(\mathbf{x}_1t), A^\dagger(\mathbf{y}_1t)] &= [B(\mathbf{x}_1t), B^\dagger(\mathbf{y}_1t)] \\
&= [C(\mathbf{x}_1t), C^\dagger(\mathbf{y}_1t)] = \delta(\mathbf{x}-\mathbf{y}) \,, \\
[A(\mathbf{x}_1t), B(\mathbf{y}_1t)] &= [A(\mathbf{x}_1t), C(\mathbf{y}_1t)] \\
&= [B(\mathbf{x}_1t), C(\mathbf{y}_1t)] = 0 \,,
\end{aligned} \tag{99}$$

all other commutators vanish.

These commutators allow one to give at each instant of time the eigenvalues of the commuting number operators, $N_{A,\alpha}(t)N_{B,\beta}(t)N_{C\gamma}(t)$. As before the A, B, C particles are "dressed" particles (and by the persistence assumption they will stay dressed the same way). One now assumes again the existence and persistence of a vacuum state $|0\rangle$ defined by

$$\begin{aligned}
A(\mathbf{x}_1t)|0\rangle &= 0 \\
B(\mathbf{x}_1t)|0\rangle &= 0 \\
C(\mathbf{x}_1t)|0\rangle &= 0 \,.
\end{aligned} \tag{100}$$

Finally, one assumes the persistence conditions

$$(\partial/\partial t)A_\alpha^\dagger|0\rangle = 0\,,$$
$$(\partial/\partial t)B_\beta^\dagger|0\rangle = 0\,,$$
$$(\partial/\partial t)C_\gamma^\dagger|0\rangle = 0\,. \tag{101}$$

Hence, a single A, B, or C particle stays single, each individual particle persists as long as it is alone.

A sufficient condition for the validity of (100) and (101) is the validity of

$$\partial A_\alpha^{(i)}/\partial t = (A^k)^\dagger F A^{(j)} \tag{102}$$

The upper indices (i), (j), (k) in (102) denotes (1), (2), or (3) and $A^{(1)} = A$, $A^{(2)} = B$, $A^{(3)} = C$. F is an arbitrary functional of A, B, C.

A very simple example of a persistent system would be one described by a Lagrangian (L_0 = sum of the free Lagrangians)

$$L = L_0 + \iint C^\dagger(x)A^\dagger(y)F(x_1y)A(x)B(y) + \iint B^\dagger(x)A^\dagger(y)G(x_1y)A(x)C(y)\,. \tag{103}$$

It is worth noting in this example, that although all single particles persist, this is no longer for the two-particle states. (103) describes the processes $A+B \to A+C$, and $A+C\to A+B$. Similarly, the persistent Lagrangian

$$L' = L_0 + \iint C^\dagger A^\dagger FABC + \iint C^\dagger B^\dagger A^\dagger F^\dagger AC \tag{103a}$$

describes the "production" processes $A+B+C \to C+A$; $A+C\to A+B+C$. A complete set of states in the present theory consists for example of *all* the *in* states:

$$|\alpha_1 \cdot \alpha_k,\beta_1 \cdots \beta_l,\gamma_1 \cdots \gamma_m in\rangle = A_{\alpha_1\,\mathrm{in}}^\dagger \cdots A_{\alpha_k\mathrm{in}}^\dagger B_{\beta_1\,\mathrm{in}}^\dagger \cdots B_{\beta l,\,\mathrm{in}}^\dagger C_{\gamma,\,\mathrm{in}}^\dagger \cdots C_{\gamma m}^\dagger|0\rangle\,. \tag{104}$$

The anti-Jost theorem is unchanged, the threefold Wightman functions are still zero. In fact, the whole formal apparatus, including the reduction formulas, carries over with only obvious modifications, and an increase in the number of subscripts and superscripts. The only slightly more significant alteration comes from the modified persistence identities. The derivation follows that given previously. Since the main problem is one of notation, the following conventions are helpful. A general time ordered product has the structure $T(A_1 \cdots A_n, A_1^\dagger \cdots A_{n1}^\dagger, B_1, B_k, B_1^\dagger \cdots B_{k'}^\dagger, C_1 \cdots C_l, C_1^\dagger \cdots C_{l'}^\dagger)$. Since each one of these field operators depends on a space time point, the product can be written as $T(a_1 \cdots a_{n1}\dot{a}_1 \cdots \dot{a}_{n1}, b_1 \cdots b_k, \dot{b}_1 \cdots \dot{b}_k, c_1 \cdots c_l, \dot{c}_1 \cdots \dot{c}_{l1})$.

The dotted variables always refer to the space-time points of the creation operators. As written T contains $n+n'+k+k'+l+l'$ operators; it will be abbreviated as $T(\cdots)$. In this notation only the presence, or absence of an operator, *different* from the original ones will be noted: $T(\cdots a_i^\star \cdots)$ is the same product but for the absence of $A(a_i)$. It is also useful to abbreviate certain sign factors:

$$\theta(a_1-a_i)\theta(a_2-a_i)\cdots\theta(b_1-a_i)\cdots\theta(c_l'-a_i) \equiv \Theta(-a_i) \tag{105a}$$

$$\theta(a_i-a_1)\cdots\theta(a_i-b_1)\cdots\theta(a_i-c_l') \equiv \Theta(+a_i)\,. \tag{105b}$$

The products in (105) contain $n+n'+k+k'+l+l'-1$ factors [the factor $\theta(a_i-a_i)$ is omitted]. Clearly $\Theta(-a_i)$ is different from zero only if $a_{i,o}$ is the smallest of *all* the times in the product. $\Theta(+a_i)$ is different from zero only if $a_{i,o}$ is the largest of all the times. With these conventions, one may write the following operator identities:

$$\begin{aligned} T(\cdots) = {}& \sum_{i=1}^{n}\Theta(-a_i)T(\cdots a_i^\star \cdots)A(a_i) \\ &+ \sum_{i=1}^{n'}\Theta(-\dot{a}_i)T(\cdots \dot{a}_i^\star \cdots)A^\dagger(\dot{a}_i) \\ &+ \sum_{i=1}^{k}\Theta(-b_i)T(\cdots b_i^\star \cdots)B(b_i) \\ &+ \sum_{i=1}^{k'}\Theta(-\dot{b}_i)T(\cdots \dot{b}_i^\star)B^\dagger(\dot{b}_i) \\ &+ \sum_{i=1}^{l}\Theta(-c_i)T(\cdots c_i^\star)C(c_i) \\ &+ \sum_{i=1}^{l'}\Theta(-\dot{c}_i)T(\cdots \dot{c}_i^\star)C^\dagger(\dot{c}_i)\,. \end{aligned} \tag{106a}$$

$$\begin{aligned} T(\cdots) = {}& \sum_{i=1}^{n}\Theta(+a_i)A(a_i)T(\cdots a_i^\star \cdots) \\ &+ \sum_{i=1}^{n'}\Theta(\dot{a}_i)A^\dagger(\dot{a}_i)T(\cdots \dot{a}_i^\star \cdots) \\ &+ \sum_{i=1}^{k}\Theta(+b_i)B(b_i)T(\cdots b_i^\star \cdots) \\ &+ \sum \Theta(+\dot{b}_i)B^\dagger(\dot{b}_i)T(\cdots \dot{b}_i^\star \cdots) \\ &+ \sum_{i=1}^{l}\Theta(+c_i)C(c_i)T(\cdots c_i^\star \cdots) \\ &+ \sum_{i=1}^{l}\Theta + (\dot{c}_i)C^\dagger(c_i)T(\cdots \dot{c}_i^\star \cdots)\,. \end{aligned} \tag{106b}$$

One now just imitates the derivation given before; take the vacuum expectation value of (106a) [just three terms remain by (100)], then insert a complete set of states in the matrix element of the product.

For the complete set of states one can use the set of *in states* given by (104). A typical term occurring in the result will be $\langle 0|T\cdot(a_i^\star\cdots)|\text{in}\rangle\langle\text{in}|A^\dagger|\dot{a}_i|0\rangle$; here $|\text{in}\rangle$ is a *general in* state containing A, B, and C particles. However, $A^\dagger(a_i)|0\rangle$ gives by the persistence condition a mixture of free A type particles. Hence, the only *in* states that give nonzero contribution are the *one-particle A in* states. Using this; as well as the analogue of the reduction formulas, one obtains

$$\begin{aligned}\tau(\cdots) = i\sum\Theta(-\dot{a}_i)\int d^4\alpha G_A(\alpha-\dot{a}_i)\\ S_{\alpha A}^\dagger\tau(\cdots\dot{a}_i^\star\cdots\alpha)\\ +i\sum\Theta(-\dot{b}_i)\int d^4\beta G_\beta(\beta-\dot{b}_i)\\ S_{\beta B}^\dagger\tau(\cdots\dot{b}_i^\star\cdots\beta)\\ +i\sum\Theta(-\dot{c}_i)\int d^4\gamma G_C(\gamma-\dot{c}_i)\\ S_{\gamma C}^\dagger\tau(\cdots\dot{c}_i^\star\cdots\gamma)\,.\end{aligned}\tag{107}$$

The S_A operator is the Schrödinger operator for the A field; $G_A\equiv\sum_\alpha f_{\alpha A}(x)f_{\alpha A}^*(y)$; similarly for the other fields. Starting from (106b) one can obtain a relation similar to (55). The linear character of the persistent identities is retained in the many particle case.

As in the case of one type of particle, the first physically interesting τ (or Wightman) function is the fourfold one. The persistence conditions allow some simplifications there, but the essential features of the interactions are contained there. It is worth emphasizing that in spite of the persistence conditions, expressions such as $\langle 0|ABA^\dagger C^\dagger|0\rangle$ do not vanish, they describe reactions of the type $A+B\to A+C$, which are nontrivial physical processes.

d. Example: Fixed Source Theory

It is interesting to observe that some extremely simple field theoretic models are not included in the class of persistent theories. The simplest example is the well-known theory of a neutral scalar field interacting with fixed sources. This theory is described by the Hamiltonian

$$\begin{aligned}H = -\frac{1}{2m}\int d^3x\psi^\dagger(\mathbf{x}_1t)\Delta\psi(\mathbf{x}_1t)\\ +\int d^3x\rho(\mathbf{x})(\psi(\mathbf{x}_1t)+\psi^\dagger(\mathbf{x}_1t)\,,\end{aligned}\tag{108}$$

ρ is the source function. The equal time commutators are the usual ones. It is easy to check that the equations of motion of the Heisenberg operators are given by

$$S_x\psi = -\rho(\mathbf{x})\,.\tag{108a}$$

The equations of evolution for the smoothed out field operators are

$$\partial\psi_\alpha/\partial t = -i\int d^3xf_\alpha^*(\mathbf{x}_1t)\rho(\mathbf{x})\equiv -i\rho_\alpha(t)\tag{108b}$$

Both of these equations can be easily solved. One should recall that $\rho(\mathbf{x})$ is a *given c* number function, $\rho_\alpha(t)$ is a known c number function of the time.

$$\psi(\mathbf{x}_1t) = \psi_{\text{free}}(\mathbf{x}_1t) - i\int d^4yG_R(x-y)\rho(y)\,,\tag{109a}$$

$$\psi_\alpha({}_t) = \psi_\alpha(0) - i\int_0^t dt'\rho_\alpha(t')\equiv\psi_\alpha(0)-iC_\alpha(t)\,.\tag{109b}$$

G_R is the function previously defined. It satisfies $iS_x'G_R(x-y)=\delta(x-y)$; ψ_{free} is an operator satisfying $S_x\psi_{\text{free}}=0$.

It is easy to see that the equation of motion (108b), is *not* of the form (25) which will guarantee a persistent theory.

That this theory is not a persistent one, can be seen, in many ways; for instance, one can calculate the second Wightman function (which is *free* in a persistent theory), using (109a)

$$\begin{aligned}W_2(x_1y) &= \langle 0|\psi(x)\psi^\dagger(y)|0\rangle\\ &= W_{\text{free}}+\iint dx'dy'\rho(x')\rho(y')\\ &\qquad G_R(x-x')G_R(y-y')\,.\end{aligned}\tag{110}$$

From (110) one sees that $W_2\neq W_{\text{free}}$. Hence, the theory described by (108) is *not* a persistent one. Actually, this fixed source theory does not even possess a persistent vacuum. [In fact since $N=\int d^3\times\psi^\dagger(\mathbf{x}_1\ t\ |\psi|\mathbf{x}_1t)$ and H as given by (108) do not commute, there are in general, no simultaneous eigenstates of N and H, hence, the number vacuum is distinct from the lowest energy state.] Suppose one has a number vacuum at time o, that is a state $|o\rangle$ so that $\psi_\alpha(o)|o\rangle=0$. If there were a *time-independent* vacuum, then one should have, using (109b) $\psi_\alpha(t)|o\rangle = 0 = (\psi_\alpha(o)-ic_\alpha(t))|o\rangle=0$. This would require $c_\alpha(t)|o\rangle=0$; in general, this is impossible. Thus the theory has no *persistent* vacuum. If the theory is to have a number vacuum at any time, (not a persistent number vacuum) one must demand at any time the existence of a state $|o_t\rangle$, which has the property that $\psi_\alpha(t)|o_t\rangle=0$. One can assume that a vacuum state $|o\rangle$ exists at time $t=o$, so that the existence of a vector having the property $\psi_\alpha(o)|o\rangle=0$ is guaranteed. Since by (109b) $\psi_\alpha(t)=\psi_\alpha(o)-ic_\alpha(t)$, the question of the existence of a vacuum state $|o_t\rangle$ boils down to the existence of an eigenvector of $\psi_\alpha(o)$ with a prescribed eigenvalue ic_α. Since the vector $|o\rangle$ is a given, $\psi_\alpha^\dagger|o\rangle$, $(\psi_\alpha^\dagger)^2|o\rangle\cdots(\psi_\alpha^\dagger)^n|o\rangle$ are all vectors in the space, they will be written as $|n\rangle_1\cdots|n+1\rangle$. The action of $\psi_\alpha(o)$ and $\psi_\alpha^\dagger(o)$ on these vectors is, of course, known. It is convenient to normalize them so that

$$\psi_\alpha(0)|n\rangle = (n)^{\frac{1}{2}}|n-1\rangle\tag{111a}$$

$$\psi_\alpha^\dagger(0)|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle\,.\tag{111b}$$

Now construct a vector $|u\rangle$ as a superposition

$$|u\rangle = \sum_n q_n|n\rangle . \qquad (112)$$

The q_n are numbers, they are to be fixed in such a way that $u\rangle$ becomes an eigenvector of $\psi_\alpha(o)$ with eigenvalue ic_α. Since $|u\rangle$ is a superposition of vectors $|n\rangle$ in the space, $|u\rangle$ is such a vector. Hence, if coefficients q_n can be found such that $\psi_\alpha(o)|u\rangle = ic_\alpha|u\rangle$ one has demonstrated the existence of a vacuum state at time t. It is easy to find such q_n. By using (112) and (111b) the requirement that $\psi_\alpha(o)|u\rangle = ic_\alpha|u\rangle$ transcribes to

$$\sum_n q_n(n)^{\frac{1}{2}}|n-1\rangle$$
$$= \sum_n (ic_\alpha)q_n|n\rangle \; q_{n+1}(n+1)^{\frac{1}{2}} = ic_\alpha q_n . \qquad (113)$$

One finds solving this recursion relation for the q' in terms of q_o that

$$|0_\alpha(t)\rangle \equiv U = \sum_n \frac{(ic_\alpha)^n}{(n!)^{\frac{1}{2}}} q_0|n\rangle \qquad (113a)$$

(113a) exhibits the vacuum state $|o_\alpha(t)\rangle$ as a linear combination (with prescribed time dependent coefficients in terms of the c's) of the states $|n\rangle$.

One can check, of course, that $|o_\alpha(t)\rangle$ has the property that $\psi_\alpha(t)|o_t\rangle = 0$. Thus in this fixed source theory one can indeed always find a state $|o_\alpha(t)\rangle$, which satisfies $\psi_\alpha(t)|o_\alpha(t)\rangle = 0$. It should be stressed however, that so far one has been dealing throughout with *one* degree of freedom, the α one. Strictly speaking, the n's in (111) and (113) should all be n_α, for $|n_\alpha\rangle$ is a state of n particles, each with a wave function f_α. One gets, of course, results similar to (113a) for all degrees of freedom. Since the vectors corresponding to n particles in state $|\alpha\rangle$, and m particles in state $|\beta\rangle$ are orthogonal (for all α,β,n,m), one obtains for the vacuum state

$$|0(t)\rangle = \prod_\alpha |0_\alpha(t)\rangle . \qquad (114)$$

Since $|o_\alpha(t)\rangle$ always exists in this theory, it appears as if (114) must exist as well. However, $|o(t)\rangle$ should be a normalizable state. If one calculates the norm of $|o_t\rangle$ [using (114) and (113)] one finds that the norm of $|o_t\rangle$ is proportional to exp $\sum_\alpha|c_\alpha|^2$. Thus the existence of a time dependent vacuum state depends directly on the convergence or divergence of $\sum|c_\alpha|^2$. In turn, the c's depend directly on the source function ρ.

e. Final Comments; Unsolved Questions

There are a number of questions, which are suggested by the present work, which so far, have been only partially solved, or which remain unsolved.

An immediate question is whether in a persistent theory one can prove analytic properties of the S matrix elements. One has the persistent identities available, in addition to the usual machinery. The use of these identities facilitates the discussion of the analytic properties, somewhat, but so far, no proof of the Mandelstam representation has been given within this framework. Since the persistent framework is rather broad, this would be a substantial generalization over the existing proofs, for potential scattering. It would be desirable to include in this formalism, the possibility of unstable particles. Clearly such particles could not satisfy a persistence condition, but it would be interesting, to see whether such particles, or resonant states, can be incorporated in this type formalism.

There are many other generalizations which one could seriously consider, such as extensions to many channel situations, bound states, etc. But the basic question which this paper raises is whether the persistence condition (in the form given here or in an alternate form) is a sensible physical condition for stable particles. It would appear to express in some crude way, that one can associate well-defined physical attributes with a single particle. This seems physically reasonable for nonrelativistic situations, and not obviously insane for *single* particles (electrons) moving at relativistic speeds. Still, within the field theoretic context, this persistence condition is compatible only with trivial relativistic theories. Part of the trouble comes no doubt from the fact that one also made certain assumptions about the persistence of the vacuum. Since it is hard (but perhaps not impossible) to construct a theory where one has persistent one-particle states, but no persistent vacuum, it is difficult to disentangle the role of the vacuum and *one*-particle persistence conditions. Perhaps at some future time a relativistic, persistent one-particle theory with a "boiling vacuum" can be obtained. Until such time the detailed applicability of the formalism here is restricted to nonrelativistic situations. However, as a testing ground for the study of general aspects of field theory, the formalism might have some value. Some of the applications, a more detailed discussion of the analytic character of S-matrix elements, as they follow from the persistence conditions will be given in a forthcoming paper.

1. A. S. Wightman, Phys. Rev. **101**, 860 (1956).
2. R. Jost, Helv. Phys. Acta **30**, 409 (1957).
3. R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 12 (1955).
4. H. Lehman, K. Symanzik, and W. Zimmerman, Nuovo cimento **2**, 429 (1955).
5. S. Schweber, *Relativistic Quantum Theory* (Row, Peterson and Company, Evanston, Illinois, 1961).
6. R. Jost, *Field Theory and the Many Body Problem* (Academic Press Inc., New York, 1961).

7. N. Burgoyne, Nuovo cimento **8**, 607 (1958).
8. Geoffrey F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1962).
9. G. F. Chew, Ann. Rev. of Nuclear Sci. **9**, 29 (1959).
10. G. Chew and S. Mandelstam, Phys. Rev. **119**, 469 (1960).
11. H. Lehman, K. Symanzik, and W. Zimmerman, Nuovo cimento **6**, 319 (1957).
12. Peter B. Kahn: Thesis, Northwestern University (1960).
13. M. Dresden, W. D. McGlinn, and P. Kahn, Bull. Am. Phys. Soc. **5**, 283 (1960).
14. M. Dresden, "Lectures on Abstract Field Theory," Summer Institute for Theoretical Physics, Boulder, Colorado, 1959.
15. R. H. Landshoff and S. A. Treiman, Nuovo cimento **19**, 1249 (1961).
16. G. Kallen and A. S. Wightman, Kgl. Danske Videnskat. Selksab. Mat.-fys. Skriffer **1**, No. 6 (1958).
17. L. Landau, Nuclear Phys. **13**, 181 (1959).
18. R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).
19. W. Zimmerman, Nuovo cimento **10**, 597 (1958).
20. A. O. Barnt and K. H. Ruei, J. Math. Phys. **2**, 181 (1961).
21. A. Klein, J. Math. Phys. **1**, 41 (1960).
22. R. Blankenbeder, M. Goldberger, N. Khuri, and S. A. Treiman, Ann. Phys. **10**, 62 (1960).
23. J. Bowcock and A. Martin, Nuovo cimento **14**, 516 (1959).
24. A. Martin, Nuovo cimento **19**, 1257 (1961).
25. V. Bargmann, Ann. Math. **59**, 1 (1954).
26. Peter J. Redmond and Jack L. Uretzky, Ann. Phys. **9**, 106 (1960); M. Hamermesh, *ibid.* **9**, 518 (1960).
27. H. Stapp, University of California Radiation Laboratory Report UCRL-9875 (unpublished).
28. L. Schwartz, *Theorie des distributions* (Hermann and Co., Paris, 1951) Actualities Scientifienes No. 1019.
29. L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., Inc., New York, 1960).
30. N. Bohr and L. Rosenfeld, Kgl. Danske Videnskat Selksab, Mat.-fys. Medd **12** (1933).
31. G. F. Chew, Revs. Modern Phys. **34**, 394 (1962); H. Stapp, *ibid.* **34**, 390 (1962).
32. L. Gårding and A. S. Wightman, Proc. Nat. Acad. Sci. U. S. **40**, 617 (1954).
33. A. S. Wightman and S. Schweber, Phys. Rev. **98**, 812 (1955).
34. W. Zimmerman, Nuovo cimento **16**, 690 (1960).

REVIEWS OF MODERN PHYSICS VOLUME 34, NUMBER 3 JULY, 1962

Kinematics of High-Energy Particles*

K. G. DEDRICK
Stanford Linear Accelerator Center, Stanford University, Stanford, California

CONTENTS

* This work was supported by the U. S. Atomic Energy Commission.

I. INTRODUCTION

EXPERIMENTAL workers in high-energy physics have frequent need of formulas and tables giving properties of the kinematics of particle production, scattering, and decay. Many excellent tables and graphs have been published,[1] and others are certain to appear as particle masses are determined more accurately, new particles are discovered, and bombarding energies are increased. The required formulas are generated using the well-known transformation laws provided by the special theory of relativity, and we note that except for the systematic treatments by Blaton[2] and others,[3–10] these formulas

[1] See Appendix B.
[2] J. Blaton, Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. **24**, No. 20 (1950).
[3] The work of Blaton[2] has been extended by Baldin *et al.*[4] In the latter, we note that many formulas are written in a variety of ways so that the reader has greater latitude for choice of an appropriate form. Collections of formulas are given by Morrison,[5] Monahan,[6] Fowler and Brolley,[7] and by Blumberg and Schlesinger.[8] Other lists of formulas are given by Marshak[9] and by Janossy.[10]
[4] A. Baldin, V. I. Goldanskiĭ, and I. L. Rozental', *Kinematics of Nuclear Reactions* (Oxford University Press, New York, 1961). Translated from the Russian by R. F. Peierls.
[5] P. Morrison in *Experimental Nuclear Physics*, edited by E. Segrè (John Wiley & Sons, Inc., New York, 1953), Vol. II, p. 3.