REVIEWS OF

MODERN PHYSICS

VOLUME 34, NUMBER 3 JULY, 1962

The Splitting of the Riemann Tensor

C. LANCZOS

Dublin Institute for Advanced Studies, Dublin, Eire

THE antidual part of the Riemann tensor R_{ijkm} in four dimensions has been investigated earlier by Rainich and Einstein; it is reducible to the contracted curvature tensor R_{ik}. Here the self-dual part of R_{ijkm} is analyzed and reduced to a new tensor of third order H_{ijk} of essentially 16 components. This tensor has the further property that it integrates the field equations of a quadratic action principle by reducing their order from 4 to 2. The relation of this tensor to Dirac's equation of the electron gives added proof of its fundamental significance.

Notations Used

(a) Einstein's sum convention (automatic summing over equal indices).

(b) The symbol "comma α" for covariant differentiation, ∂_α for ordinary differentiation (with respect to x_α).

(c) The permutation tensor

$$\delta_{ijkm} = g^{1/2}\epsilon_{ijkm}$$
$$\delta^{ijkm} = g^{-1/2}\epsilon_{ijkm},$$

where ϵ_{ijkm} is the completely antisymmetric Kronecker symbol and g the determinant of the line element. Owing to this definition we have

$$**F_{ik} = F_{ik}$$

where $F_{ik} = -F_{ki}$ is the electromagnetic field strength and the "dual field strength $*F_{ik}$" is defined by the operation

$$*F_{ik} = \tfrac{1}{2}F^{\mu\nu}\delta_{\mu\nu ik}.$$

In a universe of the signature $- - - +$, the components of δ_{ijkm} become *imaginary* if real coordinates are used (g negative). In Minkowskian coordinates the δ components remain real.

Since the contracted curvature tensor R_{ik} arises by contracting over the second and fourth indices (whereas Einstein contracts over the first and fourth indices), we stay in harmony with Einstein's definition of R_{ik} by defining the full Riemann tensor with a sign opposite to that of Einstein,

$$R_{ijkm} = \tfrac{1}{2}(\partial^2_{jm}g_{ik} + \partial^2_{ik}g_{jm} - \partial^2_{im}g_{jk} - \partial^2_{jk}g_{im}) + \left(\begin{bmatrix} ik \\ \alpha \end{bmatrix}\begin{bmatrix} jm \\ \beta \end{bmatrix} - \begin{bmatrix} jk \\ \alpha \end{bmatrix}\begin{bmatrix} im \\ \beta \end{bmatrix}\right) g^{\alpha\beta}.$$

1. INTRODUCTION

The spectacular discoveries of Einstein in the realm of gravitation were based on the use of the contracted curvature tensor R_{ik}. This led to a certain eclipse of the full Riemann tensor R_{ijkm}. Einstein upheld the fundamental significance of the contracted curvature tensor for the description of the geometrical properties of the physical universe, due to its relation to the matter tensor which could be directly interpreted in terms of momentum and energy. Since the contracted tensor did not contain anything that could be correlated to electric quantities, he was compelled to drop the classical Riemannian geometry in favor of a more restricted (addition of distance parallelism) or more general (unsymmetric line element) form of geometry.

We can hardly doubt that electricity has something to do with the specifically four-dimensional nature of the universe. The double set of Maxwellian equations

$$F^{i\alpha}{}_{,\alpha} = 0$$
$$*F^{i\alpha}{}_{,\alpha} = 0 \qquad (1.1)$$

reveal a remarkable symmetry which cannot be matched in any other number of dimensions. Dirac's equation of the electron has, similarly, a structure for which the dimension number $n = 4$ is of vital significance. The contracted tensor R_{ik} exists equally in all dimensions and shows nothing unusual for $n = 4$. If we want to stay within the confines of Riemannian geometry and yet try to arrive at a rational explanation of electricity, we have to fall back on the full Riemann tensor R_{ijkm}. Can we find something in this tensor that could be correlated to the symmetry pattern of the Maxwell equations (1.1)?

Indeed, we find that for $n = 4$, and *only* for $n = 4$, we can pass from the tensor R_{ijkm} to a dual tensor $*R_{ijkm}$ by a similar operation that leads from F_{ik} to $*F_{ik}$. If we apply the duality operation with respect to both index pairs i,j and k,m:[1]

$$*R^{ijkm} = \tfrac{1}{4}R_{\alpha\beta\mu\nu}\delta^{\alpha\beta ij}\delta^{\mu\nu km} \tag{1.2}$$

we obtain a "dual curvature tensor" that has exactly the same symmetry properties as the original Riemann tensor, with the same number of algebraically independent components. This property of the dual operation is not matched in other dimensions. In all other dimensions but four, the dual operation applied to the curvature tensor would not create an equivalent tensor.

A further remarkable property of the dual tensor is that the Bianchi identity applied to it is reduced to a pure divergence[2]

$$*R^{ijkm}{}_{,m} = 0 . \tag{1.3}$$

Hence, we can dispense with the usual cyclic permutation of three indices demanded in all other dimensions.

The new tensor has 20 algebraically independent components. But, we now form two new tensors by the following construction[3]:

$$A_{ijkm} = R_{ijkm} - *R_{ijkm} \tag{1.4}$$
$$S_{ijkm} = R_{ijkm} + *R_{ijkm}$$

so that

$$R_{ijkm} = \tfrac{1}{2}(A_{ijkm} + S_{ijkm}) . \tag{1.5}$$

We will compare these constructions with the corresponding constructions of electromagnetism. Here we can put

$$A_{ik} = F_{ik} - *F_{ik}$$
$$S_{ik} = F_{ik} + *F_{ik} . \tag{1.7}$$

[1] This notation is in harmony with that used in J. L. Synge, *Relativity, the General Theory* (North-Holland Publishing Company, Amsterdam, 1960), p. 18.

[2] Cf. C. Lanczos, Annals of Math. **39**, 842, (1938), Eq. (4.3). This paper will be quoted as (I); another paper by the author, Revs. Modern Phys. **29**, 337, (1957), will be quoted as (II).

[3] These definitions differ from those of Einstein (cf. footnote 6) by the factor 2.

These new tensors have no longer six but only three algebraically independent components, on account of the conditions

$$*A_{ik} = -A_{ik}$$
$$*S_{ik} = S_{ik} , \tag{1.8}$$

but these three components are now *complex*. For example, employing the notation

$$S_{12} = Q_3 , \quad S_{23} = Q_1 , \quad S_{31} = Q_2$$

and using Minkowskian coordinates we obtain

$$Q_1 = H_1 - iE_1 , \quad Q_2 = H_2 - iE_2 , \quad Q_3 = H_3 - iE_3 ,$$

where $\mathbf{E}$ and $\mathbf{H}$ are the customary electric and magnetic field strengths. The field equation

$$S^{i\alpha}{}_{,\alpha} = 0 \tag{1.9}$$

can now be written in quaternionic form[4]:

$$\nabla Q = \left(\frac{\partial}{\partial x_1}\mathbf{i} + \frac{\partial}{\partial x_2}\mathbf{j} + \frac{\partial}{\partial x_3}\mathbf{k} + \frac{\partial}{\partial x_4}\right) \times \left(Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k}\right) = 0 . \tag{1.10}$$

This single set of complex equations is equivalent to the double set of Maxwell equations (1.1) by separating real and imaginary parts.

The situation is different with respect to the tensors A_{ijkm} and S_{ijkm}. Here we have once more

$$*A_{ijkm} = -A_{ijkm}$$
$$*S_{ijkm} = S_{ijkm} , \tag{1.11}$$

but in view of the *double* application of the tensor S_{ijkm} *the reality of the components of* A_{ijkm} *and* S_{ijkm} *is not altered.* The Riemann tensor R_{ijkm} splits by this construction into *two independent tensors*. We call A_{ijkm} the "antidual" and S_{ijkm} the "self-dual" part of the curvature tensor.

The separation of the original 20 components of R_{ijkm} is not $10 + 10$, but $9 + 11$. We see that by introducing the local coordinates $g_{ik} = \delta_{ik}$ and investigating those components of R_{ijkm} that are different in all their four subscripts. In view of the cyclic identity, which specifically in four dimensions may be written in the scalar form

$$R_{ijkm}\delta^{ijkm} = 0 , \tag{1.12}$$

we have

$$R_{1234} + R_{1342} + R_{1423} = 0 \tag{1.13}$$

and only *two* independent components of such character exist. These two components are *self-dual* and

[4] Cf. L. Silberstein, *The Theory of Relativity* (Macmillan and Company, Ltd., London, 1924), p. 46.

thus contribute two components to the self-dual tensor S_{ijkm}, but cancel out in the case of the antidual tensor A_{ijkm}. This explains the two surplus components of the self-dual tensor. The other nine components are contributed by the six combinations of the type 1213 ± 4243 and the three combinations of the type 1212 ∓ 3434.

The splitting of the Riemann tensor into the tensors (1.4) and (1.5) was first enunciated by Rainich in a brief note[5]. Referring to this communication, Einstein[6] obtained the antidual tensor A_{ijkm} in explicit form by showing that it is reducible to the contracted tensor R_{ik} [the same relation was later found in a different context by the author, who was unaware of Einstein's previous result; cf. (I), Eq. (5.2)]. The equation found by Einstein is as follows:

$$A_{ijkm} = (R_{ik} - \tfrac{1}{4}Rg_{ik})g_{jm} + (R_{jm} - \tfrac{1}{4}Rg_{jm})g_{ik} - (R_{im} - \tfrac{1}{4}Rg_{im})g_{jk} - (R_{jk} - \tfrac{1}{4}Rg_{jk})g_{im} \,. \tag{1.14}$$

This relation can be found by direct verification in a special reference system in which the g_{ik} are reduced to their normal values δ_{ik}. However, in view of the generally covariant nature of the relation, it seems of interest to give a proof that does not depend on the choice of a special reference system. This can be done on the basis of the following property of the permutation tensor. Whereas the δ_{ijkm} in itself is not reducible to the metrical tensor g_{ik}, the *product* of two δ tensors becomes a purely *metrical* quantity. We then obtain a tensor of the order 8 of the following structure:

$$\delta_{\mu\nu ij}\delta_{\rho\sigma km} = [g_{\mu\rho}g_{\nu\sigma}g_{ik}g_{jm}] \,, \tag{1.15}$$

where the bracket [ ] refers to a *sum*, obtained as follows. We keep the subscripts $\mu\nu ij$ fixed, whereas the subscripts $\rho\sigma km$ go through all possible permutations, with a plus sign in front if the permutation is even and a minus sign in front if the permutation is odd. We have to sum over all of these 24 terms.

We now put these terms in four groups by collecting the six terms that belong to $g_{\mu\rho}$ in front, then doing the same with the six terms that belong successively to $g_{\mu\sigma}$, $g_{\mu k}$ and finally $g_{\mu m}$ in front. We then obtain the tensor

$$4{}^*R_{ijkm} = R^{\mu\nu\rho\sigma}\delta_{\mu\nu ij}\delta_{\rho\sigma km} \tag{1.16}$$

as a sum of 24 terms, distributed as follows:

first group ($g_{\mu\rho}$):

$$R\,(g_{ik}g_{jm} - g_{im}g_{jk}) - (R_{ik}g_{jm} + R_{jm}g_{ik} - R_{im}g_{jk} - R_{jk}g_{im}) \tag{1.17}$$

second group ($g_{\mu\sigma}$): the same repeated

third group ($g_{\mu k}$):

$$2R_{ijkm} - 2(R_{ik}g_{jm} - R_{jk}g_{im})$$

fourth group ($g_{\mu m}$):

$$2R_{ijkm} - 2(R_{im}g_{jk} - R_{jm}g_{ik}) \,.$$

Collecting all terms and dividing by four, the relation (1.14) follows, now established on a generally covariant basis.

This equation has a remarkable persistency, inasmuch as exactly the same relation remains valid in both two and three dimensions if the left side is replaced by the *full* Riemann tensor. The dimension $n = 4$ is the first one in which the Riemann tensor is no longer reducible to the contracted tensor. But here the *antidual* part of the Riemann tensor is still reducible to the contracted tensor, by the same formula that in the lower dimensions gave the *full* Riemann tensor.

If we employ the notation

$${}^*R_{ijkm}g^{jm} = {}^*R_{ik} \,, \quad {}^*R_{ik}g^{ik} = {}^*R \tag{1.18}$$

then contraction of (1.14) by g^{jm} yields

$${}^*R_{ik} = -(R_{ik} - \tfrac{1}{2}Rg_{ik}) \,, \tag{1.19}$$

whereas a second contraction yields

$${}^*R = R \,. \tag{1.20}$$

Einstein made use of the decomposition (1.14) in order to show that by a formal extension of the definition of the Riemann tensor one could arrive at the equation

$$R_{ik} - \tfrac{1}{4}Rg_{ik} = -\kappa T_{ik} \,, \tag{1.21}$$

where T_{ik} is Maxwell's energy-momentum tensor

$$T_{ik} = \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{ik} - F_{i\alpha}F_k{}^{\alpha} \,. \tag{1.22}$$

The unusual factor $\frac{1}{4}$ (instead of $\frac{1}{2}$) in (1.21) seemed to be justified by the automatic vanishing of the spur $T_\alpha{}^\alpha$ of the Maxwell tensor. Einstein envisaged the possibility of constructing a stable electron by this combination of gravitation and electromagnetism, assuming that the attracting gravitational effects might balance the repulsive electric forces.[7] The difficulty with Eq. (1.21) is, however, that it is *under-determined*, because the identical vanishing of the spur on both sides of the equation leads to nine instead of ten relations, with the result that *any* distribution of the electric charge density gave a possible static equilibrium. Consequently Einstein aban-

[5] G. Y. Rainich, Nature **115**, 498 (1925).

[6] A. Einstein, Math. Ann. **97**, 99 (1926).

[7] A. Einstein, Sitzber. preuss. Akad. Wiss. 349 (1919).

doned this attempt of bringing electricity within the framework of general relativity and henceforth embarked on a different course. The splitting of the Riemann tensor played no further role in his speculations.

Whereas the investigations of Rainich and Einstein brought clarity to the structure of the anti-dual tensor A_{ijkm}, reducing it to a tensor of second order, the question of the *self-dual* tensor S_{ijkm} remained open. What can we say concerning the structure of this tensor? Is it similarly reducible to some "generating function" of lower order as the antidual tensor A_{ijkm} was reducible to the tensor R_{ik}? As long as this question remains unanswered, we cannot claim that we have truly understood the structure of a four-dimensional Riemannian geometry. The present investigation is devoted to this problem. By finding its solution we arrive at the further result that *exactly those components of the Riemann tensor that are not embraced by the gravitational equations of Einstein, give us the clue toward a deeper understanding of the electromagnetic and wave-mechanical phenomena, within the framework of general relativity.*

2. THE CANONICAL LAGRANGIAN

Einstein has chosen a particularly simple Lagrangian, viz., the scalar Riemannian curvature R, for the variational derivation of his field equations

$$*R_{ik} = -(R_{ik} - \tfrac{1}{2}Rg_{ik}) = 0 . \qquad (2.1)$$

In this singular case the second derivatives of the g_{ik} remain variationally inactive and can be eliminated from the Lagrangian, with the consequence that the resulting field equations are not higher than *second* order. Any other Lagrangian leads to differential equations of *fourth* order in the g_{ik}. Einstein argued (with full justification) that the fundamental field equations of mathematical physics are either of first or of second order. Field equations of fourth order would hardly allow a reasonable physical interpretation.

In actuality, a variational principle is always solvable by field equations of not higher than *first* order. This can be accomplished by the "method of the Lagrangian multiplier," coupled with an increase of variational variables. These Lagrange multipliers are not merely mathematical quantities. Whenever they occur, they possess an important physical significance (for example in the form of a "potential energy" that maintains a given kinematic constraint, as in Hertz' "forceless mechanics,"—a forerunner of general relativity). To oversimplify a Lagrangian may entail the danger of overlooking an important physical quantity which should have come in on account of a constraint imposed as an auxiliary condition of the variational problem. By means of such constraints we can reduce every Lagrangian to first-order equations.

We consider the general problem of deducing field equations from a given Lagrangian, without specifying in advance the Lagrangian we are going to choose. We merely assume that there *exists* a Lagrangian L from which our field equations follow by variation. We consider this L as a given scalar that is constructed from the curvature components R_{ijkm} and the g_{ik}. In this assumption, both linear and quadratic action principles are included.

More specifically, we want to consider the *contravariant* components of the *dual* tensor (1.2) as the fundamental variational variables, together with the covariant g_{ik}. We thus put

$$A = \int Lg^{1/2}dx_1dx_2dx_3dx_4 \qquad (2.2)$$

with

$$L = L(*R^{ijkm}, g_{ik}) . \qquad (2.3)$$

Although in reality $*R^{ijkm}$ is a complicated differential operator of the second order in the g_{ik}, we handle it for the purposes of variation as a mere *algebraic* quantity, preserving its algebraic symmetry properties. This is permitted, provided that we take care of the limitations that are in the way of a free variation, caused by the fact that $*R^{ijkm}$ is in reality deducible by differentiation from the g_{ik}. As we have seen before, the tensor $*R^{ijkm}$ has to satisfy the Bianchi identity (1.3). This has the consequence that the *variation* of $*R^{ijkm}$ must satisfy the following linear tensor equation:

$$(\delta *R^{ijkm})_{,m} + (\delta\Gamma^{i}_{\alpha m})*R^{\alpha jkm} - (\delta\Gamma^{j}_{\alpha m})*R^{\alpha ikm} + (\delta\Gamma^{m}_{\alpha m})*R^{ijk\alpha} = 0 . \qquad (2.4)$$

On the other hand, this is the *only* condition to which the variation of $*R^{ijkm}$ is subjected. But this condition brings into evidence the variation of the Γ quantities (which have tensor character, although the Γ^{j}_{ik} themselves are not components of a tensor). These variations are again not free from their part but restricted due to the definition of the Γ^{j}_{ik} in terms of the Christoffel symbols

$$\Gamma^{j}_{ik} - \begin{Bmatrix} ik \\ j \end{Bmatrix} = 0 . \qquad (2.5)$$

We have to consider these equations as auxiliary conditions of the variation. Then the condition (2.4) involves the $\delta *R^{ijkm}$ and the δg_{ik}. These variations can again not be considered as independent from each other. If the δ^*R^{ijkm} are given, the variations of the g_{ik} are already determined and this relation has to be expressed. For this purpose, however, it is unnecessary to consider the differential expression that defines the *full* Riemann tensor. It suffices to operate with the *contracted* curvature tensor, as Einstein has done. The matter tensor can be considered as the basic field quantity from which the metrical tensor is obtainable. Although this problem is not well defined because of its highly nonlinear character, here we are only concerned with the relation between the *variation* of the matter tensor and the corresponding *variation* of the g_{ik}. This is a purely linear problem that establishes a one-to-one correspondence between the two kinds of variations; if the variation of the g_{ik} is given, the variation of $^*R_{ik}$ becomes a linear second-order differential operator of the δg_{ik}. On the other hand, if the variation of $^*R_{ik}$ is given, we can solve our differential equation with the help of the Green's function and obtain the δg_{ik} in terms of an integral operator, operating on the δ^*R_{ik}. The condition is that the *homogeneous* differential equation (with zero on the right side) shall have no nontrivial solutions under the given boundary conditions. This we want to assume (mere coordinate transformations, involving a free infinitesimal vector, remain, but are irrelevant for our purposes since they do not cause a variation of the fundamental action integral).

Hence, it suffices to introduce as an auxiliary condition of the variation Einstein's contracted tensor

$$R_{ik} \equiv F(\Gamma^j_{ik}) = \tfrac{1}{2}(\partial_i \Gamma^\alpha_{k\alpha} + \partial_k \Gamma^\alpha_{i\alpha}) - \partial_\alpha \Gamma^\alpha_{ik} + \Gamma^\alpha_{i\beta}\Gamma^\beta_{k\alpha} - \Gamma^\alpha_{ik}\Gamma^\beta_{\beta\alpha}\,, \tag{2.6}$$

which involves solely the Γ^j_{ik} and their first derivatives, whereas the Γ^j_{ik} themselves are reduced to the g_{ik}, on account of the auxiliary condition (2.5).

On the basis of this discussion we can write down the full canonical Lagrangian, which contains nothing but *first-order derivatives*, as follows:

$$L' = L(^*R^{ijkm}, g_{ik}) + H_{ijk}{}^*R^{ijkm}{}_{,m} + P^{ik}_j(\Gamma^j_{ik} - \{^{ik}_{j}\}) + \rho^{ik}[R_{ik} - F(\Gamma^j_{ik})]\,. \tag{2.7}$$

The canonical variables of our problem and their conjugates are

$$\begin{pmatrix} g_{ik}, & \Gamma^j_{ik}, & ^*R^{ijkm} \\ \rho^{ik}, & P^{ik}{}_j, & H_{ijk} \end{pmatrix}. \tag{2.8}$$

The tensors ρ^{ik} and $P^{ik}{}_j$ are symmetric in i, k, whereas the tensor H_{ijk} is antisymmetric in i, j.

We observe the peculiarity that the conjugate of $^*R^{ijkm}$ (20 independent components) is the tensor of third-order H_{ijk}, with 24 independent components. We can show, however, that the tensor H_{ijk} may be restricted without any loss of generality by the condition

$$^*H^{\alpha j}{}_\alpha = H_{\mu\nu\alpha}\delta^{\mu\nu\alpha j} = 0 \tag{2.9}$$

for which we may also write

$$H_{ijk} + H_{jki} + H_{kij} = 0\,. \tag{2.10}$$

In view of this condition, the number of algebraically independent components is reduced to 20, which exactly balances the number of independent components of the fourth-order tensor $^*R^{ijkm}$.

To perform the variation and obtain the field equations of our problem is now a mere routine procedure. Since the Lagrangian L does not depend explicitly on Γ^j_{ik}, the variation of the Γ^j_{ik} yields a relation for the tensor $P^{ik}{}_j$ that can be stated explicitly, independently of the form of L. We make a slight modification in ρ^{ik} by splitting it as follows:

$$\rho^{ik} = Q^{ik} + qg^{ik} \tag{2.11}$$

where the scalar q is defined by

$$4q = \rho^{ik}g_{ik} \tag{2.12}$$

due to which

$$Q^i_i = Q^{ik}g_{ik} = 0\,. \tag{2.13}$$

The variation of Γ^j_{ik} now yields:

$$P^{ik}{}_j = Q^{ik}{}_{,j} + q_{,j}\,g^{ik} - H_{\alpha j\beta}\,(^*R^{i\alpha k\beta} + {}^*R^{k\alpha i\beta}) - \tfrac{1}{2}\delta^i_j\,(H_{\alpha\beta\gamma}{}^*R^{\alpha\beta\gamma k} + Q^{k\alpha}{}_{,\alpha} + q_{,\alpha}\,g^{k\alpha}) - \tfrac{1}{2}\delta^k_j(H_{\alpha\beta\gamma}{}^*R^{\alpha\beta\gamma i} + Q^{i\alpha}{}_{,\alpha} + q_{,\alpha}\,g^{i\alpha})\,. \tag{2.14}$$

We see that the canonical variable $P^{ik}{}_j$ is not a genuinely new quantity, being explicitly expressible in terms of the other variables.

The variation of g_{ik} yields the following relation:

$$\partial L/\partial g_{ik} + \tfrac{1}{2}\,Lg_{ik} = Q_{\alpha\beta}{}^*R^{\alpha i\beta k} + Q^{i\alpha *}R^k_\alpha + Q^{k\alpha *}R^i_\alpha - \tfrac{1}{2}\,^*RQ^{ik} - q^*R^{ik} - \tfrac{1}{2}\,^*Rqg^{ik} - \tfrac{1}{2}\,(P^{i\alpha k} + P^{k\alpha i} - P^{ik\alpha})_{,\alpha}\,. \tag{2.15}$$

The factor of δ^*R^{ijkm} becomes

$$B_{ijkm} \equiv \partial L/\partial^*R^{ijkm} - H_{ijk,m} - Q_{ik}g_{jm} + qg_{ik}g_{jm}\,. \tag{2.16}$$

This quantity cannot be directly equated to zero because we have to take into account the algebraic symmetry properties of δ^*R^{ijkm}. The symmetrization demands the following operation:

$$\tfrac{1}{8}\,[B_{ijkm} + B_{kmij} + B_{jimk} + B_{mkji} - B_{jikm} - B_{kmji} - B_{ijmk} - B_{mkij} - \tfrac{1}{3}\,(B_{\alpha\beta\mu\nu}\delta^{\alpha\beta\mu\nu})\delta_{ijkm}]\,. \tag{2.17}$$

In our case the last term always vanishes and thus it suffices to introduce the following bracket symbol:

$$[B_{ijkm}] = \tfrac{1}{2}\,(B_{ijkm} + B_{kmij} + B_{jimk} + B_{mkji} - B_{jikm} - B_{kmji} - B_{ijmk} - B_{mkij}) \qquad (2.18)$$

(The change of the factor $\frac{1}{8}$ to $\frac{1}{2}$ is motivated by the fact that in all our applications of this formula the eight terms pair off into four terms and the numerical factor will thus disappear in the final formula. We have to remember, however, that in the case that B_{ijmk} happens to possess automatically the symmetry properties of $^*R_{ijmk}$, the bracketing amounts to a multiplication of B_{ijmk} by the factor 4.) With this notation, the resulting equation obtained by varying $^*R^{ijkm}$ becomes

$$[\partial L/\partial {}^*R^{ijkm}] = [H_{ijk,\,m} + Q_{ik}g_{jm} - qg_{ik}g_{jm}]\,. \quad (2.19)$$

Equation (2.19) allows the following conclusion. We perform the following transformation:

$$H_{ijk} = H'_{ijk} - \Phi_j g_{ik} + \Phi_i g_{jk}$$
$$Q_{ik} = Q'_{ik} + \Phi_{i,k} + \Phi_{k,i} - \tfrac{1}{2}\Phi^{\alpha}_{,\alpha} g_{ik}$$
$$q = q' - \tfrac{1}{2}\Phi^{\alpha}_{,\alpha}\,. \qquad (2.20)$$

Then the relation (2.19) remains unchanged. The freedom of choosing the vector Φ_i at will can be used for a further *normalization* of the tensor H_{ijk}. By choosing

$$\Phi_j = -\tfrac{1}{3}H_{ijk}g^{ik} \qquad (2.21)$$

we obtain

$$H'_{ijk}g^{ik} = 0\,. \qquad (2.22)$$

Assuming that this transformation has been accomplished, we *omit* the primes and submit H_{ijk} to the vectorial condition

$$H_{ijk}g^{ik} = H^{\alpha}_{j\alpha} = 0\,. \qquad (2.23)$$

Hence, the number of independent components of H_{ijk} is now reduced from 20 to 16.

It is of interest to observe that in the case that the tensor H_{ijk} happens to be the gradient of an antisymmetric tensor of second order:

$$H_{ijk} = F_{ij,k} \qquad (2.24)$$

the conditions (2.23) and (2.10) *coincide with the double set of Maxwellian equations* (1.1).

3. THE FUNDAMENTAL TENSOR H_{ijk}

A previous investigation of the author [Cf. (I), Eqs. (4.8)–(4.15)] has shown that the following Lagrangian:

$$L = R_{ijkm}{}^*R^{ijkm} \qquad (3.1)$$

yields a variation which *vanishes identically*. Hence, we cannot use this Lagrangian for the derivation of field equations. But exactly for this reason we have here a variational property which characterizes *all* Riemannian geometries of four dimensions. The consequences of this action principle are thus *universally* valid, without prejudicing the geometry by a definite set of field equations. We will see what consequences we can draw from this variational principle concerning the structure of an arbitrary Riemannian geometry of four dimensions.

We choose our L in the form

$$L = \tfrac{1}{32}\,{}^*R^{ijkm}\,{}^*R^{abcd}\delta_{abij}\delta_{cdkm} = \tfrac{1}{8}\,{}^*R^{ijkm}R_{ijkm}\,. \qquad (3.2)$$

Then

$$\partial L/\partial {}^*R^{ijkm} = \tfrac{1}{4}R_{ijkm} \qquad (3.3)$$

and Eq. (2.19) yields

$$R_{ijkm} = [H_{ijk,m} + (Q_{ik} - qg_{ik})g_{jm}]\,. \qquad (3.4)$$

We make use of the notation

$$H_{ik} = H_{i\ \ k,\alpha}^{\ \alpha} \qquad (3.5)$$

and multiply (3.4) by g^{jm}. This yields

$$R_{ik} = H_{ik} + H_{ki} + 2Q_{ik} - 6qg_{ik}\,. \qquad (3.6)$$

A second contraction by g^{ik} yields

$$R = -24q\,. \qquad (3.7)$$

We see that the tensor Q_{ik} (and the scalar q) are algebraically reducible to the other canonical variables and can thus be eliminated. If we perform this elimination and substitute back in (3.4), we obtain the following relation:

$$2R_{ijkm} - [(R_{ik} - \tfrac{1}{6}Rg_{ik})g_{jm}] = [2H_{ijk,m} - (H_{ik} + H_{ki})g_{jm}]\,. \qquad (3.8)$$

At this point we return to our previous splitting of the Riemann tensor, but with a slight modification. Instead of the 9 + 11 splitting considered in Sec. I we now make a 10 + 10 splitting by introducing the following two tensors:

$$U_{ijkm} = A_{ijkm} + \tfrac{1}{6}R(g_{ik}g_{jm} - g_{im}g_{jk})$$
$$V_{ijkm} = S_{ijkm} - \tfrac{1}{6}R(g_{ik}g_{jm} - g_{im}g_{jk})\,. \qquad (3.9)$$

Once more

$$R_{ijkm} = \tfrac{1}{2}(U_{ijkm} + V_{ijkm})\,. \qquad (3.10)$$

The added term has the effect that now

$$V_{ijkm}g^{ik} = V_{ijkm}g^{ik}g^{jm} = 0 \qquad (3.11)$$

whereas $U_{ijkm}g^{ik}g^{jm} \neq 0$. We have thus *added* one component to the previous tensor A_{ijkm} and *subtracted* one component from the tensor S_{ijkm}, thus balancing the number of components in both cases to 10.

Now the relation (1.14), if formulated for the new tensor U_{ijkm}, becomes

$$\begin{aligned} U_{ijkm} = & (R_{ik} - \tfrac{1}{6}Rg_{ik})g_{jm} + (R_{jm} - \tfrac{1}{6}Rg_{jm})g_{ik} \\ & - (R_{im} - \tfrac{1}{6}Rg_{im})g_{jk} - (R_{jk} - \tfrac{1}{6}Rg_{jk})g_{im} \\ = & [(R_{ik} - \tfrac{1}{6}Rg_{ik})g_{jm}] \,. \end{aligned} \tag{3.12}$$

Subtraction from (3.8) yields

$$V_{ijkm} = [2H_{ijk,m} - (H_{ik} + H_{ki})g_{jm}] \,. \tag{3.13}$$

We have thus obtained the self-dual tensor V_{ijkm} in terms of a "generating function" H_{ijk}, in a similar way as the tensor U_{ijkm} could be generated by means of the generating function $R_{ik} - \frac{1}{6}Rg_{ik}$. The essential difference is, however, that in the latter case the operation is *purely algebraic*, while the generation of the tensor V_{ijkm} demands that we *differentiate* the generating function H_{ijk}.

The tensor V_{ijkm} contains exactly those components of the full Riemann tensor which are not reducible to the contracted tensor R_{ik}. The generation of this tensor in terms of H_{ijk} brings into evidence the existence of a tensor of third order, (antisymmetric in i,j), which is present in every Riemannian geometry of four dimensions[8]), without adding new elements to it, or modifying it by generalizations. In his "distance parallelism" Einstein encountered a similar tensor of third order (denoted by Λ^k_{ij}), which had to be differentiated but *once* in order to obtain the basic curvature quantities. In Einstein's later theories, something similar happened with the antisymmetric part of the Γ^i_{ik} that similarly formed a tensor. Yet, it is unnecessary to abandon Riemannian geometry for the emergence of such a tensor. Although the g_{ik} form the unique basis of a Riemannian geometry, yet the tensor H_{ijk} is an added element because we cannot reduce it *locally* to the line element g_{ik} and its derivatives. It is reducible to the g_{ik} only by an *integral* operation, i.e., the value of H_{ijk} depends *globally* on the geometry of the manifold. And yet, the tensor H_{ijk} participates *locally* in the formation of the field equations.

It is clear that the 10 equations (3.13) cannot be sufficient for a unique characterization of the 16 components of H_{ijk}, if our aim is to establish a unique relation between H_{ijk} and a given Riemannian geometry. The six quantities

$$H_{ij}{}^{\alpha}{}_{,\alpha} = h_{ij} \,, \tag{3.14}$$

which form the components of an antisymmetric tensor of second order, do not participate in the operation contained on the right side of (3.4). If we normalize this tensor to zero by the condition

$$H_{ij}{}^{\alpha}{}_{,\alpha} = 0 \tag{3.15}$$

we add six equations to the previous 10 equations (3.13), and now the relation between the tensor H_{ijk} and a given Riemannian geometry becomes unique. This condition is not provided by our variational principle and for a good reason. The variational principle demands more than the fulfilment of Eqs. (3.13). Equations (2.15), obtained by varying the g_{ik}, must also be satisfied and this demands ten more relations. Without the six degrees of freedom (3.14) and the free vector Φ_i that appeared in (2.20), we would not be able to satisfy these extra conditions. But if our aim is merely to find a generating function for the tensor V_{ijkm}, we could add the condition (3.15) to the normalization of H_{ijk}, thus obtaining 16 differential equations for 16 quantities. The solution of the problem is then reducible to the invariant Laplacean operator

$$\Delta = {}_{,\alpha\beta}g^{\alpha\beta} \,. \tag{3.16}$$

Differentiating (3.13) with respect to m (after raising the subscript m to an upper position), the following result is obtained:

$$\begin{aligned} \Delta H_{ijk} & + R_{i\alpha k\beta}H^{\beta\alpha}{}_{j} - R_{j\alpha k\beta}H^{\beta\alpha}{}_{i} + \tfrac{1}{2}R_{ij\alpha\beta}H^{\alpha\beta}{}_{k} \\ & + R^{\alpha}_{k}H_{ij\alpha} + R^{\alpha}_{i}H_{k\alpha j} - R^{\alpha}_{j}H_{k\alpha i} \\ & + \tfrac{1}{2}(R_{j\alpha\beta\gamma}g_{ik} - R_{i\alpha\beta\gamma}g_{jk})H^{\beta\gamma\alpha} \\ & = \tfrac{1}{2}[(R_{ik} - \tfrac{1}{6}Rg_{ik})_{,j} - (R_{jk} - \tfrac{1}{6}Rg_{jk})_{,i}] \,. \end{aligned} \tag{3.17}$$

As an example, let us consider the highly simplified problem of *infinitesimal fields*, i.e., a metric which differs from the Euclidean (Minkowskian) values by an arbitrarily small amount throughout space

$$g_{ik} = \delta_{ik} + \gamma_{ik} \,. \tag{3.18}$$

In this case covariant and ordinary differentiations coincide, and all the second-order terms on the left of (3.17) become negligible. The resulting equation becomes

$$\Delta H_{ijk} = \tfrac{1}{2}(R_{ik} - \tfrac{1}{6}R\delta_{ik})_{,j} - \tfrac{1}{2}(R_{jk} - \tfrac{1}{6}R\delta_{jk})_{,i} \,, \tag{3.19}$$

which is solvable by putting

$$\begin{aligned} H_{ijk} &= P_{ik,j} - P_{jk,i} \\ \Delta P_{ik} &= \tfrac{1}{2}(R_{ik} - \tfrac{1}{6}R\delta_{ik}) \,. \end{aligned} \tag{3.20}$$

[8] This tensor appeared for the first time (but restricted to infinitesimal fields) in the author's paper, Revs. Modern Phys. **21**, 497, (1949).

Now, if we adopt Einstein's coordinate condition for infinitesimal fields,

$$\gamma_{i\alpha,\alpha} - \tfrac{1}{2}\gamma_{,i} = 0 \qquad (3.21)$$

(with $\gamma = \gamma_{\alpha\alpha}$), then the contracted tensor R_{ik} becomes

$$R_{ik} = \tfrac{1}{2}\,\Delta\gamma_{ik}\,, \qquad (3.22)$$

and we get

$$P_{ik} = \tfrac{1}{4}\,(\gamma_{ik} - \tfrac{1}{6}\,\gamma\delta_{ik})\,. \qquad (3.23)$$

Moreover,

$$H_{ik} = H_{ki} = H_{i\alpha k,\alpha} = \tfrac{1}{4}\,\Delta(\gamma_{ik} - \tfrac{1}{6}\,\gamma\delta_{ik}) - \tfrac{1}{12}\,\gamma_{,ik}\,. \qquad (3.24)$$

Going back to (3.8) we now obtain

$$R_{ijkm} = [H_{ijk,m} + \tfrac{1}{12}\,\gamma_{,ik}\delta_{jm}] = \tfrac{1}{2}\,(\gamma_{ik,jm} + \gamma_{jm,ik} - \gamma_{im,jk} - \gamma_{ik,jm})\,, \qquad (3.25)$$

which is, in fact, the correct expression for the curvature tensor in the case of infinitesimal fields. In this particular instance the tensor H_{ijk} becomes *locally* expressible in terms of the g_{ik}

$$H_{ijk} = \tfrac{1}{4}\,(\gamma_{ik,j} - \gamma_{jk,i} - \tfrac{1}{6}\,\gamma_{,j}\delta_{ik} + \tfrac{1}{6}\,\gamma_{,i}\delta_{jk})\,. \qquad (3.26)$$

4. THE QUADRATIC ACTION PRINCIPLE

We now employ a Lagrangian that is no longer vacuous from the standpoint of field equations. We assume that our action principle is *quadratic* in the curvature components. The most general form of such an action principle [cf. (II), Eq. (5.1)] contains one unknown numerical constant and may be put in the following form

$$L = \tfrac{1}{2}\,[R_{ik}R^{ik} - (\gamma + \tfrac{1}{3})R^2]\,, \qquad (4.1)$$

where γ is an undetermined numerical factor [the previous β being replaced by $-(\gamma + \frac{1}{3})$].

We have, however, our vacuous invariant (3.1) at our disposal, whose variation vanishes identically and which can thus be freely added (multiplied by an arbitrary constant) to our Lagrangian (4.1). It will be our aim to choose the resultant L (without encroaching on its generality) in such a form that the resulting field equations should become particularly simple.

The geometry of our manifold is determined by equating the factor of δg_{ik} to zero. This gives Eq. (2.15). In this equation the symmetric tensor Q_{ik} plays a fundamental role. Since $P^{ik}{}_j$, defined by (2.14), already contains the first derivatives of Q_{ik} and Eq. (2.15) demands the first derivatives of $P^{ik}{}_j$, we see that the fundamental metric equation depends on the *second derivatives* of Q_{ik}. If Q_{ik} depends on the curvature tensor R_{ik}, we wind up with a second-order equation for R_{ik}, which means a fourth-order equation in the g_{ik}. If, however, Q_{ik} is *independent* of R_{ik}, then the equation is greatly simplified and we obtain —apart from small correction terms which involve the first derivatives of R_{ik}—an essentially *algebraic* determination of R_{ik} in terms of the generating function H_{ijk}. Hence, we have reduced our originally fourth-order equation in g_{ik} to an essentially *second-order* equation, in harmony with Einstein's program that endeavors to express the matter tensor explicitly in other physical quantities. In this case the tensor H_{ijk} becomes a genuine *integrating function of the field equations.*

There exists, indeed, a definite Lagrangian by which this program can be achieved. We choose L in the following form:

$$L = \tfrac{1}{8}\,(R_{ijkm} + {}^*R_{ijkm})^*R^{ijkm} - \tfrac{1}{2}(\gamma + \tfrac{1}{12})^*R^2\,. \qquad (4.2)$$

Here

$$\partial L/\partial^*R^{ijkm} = \tfrac{1}{4}\,(R_{ijkm} + {}^*R_{ijkm}) - \tfrac{1}{12}\,{}^*Rg_{ik}g_{jm} - \gamma{}^*Rg_{ik}g_{jm} \qquad (4.3)$$

Now Eq. (2.19) becomes

$$R_{ijkm} + {}^*R_{ijkm} - \tfrac{1}{6}\,{}^*R(g_{ik}g_{jm} - g_{im}g_{jk}) = [H_{ijk,m} + Q_{ik}g_{jm} - (q - \gamma R)g_{ik}g_{jm}]\,, \qquad (4.4)$$

or, in view of the definition (3.9) of the self-dual tensor V_{ijkm},

$$V_{ijkm} = [H_{ijk,m} + Q_{ik}g_{jm} - (q - \gamma R)g_{ik}g_{jm}] \qquad (4.5)$$

Contraction over j, m and later over i, k yields

$$2Q_{ik} = -(H_{ik} + H_{ki}) \qquad (4.6)$$

$$q = \gamma{}^*R\,. \qquad (4.7)$$

As far as the tensor V_{ijkm} is concerned, we are back at Eq. (3.13), *but the tensor Q_{ik} does not depend anymore on R_{ik}.* The coupling between the matter tensor and the tensor H_{ijk}, expressed by Eq. (2.15), ceases to be a differential equation of second order in ${}^*R_{ik}$. It becomes *algebraic in ${}^*R_{ik}$, and because of this we can express the matter tensor ${}^*R_{ik}$ explicitly in terms of the fundamental tensor H_{ijk}.*

We have to pay closer attention to Eq. (2.15). It is this equation which determines the Riemannian metric realized in the physical universe. In view of the transformation (2.20) we have to correct the expressions obtained in (4.6) and (4.7) and substitute for Q_{ik} and q the following expressions:

$$Q_{ik} = -\tfrac{1}{2}\,(H_{ik} + H_{ki}) + \Phi_{i,k} + \Phi_{k,i} - \tfrac{1}{2}\,\Phi^{\alpha}_{,\alpha}g_{ik} \qquad (4.8)$$

$$q = \gamma{}^*R - \tfrac{1}{2}\,\Phi^{\alpha}_{,\alpha}\,, \qquad (4.9)$$

where the vector Φ_i is a free integrating function. Moreover, since our aim is to reduce $^*R_{ik}$ to the tensor H_{ijk}, we will eliminate the full Riemann tensor $^*R_{ijkm}$ by expressing it in terms of the tensor V_{ijkm} [which according to (3.13) depends solely on H_{ijk}] and $^*R_{ik}$. From the (3.12) and (3.13) we obtain

$$^*R_{ijkm} = \tfrac{1}{2} V_{ijkm} + [\tfrac{1}{2}(^*R_{ik} - \tfrac{1}{6}{}^*Rg_{ik})g_{jm}] . \qquad (4.10)$$

We will now evaluate the left side of Eq. (2.15) for our Lagrangian (4.2). We obtain

$$\partial L/\partial g_{ik} = \tfrac{1}{8}(R_{\alpha\beta\gamma\delta}{}^*R^{\alpha\beta\gamma\delta})g^{ik} + \tfrac{1}{2}{}^*R^{i\alpha\beta\gamma}{}^*R^k_{\alpha\beta\gamma} - \tfrac{1}{6}{}^*R^*R^{ik} - 2\gamma^*R^*R^{ik} . \qquad (4.11)$$

The second term on the right side can be transformed in view of the identity [cf. (I), Eq. (4.10)]

$$R^{i\alpha\beta\gamma}{}^*R^k{}_{\alpha\beta\gamma} = \tfrac{1}{4}(R_{\alpha\beta\gamma\delta}{}^*R^{\alpha\beta\gamma\delta})g^{ik} \qquad (4.12)$$

and, eliminating the full Riemann tensor with the help of (4.10), the final result becomes

$$\partial L/\partial g_{ik} = \tfrac{1}{16}V^2 g^{ik} + \tfrac{1}{2}{}^*R_{\alpha\beta}V^{i\alpha k\beta} - \tfrac{1}{6}{}^*R(^*R^{ik} - \tfrac{1}{4}{}^*Rg^{ik}) - 2\gamma^*R^*R^{ik} , \qquad (4.13)$$

while the Lagrangian L becomes

$$L = \tfrac{1}{16}V^2 - \tfrac{1}{2}\gamma^*R^2 , \qquad (4.14)$$

where we made use of the abbreviation

$$V^2 = V_{\alpha\beta\gamma\delta}V^{\alpha\beta\gamma\delta} . \qquad (4.15)$$

Hence, the left side of (2.15) becomes

$$\tfrac{3}{32}V^2 g^{ik} + \tfrac{1}{2}{}^*R_{\alpha\beta}V^{i\alpha k\beta} - \tfrac{1}{6}{}^*R(^*R^{ik} - \tfrac{1}{4}{}^*Rg^{ik}) - 2\gamma^*R^*R^{ik} - \tfrac{1}{4}\gamma^*R^2 g^{ik} . \qquad (4.16)$$

Let us rearrange Eq. (2.15) by bringing over to the left side all the terms which do *not* contain the integrating functions H_{ijk} and Φ_i, and collecting all the other terms on the right side. Then the left side becomes

$$-(\gamma + \tfrac{1}{6})^*R(^*R^{ik} - \tfrac{1}{4}{}^*Rg^{ik}) . \qquad (4.17)$$

We know from the general theory of a quadratic action principle that the scalar curvature $^*R$ satisfies the potential equation [cf. (II), Eq. (5.22)]. Since we want to exclude nontrivial solutions of a differential equation that is free of sources (right side zero), we come to the conclusion

$$^*R = \text{const} \qquad (4.18)$$

(this constant may be very large). Then we can divide the entire equation by a constant, obtaining the modified matter tensor

$$G^{ik} = {}^*R^{ik} - \tfrac{1}{4}{}^*Rg^{ik} = -(R^{ik} - \tfrac{1}{4}Rg^{ik}) \qquad (4.19)$$

in terms of the tensor H_{ijk} (and the vector Φ_i),

$$G^{ik} = \kappa T^{ik} , \qquad (4.20)$$

where

$$\kappa = -1/(\gamma + \tfrac{1}{6})^*R . \qquad (4.21)$$

The expression for T^{ik} is by far more complicated than Maxwell's energy-momentum tensor, and demands further study. But the spur $T^\alpha{}_\alpha$ of that tensor is once more zero.

The fact that the fundamental metrical equation appears in terms of the modified matter tensor (4.19), in harmony with the astonishing hunch of Einstein [cf. (1.21)], is related to the well-known general feature of the quadratic action principle that not only the solution $R_{ik} = 0$, but also the "cosmological solution,"

$$R_{ik} + \lambda g_{ik} = 0 , \qquad (4.22)$$

is an exact first integral of the field equations. At the same time we are not hampered by the under determination of our system since Eq. (4.20) is not the *only* equation we have to satisfy. The general structure of the theory provides us with the added equation $^*R =$ const by which the under determination is removed.

5. THE DIRAC EQUATION

We will study Eq. (3.13)—which is the basic field equation for the determination of H_{ijk}—by restricting ourselves to *weak* metrical fields, in which the deviation of the g_{ik} from the Euclidean normal values δ_{ik} is negligibly small. Moreover, we want to consider the *homogeneous* case when the right side of the equations *vanishes*, in analogy to the Einstein equations $R_{ik} = 0$ (we know, of course, that the right side cannot be zero *everywhere*, yet it may be zero *almost* everywhere, excepting small islands of space). We will thus consider the ten equations

$$\tfrac{1}{2}V_{ijkm} = H_{ijk,m} - H_{ijm,k} + H_{kmi,j} - H_{kmj,i} - \tfrac{1}{2}(H_{ik} + H_{ki})g_{jm} - \tfrac{1}{2}(H_{jm} + H_{mj})g_{ik} + \tfrac{1}{2}(H_{im} + H_{mi})g_{jk} + \tfrac{1}{2}(H_{jk} + H_{kj})g_{im} = 0 . \qquad (5.1)$$

To these we add the six divergence equations (3.14), again assuming that the right side vanishes. This means

$$H_{ij}{}^\alpha{}_{,\alpha} = 0 \qquad (5.2)$$

In view of the relation

$$(H_{ij\alpha} + H_{j\alpha i} + H_{\alpha ij})_{,\beta}g^{\alpha\beta} = 0 \qquad (5.3)$$

[cf. (2.10)], we see that the condition (5.2) implies

$$H_{ik} - H_{ki} = 0 , \qquad (5.4)$$

that is, the tensor $H_{ik} = H_i{}^{\alpha}{}_{k,\alpha}$ becomes *symmetric.* Hence, in Eq. (5.1) we do not have to distinguish between H_{ik} and H_{ki}.

Under the greatly simplified conditions of our problem, covariant and contravariant components coincide, and the tensor g_{ik} is replaceable by δ_{ik}. The expression for the tensor V_{ijkm} is then greatly simplified. By forming the sum

$$V_{ijkm} = \tfrac{1}{2}(V_{ijkm} + {}^*V_{ijkm}) \,,$$

we see at once that we can now write

$$V_{ijkm} = 2(H_{ijk,m} - H_{ijm,k} + H_{\mathbf{ijk},\mathbf{m}} - H_{\mathbf{ijm},\mathbf{k}}) \quad (5.5)$$

where the bold face index pair shall signify that the *dual* pair is to be taken; (e.g., **12** = 34, **13** = 42, etc.). Moreover, the "comma" now means ordinary differentiation. We observe, furthermore, that the components of H_{ijk} in our Minkowskian world are partly real, partly imaginary, according to whether the subscripts are 1,2,3, or 4; (*two* fours = real). If now we add to V_{ijkm} its dual $V_{\mathbf{ij}km}$, we obtain the sum of a real and imaginary quantity, i.e., a *complex* number. Hence, we can replace the ten *real* equations $V_{ijkm} = 0$ by the five *complex* equations

$$V_{ijkm} + V_{\mathbf{ij}km} = 0 \quad (5.6)$$

to which we add the three complex equations

$$h_{ij} + h_{\mathbf{ij}} = 0 \,. \quad (5.7)$$

The eight complex equations, (5.6) and (5.7) take now the place of the original 16 equations. (5.1) and (5.2).

Now the expressions (5.5) show that we obtain

$$\begin{aligned}(ijkm) &= V_{ijkm} + V_{\mathbf{ij}km} \\ &= C_{ijk,m} - C_{ijm,k} + C_{ij\mathbf{k},\mathbf{m}} - C_{ij\mathbf{m},\mathbf{k}} \quad (5.8)\end{aligned}$$

where the components C_{ijk} have complex values, defined as follows:

$$C_{ijk} = 2(H_{ijk} + H_{\mathbf{ij}k}) \,. \quad (5.9)$$

Similarly,

$$(ij) = 2(h_{ij} + h_{\mathbf{ij}}) = C_{ij\alpha,\alpha} \,. \quad (5.10)$$

Now in view of the properties of the tensor V_{ijkm} only the following ten index combinations need to be considered:

12 12   12 13   12 14   12 23   12 24   12 34
13 13   13 14   13 23   13 24 .   (5.11)

After the complex pairing (5.6), however, these ten combinations are reducible to the five combinations,

(12 12), (12 13), (12 14)
(13 13), (13 14) .   (5.12)

Beside putting these five complex components equal to zero, we have to add the three complex equations (12) = (13) = (23) = 0 We replace this 5 + 3 splitting by a 6 + 2 splitting in the following manner:

(12 12)   (12 13)   (12 14)   (12)
(13 12)   (13 13)   (13 14)   (13)   (5.13)

This is permitted because (12 13) − (13 12) = (23) and thus we can replace the equation (23) = 0 by (13 12) = 0.

Now, the first four equations, obtained by putting the upper row of (5.13) equal to zero, yields, if we write the equations in the sequence (12 14), − (12 13), (12 12), (12):

$$\begin{aligned} -Q_{4,1} - Q_{3,2} + Q_{2,3} + Q_{1,4} &= 0 \\ Q_{3,1} - Q_{4,2} - Q_{1,3} + Q_{2,4} &= 0 \\ -Q_{2,1} + Q_{1,2} - Q_{4,3} + Q_{3,4} &= 0 \\ Q_{1,1} + Q_{2,2} + Q_{3,3} + Q_{4,4} &= 0 \,, \quad (5.14)\end{aligned}$$

which may also be written in the form of the following quaternion equation:

$$-\left(\frac{\partial}{\partial x_1}i + \frac{\partial}{\partial x_2}j + \frac{\partial}{\partial x_3}k - \frac{\partial}{\partial x_4}\right)\left(Q_1 i + Q_2 j + Q_3 k + Q_4\right) = 0 \,, \quad (5.15)$$

where

$$Q_1 = A_{121} \,, \quad Q_2 = A_{122} \,, \quad Q_3 = A_{123} \,, \quad Q_4 = A_{124} \,. \quad (5.16)$$

The second row yields exactly the same set if the Q_i are identified with the A_{13i}.

We see that the original eight complex equations *separate into two independent groups of four equations.*[9] (This separation does not occur in covariant fashion and may be accomplished in an infinite number of ways.) We may also consider the four Q_i as the four components of a complex column vector Ψ and write Eq. (5.14) in matrix form:

$$\gamma_1\Psi_{,1} + \gamma_2\Psi_{,2} + \gamma_3\Psi_{,3} + \gamma_4\Psi_{,4} = 0 \quad (5.17)$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are four matrices defined as follows:

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \gamma_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\gamma_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \gamma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.18)$$

[9] The duplication of the equation is not accidental. One equation alone would create a right-handed or left-handed rotational sense in space time, which is not in harmony with the nature of a Riemannian geometry.

They are mathematically equivalent to the four Dirac matrices that appear in Dirac's equation of the electron in which the "mass term" is omitted. The reason for the absence of the mass term is not that it is outside the scope of the theory but rather that in our "infinitesimal" over-simplification the mass could not come into existence, since it represents a *second order superposition effect.*

The spontaneous and quite unexpected appearance of quaternion calculus and the Dirac equation (the fundamental building blocks of electromagnetism and of quantum theory) in the field equations of general relativity can hardly be considered as mere accidents. Einstein's "cosmic wisdom" hypothesis seems to have asserted itself once more on a grandiose scale. Einstein's fundamental discovery of introducing Riemannian geometry and interpreting the matter tensor in terms of the contracted tensor opened the great perspective of interpreting *all* physical phenomena in last analysis as geometrical properties of the space-time world. But the puzzling problem remained: What shall we do with the matter tensor? What is its structure? It can certainly not vanish *everywhere,* as demanded by Einstein's linear action principle. Shall we then equate the metrical matter tensor to the matter tensor of macroscopic physical events, such as Minkowski's kinetic tensor, or Maxwell's energy-momentum tensor, or a combination of both? Apart from the fact that such a procedure gives no clues concerning the atomistic structure of matter, it was in Einstein's eyes inconceivable that entirely heterogeneous quantities should be equated to each other. In his opinion this procedure could only be justified as a matter of expedience, without deeper significance.[10]

This puzzle could not be properly answered before unearthing a fundamental element of four-dimensional Riemannian geometry: a tensor of third-order H_{ijk}, (antisymmetric in ij), of 16 independent components. The quadratic action principle establishes a direct coupling between the matter tensor and this tensor H_{ijk}, while on the other hand this tensor is coupled to exactly those ten components of the Riemann tensor which are *not* included in the matter tensor. This highly involved feed-back system represents a mathematical problem which in its generality goes far beyond our present faculties and can only be handled in approximation. The problem is further complicated by the fact that the basic metrical plateau of the physical universe is far from a smooth, almost Minkowskian manifold. It is in fact an immensely agitated surface of very high frequencies which cancel out in the average and give the impression of a mirror-like surface. The "macroscopic matter" appears only as a second-order interference effect of the matter waves, (caused by the nonlinearity of the field equations) which gives a relatively stable (although very weak) *superstructure.* It is this second-order superstructure, however, which is of decisive importance from the standpoint of our physical observations, since they are solely tied to it, while the basic plateau is obtainable only by inference.

In spite of the formidable task of unraveling the mathematical consequences of the field equations, the general outlines of the Masterplan become clearly visible. Riemann's geometry remains untouched by any encroachments through additions or generalizations. The mere presence of an unadulterated Riemannian geometry of *specifically four dimensions* brings into existence a tensor of third order H_{ijk} of 16 components which bridges the gap between the second-order tensor of the line element g_{ik} and the fourth-order tensor of the Riemannian curvature R_{ijkm}. We will call it the "spintensor." We then have the hierarchy

$$\Phi_i,\ g_{ik},\ H_{ijk},\ R_{ijkm},$$

in physical interpretation: the *vector potential,* the *metrical tensor,* the *spintensor,* the *Riemann tensor.* These quantities, which emerge as inherent structural elements of a Riemannian geometry of four dimensions, seem to provide all the necessary and sufficient building blocks for a rational explanation of electricity and the quantum phenomena. Einstein's revealing essay on "Physics and Reality"[11] ends with the following words: *"It is shown, however, that the conviction to the effect that the field theory is unable to give by its methods a solution of these problems* (viz. the atomistic structure of matter and the quantum phenomena), *rests upon prejudice."* It is perhaps not too far fetched to claim that the results of the present investigation seem to corroborate Einstein's contention.

[10] Einstein expressed this thought in his characteristic style: "This equation reminds one of a palace which has two wings; the left wing is built of imperishable marble, the right wing of inferior wood." [Cf. L. de Broglie, *Nouvelles perspectives en microphysique* (A. Michel, Paris, 1956), p. 186).]

[11] A. Einstein, J. Franklin Inst. **221,** 349, (1936).