REVIEWS OF MODERN PHYSICS VOLUME 34, NUMBER 3 JULY, 1962

Axiomatic S-Matrix Theory*†

HENRY P. STAPP
Lawrence Radiation Laboratory, University of California, Berkeley, California

INTRODUCTION

IT has been the common practice to consider field theoretic axioms as the proper basis for rigor in physics. This is evidently due more to the lack of any satisfactory alternative rather than to their obvious merit. For although the axioms of field theory provide a basis for rigorous mathematics, there is considerable doubt that they are of relevance to physics. In the first place, it is not known whether the axioms admit any rigorous solutions, except trivial ones in which the scattering matrix is unity. In the second place, the axioms depart from general quantum principles by assigning a fundamental role to hypothetical space-time points, although the physically observable quantities correspond rather to Fourier components consistent with the masses of physical particles. In the third place, the specific axioms regarding positive definiteness, nondegeneracy of the vacuum, completeness, locality, and energy spectrum are all very restrictive and arbitrary, and each one eliminates interesting possibilities that appear reasonable *a priori*. Of course, it is not necessary for the axioms of physics to be reasonable *a priori*, provided they lead to practical calculations that can be tested experimentally. But this appears not to be the case; axiomatic field theory seems in fact very distant, if not totally disconnected, from most practical calculations.

This last defect is the most serious from the point of view of physics. Practical calculations are the heart of physics, and it is the job of physical axioms to specify a connection between experience and a well-defined mathematical scheme in which practical calculations have a place. Yet the axioms of field theory, while apparently stronger than necessary in many respects, are evidently not strong enough in those aspects needed for practical calculations. Specifically, while giving superfluous analyticity at unphysical values of the masses, they apparently do not ensure the mass-shell analyticity properties used in the modern dispersion theoretic approach to elementary-particle physics.[1] This approach is the basis of most recent practical calculations, at least for strong interactions, and it seems to offer the best hope for going beyond the nonrigorous, and probably divergent, perturbation solution. It is reasonable, therefore, to look for an alternative framework that will supply a formal basis for these calculations; by placing these calculations on a rigorous footing, one can hope to unite rigor with physics.

An examination of recent practical calculations shows that they are built essentially on the S matrix; they involve, essentially, only the observable physical mass-shell quantities, not hypothetical extensions to nonphysical masses. Consequently, a theory geared to these calculations would evidently avoid two of the difficulties mentioned above: It would give practical calculations and would not be based on conjectural elements. Also, the needed axioms appear less arbitrary, and more reasonable *a priori*.

The proposal that the S matrix, first defined by Wheeler,[2] might be a suitable vehicle for fundamental theory was made by Heisenberg[3] in the early forties. Heisenberg emphasized the two essential properties of unitarity and Lorentz invariance, and also recognized that analyticity would be important. He and other workers of that time were willing to assume modest analyticity properties on the basis of the Schrödinger equation. The present approach goes beyond these early efforts mainly in the more incisive analyticity requirement.

In the field theoretic approach, one of course *derives* analyticity properties from other axioms. However, taken as a group, these other axioms seem at least as doubtful as the analyticity property that will be assumed, which is a simple generalization of a property rigorously established for scattering from a sum of Yukawa potentials. This analyticity postulate is in some sense an S-matrix formulation of Yukawa's original idea.

The domain of analyticity asserted by this postulate is larger than the one so far proved from field theory, but it is also much more limited in that it

* This work was done under the auspices of the U. S. Atomic Energy Commission.
† Invited paper at the New York meeting of the American Physical Society, January 1962.

[1] For a review of the recent developments in this area see G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).
[2] J. A. Wheeler, Phys. Rev. **52**, 1107 (1937).
[3] W. Heisenberg, Z. Physik **120**, 513, 573 (1943).

does not extend off the mass shell. The two theories are therefore quite possibly not equivalent, and it is hoped that the elimination of the restrictive requirement of analyticity at nonphysical values of the masses may permit a physically relevant solution for the S-matrix theory, although none may exist in field theory.

The idea that field theory be abandoned in favor of analyticity requirements on the S matrix has, of course, been pushed vigorously in the past several years, particularly by Landau[4] and Chew.[5] The question is how to cast this idea into a satisfactory concrete formalism.

PROPOSED ASSUMPTIONS FOR AN S-MATRIX THEORY

A major problem in setting up a pure S-matrix theory is that one needs a number of properties of the relativistic S matrix that are usually derived from field theory. These are: the substitution rule, which relates each process to others involving antiparticles; the invariance conditions for charge conjugation, time reversal, etc.; the relativistic spin formalism; the symmetries under exchange of identical particles; and unitarity. One could, of course, simply extract the needed features from field theory while discarding the others. However, one can do much better. The needed properties can be derived from postulates that assert only very general physical principles, which are completely within the S-matrix framework and independent of field theoretic concepts.[6]

The first postulate is basic quantum theory. By this is meant the fundamental connection of the probabilities (of the possible outcomes of various complete sets of experiments) to squares of amplitudes that are linearly related. This linearity means that the amplitudes can be chosen so that those of any complete set of experiments are a linear combination of those of any other complete set. This relationship between probabilities is the fundamental assumption in quantum mechanics, and the assumed linearity provides the basic object of the theory, the S matrix, which is the linear transformation connecting the amplitudes of initial and final complete set of experiments. Neither commutation relations nor Planck's constant is involved in this postulate.

[4] L. D. Landau, in *Ninth International Annual Conference on High Energy Physics, Kiev, 1959* (Academy of Sciences, Moscow, 1960).

[5] G. F. Chew, Lawrence Radiation Laboratory Report UCRL-9289, June 1960 (unpublished); Revs. Modern Phys. following paper, this issue **34**, 394 (1962).

[6] H. P. Stapp, Phys. Rev. **125**, 2139 (1962); Lawrence Radiation Laboratory Report UCRL-9875 (to be published by W. A. Benjamin, Inc., New York).

The second postulate specifies that certain sets of experiments are complete. Specifically, the measurements of the momentum, the spin, and the particle type of all particles present are asserted to be a complete set of experiments. Also, the magnitudes of linear combinations of amplitudes, related by changes in the directions of energy-momentum and spin vectors, are asserted to be observable. This latter allows known interference effects to be considered observable.

The third postulate specifies that the connection of the momentum functions introduced in the first two postulates to space-time coordinates shall be given by a Fourier transformation, where Planck's constant now appears as the scale factor required by dimensional considerations. Since the momentum space variables are subject to the physical mass constraints, the coordinate space functions will represent freely moving physical particles; the S matrix transforms freely moving initial particles to freely moving final ones.

The fourth postulate is relativistic invariance, which is stated directly as a relationship between experimental observables. Correlations between probabilities of outcomes of complete sets of initial and final experiments are asserted to be invariant under Lorentz transformations.

The fifth, and last postulate not related to analyticity, is that the physical interpretation of the quantities of theory be such that translational and rotational invariance imply the conservation laws of energy-momentum and angular-momentum, respectively. This conservation-law postulate will enable us to uniquely specify the physical interpretation of quantities arising by analytic continuation.

These five postulates assert very general physical principles: basic quantum theory, particle observables, Fourier connection between the momentum-energy and space-time coordinates, relativistic invariance, and conservation laws. They are all physical principles in that they are subject to direct experimental tests. They do not have the abstract, artificial, and very specialized character of the axioms of field theory.

The analyticity postulate is formulated as follows. From the above postulates a covariant form of the unitarity relation may be deduced. With an appropriate matrix notation, this can be written in the form

$$M(E+i\epsilon) - M(E-i\epsilon) = \sum\int' M(E'+i\epsilon) \\ M(E'-i\epsilon)\, 2\pi\, \delta(E-E') \,.$$

Here the $M(E)$ are covariant scattering functions, and the integration is over the covariant momentum space elements. This equation can also be expressed in the form

$$M(E) = \int \sum{}' \frac{M(E' + i\epsilon)\, M(E' - i\epsilon)}{i(E' - E)} + M_R(E)\,,$$

where $M_R(E)$ is a remainder function that will be regular in the neighborhood of the physical region. The first term gives the contribution to $M(E)$ associated with the discontinuity across the physical cut, and $M_R(E)$ gives the contribution associated with other singularities, including the possible singularity at infinity. The essential point is that even if the M functions occurring on the right were assumed to be regular in the finite plane, or nonzero constants, the integrated expression would have singularities in the finite plane associated with the vanishing of phase-space factors. These are the simplest of the singularities called "singularities required by unitarity." If one starts with constants for the M functions on the right and then substitutes the calculated M function back into the right, neglecting the unknown M_R, and iterates a finite number of times, the singularities of the resulting functions are those called the singularities required by unitarity. Their positions depend only on the masses of the physical particles. These singularities, which come purely from kinematic phase-space factors, might more accurately be called the singularities "expected" from unitarity, since some sort of cancellation has not been precluded. But our assumption is that, on the physical sheet, no singularities except those "required by unitarity" occur; the possibility that certain of them may not occur is not forbidden.

The physical sheet is defined by allowing the singularities required by unitarity to trace out cuts, using a scale transformation on the internal masses. (This does not entail analyticity in the masses of the actual M functions.) This definition gives a physical sheet in which the scattering functions can be proved to be free of singularities not required by unitarity, for the scattering from a sum of Yukawa potentials. The analyticity postulate states that also in the relativistic many-particle case the covariant scattering functions have no singularities on this physical sheet, aside from those required by unitarity.

The locations of the singularities required by unitarity are specified by the same equations that were derived by Landau for the singularities of the terms of the perturbation solution to field theory. Thus, the analyticity postulate permits all the singularities that occur in the terms of the usual perturbation solution. One expects singularities in addition to those occurring in the terms of the perturbation solution—specifically, the resonance poles. However, in accordance with the situation in potential scattering, these are expected to occur only on unphysical sheets.

The final postulate states that all physical-type points of the physical sheet correspond to processes actually occurring in nature. A physical-type point is a point corresponding to real energy-momentum vectors, and it is to be approached with positive imaginary energy, in accordance with the potential-theory case. This postulate of physical connection requires, then, that points on the physical sheet that are susceptible to physical interpretation do in fact have a physical interpretation. The exact nature of this interpretation is not specified, however. This idea that a single function, analytically continued, will describe several related processes is of course suggested by the example of field theory. But it is also a natural companion to the analyticity postulate that could easily suggest itself to a person not familiar with field theory. That related processes should be connected via analytic continuation is certainly as natural a concept as the one given by field theory.

It might be expected that an appeal to field theory would be necessary to establish the precise way in which the various physical-type points are connected to experiment. But this connection is, in fact, uniquely specified by the other postulates; chiefly analyticity and the conservation laws. One deduces from the S-matrix postulates relationships exactly equivalent to those obtained in field theory. Specifically, the substitution rule, giving the detailed connection between the related particle and antiparticle process, follows directly from the abstract postulates just stated; field theoretic ideas are not required.

The other needed properties of the S matrix also come directly from the S-matrix postulates. Unitarity follows immediately from the first postulate. In the treatment of spin, the postulates lead to a covariant two-component formalism that is equivalent to, but considerably simpler than, the four-component one conventionally derived from field theory. The relativistic treatment of particles of arbitrary spin presents no difficulty, and the unitarity condition is easily placed in a manifestly covariant form, better suited than the usual noncovariant one to dispersion theoretic calculations. Symmetry considerations are also simplified, and the CPT theorem follows rather directly from Lorentz invariance. The symmetry or antisymmetry under interchange of identical particles also comes out.

It is perhaps rather surprising that one is able to obtain such specific results from postulates that appear so general and abstract. The details of the intimate connection between particles and antiparticles is usually thought to emerge from the local character of the basic fields, and symmetries under interchange of identical particles usually come from explicitly postulated commutation relations. Yet our postulates are essentially independent of the concept of space-time points, and nothing like commutation relations are mentioned at all. A very brief sketch of how one is able to get so much from what appears to be so little follows.

DEVELOPMENT OF THE FORMALISM

At the outset there is no condition regarding the order of the variables; there is only a correspondence between experimental results and an unordered set of variables specifying the momentums, spins, and particle types of the particles observed. Once one writes down a function, with these variables in some necessarily particular order, a certain ordering convention is established. By using the assumed analyticity this function can be analytically continued. If the original variables include two, referring to two identical particles in the same spin state, then the continuation in the momentum variables may be carried to a point on the physical sheet at which the variables describing the two particles are interchanged. The postulate of physical connection requires the function at the new point to be related to some physical process. In order to determine what this connection is, the region over which the function represents the original physical process must be specified. One is essentially free here to arbitrarily specify a well-defined ordering for which the given function represents the original processes; one simply sets the function in this region equal to the function that represents the physics. But since this original region is essentially arbitrary, analytic continuations along lines that remain at physical-type points must give functions that continue to represent the original process, with variables changed correspondingly. Consequently, the function at the two points with like variables interchanged must represent the same physical process, provided the points are connected by a curve that stays always at physical-type points. Since the function at the two points must give the same physical observables, its magnitude at the two points must be equal. A consideration of certain interference effects allows the phase factor to be restricted to plus or minus one. Thus, the usual requirement of either symmetry or antisymmetry under interchange of identical variables follows here, principally from analyticity and the postulate specifying the basic observables. Given this start, one can proceed to show that the choice between symmetry and antisymmetry depends only on the particle type, not on the particular position of the variables, or on the particular scattering function in which the variables occur.

If one analytically continues to points that are not connected to the original region by curves containing only physical-type points, the above argument breaks down and the function at the new point is expected to describe some different process. Exactly what this process is, and how it is related to the function at the new point is fixed by analyticity and the conservation laws. To show how this comes about, the analyticity postulate must be stated with somewhat greater precision. In particular, the variables in which the functions are analytic must be specified. These variables are essentially the *components* of the various energy-momentum vectors. However, the scattering functions are defined only over the manifold consistent with conservation laws and mass constraints. Thus, one must introduce new parameters representing the position in this manifold. The precise statement of the analyticity postulate is that the scattering functions are analytic functions of these new variables everywhere in and on the boundary of the physical sheet, except at singularities required by unitarity, and at singularities of the mapping between these variables and momentum-energy variables. Singularities of this second type are essentially spurious since they can be eliminated by changing the mapping.

In terms of the new variables, the mass constraints and the conservation laws are of course identically satisfied. Thus, these constraints will be *formally* maintained at all points arrived at by analytic continuation from points associated with some original process. Therefore, corresponding particles participating in the various related processes must have the same masses. In order to maintain the *physical* conservation law a momentum-energy vector having its sign reversed from what it originally was must refer to a particle in the final state, if it originally referred to a particle in the initial state, and vice versa. Moreover, the two particles referred to must carry opposite units of any additive constant of the motion; otherwise, the conservation law would be violated in one reaction or the other and the function would vanish identically. These arguments, in conjunction, allow one to specify that under the reversal of sign of the momentum-energy vector, the associated particle must be switched between the initial and final states, and also to its antiparticle. The interpretation of the spin states follows from the conservation of angular momentum in a similar way.

This connection between particle and antiparticle reactions, though equivalent to the field theoretic one, comes out in a much simpler form for the case of particles with spin. Usually some rather awkward manipulations involving multiplications by appropriate spinors are required. These manipulations, in effect, eliminate the redundant variables associated with the use of, say, four-component Dirac fields to describe spin 1/2 particles. Only two components are really needed, and the S-matrix approach leads directly to a simple covariant two-component formalism for spin 1/2 particles. For higher spins one gets a covariant description involving only the necessary $(2S + 1)$ components.

A principal triumph of axiomatic field theory is the proof of the normal connection between spin and statistics. This connection follows also from the S-matrix postulates, provided, in addition, that the magnitudes of self-conjugate combinations of particle-antiparticle amplitudes are not in principle unobservable. We know experimentally of certain combinations, the K_1 and K_2, that are in fact observable. This added assumption, which is analogous to one needed until recently in field theoretic proofs, can probably be eliminated if the full power of the analyticity postulate is utilized.

SUMMARY

The general properties of the S matrix usually deduced from field theory can be derived from postulates expressing very general physical principles. This provides a basis for the establishment of S-matrix theory as an independent and self-contained framework for describing elementary-particle physics, a framework suited to the modern practical calculations in this field.

REVIEWS OF MODERN PHYSICS VOLUME 34, NUMBER 3 JULY, 1962

S-Matrix Theory of Strong Interactions without Elementary Particles*†

GEOFFREY F. CHEW

Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California

1. INTRODUCTION

IN this paper I present an indecently optimistic view of strong interaction theory. My belief is that a major breakthrough has occurred and that within a relatively short period we are going to achieve a depth of understanding of strong interactions that a few years ago I, at least, did not expect to see within my lifetime. I know that few of you will be convinced by the arguments given here, but I would be masking my feelings if I were to employ a conventionally cautious attitude in this talk. I am bursting with excitement, as are a number of other theorists in this game.

I present my view of the current situation entirely in terms of the analytically continued S matrix, because there is no other framework that I understand for strong interactions. My oldest and dearest friends tell me that this is a fetish, that field theory is an equally suitable language, but to me the basic strong-interaction concepts, simple and beautiful in a pure S-matrix approach, are weird, if not impossible, for field theory. It must be said, nevertheless, that my own awareness of these concepts was largely achieved through close collaboration with three great experts in field theory, M. L. Goldberger, Francis Low, and Stanley Mandelstam. Each of them has played a major role in the development of the strong interaction theory that I describe,[1] even though the language of my description may be repugnant to them. Murray Gell-Mann, also, although he has not actually published a great deal on the analyticity aspects of strong interactions, has for many years exerted a major positive influence both on the subject and on

* This work was done under the auspices of the U. S. Atomic Energy Commission.
† Invited paper at the New York meeting of the American Physical Society, January 1962.

[1] A brief review of the development of S-matrix theory, with references, may be found in G. F. Chew, *The S-Matrix Theory of Strong Interactions* (W. A. Benjamin and Company, Inc., New York, 1961).