Explicit Representation of Discrete-Symmetry Operators in Quantum Theory of Free Fields*

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INTRODUCTION

THE importance of the discrete-symmetry transformations in a relativistic local-quantum-field theory is well-known; their possibilities beyond the limitations of that theory are increasingly recognized.[1] Particularly, since the discovery of the $T\mathcal{C}\mathcal{P}$ theorem, a large number of papers have been published on this subject; however, little attention has been given to the explicit representation of these symmetry operators which is so helpful from the point of view of a systematic study of the subject.

The purpose of this paper is to consider the discrete-symmetry operators of the free fields from the point of view of their explicit representation in terms of creation operators (CO's) and annihilation operators (AO's) of the fields in question; an attempt is made to clarify certain aspects that have not been carefully treated in the literature.

The phrase "discrete transformation" implies that it cannot be generated continuously from the identity; however, note that this is no longer the case, if one admits complex transformations.[2] The $T\mathcal{C}\mathcal{P}$ transformation or strong reflection (as Pauli calls it) is such a discrete transformation, invariance under which of the local-quantum-field theory is assured if one assumes Lorentz invariance (the inhomogeneous Lorentz group) and the connection between spin and statistics (the $T\mathcal{C}\mathcal{P}$ theorem). Under a strong reflection transformation, one replaces an interaction referred to a left-hand frame by the one referred to a right-hand frame together with the interchange of particles and antiparticles and the reversal of spins and the order of events in the original interaction. For electromagnetic and strong interactions, one can look upon strong reflection as a product of time reversal (T), space inversion ($\mathcal{P}$), and particle conjugation ($\mathcal{C}$); the last two of these do not hold seperately for systems of lower symmetry (weak interactions), but only their product does. The various possible discrete-symmetry operators obtained from the products of T, $\mathcal{C}$, $\mathcal{P}$ and their effect on observables—charge (used in the generalized sense, includes electrically neutral particles such as K mesons), and linear and angular momenta—are summarized in Table I. Together with the identity, these are in all eight: $\mathcal{P}$, $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{E}$ are linear, unitary operators, while T, W, I, and S are antilinear, unitary operators. In literature, only the representations for $\mathcal{P}$, $\mathcal{C}$, and T have been considered. In early attempts,[3–5] the phase factors have been generally disregarded, Watanabe's[6] work being an exception. There is also a recent paper by Kaempffer,[7] *who employs certain singular operators; the algebra of these operators is however not clear and leads to contradictions.* Representations have also been treated by Federbush[8] and Sudarshan.[9]

TABLE I. Definitions of discrete-symmetry operators in terms of their effect on coordinates ($\mathbf{x}$,t), charge (Q), and linear ($\mathbf{P}$) and angular momenta ($\mathbf{J}$).

	U	$\mathbf{x}'$	t'	$\mathbf{P}'$	Q'	$\mathbf{J}'$
1. Identity	$\mathcal{E}$	$+\mathbf{x}$	$+t$	$+\mathbf{P}$	$+Q$	$+\mathbf{J}$
2. Space inversion	$\mathcal{P}$	$-\mathbf{x}$	$+t$	$-\mathbf{P}$	$+Q$	$+\mathbf{J}$
3. Particle conjugation	$\mathcal{C}$	$+\mathbf{x}$	$+t$	$+\mathbf{P}$	$-Q$	$+\mathbf{J}$
4. Reflection	$\mathcal{R}$	$-\mathbf{x}$	$+t$	$-\mathbf{P}$	$-Q$	$+\mathbf{J}$
5. Time reversal	T	$+\mathbf{x}$	$-t$	$-\mathbf{P}$	$+Q$	$-\mathbf{J}$
6. Inversion	I	$-\mathbf{x}$	$-t$	$+\mathbf{P}$	$+Q$	$-\mathbf{J}$
7. Weak reflection	W	$+\mathbf{x}$	$-t$	$-\mathbf{P}$	$-Q$	$-\mathbf{J}$
8. Strong reflection	S	$-\mathbf{x}$	$-t$	$+\mathbf{P}$	$-Q$	$-\mathbf{J}$

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[1] See in this connection the proof of $T\mathcal{C}\mathcal{P}$ theorem in the S-matrix approach, H. P. Stapp, Phys. Rev. (to be published).

[2] See, e.g., R. Jost, Helv. Phys. Acta **30**, 409, 1957.

[3] R. G. Sachs, Phys. Rev. **87**, 1100 (1952): Representations of $\mathcal{P}$ and T for the Klein-Gordon (K-G) field in angular momentum expansion.

[4] L. Wolfenstein and D. Ravenhall, Phys. Rev. **88**, 279 (1952): Representation of $\mathcal{C}$ for the K-G and Dirac fields.

[5] B. P. Nigam and L. L. Foldy, Phys. Rev. **102**, 1410 (1956): Representation of $\mathcal{C}$ for the K-G and Dirac fields.

[6] S. Watanabe, Revs. Modern Phys. **27**, 40 (1955): Representation of P, C, and T for various fields.

[7] F. A. Kaempffer, Can. J. Phys. **39**, 22 (1961): Representation of P, C, and T for the K-G and Dirac fields. The reader is warned against drawing certain conclusions from this paper: (1) Spin changes sign under charge conjugation; (2) eigenvalue of an antilinear time reversal operator can be uniquely defined; (3) the previous work[3–6,8,9] on representations of DST's is incorrect; (4) P^2, T^2, are c numbers for the Dirac field.

[8] P. G. Federbush and M. T. Grisaru, Nuovo cimento **9**, 890 (1958).

[9] E. C. G. Sudarshan, Proc. Indian Acad. Sci. **49**, 66 (1959). See also R. E. Marshak and E. C. G. Sudarshan, *Introduction to Elementary Particle Physics* (Interscience tracts on Phys. and Ast. No. 11 (1961).

The subject matter considered in this paper is divided into two sections. In the first section, the formalism and notation for the Klein-Gordon (K-G), Dirac, and electromagnetic fields are summarized, and the transformation properties of the field functions and the corresponding AO's under various discrete-symmetry transformations (DST) are tabulated; the conditions on the phase factors introduced under DST's are also summarized in a table. In deriving the transformation properties of AO's of the Dirac field, it has been possible to avoid an explicit representation of the Dirac matrices or spinor functions. In Sec. II, the representations of all the discrete-symmetry operators are given for the K-G, Dirac, and electromagnetic fields.

I. FORMALISM AND NOTATION

We wish to find representation of all discrete-symmetry operators in terms of creation operators (CO's) and annihilation operators (AO's) of the free fields in question. For this purpose, the free-field function is expanded in terms of a certain complete set of orthonormal functions with constant coefficients which, on quantization, become CO's and AO's. The operators thus introduced on quantization are referred to as q numbers. Unless otherwise stated, the field functions will always be expanded in discrete-linear-momentum representation in a cubic box of volume V.

For the non-Hermitian K-G field, the expansion is[10]

$$\phi(\mathbf{x},t)=\sum_{\mathbf{k}}\frac{1}{(2V\omega_k)^{\frac{1}{2}}}(b_{\mathbf{k}}(1)e^{ikx}+b_{\mathbf{k}}{}^{\dagger}(2)e^{-ikx}), \quad (1.1)$$

where

$$kx=\mathbf{k}\cdot\mathbf{x}-\omega t, \quad \omega=\omega_{\mathbf{k}}=+(\mathbf{k}^2+m^2)^{\frac{1}{2}} \quad (1.2)$$

and $b_{\mathbf{k}}(r)$, $b_{\mathbf{k}}{}^{\dagger}(r)$ are, respectively, annihilation and creation operators of particles ($r=1$) and antiparticles ($r=2$), and obey the commutation relations

$$[b_{\mathbf{k}}(r),b^{\dagger}{}_{\mathbf{k}'}(r')]=\delta_{\mathbf{k}\mathbf{k}'}\delta_{rr'}, \quad (1.3)$$

and all other commutators $\equiv 0$. Accordingly, the representation for $b_{\mathbf{k}}$ is written as a direct product, arbitrary up to a phase factor ξ:

$$b_{\mathbf{k}}=b_{\mathbf{k}_n}=\xi_{\mathbf{k}}(1\times1\times1\cdots\times1\times b\times1\cdots), \quad (1.4)$$

where b occurs at the nth position in the direct product. The vacuum is defined as

$$b_{\mathbf{k}}(r)|0\rangle=0, \quad (1.5)$$

and is written as a direct product

$$|0\rangle=\prod_{\mathbf{k}}\prod_{r}|0_{\mathbf{k}}(r)\rangle; \quad |0_{\mathbf{k}}(r)\rangle=\zeta_{\mathbf{k}}\begin{pmatrix}1\\ \vdots\end{pmatrix}_{\mathbf{k}}(r), \quad (1.6)$$

where $\zeta_{\mathbf{k}}$ is an arbitrary-phase factor. The Hermitian conjugate of a ket is a bra[11]; since the operators for observables are Hermitian, the theory is symmetrical in ket's and bra's. The arbitrariness of phase in the representation of b and $|\ \rangle$ will be of importance when considering antilinear transformations. The observables, Hamiltonian (H) linear momentum and charge (the word charge is used in a generalized sense), are given by

$$H=\sum_{\mathbf{k}}\sum_{r}\omega_k b_{\mathbf{k}}{}^{\dagger}(r)b_{\mathbf{k}}(r)=\sum_{\mathbf{k},r}\omega_k n_{\mathbf{k}}(r), \quad (1.7)$$

$$\mathbf{P}=\sum_{\mathbf{k},r}\mathbf{k}n_{\mathbf{k}}(r), \quad (1.8)$$

and

$$Q=\sum_{\mathbf{k},r}(n_{\mathbf{k}}(1)-n_{\mathbf{k}}(2)). \quad (1.9)$$

We recall[12] that the important physical quantity in quantum mechanics is the square of the absolute value of a scalar product, which has a meaning independent of the observer. Those transformations that leave the scalar product unchanged also preserve the absolute value; these are linear unitary transformations. However, there are nonlinear transformations which transform a scalar product into its complex conjugate, so that the absolute value is again preserved; such nonlinear transformations are called antilinear, unitary, or simply antiunitary transformations. Explicity, the antiunitary transformations are expressed by

$$(f,g)\rightarrow(g',f')=(f',g')^* \quad (1.10a)$$

and

$$(f,ABg)\rightarrow(A'B'g',f')=(g',B'^{\dagger}A'^{\dagger}f')$$

or, in Dirac notation,

$$\langle f|g\rangle\rightarrow\langle g'|f'\rangle=\langle\bar{f}'|\bar{g}'\rangle, \quad (1.10b)$$

$$\langle f|AB|g\rangle\rightarrow\langle g'|(A'B')^{\sim}|f'\rangle=\langle\bar{f}'|A'^*B'^*|\bar{g}'\rangle,$$

where the bar on f and g indicates the complex conjugate vectors, and the primed and unprimed quantities are related by linear unitary transformations. It is apparent from (1.10) that one can represent the antilinear operation in two alternate ways:

(1) antiunitary α_1= unitary operator (U) $\times$ complex conjugation (L),

(2) antiunitary α_2= transposition (K) $\times$ unitary operator ($V=U^*$).

Thus,

$$\alpha_1(AB)\alpha_1{}^{\dagger}=UA^*B^*U^{\dagger}=A^TB^T, \quad (1.11a)$$

$$\alpha_2(AB)\alpha_2{}^{\dagger}=K(VABV^{\dagger})K^{-1}=B^{T\dagger}A^{T\dagger}; \quad (1.11b)$$

[10] The following notation is used throughout: $*$=complex conjugate, $\sim$=transposed, $+$=complex conjugate-transposed =Hermitian conjugate, subscript T=transposed of spinors and γ_μ matrices only, superscript T=transformed operator or vector. Units have been so chosen that $\hbar=c=1$.

[11] However, note that bras and kets cannot be added together; see P. A. M. Dirac; *The Principles of Quantum Mechanics* (Oxford University Press, New York, 1958), 4th edition, pp. 26–28.

[12] For discussion in this paragraph, see: E. Wigner, Gottinger Nach. **31**, 546 (1932); or *Group Theory* (Academic Press Inc., New York, 1959), revised edition; J. Schwinger, Phys. Rev. **82**, 914 (1951); G. Grawert, G. Lüders, and H. Rollnik, Fortschr. Physik **7**, 291–328 (1959). See also reference 11.

the state vectors transform as

$$\alpha_1|\,\rangle = U|\,\rangle^* = |\,\rangle^{*\prime}, \tag{1.12a}$$

and

$$\alpha_2|\,\rangle = KV|\,\rangle = \langle|^* U^\dagger = (U|\,\rangle^*)^\dagger. \tag{1.12b}$$

On account of the reverse of factor in (1.11b), α_2 is called antiautomorphism.[13] It is clear that the two representations are related by Hermitian conjugation. It can be seen that howsoever one defines the antilinear operator, the concept of an eigenvalue for an antilinear operator makes no sense. Furthermore, under an arbitrary unitary transformation β, the unitary factor U of the antiunitary α transforms as $\beta U \beta^{*-1} = U'$, so that even if one chooses a particular representation as real, the eigenvalues thus obtained for an antiunitary α will not be invariant under a unitary transformation. We are now ready to consider the time-reversal transformation. If one attempts to define the time-reversal transformation by

$$T\phi(\mathbf{x},t)T^{-1} = \eta_T \phi(\mathbf{x}, -t), \tag{1.13a}$$

where T is a linear (see Appendix) unitary operator, one obtains, on using the development (1.1),

$$Tb_{\mathbf{k}}(1)T^\dagger = \eta_T b^\dagger_{-\mathbf{k}}(2), \tag{1.13b}$$

which contradicts the commutation relations (1.3). The following three, entirely equivalent ways are used to remedy the situation:

(a) T=unitary $(U_T)\times$complex conjugation (L), then,

$$T\phi(\mathbf{x},t)T^\dagger = \eta_T \phi(x, -t),$$

and

$$Tb_{\mathbf{k}}(1)T^\dagger = \eta_T b_{-k}(1). \tag{1.14}$$

(b) θ=transposition $(K)\times$unitary $(V_\theta = U_T^*)$, and time reversal is defined as

$$\theta\phi(\mathbf{x},t)\theta^{-1} = \eta_\theta \phi^\dagger(\mathbf{x}, -t). \tag{1.15a}$$

It follows that

$$\theta b_{\mathbf{k}}(1)\theta^\dagger = \eta_\theta^* b^\dagger_{-\mathbf{k}}(1) \tag{1.15b}$$

and

$$\theta\phi(\mathbf{x},t)\phi^\dagger(\mathbf{x}',t')\theta^\dagger = \phi(\mathbf{x}', -t')\phi(\mathbf{x}, -t). \tag{1.15c}$$

(c) The equation (1.15c) suggests the following prescription for expressions of the type $\phi\phi^\dagger$ that occur so often in all operator expressions (e.g., commutation relations and Lagrangian): Replace t by $-t$ and read all operator relations from right to left. From (1.14) or (1.15b), we see that

$$U_T b_{\mathbf{k}}(1) U^\dagger{}_T = \xi^2 \eta_T b_{-\mathbf{k}}(1), \tag{1.14a}$$

where ξ is the arbitrary factor associated with the representation of $b_{\mathbf{k}}$ (see 1.4); thus, the phase factor $\eta_{UT} = \xi^2 \eta_T$ associated with the unitary U_T is doubly arbitrary. In case of Hermitian fields, where η_T is real, the phase factor $\eta_{UT} = \pm\xi^2$ will still be arbitrary.

A superposition of antiparticle states may be written as

$$\sum_{\mathbf{k}} \frac{1}{(2V\omega_k)^{\frac{1}{2}}} b_{\mathbf{k}}^\dagger(2) e^{-ikx}|0\rangle = \phi(\mathbf{x},t)|0\rangle. \tag{1.16a}$$

Under particle conjugation, the above state should be transformed into a superposition of particle states:

$$\sum_{\mathbf{k}} \frac{1}{(2V\omega_k)^{\frac{1}{2}}} b_{\mathbf{k}}^\dagger(1) e^{-ikx}|0\rangle = \phi^\dagger(\mathbf{x},t)|0\rangle; \tag{1.16b}$$

this yields the definition of particle conjugation as

$$\mathcal{C}\phi(\mathbf{x},t)\mathcal{C}^\dagger = \eta_C \phi^\dagger(\mathbf{x},t). \tag{1.16c}$$

The transformation properties of $\phi(\mathbf{x},t)$ and $b_{\mathbf{k}}(r)$ under various DST's are summarized in Table II; it is easily checked that these obey the definitions in Table I.

Angular Momentum Representation

In angular momentum representation, $\phi(\mathbf{x},t)$ is expanded in a sphere of radius R

$$\phi(\mathbf{x},t) = \sum_{\mathbf{k}} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{g_{lk}|\mathbf{x}|}{(2\omega)^{\frac{1}{2}}} (b(k,l,m,1) y_l^m(\mathbf{x}) e^{-i\omega t} + b^\dagger(k,l,m,2) y_l^{m*}(\mathbf{x}) e^{-i\omega t}), \tag{1.17}$$

where

$$y_l^m(\mathbf{x}) = (-1)^l y_l^m(-\mathbf{x}) = (-1)^m y_l^{-m*}(\mathbf{x}) \tag{1.18}$$

are spherical harmonics, satisfying

$$\int y_l^{m*}(\mathbf{x}) y_{l'}^{m'}(\mathbf{x}) d\Omega(\mathbf{x}) = \delta_{mm'}\delta_{ll'},$$
$$\sum_{l,m} y_l^m(\theta,\varphi) y_{l'}^{m'*}(\theta',\varphi') = \delta(\cos\theta' - \cos\theta)\delta(\varphi' - \varphi), \tag{1.18a}$$

and

$$g_{lk}|\mathbf{x}| = (\pi k/|\mathbf{x}|R)^{\frac{1}{2}} J_{l+\frac{1}{2}}(k|\mathbf{x}|), \tag{1.19}$$

$J_{l+\frac{1}{2}}$ being half-integral Bessel functions, are normalized in a sphere of large radius $R(kR \gg 1)$:

$$\int_0^R g_{lk}|\mathbf{x}| g_{lk'}|\mathbf{x}|\ |\mathbf{x}|^2 d|\mathbf{x}| = \frac{\pi}{R}\delta(\mathbf{k}-\mathbf{k}') = \delta_{kk'} \tag{1.19a}$$

$b_{\mathbf{k}}(r)$ and $b(k,l,m,r)$ are connected by the relations

$$b_{\mathbf{k}}(r) = \left(\frac{8\pi^2 R}{Vk^2}\right)^{\frac{1}{2}} \sum_{l,m} (-i)^l b(k,l,m,r) y_l^m(\mathbf{k}). \tag{1.20}$$

The commutation relations are

$$[b(k,l,m,r), b^\dagger(k',l',m',r')] = \delta_{mm'}\delta_{kk'}\delta_{ll'}\delta_{rr'} \tag{1.21}$$

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[13] This use is due to E. C. G. Sudarshan (private communication). We recall that in modern algebra, a (biunique) correspondence $A \leftrightarrow A^1$, $B \leftrightarrow B^1$ is called automorphism if $(A+B)^1 = A^1 + B^1$ and $(AB)^1 = A^1B^1$, and antiautomorphism if $(A+B)^1 = A^1 + B^1$, but $(AB)^1 = B^1A^1$. The operation of transposition and reading from right to left are clearly antiautomorphisms.

Table II. Transformation properties under various discrete-symmetry transformations (defined in Table I) of the field functions and annihilation operators (AO's) of particles. $b_{\mathbf{k}}(1)$, $b(k,l,m,1)$ are AO's of particles for the K-G field $\phi(\mathbf{x},t)$ in linear and angular momentum representations, respectively; $b_{\mathbf{k}}(\mathbf{s})$ is AO of polarization index $\mathbf{s}$ for the electromagnetic field; $a_{\mathbf{k}}(s,1)$, $a_{\mathbf{k}}(\lambda,1)$ are AO's of particles for the Dirac field with spin $+\frac{1}{2}$, $-\frac{1}{2}$ ($s=1,2$) along k_3 and spin parallel or antiparallel ($\lambda=\uparrow,\downarrow$) to $\mathbf{k}$, respectively.[a,b]

$U\beta U^{-1}$ \ β	$\phi(\mathbf{x},t)$	$b_{\mathbf{k}}(1)$	$b(k,l,m,1)$	$A_\mu(\mathbf{x},t)$	$b_{\mathbf{k}}(\mathbf{s})$	$\psi(\mathbf{x},t)$	$\bar{\psi}(\mathbf{x},t)$	$a_{\mathbf{k}}(s,1)$	$a_{\mathbf{k}}(\lambda,1)$
$\mathcal{P}$	$\eta'_P\phi(-\mathbf{x},t)$	$\eta'_P b_{-\mathbf{k}}(1)$	$\eta'_P(-1)^l$ $b(k,l,m,1)$	$-(\mu 4)$ $A_\mu(-\mathbf{x},t)$	$(\mathbf{s}2)$ $b_{-\mathbf{k}}(\mathbf{s})$	η''_P $\gamma_4\psi(-\mathbf{x},t)$	η''^*_P $\bar{\psi}(-\mathbf{x},t)\gamma_4$	η''_r $a_{-\mathbf{k}}(s,1)$	$-i\eta''_P$ $a_{-\mathbf{k}}(\lambda',1)$
$\mathcal{C}$	$\eta'_C\phi^\dagger(\mathbf{x},t)$	$\eta'_C b_{\mathbf{k}}(2)$	η'_C $b(k,l,m,2)$	$-A_\mu(-\mathbf{x},t)$	$-b_{\mathbf{k}}(\mathbf{s})$	η''_C $C\bar{\psi}_T(\mathbf{x},t)$	$-\eta''^*_C$ $\psi_T C^\dagger$	η''_C $a_{\mathbf{k}}(s,2)$	η''_C $a_{\mathbf{k}}(\lambda,2)$
T	$\eta'_T\phi(\mathbf{x},-t)$	$\eta'_T b_{-\mathbf{k}}(1)$	$\eta'_T(-1)^m$ $b(k,l,-m,1)$	$-(\mu 4)$ $A_\mu(\mathbf{x},-t)$	$(\mathbf{s}2)$ $b_{-\mathbf{k}}(\mathbf{s})$	η''_T $\tau\psi(\mathbf{x},-t)$	η''^*_T $\bar{\psi}(\mathbf{x},-t)\tau^\dagger$	$\eta''_T(-1)^s$ $a_{-\mathbf{k}}(s',1)$	$i\eta''_T(\lambda\downarrow)$ $a_{-\mathbf{k}}(\lambda,1)$
$\mathcal{R}$	$\eta'_R\phi^\dagger(-\mathbf{x},+t)$	$\eta'_R b_{-\mathbf{k}}(2)$	$\eta'_R(-1)^l$ $b(k,l,m,2)$	$(\mu 4)$ $A_\mu(-\mathbf{x},t)$	$-(\mathbf{s}2)$ $b_{-\mathbf{k}}(\mathbf{s})$	η''_R $\gamma_4 C\bar{\psi}_T(-\mathbf{x},t)$	$-\eta''^*_R$ $\psi_T(-\mathbf{x},t)C^\dagger\gamma_4$	η''_R $a_{-\mathbf{k}}(s,2)$	$-i\eta''_R$ $a_{-\mathbf{k}}(\lambda',2)$
I	$\eta'_I\phi(-\mathbf{x},-t)$	$\eta'_I b_{\mathbf{k}}(1)$	$\eta'_I(-1)^{l+m}$ $b(k,l,-m,1)$	$A_\mu(-\mathbf{x},-t)$	$b_{\mathbf{k}}(\mathbf{s})$	η''_I $\tau\gamma_4\psi(-\mathbf{x},-t)$	η''^*_I $\bar{\psi}(-\mathbf{x},-t)\gamma_4\tau^\dagger$	$\eta''_I(-1)^s$ $a_{\mathbf{k}}(s',1)$	$\eta''_I(\lambda\downarrow)$ $a_{\mathbf{k}}(\lambda',1)$
W	$\eta'_W\phi^\dagger(\mathbf{x},-t)$	$\eta'_W b_{-\mathbf{k}}(2)$	$\eta'_W(-1)^m$ $b(k,l,-m,2)$	$(\mu 4)$ $A_\mu(\mathbf{x},-t)$	$-(\mathbf{s}2)$ $b_{-\mathbf{k}}(\mathbf{s})$	η''_W $\tau C\bar{\psi}_T(\mathbf{x},-t)$	$-\eta''^*_W$ $\psi_T(\mathbf{x},-t)\tau C$	$\eta''_W(-1)^s$ $a_{-\mathbf{k}}(s',2)$	$-i\eta''_W$ $a_{-\mathbf{k}}(\lambda,2)$
S	$\eta'_S\phi^\dagger(-\mathbf{x},-t)$	$\eta'_S b_{\mathbf{k}}(2)$	$\eta'_S(-1)^{l+m}$ $b(k,l,-m,2)$	$-A_\mu(-\mathbf{x},-t)$	$-b_{\mathbf{k}}(\mathbf{s})$	η''_S $\tau\gamma_4 C\bar{\psi}_T(-\mathbf{x},-t)$	$-\eta''^*_S$ $\psi_T(-\mathbf{x},-t)(\tau\gamma_4 C)^\dagger$	$\eta''_S(-1)^s$ $a_{\mathbf{k}}(s',2)$	$\eta''_S(\lambda\downarrow)$ $a_{\mathbf{k}}(\lambda',2)$

[a] In the transformation of antiparticle AO's $b_{\mathbf{k}}(2)$, $b(k,l,m,2)$, and $a_{\mathbf{k}}(\lambda,2)$, the phase factors are complex conjugates of those for particle AO's; for $a_{\mathbf{k}}(s,2)$, there is an additional negative sign in case of operators $\mathcal{P}$, $\mathcal{R}$, I, and S.
[b] The notation $(\mu 4)$, $(\lambda\downarrow)$, etc., stands for $\exp(i\pi\delta_{\mu 4})$, $\exp(i\pi\delta\lambda\downarrow)$, respectively.

Unlike $b_{\mathbf{k}}(r)$, $b(k,l,m,r)$ cannot be chosen real; in fact

$$b^*(k,l,m,r)=(-1)^{m+l}b(k,l,-m,r). \tag{1.22}$$

The transformation properties of $b(k,l,m,r)$ under various DST's are summarized in Table II.

Dirac Field

The field function $\psi(\mathbf{x},t)$ of the four-component-spinor field satisfying the Dirac equation

$$(\gamma_\mu\partial_\mu+m)\psi=0 \tag{1.23}$$

may be developed as

$$\psi(\mathbf{x},t)=\sum_{\mathbf{k}}\sum_{s=1}^{2}\frac{1}{(V\omega)^{\frac{1}{2}}}[a_{\mathbf{k}}(s,1)u_{\mathbf{k}}(s)e^{ikx}+a_{\mathbf{k}}^\dagger(s,2)v_{\mathbf{k}}(s)e^{-ikx}]; \tag{1.24a}$$

the adjoint equation is

$$\bar{\psi}(\mathbf{x},t)=\psi^\dagger(\mathbf{x},t)\gamma_4=\sum_{\mathbf{k},s}\frac{1}{(V\omega)^{\frac{1}{2}}}[a_{\mathbf{k}}^\dagger(s,1)\bar{u}_{\mathbf{k}}(s)e^{-ikx}+a_{\mathbf{k}}(s,2)\bar{v}_{\mathbf{k}}(s)e^{ikx}]. \tag{1.24b}$$

The normalization has been so chosen that the *spinors u and v and the creation and annihilation operators* $a^\dagger_{\mathbf{k}}(s,r)$, $a_{\mathbf{k}}(s,r)$ *are dimensionless; the latter satisfy the anticommutation relations*:

$$[a^\dagger_{\mathbf{k}}(s,r),a_{\mathbf{k}'}(s',r')]_+=\delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}\delta_{rr'}. \tag{1.25}$$

The index s denotes spin; however, independently of a more specific interpretation, u and v must satisfy

$$\textstyle\sum_s v_{\mathbf{k}\alpha}(s)v_{\mathbf{k}\beta}(s)=\Delta\cdot[ik_\mu\gamma_\mu+m]_{\alpha\beta}. \tag{1.26a}$$

The normalization Δ may be taken to be $-1/2\omega$, so that

$$\textstyle\sum_s (u_{\mathbf{k}}(s)\bar{u}_{\mathbf{k}}(s)-v_{\mathbf{k}}(s)\bar{v}_{\mathbf{k}}(s))=-2\Delta\cdot m=m/\omega \tag{1.26b}$$

and

$$\bar{u}_{\mathbf{k}}(s)u_{\mathbf{k}}(s')=-v_{\mathbf{k}}(s)v_{\mathbf{k}}(s')=(m/\omega)\delta_{ss'}. \tag{1.26c}$$

In (24–25), $a_{\mathbf{k}}(s,r)$ is the annihilation operator of a particle ($r=1$) or an antiparticle ($r=2$) of spin $+\frac{1}{2}$ ($s=1$) or $-\frac{1}{2}$ ($s=2$) in the direction k_3. One could also use the representation in which the spin is taken parallel and antiparallel to the direction $\mathbf{k}$. The development of $\psi(\mathbf{x},t)$ in this case is

$$\psi(\mathbf{x},t)=\sum_{\mathbf{k},\lambda=\downarrow\uparrow}\frac{1}{(V\omega)^{\frac{1}{2}}}(a_{\mathbf{k}}(\lambda,1)u_{\mathbf{k}}(\lambda)e^{ikx}+a_{\mathbf{k}}^\dagger(\lambda,2)v_{\mathbf{k}}(\lambda)e^{-ikx}), \tag{1.24c}$$

where $u_{\mathbf{k}}(\lambda)$ and $v_{\mathbf{k}}(\lambda)$ are spinors with momentum $\mathbf{k}$ and spin parallel ($\lambda=\uparrow$) and antiparallel ($\lambda=\downarrow$) to $\mathbf{k}$ and momentum $-\mathbf{k}$ and spin parallel ($\lambda=\downarrow$) and antiparallel ($\lambda=\uparrow$) to $-\mathbf{k}$, respectively. The two representations are connected by the unitary transformation

$$A=\exp(-\tfrac{1}{2}i\hat{\sigma}_3\varphi)\exp(-\tfrac{1}{2}i\hat{\sigma}_2\theta)=\begin{pmatrix}\alpha & -\beta^*\\ \beta & \alpha^*\end{pmatrix}, \tag{1.27a}$$

where $\hat{\sigma}_2$, $\hat{\sigma}_3$ are Pauli spin matrices and $\alpha=\cos(\frac{1}{2}\theta)\times\exp(-\frac{1}{2}i\varphi)$, $\beta=\sin(\frac{1}{2}\varphi)\exp(\frac{1}{2}i\varphi)$, so that

$$\begin{pmatrix} a_{\mathbf{k}}(1,1) \\ a_{\mathbf{k}}(2,1) \end{pmatrix} = A^{\dagger} \begin{pmatrix} a_{\mathbf{k}}(\uparrow,1) \\ a_{\mathbf{k}}(\downarrow,1) \end{pmatrix}, \quad \begin{pmatrix} u_{\mathbf{k}}(1) \\ u_{\mathbf{k}}(2) \end{pmatrix} = A^{*} \begin{pmatrix} u_{\mathbf{k}}(\uparrow) \\ u_{\mathbf{k}}(\downarrow) \end{pmatrix} \tag{1.27b}$$

and

$$a_{\mathbf{k}}(\uparrow,1)u_{\mathbf{k}}(\uparrow)+a_{\mathbf{k}}(\downarrow,1)u_{\mathbf{k}}(\downarrow) = a_{\mathbf{k}}(1,1)u_{\mathbf{k}}(1)+a_{\mathbf{k}}(2,1)u_{\mathbf{k}}(2). \tag{1.27c}$$

Obviously, the two representations coincide for $k_1=k_2=0$. It may be remarked in passing that in the unquantized theory, while the spinors $u_{\mathbf{k}}(\lambda)$, $v_{\mathbf{k}}(\lambda)$ are the eigenstates of the operator

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{|\mathbf{k}|}=\frac{-i\gamma_5\gamma_4\boldsymbol{\gamma}\cdot\mathbf{k}}{|\mathbf{k}|}, \tag{1.28}$$

the spinors $u_{\mathbf{k}}(s)$, $v_{\mathbf{k}}(s)$ are not the eigenstates of the operator σ_3; this is connected with the fact that $\boldsymbol{\sigma}$ and σ_3 are not constants of the motion. One can, however, define an operator $\boldsymbol{\Sigma}$, such that $\boldsymbol{\Sigma}\cdot\mathbf{k}=\boldsymbol{\sigma}\cdot\mathbf{k}$ and $u_{\mathbf{k}}(s)$, $v_{\mathbf{k}}(s)$ are the eigenstates of Σ_3. This may be achieved by noting that since

$$(ik_\mu\gamma_\mu+m)u_{\mathbf{k}}=0, \tag{1.23'}$$

one can write

$$\Gamma^{\dagger}u_{\mathbf{k}}=\left(1+i\frac{\mathbf{k}\cdot\boldsymbol{\gamma}}{\omega+m}\right)\left(\frac{\omega+m}{2\omega}\right)^{\frac{1}{2}}u_{\mathbf{k}} =\left(\frac{2\omega}{\omega+m}\right)^{\frac{1}{2}}\frac{1+\gamma_4}{2}u_{\mathbf{k}}; \tag{1.29a}$$

similarly,

$$\Gamma v_{\mathbf{k}}=\left(\frac{2\omega}{\omega+m}\right)^{\frac{1}{2}}\frac{1-\gamma_4}{2}v_{\mathbf{k}}. \tag{1.29b}$$

The unitary transformation Γ then transforms $\boldsymbol{\sigma}$ into

$$\boldsymbol{\Sigma}=\boldsymbol{\sigma}+\Gamma[2/\omega(\omega+m)]^{\frac{1}{2}}\boldsymbol{\gamma}\times\mathbf{k}. \tag{1.30}$$

In the c-number theory, $\boldsymbol{\Sigma}$ may be looked upon as mean spin.[14]

Irrespective of the representation chosen, u and v are connected by the relation

$$v_{\mathbf{k}}=C_1u_{\mathbf{k}}^{*}=C(\bar{u}_{\mathbf{k}})^{\sim}, \tag{1.31}$$

where the symmetric matrix $C_1=-\gamma_4C$, and the skew-symmetric matrix C transform the γ_μ as follows:

$$C_1\gamma_\mu^{*}C_1^{-1}=\gamma_\mu\exp(i\pi\delta_{\mu4}) \tag{1.32a}$$

and

$$C\gamma_\mu^{*}C^{-1}=-\gamma_\mu. \tag{1.32b}$$

The relation (1.31) is in agreement with the concept of charge conjugation as an operation that transforms an antiparticle state

$$\sum_{\mathbf{k},s}\exp(-ikx)v_{\mathbf{k}}(s)a^{\dagger}{}_{\mathbf{k}}(s,2)|0\rangle=\psi(\mathbf{x},t)|0\rangle$$

into the particle state

$$\sum_{\mathbf{k},s}\exp(-ikx)v_{\mathbf{k}}(s)a^{\dagger}{}_{\mathbf{k}}(s,1)|0\rangle=\psi_c(\mathbf{x},t)|0\rangle,$$

so that

$$\psi_C(\mathbf{x},t)=\mathcal{C}\psi(\mathbf{x},t)\mathcal{C}^{-1}=\eta_CC\bar{\psi}_T(\mathbf{x},t)=\eta_CC_1\psi_T{}^{\dagger}(\mathbf{x},t). \tag{1.33}$$

The time reversal is analogously defined as

$$T\psi(\mathbf{x},t)T^{-1}=\eta_T\mathcal{T}\psi(\mathbf{x},-t), \tag{1.34}$$

where the skew-symmetric matrix $\mathcal{T}=C^{\dagger}\gamma_5$ transforms γ_μ as

$$\mathcal{T}\gamma_\mu\mathcal{T}^{-1}=\tilde{\gamma}_\mu. \tag{1.35}$$

As γ_μ are always taken Hermitian, so as to obtain proper normalization of spinor functions, C_1, C, $\mathcal{T}$ are chosen unitary. Since

$$(\gamma_\mu\partial_\mu+m)\sum_{\mathbf{k},s}u_{\mathbf{k}}(s)\exp(ikx)=0, \tag{1.36a}$$

under space inversion, one obtains on multiplying (1.36a) by γ_4 and changing $\mathbf{k}$ to $-\mathbf{k}$

$$(\gamma_\mu\partial_\mu+m)\gamma_4\sum_{\mathbf{k},s}u_{-\mathbf{k}}(s)\exp(ikx)=0; \tag{1.36b}$$

comparing

$$u_{\mathbf{k}}(s)=\gamma_4u_{-\mathbf{k}}(s),\quad v_{\mathbf{k}}(s)=-\gamma_4v_{-\mathbf{k}}(s); \tag{1.37}$$

also,

$$u_{-\mathbf{k}}(\lambda)=i\gamma_4u_{\mathbf{k}}(\lambda'),\quad v_{-\mathbf{k}}(\lambda)=i\gamma_4v_{\mathbf{k}}(\lambda'), \tag{1.38}$$

where $\lambda\neq\lambda'$. To fix the phase between u and v, for $s\neq s'$, we set in the strong-reflection transformation

$$S\psi(x_\mu)S^{-1}=\eta_s[\psi^{\dagger}(-x_\mu)\gamma_5]_T, \tag{1.39}$$

$$\gamma_5u_{\mathbf{k}}(1)=v_{\mathbf{k}}(2); \tag{1.40}$$

whence

$$\mathcal{T}u_{-\mathbf{k}}(s)=(-1)^su_{\mathbf{k}}^{*}(s'). \tag{1.41}$$

Alternately, since

$$(\boldsymbol{\sigma}\cdot\mathbf{k}/|\mathbf{k}|)u_{\mathbf{k}}(\lambda)=\exp(i\pi\delta_{\lambda\downarrow})u_{\mathbf{k}}(\lambda),\quad (\boldsymbol{\sigma}\cdot\mathbf{k}/|\mathbf{k}|)v_{\mathbf{k}}(\lambda)=\exp(i\pi\delta_{\lambda\uparrow})v_{\mathbf{k}}(\lambda), \tag{1.42}$$

and,

$$(i\boldsymbol{\gamma}\cdot\mathbf{k}-k_0\gamma_4+m)u_{\mathbf{k}}(\lambda)=0,\quad (-i\boldsymbol{\gamma}\cdot\mathbf{k}+k_0\gamma_4+m)v_{\mathbf{k}}(\lambda)=0. \tag{1.43}$$

One obtains on combining these two

$$-(k_0+m\gamma_4/|\mathbf{k}|)\gamma_5u_{\mathbf{k}}(\lambda)=\tfrac{1}{2}\exp(i\pi\delta_{\lambda\downarrow})u_{\mathbf{k}}(\lambda),$$
$$-(k_0+m\gamma_4/|\mathbf{k}|)v_{\mathbf{k}}(\lambda)=\tfrac{1}{2}\exp(i\pi\delta_{\lambda\uparrow})\gamma_5v_{\mathbf{k}}(\lambda);$$

comparison yields,

$$\gamma_5u_{\mathbf{k}}(\lambda)=\alpha\exp(i\pi\delta_{\lambda\downarrow})v_{\mathbf{k}}(\lambda'), \tag{1.44}$$

where $\lambda\neq\lambda'$ and α is a phase factor which we choose

[14] See, e.g., L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

TABLE III. Phase factors introduced by the DST's, operating on the field functions of the non-Hermitian, Klein-Gordon (K-G) field and the Dirac field.[a]

U / η_u	$\mathcal{R}$: $\mathcal{P}\mathcal{C}$	$\mathcal{R}$: $\mathcal{C}\mathcal{P}$	I: $\mathcal{P}T$	I: $T\mathcal{P}$	W: $\mathcal{C}T$	W: $T\mathcal{C}$	
K-G field	$\eta_P{}^*\eta_C$	$\eta_P\eta_C$	$\eta_P\eta_T$	$\eta_P{}^*\eta_T$	$\eta_C\eta_T$	$\eta_C{}^*\eta_T{}^*$	
Dirac field	$-\eta_P{}^*\eta_C$	$\eta_P\eta_C$	$\eta_P\eta_T$	$\eta_P{}^*\eta_T$	$\eta_C\eta_T$	$\eta_C{}^*\eta_T{}^*$	
S / η_s	$\mathcal{P}T\mathcal{C}$	$T\mathcal{C}\mathcal{P}$	$\mathcal{C}\mathcal{P}T$	$T\mathcal{P}\mathcal{C}$	$\mathcal{P}\mathcal{C}T$	$\mathcal{C}T\mathcal{P}$	
K-G field	$\eta_P{}^*\eta_C{}^*\eta_T{}^*$	$\eta_P{}^*\eta_C{}^*\eta_T{}^*$	$\eta_P\eta_C\eta_T$	$\eta_P\eta_C{}^*\eta_T{}^*$	$\eta_P{}^*\eta_C\eta_T$	$\eta_P{}^*\eta_C\eta_T$	
Dirac field	$-\eta_P{}^*\eta_C{}^*\eta_T{}^*$	$\eta_P{}^*\eta_C{}^*\eta_T{}^*$	$\eta_P\eta_C\eta_T$	$-\eta_P\eta_C{}^*\eta_T{}^*$	$-\eta_P{}^*\eta_C\eta_T$	$\eta_P{}^*\eta_C\eta_T$	
U^2	$\mathcal{P}^2$	$\mathcal{C}^2$	T^2	$\mathcal{R}^2$	I^2	W^2	S^2
K-G field	$\eta_P{}^2$	$+1$	$+1$	$+1$	$+1$	$\eta_W{}^2$	$\eta_s{}^{*2}$
Dirac field	$\eta_P{}^2$	$+1$	-1	-1	-1	$-\eta_W{}^2$	$\eta_s{}^{*2}$

[a] The phase factors for the Hermitian fields are necessarily real.

equal to unity. A simple calculation then gives

$$\mathcal{T}u_{\mathbf{k}}(\lambda)=i\exp(i\pi\delta_{\lambda\downarrow})u^*_{-\mathbf{k}}(\lambda). \qquad (1.45)$$

For spin in the direction of z axis, one similarly obtains (1.40) and (1.41).

The knowledge of the transformation properties of u and v enables one to find the transformation properties of a's under various DST's; the transformation properties of $a_{\mathbf{k}}(s,r)$ are summarized in Table II.

The observables, momentum, charge, and spin, are given by

$$\mathbf{P}=\sum_{\mathbf{k},s,r}\mathbf{k}a^\dagger{}_{\mathbf{k}}(s,r)a_{\mathbf{k}}(s,r)=\sum_{\mathbf{k},s,r}\mathbf{k}N_{\mathbf{k}}(s,r), \qquad (1.46)$$

$$Q=\sum_{\mathbf{k},s}(N_{\mathbf{k}}(s,1)-N_{\mathbf{k}}(s,2)), \qquad (1.47)$$

$$S_3=\tfrac{1}{2}\sum_{\mathbf{k},r}(N_{\mathbf{k}}(1,r)-N_{\mathbf{k}}(2,r)) \quad (k_1=k_2=0), \qquad (1.48\text{a})$$

$$S_3=\tfrac{1}{2}\sum_{\lambda,\mathbf{k},r}\exp(i\pi\delta_{\lambda\downarrow})\frac{\mathbf{k}}{|\mathbf{k}|}N_{\mathbf{k}}(\lambda,r). \qquad (1.48\text{b})$$

Since a many-particle state is a direct product of one-particle states, a general n-particle state in the case of the K-G field can be written as

$$\overline{\langle n|=\langle\mathbf{k},1|\langle-\mathbf{k},1|\langle-\mathbf{k},2|\langle\mathbf{k},2|}, \qquad (1.46)$$

where $r=1$, 2 are, respectively, particle and antiparticle labels. The operators $\mathcal{P}$, $\mathcal{C}$, T may then be represented in the form

$$\mathcal{C}=\begin{vmatrix} \cdot & \cdot & \cdot & \eta_c \\ \cdot & \cdot & \eta_c & \cdot \\ \cdot & \eta_c{}^* & \cdot & \cdot \\ \eta_c{}^* & \cdot & \cdot & \cdot \end{vmatrix}, \quad \mathcal{P}=\begin{vmatrix} \eta_p & \cdot & \cdot & \cdot \\ \cdot & \eta_p & \cdot & \cdot \\ \cdot & \cdot & \eta_p{}^* & \cdot \\ \cdot & \cdot & \cdot & \eta_p{}^* \end{vmatrix},$$

$$T=\begin{vmatrix} \cdot & \eta_u & \cdot & \cdot \\ \eta_u & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \eta_u{}^* \\ \cdot & \cdot & \eta_u{}^* & \cdot \end{vmatrix}L. \qquad (1.47)$$

Analogously, for the Dirac spinor field, we have

$$\langle\mathbf{k},r,s|=\overline{\langle\mathbf{k},1,1|\langle-\mathbf{k},1,1|\langle+\mathbf{k},1,2|\langle-\mathbf{k},1,2|}$$
$$\overline{\langle-\mathbf{k},2,2|\langle\mathbf{k},2,2|\langle-\mathbf{k},2,1|\langle\mathbf{k},2,1|}. \qquad (1.48)$$

The corresponding symmetry operators are

$$\mathcal{C}=\begin{vmatrix} \cdot & \cdot & \cdot & \eta_c\bar{\delta} \\ \cdot & \cdot & \eta_c\bar{\delta} & \cdot \\ \cdot & \eta_c{}^*\bar{\delta} & \cdot & \cdot \\ \eta_c{}^*\bar{\delta} & \cdot & \cdot & \cdot \end{vmatrix},$$

$$\mathcal{P}=\begin{vmatrix} \eta_p\bar{\delta} & \cdot & \cdot & \cdot \\ \cdot & \eta_p\bar{\delta} & \cdot & \cdot \\ \cdot & \cdot & -\eta_p{}^*\bar{\delta} & \cdot \\ \cdot & \cdot & \cdot & -\eta_p{}^*\bar{\delta} \end{vmatrix},$$

$$T=\begin{vmatrix} \cdot & -\eta_u\bar{\delta} & \cdot & \cdot \\ \eta_u\bar{\delta} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\eta_u{}^*\bar{\delta} \\ \cdot & \cdot & \eta_u{}^*\bar{\delta} & \cdot \end{vmatrix}L, \qquad (1.49)$$

and their products, where

$$\bar{\delta}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \delta=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \qquad (1.50)$$

The phase factors for the squares of these operators and of products of $\mathcal{P}$, $\mathcal{C}$, T are summarized in Table III.

Electromagnetic Field

We assume that the fourth component A_4 of the electromagnetic 4-potentials A_μ is quantized with indefinite metric,[15] so that all the four components are Hermitian ($A_0=iA_4$ is skew-Hermitian). A_μ may be

[15] This procedure was originally introduced by Suraj Gupta, Proc. Phys. Soc. (London) **63**, 681, (1950); we here use the convention of K. Bleuler and W. Heitler, Prog. Theoret. Phys. (Kyoto) **5**, 600 (1950).

expanded as

$$A_\mu(x_\mu)=\sum_{\mathbf{k}}\sum_{s=1}^{4}\frac{\epsilon_\mu(s,\mathbf{k})}{(2\omega V)^{\frac{1}{2}}}(b_{\mathbf{k}}(s)\exp(ikx)+b^\dagger{}_{\mathbf{k}}(s)\exp(-ikx)), \quad (1.51)$$

where $\omega=|\mathbf{k}|$ and the polarization vectors

$$\begin{aligned}\epsilon_\mu(j,\mathbf{k})&=(\boldsymbol{\varepsilon}(j,\mathbf{k}),0), \quad j=1,2,3\\ \epsilon_\mu(4,\mathbf{k})&=(0,1)\end{aligned} \quad (1.52)$$

are assumed real, and satisfy the relations

$$\left.\begin{aligned}\epsilon_\mu(s,\mathbf{k})\epsilon_\mu(s',\mathbf{k})&=\delta_{ss'}\\ \textstyle\sum_s \epsilon_\mu(s,\mathbf{k})\epsilon_\nu(s,\mathbf{k})&=\delta_{\mu\nu}\\ \epsilon_\mu(s,\mathbf{k}).k_\mu&=0 \quad \text{for} \quad s=1,2\\ &=\omega \quad \text{for} \quad s=3\\ &=i\omega \quad \text{for} \quad s=4.\end{aligned}\right\}. \quad (1.53)$$

The commutation relations are

$$[b_{\mathbf{k}}(s)b^\dagger{}_{\mathbf{k}'}(s')]=\delta_{ss'}\delta_{\mathbf{kk}'}. \quad (1.54)$$

Physically, it is more instructive to consider the so-called circular components of the transverse field; these are defined by

$$\begin{aligned}\boldsymbol{\varepsilon}(\mathbf{k},\mathrm{L})&=(\boldsymbol{\varepsilon}(1,\mathbf{k})+i\boldsymbol{\varepsilon}(2,\mathbf{k}))/\sqrt{2}=\boldsymbol{\varepsilon}^*(\mathbf{k},\mathrm{R}),\\ b_{\mathbf{k}}(\mathrm{L})&=(b_{\mathbf{k}}(1)-ib_{\mathbf{k}}(2))/\sqrt{2}=b_{\mathbf{k}}^*(\mathrm{R}).\end{aligned} \quad (1.55)$$

The transformation properties of the field functions and creation and annihilation operators are summarized in Table II; for the calculation of the latter, the convention

$$\boldsymbol{\varepsilon}_\mu(s,\mathbf{k})=(-1)^s\boldsymbol{\varepsilon}_\mu(s,-\mathbf{k}) \quad (1.56)$$

has been adopted.

II. REPRESENTATION OF SYMMETRY OPERATORS

Since symmetry operators are unitary (or have unitary factors in case of antiunitary operators), the representations are constructed in the form $U=\exp(i\alpha\Omega)$, where α is a real number and the generator Ω is Hermitian and is bilinear in creation and annihilation operators analogous to the observables momentum, charge, etc.; under this transformation, an operator β transforms as

$$U\beta U^\dagger=\sum_{n=0}^{\infty}\left\{(-1)^n\frac{\alpha^{2n}}{(2n)!}[\Omega_{2n},\beta]\right\}+i\sum_{n=0}^{\infty}\left\{(-1)^n\frac{\alpha^{2n+1}}{(2n+1)!}[\Omega_{2n+1},\beta]\right\}, \quad (2.1)$$

where, for example, $[\Omega_2,\beta]=[\Omega[\Omega,\beta]_-]_-$ etc. Equation (2.1) in conjunction with the operator identities

$$\begin{aligned}[AB,C]_-&=A[B,C]_-+[A,C]_-B\\ &=A[B,C]_+-[A,C]_+B\end{aligned} \quad (2.2)$$

enable us to construct the appropriate Ω.

The simplest case is the Hermitian K-G field, for which particle and antiparticle are alike [replace $b_{\mathbf{k}}(1)$, $b_{\mathbf{k}}(2)$ by $b_{\mathbf{k}}$, and $\delta_{rr'}$ by 1 in (1.1–1.9)]; here, one has four operators: $b_{\mathbf{k}}$, $b_{-\mathbf{k}}$, $b^\dagger{}_{\mathbf{k}}$, $b^\dagger{}_{-\mathbf{k}}$, which give four bilinears; $b_{\mathbf{k}}^2$, $b_{\mathbf{k}}b_{-\mathbf{k}}$, $B_{\mathbf{k}}=b^\dagger{}_{\mathbf{k}}b_{-\mathbf{k}}$, and $n_{\mathbf{k}}=b^\dagger{}_{\mathbf{k}}b_{\mathbf{k}}$. The unitary transformations constructed from the first two transform a CO into a linear combination of CO's and AO's, and are, therefore, not useful for our purpose (see Appendix); on the other hand, $n_{\mathbf{k}}$, $B_{\mathbf{k}}$ have the following properties

$$[B_{\mathbf{k}},b_{\mathbf{k}}]_-=-b_{-\mathbf{k}}, \quad p_{\mathbf{k}}\equiv[B_{\mathbf{k}},B^\dagger{}_{\mathbf{k}}]_-=n_{\mathbf{k}}-n_{-\mathbf{k}}, \quad (2.3)$$

$$[n_{\mathbf{k}},b_{\mathbf{k}}]_-=-b_{\mathbf{k}}, \qquad [n_{\mathbf{k}},n^\dagger{}_{\mathbf{k}}]_-=0. \quad (2.4)$$

Note that $h_{\mathbf{k}}=[(b_{\mathbf{k}}b_{-\mathbf{k}}),(b_{\mathbf{k}}b_{-\mathbf{k}})^\dagger]=1+n_{\mathbf{k}}+n_{-\mathbf{k}}$, so that

$$\mathbf{P}={\sum}'_{\mathbf{k}}\,\mathbf{k}p_{\mathbf{k}} \quad \text{and} \quad H={\sum}'_{\mathbf{k}}\,\omega_{\mathbf{k}}(h_{\mathbf{k}}-1), \quad (2.5)$$

where the prime on $\sum$ denotes summation over half the k space only (say $k_3>0$).

A Hermitian Ω constructed from $n_{\mathbf{k}}$'s is a generator of unitary phase transformations and space-time displacements, and gives rise to the so-called multiplicative quantum numbers.[16] An operator $\exp(i\Omega)$ constructed from $B_{\mathbf{k}}$'s transforms $b_{\mathbf{k}}$ into $b_{-\mathbf{k}}$ up to a phase transformation, and is, therefore, essentially a generator of DST. The maximum number of such bilinears depends upon the number of DST's possible, and these are enumerated in the following:

1. K-G Hermitian: 1;

$$B_{\mathbf{k}}=b^\dagger{}_{\mathbf{k}}b_{-\mathbf{k}}, \quad n_{\mathbf{k}}=b^\dagger{}_{\mathbf{k}}b_{\mathbf{k}} \quad (2.6)$$

2. K-G non-Hermitian: 3;

$$\left.\begin{aligned}B_{\mathbf{k}}(r)&=b^\dagger{}_{\mathbf{k}}(r)b_{-\mathbf{k}}(r)\\ A_{\mathbf{k}}&=b_{\mathbf{k}}(1)b^\dagger{}_{-\mathbf{k}}(2)\\ C_{\mathbf{k}}&=b^\dagger{}_{\mathbf{k}}(2)b_{\mathbf{k}}(1)\\ n_{\mathbf{k}}(r)&=b^\dagger{}_{\mathbf{k}}(r)b_{\mathbf{k}}(r)\end{aligned}\right\}. \quad (2.7)$$

3. Dirac field: 7;

$$\left.\begin{aligned}B_{\mathbf{k}}(s,r)&=a^\dagger{}_{\mathbf{k}}(s,r)a_{-\mathbf{k}}(s,r)\\ \beta_{\mathbf{k}}(r)&=a^\dagger{}_{\mathbf{k}}(1,r)a_{-\mathbf{k}}(2,r)\\ C_{\mathbf{k}}(s)&=a^\dagger{}_{\mathbf{k}}(s,1)a_{\mathbf{k}}(s,2)\\ A_{\mathbf{k}}(s)&=a^\dagger{}_{\mathbf{k}}(s,1)a_{-\mathbf{k}}(s,2)\\ D_{\mathbf{k}}(r)&=a^\dagger{}_{\mathbf{k}}(2,r)a_{\mathbf{k}}(1,r)\\ G_{\mathbf{k}}(r)&=a^\dagger{}_{\mathbf{k}}(1,r)a_{-\mathbf{k}}(2,r')\\ F_{\mathbf{k}}(s)&=a^\dagger{}_{\mathbf{k}}(s,1)a_{\mathbf{k}}(s',2)\\ N_{\mathbf{k}}(s,r)&=a^\dagger{}_{\mathbf{k}}(s,r)a_{\mathbf{k}}(s,r)\end{aligned}\right\}. \quad (2.8)$$

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[16] Compare, E. C. G. Sudarhan, Proc. Ind. Acad. Sci. **49**, 66 (1959); G. Feinberg and S. Weinberg, Nuovo cimento **14**, 571 (1959).

Thus, there can be a maximum of $2^3-1=7$ DST's corresponding to 3 observables: linear momentum, charge, and spin. The representations of various DST's for the K-G, the electromagnetic, and the Dirac fields are given in the following (note: prime on the summation $\sum_{(\mathbf{k})}$ in the following indicates summation over half the $\mathbf{k}$ space only, i.e., $k_3>0$).

Hermitian K-G Field

1. *Space Inversion:* $\mathcal{P}$

$$\text{(a)}\quad \exp(i\pi \textstyle\sum'_{\mathbf{k}} n_{\mathbf{k}}) \exp\{\frac{1}{2}\pi \sum'_{\mathbf{k}} \eta_p(B^\dagger_{\mathbf{k}}-B_{\mathbf{k}})\}$$

$$\text{(b)}\quad \exp(i\pi \textstyle\sum'_{\mathbf{k}} n_{-\mathbf{k}}) \exp\{\frac{1}{2}\pi \sum'_{\mathbf{k}} \eta_p(B_{\mathbf{k}}-B^\dagger_{\mathbf{k}})\} \tag{2.9}$$

$$\text{(c)}\quad \exp\{\tfrac{1}{2}i\pi \textstyle\sum'_{\mathbf{k}}(-n_{\mathbf{k}}-n_{-\mathbf{k}}+\eta_pB_{\mathbf{k}}+\eta_pB^\dagger_{\mathbf{k}})\}.$$

2. *Unitary Factor of Time Reversal:* U_T

Though η_T is real for a Hermitian field, $\eta_{UT}=\exp i\delta\equiv\xi^2$ is not real because of the arbitrariness of the phase of $b_{\mathbf{k}}$ [see Eq. (1.5)].

$$\text{(a)}\ \exp[i(\pi-2\delta) \textstyle\sum'_{\mathbf{k}}n_{\mathbf{k}}]\times\exp[\frac{1}{2}\pi \sum'_{\mathbf{k}} (\eta_{UT}^*B^\dagger_{\mathbf{k}}-\eta_{UT}B_{\mathbf{k}})]$$

$$\text{(b)}\ \exp[i(\pi-2\delta) \textstyle\sum'_{\mathbf{k}} n_{-\mathbf{k}}]\times\exp[\frac{1}{2}\pi \sum'_{\mathbf{k}} (\eta_{UT}B_{\mathbf{k}}-\eta_{UT}^*B^\dagger_{\mathbf{k}})] \tag{2.10}$$

$$\text{(c)}\ \exp[\tfrac{1}{2}i\pi \textstyle\sum'_{\mathbf{k}} (\eta_{UT}B_{\mathbf{k}}+\eta_{UT}^*B^\dagger_{\mathbf{k}})]\times\exp[i(\frac{1}{2}\pi-2\delta) \sum_{\mathbf{k}} n_{\mathbf{k}}].$$

Non-Hermitian K-G Field; Linear-Momentum Representation

(*i*) *Unitary Factor of Time Reversal:* $U_T(\exp i\delta=\eta_{UT};\ \eta_r:\eta_1=\eta,\ \eta_2=\eta^*)$

$$\text{a.}\ \exp[i(\pi-2\delta)\sum_{\mathbf{k},r}{}' (-1)^{r+1}n_{\mathbf{k}}(r)]\times\exp\{\tfrac{1}{2}\pi\sum_{\mathbf{k},r}{}' (-1)^r[\eta_rB_{\mathbf{k}}(r)-\eta_\gamma^*B^\dagger_{\mathbf{k}}(r)]\},$$

$$\text{b.}\ \exp[i(\pi-2\delta)\sum_{\mathbf{k},r}{}' (-1)^{r+1}n_{\mathbf{k}}(r)]\times\exp\{\tfrac{1}{2}\pi\sum_{\mathbf{k},r}{}' (-1)^r[\eta_rB_{\mathbf{k}}^\dagger(r)-\eta_r^*B_{\mathbf{k}}(r)]\}, \tag{2.11}$$

$$\text{c.}\ \exp\{\tfrac{1}{2}i\pi\sum_{\mathbf{k},r}[\eta_{UT}B_{\mathbf{k}}(r)+\eta_{UT}^*B_{\mathbf{k}}(r)]\}\times\exp\{\tfrac{1}{2}i\pi\sum_{\mathbf{k},r} n_{\mathbf{k}}(r)\}\exp\{2i\delta\sum_{\mathbf{k},r}(-1)^rn_{\mathbf{k}}(r)\}.$$

The above expressions are also representations for P, if one puts $\eta_{UT}=\eta_P=\mp1$.

(*ii*) *Particle Conjugation:* $\mathcal{C}$

$$\text{a.}\ \exp[i\pi \textstyle\sum_{\mathbf{k}} n_{\mathbf{k}}(1)]\times\exp[\frac{1}{2}\pi \sum_{\mathbf{k}} (\eta_C^*C_{\mathbf{k}}-\eta_CC^\dagger_{\mathbf{k}})],$$

$$\text{b.}\ \exp[i\pi \textstyle\sum_{\mathbf{k}} n_{\mathbf{k}}(2)]\times\exp[\frac{1}{2}\pi \sum_{\mathbf{k}} (\eta_CC^\dagger_{\mathbf{k}}-\eta_C^*C_{\mathbf{k}})], \tag{2.12}$$

$$\text{c.}\ \exp\{\tfrac{1}{2}\pi \textstyle\sum_{\mathbf{k}} [-n_{\mathbf{k}}(1)-\eta_{\mathbf{k}}(2)+\eta_CC^\dagger_{\mathbf{k}}+\eta_C^*C_{\mathbf{k}}]\}.$$

These are also the representations for the unitary factor U_s of strong reflection, if we replace η_C by η_{US}; note, however, that unlike $\mathcal{C}^2$, S^2 is not a unit operator. Even if one assumes η_S real, $\eta_{US}=\eta_S\xi^2$ will still be arbitrary.

(*iii*) *Reflection:* $\mathcal{R}$

$$\text{a.}\ \exp[i\pi \textstyle\sum_{\mathbf{k}} n_{\mathbf{k}}(1)]\times\exp[\frac{1}{2}i\pi \sum_{\mathbf{k}} (\eta_RA^\dagger_{\mathbf{k}}-\eta_R^*A_{\mathbf{k}})],$$

$$\text{b.}\ \exp[i\pi \textstyle\sum_{\mathbf{k}} n_{\mathbf{k}}(2)]\times\exp[\frac{1}{2}\pi \sum_{\mathbf{k}} (\eta_R^*A_{\mathbf{k}}-\eta_RA^\dagger_{\mathbf{k}})], \tag{2.13}$$

$$\text{c.}\ \exp[\tfrac{1}{2}i\pi \textstyle\sum_{\mathbf{k}} (n_{\mathbf{k}}(1)+n_{\mathbf{k}}(2)+\eta_RA^\dagger_{\mathbf{k}}+\eta_R^*A_{\mathbf{k}})].$$

These are also the representations for the unitary factor of weak reflection; the phase factor η_{UW} is arbitrary even if W^2 is required to be a unit operator.

(*iv*) *Inversion:* I

The representation of the unitary factor U_I is given by any operator of phase transformation; the phase factors η_{UI} and η_I are both arbitrary, for I^2 is an identity operator.

Non-Hermitian K-G Field, Angular-Momentum Representation

The transformation properties of $b(k,l,m,r)$ are given in Table II; however, representations can be considerably simplified by noting that

$$b^*(k,l,m,r)=(-1)^{m+l}b(k,l,-m,r);$$

the representations then are

$$\left.\begin{aligned}&\text{(i)}\quad \exp[i\sum_{k,l,m,r}\alpha(-1)^rn(k,l,m,r)],\\ &\text{where,}\\ &\alpha=\beta_P+l\pi,\quad \exp(i\beta_P)=\eta_P\quad \text{for}\ \mathcal{P}\\ &\quad=\beta_{UT}+l\pi,\quad \exp(i\beta_{UT})=\eta_{UT}\quad \text{for}\ U_T\\ &\quad=\beta_{UI},\quad \exp(i\beta_{UI})=\eta_{UI}\quad \text{for}\ U_I\end{aligned}\right\} \tag{2.14}$$

and, (ii)

$$\left.\begin{aligned}
&\text{a. } \exp[i\pi \sum_{k,l,m,r} \alpha_r n(k,l,m,r)] \\
&\qquad \times \exp\left\{\frac{\pi}{2}\sum_{k,l,m}[\eta^* C(k,l,m) - \eta C\ (k,l,m)]\right\}, \\
&\text{b. } \exp\left[-i\frac{\pi}{2}\sum_{k,l,m,r}\beta n(k,l,m,r)\right] \\
&\qquad \times \exp\left\{i\frac{\pi}{2}\sum_{k,l,m}[\eta^* C(k,l,m) + \eta C^\dagger(k,l,m)]\right\}, \\
&\text{where} \\
&\qquad C(k,l,m) = b^\dagger(k,l,m,2)b(k,l,m,1); \\
&\alpha_1 = \beta = 1, \quad \alpha_2 = 0 \qquad \text{for } \mathcal{C}, U_S \\
&\alpha_1 = l+1, \quad \beta = 2l+1, \quad \alpha_2 = l \quad \text{for } \mathcal{R}, U_W
\end{aligned}\right\}. \quad (2.15)$$

These representations can also be obtained from those in linear momentum by direct substitution of Eq. (1.20).

Electromagnetic Field

In terms of $b_{\mathbf{k}}(s)$ $(s=1,2,3,4)$ the representations are

$$\left.\begin{aligned}
&\text{(i) } \exp\{i\pi \sum_{\mathbf{k},s}{}' n_{\mathbf{k}}(s)\} \\
&\qquad \times \exp\left\{\frac{\pi}{2}\sum_{\mathbf{k},s}{}' \alpha[\beta^* B^\dagger_{\mathbf{k}}(s) - \beta B_{\mathbf{k}}(s)]\right\}, \\
&\text{where} \\
&B_k(s) = b^\dagger_{\mathbf{k}}(s) b_{-\mathbf{k}}(s); \quad n_{\mathbf{k}}(s) = b^\dagger_{\mathbf{k}}(s) b_{\mathbf{k}}(s); \\
&\alpha = +\exp(i\pi\delta_{s2}) \quad \text{for} \quad P, U_T; \\
&\quad -\exp(i\pi\delta_{s2}) \quad \text{for} \quad R, U_W \\
&\beta = +1 \text{ for } \mathcal{P}, \mathcal{R}; \text{ real, arbitrary for } U_T, U_W
\end{aligned}\right\} \quad (2.16)$$

$$\left.\begin{aligned}
&\text{(ii) } \exp(i\alpha \sum_{\mathbf{k},s} n_{\mathbf{k}}(s)), \\
&\text{where} \\
&\alpha = \pi, \pi+\beta, \beta \quad \text{for} \quad \mathcal{C}, U_s, U_I, \text{ respectively;} \\
&\qquad\qquad \exp i\beta = \xi
\end{aligned}\right\}. \quad (2.17)$$

The DST's for $s=1, 2$, when expressed in circular components $(\lambda=\uparrow,\downarrow)$, clearly bring out the spin property of the photon. The transformation properties are:

$$\left.\begin{aligned}
\mathcal{P} b_{\mathbf{k}}(\uparrow)\mathcal{P}^\dagger &= b_{-\mathbf{k}}(\downarrow) \\
T b_{\mathbf{k}}(\uparrow)\mathcal{P}^\dagger &= b_{-\mathbf{k}}(\uparrow) \\
S b_{\mathbf{k}}(\uparrow) S^\dagger &= -b_{\mathbf{k}}(\downarrow)
\end{aligned}\right\}. \quad (2.18)$$

These show that the spin of photon is all in the direction of momentum. The representations of DST's are:

$$\text{(i)} \quad \mathcal{P} = \exp\left\{i\frac{\pi}{2}\sum_{\mathbf{k},\lambda} n_k(\lambda)\right\}$$

$$\times \exp\left\{i\frac{\pi}{2}\sum_{\mathbf{k},\lambda} b^\dagger_{\mathbf{k}}(\lambda) b_{-k}(\lambda')\right\}, \quad (2.19)$$

$$\text{(ii)} \quad U_T = \exp\left\{-i\left(\frac{\pi}{2}-\beta\right)\sum_{\mathbf{k},\lambda} n_k(\lambda)\right\}$$

$$\times \exp\left\{i\frac{\pi}{2}\sum_{k,\lambda} b^\dagger_{\mathbf{k}}(\lambda) b_{-k}(\lambda)\right\}, \quad (2.20)$$

$$\text{(iii)} \quad U_S = \exp\left\{-i\left(\frac{\pi}{2}-\beta\right)\sum_{\mathbf{k},\lambda} n_{\mathbf{k}}(\lambda)\right\}$$

$$\times \exp\left\{i\frac{\pi}{2}\sum_{\mathbf{k},\lambda} b^\dagger_{\mathbf{k}}(\lambda) b_k(\lambda')\right\}, \quad (2.21)$$

where $\exp(i\beta) = \xi$ is the arbitrary phase factor associated with the antiunitarity of T and S. The operators $\mathcal{R}, W, I$ may be obtained from the above 3 respectively by multiplying them with

$$\mathcal{C} = \exp\{i\pi \sum_{\mathbf{k},\lambda} n_{\mathbf{k}}(\lambda)\}. \quad (2.22)$$

V. Dirac Field

Here, because of anticommutation relations squares of CO's and AO's vanish, so that one can obtain $\exp(\alpha\Omega)$ in the form $1+G$; in fact in cases of interest, one obtains $\Omega^2 = -\Omega_n$, $\Omega^2{}_n = \Omega_n$, $\Omega_n\Omega = \Omega\Omega_n = \Omega$

$$\exp(\alpha\Omega) = 1 + \Omega^2(1-\cos\alpha) + \Omega \sin\alpha. \quad (2.23)$$

The representations are as follows:

(i) *Space Inversion* $\mathcal{P}$: $(\exp i\delta = \eta_1 = \eta_p;\ \eta_2 = \eta_p^*)$

$$\text{a. } \exp[i(2\delta-\pi)\sum_{\mathbf{k},r,s}{}' (-1)^r N_{\mathbf{k}}(s,r)]$$

$$\times \exp\left\{\frac{\pi}{2}\sum_{\mathbf{k},r,s}{}' (-1)^r[\eta_r B_{\mathbf{k}}(s,r) - \eta_r^* B^\dagger_{\mathbf{k}}(s,r)]\right\} \quad (2.24)$$

$$\text{b. } \prod_{\mathbf{k},s,r}{}' \{1 - N_{\mathbf{k}}(s,r) - N_{-\mathbf{k}}(s,r) + (1-\eta_r^{*2}) N_{-\mathbf{k}}(s,r)$$

$$\times N_{\mathbf{k}}(s,r) - (-1)^r \eta_r^*[B_{\mathbf{k}}(s,r) + B^\dagger_{\mathbf{k}}(s,r)]\},$$

where the prime on $\prod$ indicates product over half the

k space only. The representation for $\mathcal{P}^2$ is

$$\mathcal{P}^2=\prod_{\mathbf{k},s,r}{}' \{1+(\eta_r^2-1)[N_\mathbf{k}(s,r)+N_{-\mathbf{k}}(s,r)] +(\eta_r^2-1)^2N_\mathbf{k}(s,r)N_{-\mathbf{k}}(s,r)\} \quad (2.25)$$

$$=\prod_{\mathbf{k},s,r}{}' \{1-2[N_\mathbf{k}(s,r)-N_{-\mathbf{k}}(s,r)]^2\} \text{ for } \eta_p=\mp i$$

and identity for $\eta_p=\mp 1$.

(ii) Unitary Factor of Time Reversal: $U_T(\eta_1=\eta_{UT}=\eta_2^*=\exp i\delta)$

$$\text{a. } \prod_{\mathbf{k},r} \{1-N_\mathbf{k}(1,r)-N_{-\mathbf{k}}(2,r) +(1-\eta_r^{*2})N_\mathbf{k}(1,r)N_{-\mathbf{k}}(2,r) +\eta_r^*[\beta_\mathbf{k}(r)-\beta_\mathbf{k}^\dagger(r)]\},$$

$$\text{b. } \exp\{2i\delta\sum_{\mathbf{k},r}(-1)^rN_\mathbf{k}(1,r)\} \times\exp\left\{\frac{\pi}{2}\sum_{\mathbf{k},r}[\eta_r\beta_\mathbf{k}(r)-\eta_r^*\beta^\dagger{}_\mathbf{k}(r)]\right\}. \quad (2.26)$$

(iii) Particle Conjugation: $\mathcal{C}$

$$\text{a. } \exp[i\pi\sum_{\mathbf{k},s}N_\mathbf{k}(s,1)] \times\exp\left\{\frac{\pi}{2}\sum_{\mathbf{k},s}[\eta_c^*C^\dagger{}_\mathbf{k}(s)-\eta_cC_\mathbf{k}(s)]\right\}, \quad (2.27)$$

$$\text{b. } \prod_{\mathbf{k},s}\{1-N_\mathbf{k}(s,1)-N_\mathbf{k}(s,2) +\eta_cC_\mathbf{k}(s)+\eta_c^*C^\dagger{}_\mathbf{k}(s)\}.$$

(iv) Reflection: $\mathcal{R}$

$$\text{a. } \exp\left\{\frac{\pi}{2}\sum_{\mathbf{k},s}[\eta_R^*A^\dagger{}_\mathbf{k}(s)-\eta_RA_\mathbf{k}(s)]\right\},$$

$$\text{b. } \prod_{\mathbf{k},s}\{1-N_\mathbf{k}(s,1)-N_{-\mathbf{k}}(s,2)+2N_\mathbf{k}(s,1)N_{-\mathbf{k}}(s,2) +\eta_R^*A^\dagger{}_\mathbf{k}(s)-\eta_RA_\mathbf{k}(s)\}. \quad (2.28)$$

(v) Unitary Factor of Inversion: $I(\eta_1=\eta_{UI}=\eta_2^*=\exp i\delta)$

$$\text{a. } \exp[2i\delta\sum_{\mathbf{k},r}(-1)^rN_\mathbf{k}(1,r)] \times\exp\left\{\frac{\pi}{2}\sum_{\mathbf{k},r}[\eta_{UI}D_\mathbf{k}(r)-\eta_{UI}^*D^\dagger{}_\mathbf{k}(r)]\right\}, \quad (2.29)$$

$$\text{b. } \prod_{\mathbf{k},r}\{1-N_\mathbf{k}(1,r)-N_{-\mathbf{k}}(2,r) +(\eta_r^{*2}+1)N_\mathbf{k}(1,r)N_\mathbf{k}(2,r) +\eta_r^*[D_\mathbf{k}(r)-D^\dagger{}_\mathbf{k}(r)]\}.$$

(vi) Unitary Factor of Weak Reflection: $W(\eta_1=\eta_{UW}=\eta_2^*)$

$$\text{a. } \exp\left\{\frac{\pi}{2}\sum_{\mathbf{k},r}[\eta_rG_\mathbf{k}(r)-\eta_r^*G^\dagger{}_\mathbf{k}(r)]\right\},$$

$$\text{b. } \prod_{\mathbf{k},r}\{1-N_\mathbf{k}(1,r)-N_{-\mathbf{k}}(2,r') +2N_\mathbf{k}(1,r)N_{-\mathbf{k}}(2,r')+\eta_rG_\mathbf{k}(r)-\eta_r^*G^\dagger{}_\mathbf{k}(r)\}. \quad (2.30)$$

(vii) Unitary Factor of Strong Reflection: U_S

$$\text{a. } \exp[i\pi\sum_{\mathbf{k},s}N_\mathbf{k}(s,1)] \times\exp\left\{\frac{\pi}{2}\sum_{\mathbf{k},s}(-1)^s[\eta_{US}^*F_\mathbf{k}^\dagger(s)-\eta_{US}F_\mathbf{k}(s)]\right\},$$

$$\text{b. } \prod_{\mathbf{k},s}\{1-N_\mathbf{k}(s,1)-N_\mathbf{k}(s',2) +(-1)^s[\eta_{US}^*F_\mathbf{k}^\dagger(s)+\eta_{US}F_\mathbf{k}(s)]\}. \quad (2.31)$$

Equation (2.25) also gives representations of W^2 and S^2 if one replaces η_p by $-i\eta_W^*$ and η_S^*, respectively.

CONCLUDING REMARKS

An examination of Table III reveals that for the non-Hermitian fields, the phase factors due to the squares of the operators $\mathcal{C}$, $\mathcal{R}$, T, I are independent of the phase factors due to the operators themselves; these latter phases are therefore unmeasurable. An analogous conclusion is obtained by Feinberg and Weinberg[16] from the point of view of multiplicative operators; squares of DST's are examples of such operators. This unmeasurability is connected with the existence of more than one interacting Hamiltonian corresponding to the same set of observables.[16,17]

Unlike the generators of some continuous symmetry transformations (such as space time translations)[18] that give rise to additive quantum numbers, there does not exist an obvious physical interpretation of the generators of DST's (or their unitary factors). One can however decompose every DST into suitable factors, whose generators can be separately interpreted. The representations factored for the non-Hermitian K-G and Dirac fields are given in Table IV. We make the following remarks.

[17] W. Pauli, Nuovo cimento **6**, 204 (1957). D. Pursey, *ibid.* **6**, 266 (1957).

[18] See, e.g., M. A. Melvin, Revs. Modern Phys. **32**, 477, (1960).

TABLE IV. Representations of the discrete symmetry transformations of the K-G and Dirac fields.[a,b,c]

U	Klein-Gordon field	Dirac field
$\mathcal{P}$	$e^{i(\pi/2)\Omega_P}e^{-i(\pi/2)n}e^{-i\alpha_P Q}$	$e^{i(\pi/2)\Omega_P}e^{-i(\pi/2)Q}e^{-i\alpha_P Q}$
$\mathcal{R}$	$e^{i(\pi/2)\Omega_P}e^{-i(\pi/2)n}e^{-i\alpha_R Q}$	$e^{i(\pi/2)\Omega_R}e^{-i(\pi/2)Q}e^{-i\alpha_R Q}$
$\mathcal{C}$	$e^{i(\pi/2)\Omega_C}e^{-i(\pi/2)n}e^{-i\alpha_C Q}$	$e^{i(\pi/2)\Omega_G}e^{-i(\pi/2)N}e^{-i\alpha_G Q}$
U_T	$e^{i(\pi/2)\Omega_P}e^{-i(\pi/2)n}e^{-i\alpha_{UT} Q}$	$e^{i(\pi/2)\Omega_T}e^{i\pi S_3}e^{-i\alpha_{UT} Q}$
U_W	$e^{i(\pi/2)\Omega_R}e^{-i(\pi/2)n}e^{-i\alpha_{UW} Q}$	$e^{i(\pi/2)\Omega_W}e^{i\pi S_3}e^{-i\alpha_{UW} Q}$
U_I	Any phase transformation	$e^{i(\pi/2)\Omega_I}e^{i\pi S_3}e^{-i(\pi/2)N}e^{i[(\pi/2)-\alpha_{UI}]Q}$
U_S	$e^{i(\pi/2)\Omega_C}e^{-i(\pi/2)n}e^{-i\alpha_{US} Q}$	$e^{i(\pi/2)\Omega_S}e^{i\pi S_3}e^{-i(\pi/2)N}e^{i[(\pi/2)-\alpha_{US}]Q}$

[a] The operators Q for K-G field and Q and S_3 for the Dirac field are given by Eqs. (1.9), (1.47), and (1.48), respectively.

[b] n and N are, respectively, total particle number operators of the K-G and Dirac fields.

[c] For the K-G field Ω_P, Ω_R, Ω_C are, respectively, the bilinears B, A, C of (2.7) summed over all $\mathbf{k}$ and r; for the Dirac field Ω_P, Ω_R, Ω_C, Ω_T, Ω_W, Ω_I, Ω_S are the bilinears B, A, C, β, G, D, F, of (2.8) summed over all $\mathbf{k}$, s, r.

(1) There is a factor $e^{i\alpha_U Q}$ in every DST and is responsible for, in general, the complex phase factor $\eta_U = e^{i\alpha_U}$; for Hermitian fields, $Q=0$, and η_U is real. The multiplicative operators such as $\mathcal{P}^2$ have Q for their generators.

(2) A factor $e^{i\pi S_3}$ (S_3 spin in z direction) occurs for every antilinear DST of a spin nonzero field; this is associated with the change of sign of spin in an antilinear DST.

(3) In the representation of the operators $\mathcal{P}$, $\mathcal{R}$, U_I, U_S of the Dirac field, the factor $e^{i(\pi/2)N}$ of the corresponding operators of the K-G field is replaced by the factor $e^{i(\pi/2)Q}$; this is because the parities of particle and antiparticle are opposite for the Dirac field.

(4) Apart from these there is a factor $e^{i(\pi/2)\Omega_U}$—the discrete factor of the transformation U ($\Omega_U \equiv$ discrete generator), which is different for every DST U, unless one or more of the observables $\mathcal{P}$, Q, S_3 vanish; i.e., every discrete symmetry operator has a characteristic discrete generator, or what is the same thing—a bilinear in CO's and AO's. Also there are in general, corresponding to the three observables P, Q, J, 2^3-1 = seven DST's and the same number of unitary operators. Thus for the Dirac field there are eight distinct bilinears. In the absence of spin one obtains 2^2 distinct bilinears and therefore only three distinct discrete factors; the antiunitary DST's in this case differ from the unitary ones only by a possible phase transformation and the factor of complex conjugation; this is the case for non-Hermitian K-G field. For Hermitian K-G field, since $Q=S_3=0$, there are 2^1 distinct bilinears and therefore only one discrete generator. The connection between the bilinears that are generators of the discrete factors of DST's U and the observables that change sign under U is also contained in equations like (2.3) and (2.5).

Regarding eigenstates of Ω_U, we remark that since Ω_U consists of bilinears that annihilate a particle with one physical property and create it with the opposite physical property, only those one-particle states that are superpositions of states with opposite physical properties in question will be its eigenstates. Thus if $\Omega_c = \sum_{\mathbf{k},r} b^\dagger{}_{\mathbf{k}}(r) b_{-\mathbf{k}}(r')$, its eigenstates are

$$|K_1\rangle = \frac{b^\dagger{}_{\mathbf{k}}(4) + b^\dagger{}_{-\mathbf{k}}(2)}{\sqrt{2}}|0\rangle, \quad |K_2\rangle = \frac{b^\dagger{}_k(1) - b^\dagger{}_{-k}(2)}{\sqrt{2}}|0\rangle$$

with eigenvalues $+1$, -1, respectively. If we identify $|K^0\rangle$, $|\bar{K}^0\rangle$ with $b^\dagger{}_k(r)|0\rangle$, for $r=1$, 2, respectively, we see that during the production of K^0 by the strong interaction such as $P+\pi^- \rightarrow \Lambda^0 + K^0$, $|K^0\rangle$ is the eigenstate of Q; but during the latter's weak decay we get the eigenstates $|K_1\rangle$ and $|K_2\rangle$ of Ω_C (this example is due to Grawert[12]).

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APPENDIX

In case of a Hermitian K-G field, one can write a general, Hermitian, operator bilinear in creation and annihilation operators as

$$\Omega = \sum_{i,j} \alpha_{ij} b_{\mathbf{k}_i} b^\dagger{}_{\mathbf{k}_j} + \sum_{i,j} (\beta_{ij} b_{\mathbf{k}_i} b_{\mathbf{k}_j} + \beta_{ij}{}^* b^\dagger{}_{\mathbf{k}_i} b^\dagger{}_{\mathbf{k}_j}), \quad \text{(A1)}$$

where $\alpha_{ij} = \alpha_{ji}$ and β_{ij} are c-number coefficients; then, $\exp(i\Omega)$ transforms $b_{\mathbf{k}}$ as

$$V b_{\mathbf{k}_r} V^\dagger = \sum_i (A_{ir} b_{\mathbf{k}_i} + B_{ir} b^\dagger{}_{\mathbf{k}_i}). \quad \text{(A2)}$$

The terms in $b^\dagger{}_{\mathbf{k}}$ on the right are due essentially to the second member of Ω (i.e., in the parantheses). A free-particle Hamiltonian H_0 under this transformation takes the form analogous to Ω; conversely, a general Hamiltonian H of the form Ω can be diagonalized by a unitary transformation of the type $\exp(i\Omega)$. The transformed Hamiltonian thus obtained represents the same interaction as before, provided the eigenstates also undergo the same transformation. In case of perturbation calculations, one expands $\exp(i\Omega)$ to obtain approximate states.

However, it may be remarked that transformation of the type $\exp(i\Omega)$ does not transform a creation operator into an annihilation operator or vice versa; i.e., a unitary transformation (1.13) is not possible. This has to do with unitary invariance of the theory and antilinear nature of the time-reversal transformation, the latter being connected with the positive definiteness of energy. However, if one admits nonunitary transformations, e.g., Hermitian $h=\exp(\alpha\Omega)$ (α real and Ω Hermitian), one can construct representations that leave the commutation relations unchanged; for example,

$$\mathbf{h}_1=\exp\{\tfrac{1}{2}\pi \sum_k [\eta b_k^2+\eta^* b^\dagger{}_k{}^2]\} \tag{3}$$

transforms $b^\dagger{}_k$ into $-\eta^* b_k$ and n_k into $-(1+n_k)$; the commutation relations are preserved, but one obtains

$$[\mathbf{h}_1,H]_+=\sum_k \omega_k h_1, \quad [\mathbf{h}_1,\mathbf{P}]_+=0; \tag{4}$$

furthermore, the vacuum state is no longer unchanged. Disregard of such bilinears in constructions of symmetry operators is thus justified.

We remark that one can construct an operator $\mathbf{a}$, such that $\mathbf{a}b^\dagger=b\mathbf{a}$; in fact, $\mathbf{a}$ is symmetric and has matrix elements

$$\mathbf{a}_{\mu\nu}=\left[\begin{pmatrix}\mu+\nu-2\\ \mu-1\end{pmatrix}\right]^{\frac{1}{2}}\mathbf{a}_{1,\ \mu+\nu-1}; \tag{5}$$

it does not seem possible, however, to construct an inverse $\mathbf{c}=\mathbf{a}^{-1}$, as the equation $b^\dagger\mathbf{c}=\mathbf{c}b$ seems difficult to satisfy simultaneously with (5), though it would appear from the theorem of po′ lya[19] that there exist infinitely many left and right inverses.

Finally, it is interesting to mention that there exist a nonunitary, non-Hermitian operator A such that

$$A^\dagger b^\dagger=b, \quad bA=b^\dagger \tag{6}$$

and

$$A^\dagger|0\rangle=|0\rangle, \quad \langle 0|A=\langle 0|. \tag{7}$$

One can thus construct A_k by the method of direct products.[20] The general transformation may then be written as $\mathcal{A}=\prod_k A_k$. The prescription then is that $\mathcal{A}$ operates on all b_k and bra's while $\mathcal{A}^\dagger$ operates on $b^\dagger{}_k$ and ket's; this operation should be followed by reading all expressions from right to left. The appropriate representation of A is:

$$\begin{Bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\left(\frac{1}{2}\right)^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \left(\frac{2}{3}\right)^{\frac{1}{2}} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \left(\frac{3}{4}\right)^{\frac{1}{2}} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \left(\frac{4}{5}\right)^{\frac{1}{2}} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \left(\frac{5}{6}\right)^{\frac{1}{2}} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \left(\frac{6}{7}\right)^{\frac{1}{2}} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots
\end{Bmatrix}. \tag{8}$$

[19] See: R. G. Cooke, *Infinite Matrices and Sequence Spaces* (Macmillan and Company, Ltd., London, England, 1950).

[20] F. D. Murnagham, *Theory of Group Representations* (Johns Hopkins Press, Baltimore, Maryland, 1938).