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Simple Groups and Strong Interaction Symmetries*

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CONTENTS

INTRODUCTION

ONE of the most natural questions when one looks at the mass of uncorrelated data on elementary particle interactions[1] is whether a systematic pattern is emerging from this complexity. The penetration of controlled laboratory experiments into the multi-Bev energy region can only make such a question more acute. Several attempts[2] have already been made to unfurl the underlying symmetry of strong interactions, such as might exist above and beyond those symmetries, e.g., isotopic symmetry,[3] which have already survived experimental tests.

In this article, we sharpen some tools which prove useful in formulating the consequences of proposed symmetries of a rather special type, namely, those symmetries which are characteristic of the simple Lie groups. Since it is as yet too early to establish a definite

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[1] See, for example, the *Proceedings of the Tenth Annual Conference on High Energy Nuclear Physics, Rochester, 1960, University of Rochester* (Interscience Publishers, Inc., New York, 1960).

[2] See, for example, B. d'Espagnat and J. Prentki, Nuclear Phys. **1**, 33 (1956); J. Schwinger, Ann. Phys. **2**, 407 (1957); M. Gell-Mann, Phys. Rev. **106**, 1296 (1957); A. Pais, *ibid.* **110**, 574 (1958); J. Tiomno, Nuovo cimento **6**, 69 (1957); R. E. Behrends, *ibid.* **11**, 424 (1959); D. C. Peaslee, Phys. Rev. **117**, 873 (1960); J. J. Sakurai, *ibid.* **115**, 1304 (1959).

[3] See, for example, W. Heisenberg, Z. Physik **77**, 1 (1932); B. Cassen and E. U. Condon, Phys. Rev. **50**, 846 (1936); G. Breit, E. U. Condon, and R. D. Present, *ibid.* **50**, 825 (1936); G. Breit and E. Feenberg, *ibid.* **50**, 850 (1936).

symmetry of the strong interactions, both because of the lack of experimental data and the theoretical uncertainties about the way in which the symmetries will manifest themselves, the formalism developed is left quite flexible in order to accommodate a wide range of conceivable symmetries.

Much of the material is an exposition of the theory of Lie groups and, although most of the results have been known for many years, several new features appear. Thus the material on the composition and decomposition of Lie algebras by point set theory, the explicit construction of the Lie algebras, the tensor analysis of the groups B_2 and G_2, and the possible physics associated with the group B_2 is believed to be novel. A large portion of the remaining material is possibly unfamiliar to many physicists (as it was to us), and so is pedagogical in nature. Although the discussions are directed primarily to applications in elementary particle physics, many of the techniques have been used before in group theoretical treatments of atomic and nuclear spectroscopy.[4]

An admirable summary of the elementary properties of semi-simple Lie algebras is contained in the lecture notes of Racah,[5] which treat both the classification of semi-simple groups, following Cartan,[6] and their linear representations. A complete and rigorous derivation of the properties of semi-simple Lie algebras can be found in the work of Dynkin,[7] while Weyl's original work[8] remains the standard reference on the representation theory of semi-simple groups. For the tensor analysis associated with particular groups and with the Young tableaux, Weyl's *Classical Groups*[9] and *Group Theory and Quantum Mechanics*[10] is recommended. We assume that the reader is mildly conversant with the group theoretical treatment of angular momentum as given by Wigner,[11] for example. Finally, we give various references[12] to the basic mathematical literature.

As far as the physical application of the group theoretical methods is concerned, we are immediately faced with the problem of justifying the specific course which we pursue in attributing symmetries to strong particle interactions. The hope that symmetries exist, other than those associated with space-time structure, is kindled by the observation that some such "internal" symmetries are already apparent. First of all, charge independence has so far run the gauntlet of experimental tests[13] and has become a commonly accepted symmetry. In addition, a second kind of symmetry, slightly more mysterious than the former, is afforded by the electrodynamic[14] and weak-dynamic equivalence[15] of the muon and electron. Both of these symmetries call for a closer discussion.

It is well known that particles belonging to the same isotopic multiplet exhibit a remarkable similarity in their strong-interaction dynamics. Differences in behavior and in mass of isotopic spin multiplet members are quite naturally attributed to the charge-dependent electromagnetic interaction, which acts as a weak perturbation on the strong-interaction dynamics. Indeed, the breakdown of isotopic symmetry is evidenced in the high Z nuclear species where the coherent Coulomb field no longer can be treated as a perturbation. By analogy, we may conjecture that a basic symmetry exists among, say, baryon-baryon interactions, but that the full force of this symmetry is diluted by a relatively weak symmetry-breaking interaction. The answer to the question "under what circumstances will the symmetry-masking interaction be minimized?" is not yet clear, since the answer undoubtedly depends on the specific nature of the symmetry-breaking interaction. Of course, the latter interaction would most likely, produce the baryon mass differences besides its other effects.

In the case of the dynamic symmetry of muon and electron, no interaction is known which can serve to break the symmetry and account for the mass difference. Most physicists seem to feel that a specific difference in muon and electron interactions will ultimately emerge even if present experimental circumstances have not revealed it. If the proposed strong interaction symmetry resembles that of the muon and electron, it could conceivably be discernible even in the presence

[4] See, for example, G. Racah, Phys. Rev. **61**, 186 (1942); **62**, 438 (1942); **63**, 367 (1943); **76**, 1352 (1949). T. H. R. Skyrme, "Lectures in Nuclear Structure (I), General Theory and Shell Model," Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania, 1958.

[5] G. Racah, "Group Theory and Spectroscopy," Institute for Advanced Study, Lecture notes, Princeton, New Jersey, 1951.

[6] E. Cartan, *Thèse Paris* (1894) reprinted in E. Cartan, *Oeuvres Complètes* (Gauthiers-Villars, Paris, France, 1952). E. Cartan, Bull. Soc. Math. de France **41**, 53 (1913).

[7] E. B. Dynkin, Am. Math. Soc. Translations, No. 17 (1950).

[8] H. Weyl, Z. Math. **24**, 328, 377 (1925), reprinted in H. Weyl, *Selecta* (Birkhauser Verlag, Basel und Stuttgart, Germany, 1956), p. 262.

[9] H. Weyl, *Classical Groups* (Princeton University Press, Princeton, New Jersey, 1946), 2nd ed.

[10] H. Weyl, *Group Theory and Quantum Mechanics* (reprint, Dover Publications, New York).

[11] E. P. Wigner, *Group Theory and Its Applications to Atomic Structure* (Academic Press, Inc., New York, 1960).

[12] S. Lie and F. Engels, Theorie der Transformationsgruppen (B. G. Teubner, Leipzig, Germany, 1888–1893). V. Killing, Math. Ann. **31**, 252 (1888); **33**, 1 (1889); **34**, 57 (1889); **36**, 161 (1890). L. P. Eisenhart, *Groups of Continuous Transformations* (reprint, Dover Publications, New York, 1961). L. S. Pontrjagin, Topological Groups (Princeton University Press, Princeton, New Jersey, 1958). H. Freudenthal, "Lie Groups," Lecture notes, Department of Mathematics, Berkeley, California, 1960). D. Montgomery, "Topological Groups," Lecture notes, Haverford College, Haverford, Pennsylvania, 1956.

[13] See, for example, J. M. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

[14] J. Garwin, L. Lederman, and M. Weinrich, Phys. Rev. **105**, 1415 (1957). G. Charpak, F. Farley, R. Garwin, T. Muller, J. Sens, V. Telegdi, and A. Zichichi, Phys. Rev. Letters **6**, 128 (1961).

[15] M. Ruderman and R. J. Finkelstein, Phys. Rev. **76**, 1458 (1949); J. A. Wheeler and J. Tiomno, Revs. Modern Phys. **21**, 144 (1949); O. Klein, Nature **161**, 897 (1948); E. Clementel and G. Puppi, Nuovo cimento **5**, 505 (1948); T. D. Lee, M. Rosenbluth, and C. N. Yang, Phys. Rev. **75**, 905 (1949).

of baryon mass differences, just as is the case with muon and electron interactions.

In summary, we are unable to give any *a priori* justification for the existence of strong interaction symmetries, but share the widespread feeling that such symmetries are plausible and not entirely unprecedented.

In Sec. I, the embryonic elements of the application of symmetry considerations to elementary particle interactions are presented to motivate physically the following sections. Section II is devoted to a necessarily abbreviated form of the theory of Lie groups, in which an attempt is made to appeal as much as possible to a physicist's intuition. There then follows (Sec. III) the properties and the construction of linear representations of Lie groups, of which it is hoped that elementary particles provide an instance. The next two sections (Secs. IV and V) solve the problem of finding certain properties of the Lie algebra representations, in particular, the "weights" of the representations and the decomposition of direct products of representations (generalized Clebsch-Gordan series). Two approaches are employed; one predominantly geometric (Sec. IV), the second predominantly algebraic (Sec. V). Section V is essentially the tensor analysis associated with simple groups. All roads lead to Sec. VI which is concerned with physical applications of the mathematical complex of the previous sections. From this summit, we briefly view the expanding vistas of possible strong interaction symmetries.

I. SYMMETRIES OF THE LAGRANGIAN

The basic idea behind Heisenberg's introduction of the concept of isotopic spin[3] was the realization that the neutron and the proton are, after all, quite similar. The differences in mass and in electromagnetic interactions are small in the context of the strong interactions. The fact that only two baryons were known led unambiguously to the assignment of a doublet structure to the "nucleon." Later, when strange particles were discovered, these were found, as is reflected in the name, to have properties so widely different from the nucleons that the assignment of the proton and the neutron to a doublet was retained without question. When one attempts to introduce symmetries which treat particles of widely different masses as states of the same field, however, it is not wise to be so categorical about the number of particles to be included in the scheme. Specific conjectures are made in the last section; for the present let n be the number of baryons treated as states of the same field, i.e., as belonging to the same supermultiplet. A favorite choice for n is 8, if all the observed baryons are included.[16–18] It could be less than eight if the baryons separate into two or more supermultiplets,[19] or it could be larger than 8 if some hypothetical baryons not yet discovered are included.

Let ψ_a, $a=1, 2, \cdots, n$, denote the n-component baryon field, where each component is a Dirac four-spinor, and let $\bar{\psi}^a=\psi_a{}^\dagger\gamma_4$. The free Lagrangian is

$$\mathcal{L}_0=\left(\frac{1}{2i}\right)\sum_{a=1}^{n}\bar{\psi}^a(p-im_a)\psi_a.$$

In the introduction we mentioned several different points of view, according to which the mass differences may be argued to be nonessential in the first analysis. When the n masses are put equal, $m_1=m_2=\cdots=m_n$, $\mathcal{L}_0$ is invariant under a set of linear transformations, acting on the set $\psi_a=\{\psi_1\cdots\psi_n\}$. In fact, let $U_a{}^b$ be a square $n\times n$ matrix, and consider the transformation

$$\psi_a\rightarrow\psi_a{}'=U_a{}^b\psi_b,$$
$$\bar{\psi}^a\rightarrow\bar{\psi}'^a=(U\psi)^{\dagger a}\gamma_4=\bar{\psi}^b(U^\dagger)_b{}^a.$$

Clearly, $\mathcal{L}_0$ is invariant if and only if U is unitary, i.e.,

$$(U^\dagger)_a{}^c(U)_c{}^b=\delta_a{}^b.$$

Hence, in matrix notation

$$\psi\rightarrow U\psi,\quad \bar{\psi}\rightarrow\bar{\psi}U^{-1},\quad UU^\dagger=U^\dagger U=1. \tag{I.1}$$

The set of all $n\times n$ unitary matrices forms a *group*.[20] That is, if U, V are unitary, so are UV and U^{-1}. Hence $\mathcal{L}_0$ is invariant under the group of unitary transformations (I.1). This group contains an invariant subgroup, which is usually called the baryon gauge group. Any unitary matrix U may be written

$$U=e^{i\varphi}\mathcal{U},$$

where φ is real and $\mathcal{U}$ is unitary and unimodular:

$$\mathcal{U}^\dagger\mathcal{U}=\mathcal{U}\mathcal{U}^\dagger=1,\quad \det\mathcal{U}=1. \tag{I.2}$$

Invariance under the gauge transformation, represented by the factor $e^{i\varphi}$, corresponds to the conservation of baryons. This conservation law is taken for granted, and it is therefore unnecessary to include the gauge transformations in our analysis. From now on we deal with transformation matrices that are unimodular as well as unitary. The set of all such matrices forms a group[21] which is denoted $\mathcal{S}U_n$.

In general, the interaction between the fields will break part of the symmetry of the free Lagrangian. Invariance under $\mathcal{S}U_n$ represents the maximum symmetry between the n baryons, and any group of transformations admitted by the fields in interaction is a subgroup of $\mathcal{S}U_n$. In order to explore, in a systematic manner, the various groups of interest, it is helpful to review some topics from the theory of Lie groups. The basic concepts of the theory of Lie groups and of their

[16] R. E. Behrends and D. C. Peaslee, reference 2.
[17] T. D. Lee and C. N. Yang, Phys. Rev. **122**, 1954 (1961).
[18] M. Gell-Mann, Phys. Rev. (to be published).
[19] R. E. Behrends and A. Sirlin, Phys. Rev. **121**, 324 (1961).
[20] This group is called the *unitary group* U_n.
[21] It is the *factor group* of U_n with respect to the gauge group.

representations are reviewed in the next two sections. Before that, however, we say a few words about the problem of writing down interactions. It is convenient to deal with a simple specific example only, without any implication that the problems and their solution are peculiar to this case, or to this point of view. By way of an example, let us treat the case of a Yukawa-type interaction, invariant under $\mathcal{S}U_8$, between the 8 baryons and a number m of bosons. The interaction Lagrangian is of the form

$$\mathcal{L}'=\bar{\psi}^a(\Gamma_\sigma)_a{}^b\psi_b\varphi^\sigma,$$

where the sum over σ runs from 1 to m. The φ^σ may or may not transform under $\mathcal{S}U_8$, but once the transformation character of the φ^σ is fixed, generally it is not possible, to find matrices $(\Gamma_\sigma)_a{}^b$ such that $\mathcal{L}'$ is invariant. In order to answer questions of this kind, it is necessary to know the theory of direct products and reduction of representations. This is taken up in Sec. IV by one method, and in Sec. V by another. The answer, in the special case mentioned, is that there exist matrices $(\Gamma_\sigma)_a{}^b$ that make $\mathcal{L}'$ invariant in two cases only. Either all the φ^σ are invariant under $\mathcal{S}U_8$, or there are at least 63 of them.[22]

II. LIE ALGEBRAS OF SIMPLE GROUPS

An important tool in the study of groups is the concept of an infinitesimal transformation. Since $\mathfrak{U}$ is unitary, it can be written $\exp(i\epsilon^A L_A)$ with L_A Hermitian, where the ϵ^A are a set of real continuous parameters.[23] For an infinitesimal transformation the exponential may be approximated by[24]

$$\mathfrak{U}=1+i\epsilon^A L_A,$$

or

$$\mathfrak{U}_a{}^b=\delta_a{}^b+i\epsilon^A(L_A)_a{}^b. \tag{II.1}$$

The set of linear combinations, with arbitrary complex coefficients, of the Hermitian matrices L_A, associated with the transformations $\mathfrak{U}$, form the *Lie algebra* of the group. The $\mathfrak{U}_a{}^b$ determine the $(L_A)_a{}^b$ uniquely, and the converse is almost true. In fact, the L_A determine the $\mathfrak{U}$ up to a discrete set of transformations which commute with all the $\mathfrak{U}$.[25] We have taken $\mathfrak{U}$ to be unimodular, and this requires L_A to be traceless:

$$(L_A)_a{}^a=0. \tag{II.2}$$

According to the fundamental theorem proved by Lie and Engels,[12] the structure of the group is completely specified by the commutation relations among the generators L_A of infinitesimal transformations,

$$[L_A,L_B]=C_{AB}{}^D L_D \tag{II.3}$$

where the $C_{AB}{}^D$ are called the *structure constants* and satisfy the conditions

$$C_{AB}{}^D=-C_{BA}{}^D \quad \text{(Antisymmetry)}$$

$$C_{AB}{}^E C_{EF}{}^G+C_{BF}{}^E C_{EA}{}^G+C_{FA}{}^E C_{EB}{}^G=0 \quad \text{(Jacobi identity)}. \tag{II.4}$$

Many different sets of matrices may be found that satisfy the same commutation relations (II.3), with the same structure constants. Such matrix sets may be regarded as different realizations (or representations, see next section) of the same set of *abstract operators*. The latter, whose only properties are the commutation relations, is designated by a caret, as $\hat{H}_i$, $\hat{E}_\alpha$, etc., in order to emphasize that we are not dealing with any particular realization.

A group is *simple* if it has no invariant subgroups[26] except the unit element. A group is *semi-simple* if it has no Abelian (commutative) invariant subgroups. We have disposed of an Abelian invariant subgroup which is the baryon gauge group at the beginning. The distinction between groups which have Abelian invariant subgroups, and those which do not, rests upon the fact that the Abelian subgroups are most troublesome to handle from the viewpoint of representations.[27] We therefore restrict ourselves to the study of simple groups.[28] There are certain cases of simple or semi-simple groups with discrete transformations added, such as that discussed by Lee and Yang,[17] which have equal claim for attention, but these are not discussed in this paper.

It is worthwhile to draw an analogy between the possible symmetries of elementary particles and the three dimensional rotation group in ordinary quantum mechanics.[11,29] In quantum mechanics, one observes that when the potential is spherically symmetric, the angular-momentum operators, which are the generators of infinitesimal rotations, commute with the Hamiltonian. Since the three angular momentum operators do not commute among themselves one can diagonalize only one of them at a time, call it H_1. This is a linear operator, and so the eigenvalue of H_1

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[22] D. Speiser and J. Tarski (to be published).

[23] Unitarity of $\mathfrak{U}$ requires the $(\epsilon^A L_A)$ be Hermitian; sometimes we shall use non-Hermitian L_A, in which case it is implied that the ϵ^A have appropriate reality properties.

[24] For a more complete discussion see reference 12, or reference 5, Chap. I.

[25] L. S. Pontrjagin, reference 12, Chap. IX, Sec. 54. In the case of $\mathcal{S}U_2$, for example, we find that it has the same Lie algebra as the three dimensional rotation group R_3, although the two groups differ in that a rotation by 2π is the identity transformation in R, while it is -1 in $\mathcal{S}U_2$.

[26] A *subgroup* is a subset of the elements of the group that has the group property. A subgroup S of a group G is an *invariant subgroup* if gSg^{-1} is in S for every g in G and s in S. In keeping with convention, we shall call a group simple if the only invariant subgroup is discrete. The reason for this is that the Lie algebra of such groups are often simple. (An algebra is *simple* if it has no invariant subalgebra.)

[27] See reference 5, p. 55.

[28] The study of semi-simple groups can be reduced in a trivial manner to that of simple groups.

[29] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, New York, 1935); A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957); M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

for a compound state is the sum of the eigenvalues associated with the component states (to be contrasted with the properties of L^2, say).

The conservation of the additive quantum numbers charge and strangeness[30] (or, equivalently, the third component of the isotopic spin and the hypercharge) in strong interactions is so well established that any group of practical interest must contain at least two commuting linear operators whose eigenvalues are the isotopic spin and the hypercharge. Let us denote these two operators by H_1 and H_2. Since the group is assumed to be the group of the Hamiltonian, i.e., every element of the group commutes with the Hamiltonian, one can diagonalize H_1 and H_2 simultaneously with the Hamiltonian, so that the eigenstates of the Hamiltonian have definite eigenvalues of H_1 and H_2, proportional to the I_3 and hypercharge quantum numbers.

The number of mutually-commuting linear operators[31] is called the *rank* of the group. Hence the rank of the three-dimensional rotation group is one. If the rank of the group is larger than two, there exists at least one more operator, H_3 say, which commutes with H_1 and H_2. But such an operator can mix states which are degenerate with respect to H_1 and H_2 only. Among the eight baryons, only the Λ and Σ^0 have equal charge and strangeness. Among the seven mesons no such degeneracy occurs. Thus, if H_3 is independent of H_1 and H_2, one or more of the following four possibilities can be considered:

(1) H_3 is the same for all eight baryons[32] and has a different value on a set of other baryons or physical states;

(2) H_3 mixes observed baryons with other physical states;

(3) H_3 splits the Λ, Σ^0 degeneracy, but is of the form aH_1+bH_2 for the other six baryons[33];

(4) The eight baryons are eigenstates of H_3 with eigenvalues which cannot be written in the form aH_1+bH_2+c.

Sometimes (3) and (4) leads to the forbidding of certain observed processes.[22] Although we can offer no arguments against the first two possibilities, we note that for these cases any group of rank three which accommodates the baryons will have a subgroup of rank two whose predictions will be less restrictive. If any of these should be acceptable, the "parent" groups of higher ranks should be investigated.

[30] T. Nakano and K. Nishijima, Progr. Theoret. Phys. (Kyoto) **10**, 581 (1953); M. Gell-Mann, Phys. Rev. **92**, 833 (1953).

[31] To be contrasted with commuting operators of the group, e.g., Casimir operators which are non-linear in the L_A.

[32] Or, what is equivalent, of the form $H_3=aH_1+bH_2+c$.

[33] The eigenstates of H_3 would then be linear combinations of Λ and Σ, as in the doublet symmetry of Pais, reference 2. A model based on the seven-dimensional rotation group comes under this category, see R. E. Behrends, and D. C. Peaslee, reference 2. Pais has shown that the doublet symmetry scheme leads to difficulties, (which are shared by the R_7 model), A. Pais, Phys. Rev. **110**, 574 (1958).

In the case of the angular momentum, the commutation relations, or the Lie algebra of the angular momentum operators, are sufficient to specify the physical content of the spherical symmetry of the system as in the classification of states and deduction of selection rules, etc. We now present a way of constructing the algebra of all simple groups, specializing later to those of rank two.

We call the number of independent elements of the algebra the *order* (r) of the group, or the *dimension* of the algebra. A particular choice of r linearly independent operators forms a *basis* of the Lie algebra. As an illustration, let us take the three dimensional rotation group R_3. The order of the group is three and the usualy choice of the basis is $\hat{T}_x$, $\hat{T}_y$, and $\hat{T}_z$. Instead, we may choose a basis as follows. Take an operator $\hat{H}_1=\hat{T}_z$ and consider an "eigenvalue" problem:

$$[\hat{T}_z,\hat{E}_\alpha]=r(\alpha)\hat{E}_\alpha.$$

The "eigenvectors" are $\hat{E}_{\pm 1}=\hat{T}_\pm=\hat{T}_x\pm i\hat{T}_y$ with "eigenvalues," $r(\pm 1)=\pm 1$ ($\hat{T}_+$ and $\hat{T}_-$ are the "raising and lowering" operators). Here $\hat{T}_+$, $\hat{T}_-$ and $\hat{T}_z$ form an alternative basis of the algebra. Note that, while $\hat{T}_x$ and $\hat{T}_y$ are Hermitian in the usual representation, $\hat{T}_+$ and $\hat{T}_-$ are not; instead they are related by Hermitian conjugation.[23]

For simple groups of rank l, the basis of the algebra may be so chosen that $\hat{H}_1, \cdots, \hat{H}_l$ are l elements of the basis and

$$[\hat{H}_i,\hat{H}_j]=0. \quad i,j=1,2,\cdots l. \qquad \text{(II.5)}$$

The rest of the basis may be chosen to be the $r-l$ elements E_α of the algebra satisfying

$$[\hat{H}_i,\hat{E}_\alpha]=r_i(\alpha)\hat{E}_\alpha, \qquad \text{(II.6)}$$

where $r_i(\alpha)$ is the ith component of the *root* $\mathbf{r}(\alpha)$, that is, the $r_i(\alpha)$ form a "vector" in an l-dimensional *root space*. If $\mathbf{r}(\alpha)$ is a root, then $-\mathbf{r}(\alpha)\equiv\mathbf{r}(-\alpha)$ is also a root, and we denote the corresponding operator by $\hat{E}_{-\alpha}$. Then it can be shown that

$$[\hat{E}_\alpha,\hat{E}_{-\alpha}]=C_{\alpha,-\alpha}{}^i\hat{H}_i, \quad \alpha=\pm 1, \pm 2, \cdots \pm\tfrac{1}{2}(r-l), \qquad \text{(II.7)}$$

and that

$$[\hat{E}_\alpha,\hat{E}_\beta]=C_{\alpha,\beta}{}^\gamma\hat{E}_\gamma, \quad \text{(not summed)}, \qquad \text{(II.8)}$$

if $\mathbf{r}(\gamma)\equiv\mathbf{r}(\alpha)+\mathbf{r}(\beta)$ is a nonvanishing root and $[\hat{E}_\alpha,\hat{E}_\beta]=0$, otherwise. These statements can be easily verified for R_3. It is possible to normalize the $\hat{H}_i$, such that

$$\textstyle\sum_\alpha r_i(\alpha)r_j(\alpha)=\delta_{ij}. \qquad \text{(II.9)}$$

Then it can be shown that

$$C_{\alpha,-\alpha}{}^i\equiv r^i(\alpha)=r_i(\alpha), \qquad \text{(II.10)}$$

so that

$$[\hat{E}_\alpha,\hat{E}_{-\alpha}]=r^i(\alpha)\hat{H}_i. \qquad \text{(II.11)}$$

Collecting these results, we have the standard form of the commutation relations:

$$\begin{aligned}[\hat{H}_i,\hat{H}_j]&=0,\\ [\hat{H}_i,\hat{E}_\alpha]&=r_i(\alpha)\hat{E}_\alpha,\\ [\hat{E}_\alpha,\hat{E}_{-\alpha}]&=r^i(\alpha)\hat{H}_i,\\ [\hat{E}_\alpha,\hat{E}_\beta]&=N_{\alpha\beta}\hat{E}_\gamma,\end{aligned} \tag{II.12}$$

if $\mathbf{r}(\gamma)=\mathbf{r}(\alpha)+\mathbf{r}(\beta)$ is a nonvanishing root; $N_{\alpha\beta}=C_{\alpha,\beta}{}^{\gamma}$. The explicit form of $N_{\alpha\beta}$ is given in Eq. (II.14).

The graphical representation of the root vectors is called a *root diagram.* All simple groups can be classified by root diagrams.[34] Since roots and structure constants $N_{\alpha\beta}$ can be deduced simply from the vector diagram for all simple groups, we describe the vector diagrams for simple groups of rank two in some detail. The following theorem plays a central role in the construction of the vector diagram:

Theorem[35]: If $\mathbf{r}(\alpha)$ and $\mathbf{r}(\beta)$ are two roots, then $2[\mathbf{r}(\alpha)\cdot\mathbf{r}(\beta)]/[\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)]$ is an integer and $\mathbf{r}(\beta)-2\mathbf{r}(\alpha)\times[\mathbf{r}(\alpha)\cdot\mathbf{r}(\beta)]/|\mathbf{r}(\alpha)|^2$ is also a root.

Graphically, this means that a new root $\mathbf{r}(\beta)-2\mathbf{r}(\alpha)\times[\mathbf{r}(\alpha)\cdot\mathbf{r}(\beta)]/|\mathbf{r}(\alpha)|^2$ can be obtained from $\mathbf{r}(\beta)$ by reflection with respect to a hyperplane perpendicular to $\mathbf{r}(\alpha)$.

Suppose we have two roots, $\mathbf{r}(\alpha)$ and $\mathbf{r}(\beta)$, and let φ be the angle between them. Then it follows from the theorem that

$$\mathbf{r}(\alpha)\cdot\mathbf{r}(\beta)=\tfrac{1}{2}m|\mathbf{r}(\alpha)|^2=\tfrac{1}{2}n|\mathbf{r}(\beta)|^2, \tag{II.13a}$$

where m and n are integers. From this we further obtain

$$\cos^2\varphi=\tfrac{1}{4}mn. \tag{II.13b}$$

We see that φ can have only the values 0°, 30°, 45°, 60°, and 90°. From Eq. (II.13a) one deduces that the ratios of the lengths of the two vectors are $\sqrt{3}$ for 30°, $\sqrt{2}$ for 45°, 1 for 60°, and undetermined for 90°.

It is easy to see that the only possible two dimensional diagrams corresponding to simple groups of rank two, compatible with Eqs. (13a) and (13b), are those drawn in Fig. 1. The first one corresponds to the three-dimensional special unitary group $SU_3(A_2)$; the second to the five-dimensional orthogonal group $O_5(B_2)$, which is also isomorphic to the two-dimensional symplectic group $Sp_2(C_2)$; the last to the exceptional group G_2. The notations in parenthesis are those used by Cartan. The number of parameters of a group (order) is equal to the sum of the number of root vectors and the rank of the group: SU_3 is a 8(=6+2) parameter group; O_5 a 10 parameter group; G_2 a 14 parameter group.

Once the vector diagram of a simple group is known, it is a trivial matter to construct the standard form of the commutation relations (12). This is due to the theorem:

Theorem[36]: Form $[\hat{E}_\beta,\hat{E}_\alpha]$, $[[\hat{E}_\beta,\hat{E}_\alpha],\hat{E}_\alpha]$, $\cdots$ and $[\hat{E}_\beta,\hat{E}_{-\alpha}]$, $[[\hat{E}_\beta,\hat{E}_{-\alpha}],\hat{E}_{-\alpha}]$, $\cdots$, where $\mathbf{r}(\beta)\neq\pm\mathbf{r}(\alpha)$ These series must terminate. A series of $\hat{E}_\lambda$'s are generated in this manner. Let

$$\mathbf{r}(\lambda)=\mathbf{r}(\beta)-m\mathbf{r}(\alpha),\quad \mathbf{r}(\beta)-(m-1)\mathbf{r}(\alpha),\ \cdots\mathbf{r}(\beta),\ \cdots,\ \mathbf{r}(\beta)+n\mathbf{r}(\alpha)$$

be the corresponding nonvanishing roots. Then

$$N_{\alpha\beta}=\pm[\tfrac{1}{2}(m+1)n|\mathbf{r}(\alpha)|^2]^{\frac{1}{2}}. \tag{II.14}$$

Here the signatures of $N_{\alpha\beta}$ must be chosen so that

$$N_{\alpha\beta}=-N_{\beta\alpha}=-N_{-\alpha,-\beta}, \tag{II.15}$$

$$N_{\alpha\beta}=N_{\beta,-\alpha-\beta}=N_{-\alpha-\beta,\alpha}. \tag{II.16}$$

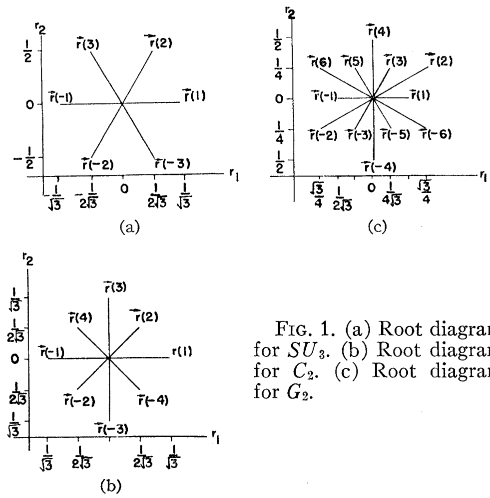

FIG. 1. (a) Root diagram for SU_3. (b) Root diagram for C_2. (c) Root diagram for G_2.

As an example, let us construct the standard commutation relations for SU_3. Label the root vectors as in Fig. 1(a), where the lengths of the $\mathbf{r}(\alpha)$ are normalized according to Eq. (10):

$$\textstyle\sum_\alpha \mathbf{r}_i(\alpha)\mathbf{r}_j(\alpha)=\delta_{ij}.$$

Let us consider $[\hat{E}_1,\hat{E}_3]$. Since $\mathbf{r}(3)+\mathbf{r}(1)$ is a root while $\mathbf{r}(3)+2\mathbf{r}(1)$ is not, we have $n=1$; since $\mathbf{r}(3)-\mathbf{r}(1)$ is not a root, $m=0$. We choose the sign such that[37]

$$N_{13}=-N_{31}=\sqrt{\tfrac{1}{6}}.$$

Equations (15) and (16) give 5 other constants:

$$N_{-3,-1}=N_{3,-2}=N_{-2,1}=N_{2,-3}=N_{-1,2}=\sqrt{\tfrac{1}{6}}. \tag{II.17}$$

The roots are

$$\begin{aligned}\mathbf{r}(1)&=(1/\sqrt{3})(1,0);\\ \mathbf{r}(2)&=(1/2\sqrt{3})(1,\sqrt{3});\\ \mathbf{r}(3)&=(1/2\sqrt{3})(-1,\sqrt{3}).\end{aligned} \tag{II.18}$$

The $N_{\alpha\beta}$ and the roots listed above give a complete set of commutation relations when inserted in Eq. (14).

We summarize a choice of the $N_{\alpha\beta}$ for C_2, and for G_2.

[34] B. L. van der Waerden, Math. Z. **37**, 446 (1933).
[35] See, for example, G. Racah, reference 5, p. 21.
[36] See, for example, G. Racah, reference 5, p. 24.

[37] The number of signs that can be chosen independently is the number of different pairs of roots with positive α's whose sums are roots.

[The roots can be read off immediately from Figs. 1(b) and (c).]

C_2:

$$N_{24}=N_{4,-2}=N_{-4,-2}=N_{2,-4}=N_{1,4}=N_{-2,1}=N_{-2,3}=N_{3,-4}=N_{-1,2}=N_{-4,-1}=N_{4,-3}=N_{-3,2}=\sqrt{\tfrac{1}{6}}. \quad \text{(II.19)}$$

G_2:

$$N_{26}=N_{4,-6}=N_{-2,4}=N_{2,-1}=N_{3,1}=N_{-2,3}=N_{5,-6}=N_{1,6}=N_{-1,5}=N_{3,-4}=N_{-5,4}=N_{-3,-5}=1/2\sqrt{2};$$

$$N_{1,5}=N_{3,-5}=N_{-1,3}=\sqrt{\tfrac{1}{6}}. \quad \text{(II.20)}$$

III. REPRESENTATIONS OF LIE ALGEBRAS

A. General Properties of Representations

In a previous section we discussed the r infinitesimal operators of a Lie group and their commutation relations from an abstract point of view, without using an explicit form of the operators. In order to make connection with physical situations, it is necessary to introduce specific realizations of these operators. If we associate a matrix with each operator $\hat{H}_i$ and $\hat{E}_\alpha$ such that these r matrices satisfy the commutation relations of the r operators, then the matrices are said to constitute a *representation* of the group.[38] In what follows, the symbols H_i and E_α denote a matrix representation. The dimension of these matrices N is called the dimension (or degree) of the representation. If the r matrices of a particular representation can be simultaneously brought into block diagonal form, by a similarity transformation, the representation is said to be *decomposable* (or fully reducible) into lower dimensional representations. When this is not possible, the representation is called *irreducible*.[39]

From the commutation relations, we see that the H_i commute among themselves, so that it is possible to diagonalize simultaneously these l matrices. We choose a representation in which the H_i are diagonal, and write ψ for an N-component basis vector. The eigenfunctions and eigenvalues of $\hat{H}_i$ are defined by

$$H_i\psi=m_i\psi.$$

The l-component vector $\mathbf{m}=(m_1,m_2,\cdots,m_l)$ is called the *weight*,[6] and the l-dimensional vector space spanned by the set of weights is called the *weight space*.

In order to develop some physical intuition for what we are doing, consider the isotopic-spin rotation group. The commutation relations are the usual angular-momentum set. We know that only one of the three matrices can be diagonalized at a time (it then corresponds to H_1, $l=1$ for this group), and the eigenvalues of this matrix are the components of isotopic spin. The E_1 and E_{-1} in this case are proportional to the usual isotopic spin raising and lowering operators. This algebra is the only simple or semi-simple Lie algebra of rank one. The three groups of rank two ($l=2$) were given in a previous section, i.e., B_2, G_2 and SU_3. For these groups we might identify the eigenvalues of $\hat{H}_1$ and $\hat{H}_2$ with the third component of isotopic spin and with the hypercharge,[40] $Y=N+S$, two good quantum numbers for the strong interactions as well as the electromagnetic interactions. The ψ's which would represent the various particles or states, would then be labeled by their eigenvalues of $\hat{H}_i$, i.e., weights $\mathbf{m}$. The ψ's having different weights are obviously linearly independent, so that there are at most N different weights. If a weight belongs to only one eigenvector, it is called *simple* (for groups of rank greater than one, not all weights are simple).

Let us consider the weights more closely. The following powerful theorem is very useful.

Theorem[41]: For any weight $\mathbf{m}$ and root $\mathbf{r}(\alpha)$, the quantity $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ is an integer and $\mathbf{m}'=\mathbf{m}-\mathbf{r}(\alpha)2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ is also a weight, and has the same multiplicity as $\mathbf{m}$. It can be easily verified that this prescription for obtaining $\mathbf{m}'$ from $\mathbf{m}$ corresponds geometrically, in the weight space, to reflecting $\mathbf{m}$ through a hyperplane perpendicular to the root $\mathbf{r}(\alpha)$. Weights that are related by a reflection or a product of reflections are said to be *equivalent*. Reflections and the product of reflections give the set of all equivalent weights. We denote by S the group generated by these reflections.[42]

A weight $\mathbf{m}$ is said to be *higher* than a weight $\mathbf{m}'$ if $\mathbf{m}-\mathbf{m}'$ has a positive number for its first non-vanishing component, e.g., if $m_1-m_1'=0$ and $m_2-m_2'>0$, then $\mathbf{m}$ is higher than $\mathbf{m}'$. A *dominant* weight is the highest member of a set of equivalent weights, and the ***highest*** weight is the dominant weight which is higher than any other dominant weight in a representation. For an irreducible representation, the highest weight is simple.[43] This concept of a highest weight is useful because two irreducible representations which are related by a similarity transformation (the representations are called *equivalent*) have the same highest weight, and vice versa.

With regard to dominant weights, Cartan[6] has proved that for every simple group of rank l there are l *fundamental* dominant weights $\mathbf{M}^{(1)}\cdots\mathbf{M}^{(l)}$ such that

[38] A representation is faithful if the correspondence between $\hat{L}_A$ and L_A is one-to-one. For simple algebras, all except the identity representation ($L_A=0$) are faithful.

[39] A noncompact group has no finite dimensional unitary representations (see Pontrjagin, reference 12, Chap. III). Therefore all admissible groups are compact. Representations of compact groups are either irreducible or fully reducible.

[40] Here S is the strangeness quantum number and N is the baryon number. This is the usual definition of hypercharge, although some authors define it as $\frac{1}{2}(N+S)$.

[41] See, for example, G. Racah, reference 5, p. 35.

[42] This group was first introduced by H. Weyl, reference 8, (*Selecta*), p. 338.

[43] See, G. Racah, reference 5, p. 37.

any other dominant weight $\mathbf{M}$ is a linear combination

$$\mathbf{M}=\sum_{i=1}^{l}\lambda_i\mathbf{M}^{(i)}\equiv\mathbf{M}(\lambda_1\cdots\lambda_l),\qquad\text{(III.1)}$$

with λ_i as non-negative integral coefficients, and that there exist l *fundamental* irreducible representations which have the fundamental weights as their highest weights.[44]

Let us return to the isotopic spin rotation group. The weights m are $\pm I_3$ (weight space is one-dimensional, in this case, since the group is of rank $l=1$). The weight $-I_3$ is obtained from I_3 by reflection through the "plane" perpendicular to the root $\mathbf{r}(1)$ and I_3 is the dominant weight. For each I_3 which appears in an irreducible representation, there will be a $-I_3$, which is equivalent and has the same multiplicity. The fundamental dominant weight is $\frac{1}{2}$ in order that $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ be an integer for all weights. The highest weight is $I=\lambda\frac{1}{2}$, where λ is a non-negative integer, and is simple in an irreducible representation. The corresponding statements for the groups of rank two are postponed until later.

In order to distinguish the different irreducible representations of a group. Weyl has utilized extensively a quantity called the *character*. This is a function of l real variables $\varphi^1, \cdots, \varphi^l$ defined by

$$\chi(\varphi^1,\cdots,\varphi^l)\equiv\text{trace}\exp(iH_i\varphi^i)$$
$$=\textstyle\sum_a\exp[i(H_i\varphi^i)_a{}^a],$$

where, in the last expression, $(H_i)_a{}^b$ has been assumed to be in diagonal form. Since the trace of a matrix is invariant under a similarity transformation, the characters of two representations are equal if and only if the two representations are equivalent. In particular, a representation and its complex conjugate are equivalent, if and only if the trace is real.

Weyl[8] has given an explicit formula for calculating the character of any representation of any simple group, namely,

$$\chi(\lambda_i,\boldsymbol{\varphi})=\frac{\xi(\lambda_i)}{\xi(0)},\quad \xi(\lambda_i)\equiv\textstyle\sum_S\delta_S\exp[i(S\mathbf{K})\cdot\boldsymbol{\varphi}],\qquad\text{(III.2)}$$

where the sum is over the reflection operations S defined above and $\delta_S=+1$ for an even number of reflections and -1 for an odd number. If $\mathbf{R}$ is defined by

$$\mathbf{R}=\tfrac{1}{2}\sum_{\alpha,+}\mathbf{r}(\alpha)\qquad\text{(III.3)}$$

where the sum is over the positive roots, i.e., those roots which have a positive first nonvanishing component, then $\mathbf{K}$ is $\mathbf{R}$ plus the highest weight of the representation, $\mathbf{M}$

$$\mathbf{K}=\mathbf{R}+\mathbf{M}(\lambda_1,\cdots,\lambda_l).\qquad\text{(III.4)}$$

[44] In fact, every $\mathbf{M}$ determines uniquely an irreducible representation with $\mathbf{M}$ as the highest weight.

It is obvious from the above definition of the character as a trace that it may also be written as

$$\chi(\lambda_i,\boldsymbol{\varphi})=\textstyle\sum_{\mathbf{m}}\gamma_{\mathbf{m}}\exp(i\mathbf{m}\cdot\boldsymbol{\varphi}),\qquad\text{(III.5)}$$

where the sum is over all the weights and $\gamma_{\mathbf{m}}$ is the number of times a weight $\mathbf{m}$ occurs, i.e., the multiplicity of the weight. For $\boldsymbol{\varphi}=0$, the character is just the dimensionality of the representation, i.e.,

$$N(\lambda_i)=\textstyle\sum_{\mathbf{m}}\gamma_{\mathbf{m}}=\chi(\lambda_i,0).\qquad\text{(III.6)}$$

The above can be exemplified by referring once again to the isotopic-spin-rotation group. There is one positive root, $\mathbf{r}(1)=1$, therefore, $\mathbf{R}=\frac{1}{2}$; $\mathbf{M}=I=\lambda\frac{1}{2}$. Thus,

$$\mathbf{K}=\tfrac{1}{2}(\lambda+1)=I+\tfrac{1}{2}.$$

Since there is only one reflection,

$$\xi(\lambda)=e^{i(I+\frac{1}{2})\varphi}-e^{-i(I+\frac{1}{2})\varphi},$$

and

$$\chi(\lambda,\varphi)=(e^{i(I+\frac{1}{2})\varphi}-e^{-i(I+\frac{1}{2})\varphi})/(e^{\frac{1}{2}i\varphi}-e^{-\frac{1}{2}i\varphi}),\quad I=\tfrac{1}{2}\lambda.$$

This may easily be shown to be

$$\chi(\lambda,\varphi)=\sum_{I_3=-I}^{+I}e^{iI_3\varphi},$$

so that the multiplicity of each weight is one, $\gamma_{\mathbf{m}}=1$. The dimensions of the irreducible representations are $N=\chi(\lambda,0)=2I+1$.

So far, in order to distinguish the various eigenvectors, or bases, we have the l integers $(\lambda_1,\cdots,\lambda_l)$ which are necessary to form the highest weight $\mathbf{M}$. These numbers distinguish between representations of different dimensionalities as well as inequivalent representations of the same dimensionality. However, within an irreducible representation, in addition to the weights we still need $\frac{1}{2}(r-3l)$ more numbers, $\mu=(\mu_1,\mu_2,\cdots,\mu_{\frac{1}{2}(r-3l)})$ in order to distinguish the various eigenvectors of the same weight.[5] Given these numbers, it would then be possible to determine the explicit form of the matrix element

$$\psi^\dagger(\mathbf{M},\mathbf{m},\mu)\hat{E}_\alpha\psi(\mathbf{M},\mathbf{m}',\mu')=f(\mathbf{M},\mathbf{m},\mathbf{m}',\mu,\mu').$$

For example, in the isotopic spin rotation group $\frac{1}{2}(r-3l)=0$, so that we need no additional numbers. This matrix element is then the well known[29]

$$\psi^\dagger(I,I_3)I_+\psi(I,I_3')=[\tfrac{1}{2}(I-I_3')(I+I_3)]^{\frac{1}{2}}\delta_{I_3,I_3'+1}.$$

We shall show how to circumvent the task of finding the operators whose eigenvalues are the μ's for groups of higher rank.

Thus far we have used the isotopic spin rotation group as an example. Let us now demonstrate the method with the rank two groups SU_3, G_2, and C_2.

B. Characters of Representations of SU_3

In order to satisfy the condition that $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ be an integer for an arbitrary weight $\mathbf{m}=(m_1,m)_2$ and any root, $\mathbf{r}(\alpha)$, it is necessary that $m_1=(1/2\sqrt{3})(a+b)$ and $m_2=\frac{1}{6}(a-b)$, where a and b are integers. Thus $\mathbf{m}=\frac{1}{6}a(\sqrt{3},1)+\frac{1}{6}b(\sqrt{3},-1)$. By noting that $\frac{1}{6}(\sqrt{3},1)$ and $\frac{1}{6}(\sqrt{3},-1)$ each lie in a plane perpendicular to a root, we see that each belongs to a set of 3 equivalent weights and that each is a dominant weight of its set, in fact, a fundamental dominant weight. Thus

$$\mathbf{M}(\lambda_1\lambda_2)=\tfrac{1}{6}\lambda_1(\sqrt{3},1)+\tfrac{1}{6}\lambda_2(\sqrt{3},-1).$$

The quantity $\mathbf{R}$ for SU_3 is

$$\mathbf{R}=\tfrac{1}{2}\sum_{\alpha,+}\mathbf{r}(\alpha)=(1/\sqrt{3})(1,0),$$

so that $\mathbf{K}$ is

$$\mathbf{K}=\mathbf{M}+\mathbf{R}=\tfrac{1}{6}(\sqrt{3}\lambda_1+\sqrt{3}\lambda_2+2\sqrt{3},\ \lambda_1-\lambda_2).$$

Thus, $\xi(\lambda_1,\lambda_2)$ may be written

$$\begin{aligned}\xi(\lambda_1,\lambda_2)=&\exp\tfrac{1}{6}i[(\lambda_1+\lambda_2+2)\sqrt{3}\varphi_1+(\lambda_1-\lambda_2)\varphi_2]\\&-\exp\tfrac{1}{6}i[-(\lambda_1+\lambda_2+2)\sqrt{3}\varphi_1+(\lambda_1-\lambda_2)\varphi_2]\\&-\exp\tfrac{1}{6}i[(\lambda_2+1)\sqrt{3}\varphi_1-(2\lambda_1+\lambda_2+3)\varphi_2]\\&+\exp\tfrac{1}{6}i[-(\lambda_2+1)\sqrt{3}\varphi_1-(2\lambda_1+\lambda_2+3)\varphi_2]\\&-\exp\tfrac{1}{6}i[(\lambda_1+1)\sqrt{3}\varphi_1+(\lambda_1+2\lambda_2+3)\varphi_2]\\&+\exp\tfrac{1}{6}i[-(\lambda_1+1)\sqrt{3}\varphi_1+(\lambda_1+2\lambda_2+3)\varphi_2].\end{aligned}$$

It should be apparent that dividing $\xi(\lambda_1,\lambda_2)$ by $\xi(0,0)$ in order to obtain the character in the form $\sum\gamma_\mathbf{m}\exp(i\mathbf{m}\cdot\boldsymbol{\varphi})$ is no trivial matter for this group. In the next section, we develop a technique for handling this problem. First let us find the dimensions N of the irreducible representations. In terms of the character, $N=\chi(\lambda_1,\lambda_2,\ \varphi_1=\varphi_2=0)$. Since $\xi(\lambda_1,\lambda_2)$ is zero for $\varphi_1=\varphi_2=0$, we use L'Hôpital's rule[45] to find

$$N=[1+\tfrac{1}{2}(\lambda_1+\lambda_2)](1+\lambda_1)(1+\lambda_2).$$

The numbers $\lambda_1,\ \cdots,\ \lambda_l$ are sufficient to identify a representation. For this reason we label the representations by $D^{(N)}(\lambda_1,\lambda_2)$ [or occasionally by just $D(\lambda_1,\lambda_2)$ or $D^{(N)}$]. Thus $D^{(3)}(1,0)$ denotes one of the 3-dimensional representations, while $D^{(3)}(0,1)$ denotes the complex conjugate (χ^*) inequivalent 3-dimensional representation.

We note that $\chi^*=\chi$ only for values of $\lambda_1=\lambda_2$.[46] Thus, only in this case are the complex conjugate representations equivalent. In Fig. 2 we have drawn the weight diagrams for a few of the lower dimensional representations of SU_3. The solid lines with arrows represent the weight vectors while the dotted lines which are perpendicular to the roots represent the

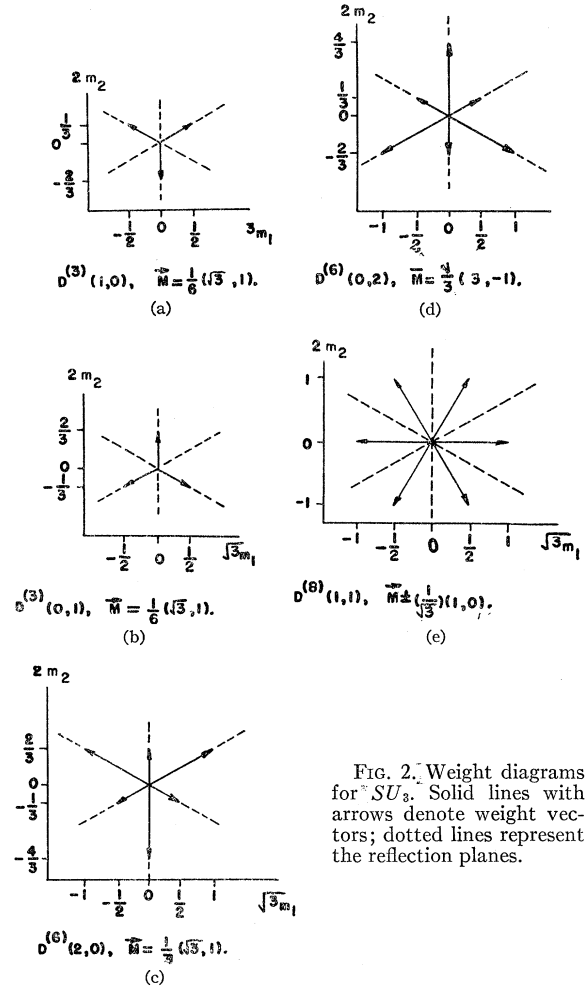

FIG. 2. Weight diagrams for SU_3. Solid lines with arrows denote weight vectors; dotted lines represent the reflection planes.

planes of reflection that leave the weight diagram unchanged (the set of operations S defined above). The 3-dimensional representations $D^{(3)}(1,0)$ and $D^{(3)}(0,1)$ are the fundamental irreducible representations, $D^{(8)}(1,1)$ is the regular representation.[47]

C. Characters of Representations of G_2

In order that $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ be an integer for an arbitrary weight $\mathbf{m}=(m_1,m_2)$ and any root, $\mathbf{r}(\alpha)$, it is necessary that $m_1=(1/4\sqrt{3})(2a+3b)$ and $m_2=\frac{1}{4}b$, where a and b are integers. Thus $\mathbf{m}=(a/2\sqrt{3})(1,0)+(b/2\sqrt{3})\times(3/2,\sqrt{3}/2)$. By noting that $(1/2\sqrt{3})(1,0)$ and $(1/2\sqrt{3})(3/2,\sqrt{3}/2)$ each lie in a plane perpendicular to a root, we see that each belongs to a set of 6 equivalent weights and that each is a dominant weight of its set,

[45] Marquis G. F. A. de l'Hôpital, *Analyse des Infiniement Petits* (Paris, 1730).

[46] This follows from the identity $\chi^*(\lambda_1,\lambda_2)=\chi(\lambda_2,\lambda_1)$ satisfied by the SU_3 characters.

[47] The regular representation is very important and plays a prominent role in later sections. It is defined by $L_A\to -C_A$, where the components of the matrix C_A are the structure constants $C_{AB}{}^D$. That this is a representation can be seen by rewriting the Jacobi identity (II.4) in the form $C_{AE}{}^F C_{BF}{}^G - C_{BE}{}^F C_{AF}{}^G = -C_{AB}{}^F C_{EF}{}^G$. It can easily be proved that the regular representation is irreducible if and only if the group is simple.

in fact, a fundamental dominant weight. Thus

$$\mathbf{M}(\lambda_1,\lambda_2)=(\lambda_1/2\sqrt{3})(1,0)+(\lambda_2/4\sqrt{3})(3,\sqrt{3}).$$

The quantity $\mathbf{R}$ for G_2 is

$$\mathbf{R}=(1/4\sqrt{3})(5,\sqrt{3}),$$

so that

$$\mathbf{K}=(1/4\sqrt{3})(2\lambda_1+3\lambda_2+5,\ \sqrt{3}\lambda_2+\sqrt{3}).$$

Then, $\xi(\lambda_1,\lambda_2)$ may be written

$$\begin{aligned}\xi(\lambda_1,\lambda_2)&\\ =\{&\exp[i(2\lambda_1+3\lambda_2+5)\varphi_1/4\sqrt{3}]\\ &-\exp[-i(2\lambda_1+3\lambda_2+5)\varphi_1/4\sqrt{3}]\}\\ &\times\{\exp[i(\lambda_2+1)\varphi_2/4]-\exp[-i(\lambda_2+1)\varphi_2/4]\}\\ &-\{\exp[i(\lambda_1+3\lambda_2+4)\varphi_1/4\sqrt{3}]\\ &-\exp[-i(\lambda_1+3\lambda_2+4)\varphi_1/4\sqrt{3}]\}\\ &\times\{\exp[i(\lambda_1+\lambda_2+2)\varphi_2/4]\\ &-\exp[-i(\lambda_1+\lambda_2+2)\varphi_2/4]\}\\ &+\{\exp[i(\lambda_1+1)\varphi_1/4\sqrt{3}]-\exp[-i(\lambda_1+1)\varphi_1/4\sqrt{3}]\}\\ &\times\{\exp[i(\lambda_1+2\lambda_2+3)\varphi_2/4]\\ &\qquad-\exp[-i(\lambda_1+2\lambda_2+3)\varphi_2/4]\}.\end{aligned}$$

The dimensions N of the irreducible representations are $N=\chi(\lambda_1,\lambda_2,\ \varphi_1=\varphi_2=0)$. The result is

$$\begin{aligned}N=(1+\lambda_1)(1+\lambda_2)&[1+\tfrac{1}{2}(\lambda_1+\lambda_2)][1+\tfrac{1}{3}(\lambda_1+2\lambda_2)]\\ &\times[1+\tfrac{1}{4}(\lambda_1+3\lambda_2)][1+\tfrac{1}{5}(2\lambda_1+3\lambda_2)].\end{aligned}$$

We note that $\chi^*=\chi$, so that representations related by complex conjunction are always equivalent.

In Fig. 3 we have drawn the weight diagram for the 7- and 14-dimensional representations of G_2. The solid lines with arrows denote the weight vectors while the dotted lines, which are perpendicular to the roots, represent the planes of reflection that leave the weight diagram unchanged (the set of reflections S defined above). These two representations are the fundamental irreducible representations of G_2, and $D^{(14)}(0,1)$ is the regular representation.[47]

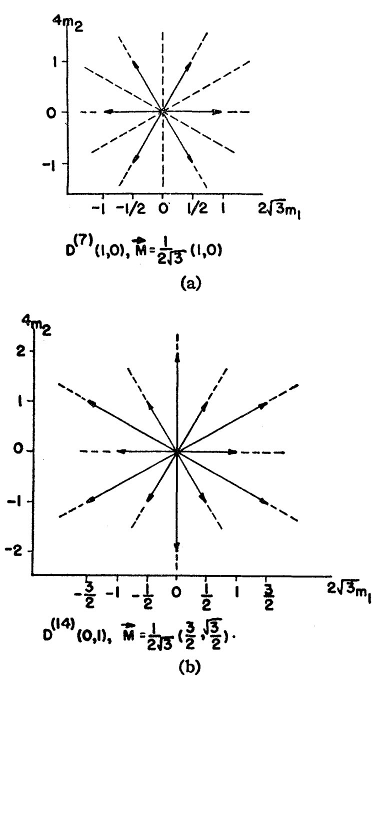

FIG. 3. Weight diagrams for G_2. Solid lines with arrows denote weight vectors; dotted lines represent the reflection plane.

D. Characters of Representations of C_2

In order that $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ be an integer for an arbitrary weight $\mathbf{m}=(m_1,m_2)$ and any root $\mathbf{r}(\alpha)$, it is necessary that $m_1=(2\sqrt{3})^{-1}(a+b)$ and $m_2=b/2\sqrt{3}$, where a and b are integers. Thus, $\mathbf{m}=(a/2\sqrt{3})(1,0)+(b/2\sqrt{3})(1,1)$. By noting that $(1/2\sqrt{3})(1,0)$ and $(1/2\sqrt{3})(1,1)$ each lie in a plane perpendicular to a root, we see that each belongs to a set of 4 equivalent weights and that each is a dominant weight of its set, in fact, a fundamental dominant weight. Thus

$$\mathbf{M}(\lambda_1,\lambda_2)=(\lambda_1/2\sqrt{3})(1,0)+(\lambda_2/2\sqrt{3})(1,1).$$

The quantity $\mathbf{R}$ for C_2 is

$$\mathbf{R}=(1/2\sqrt{3})(2,1),$$

so that

$$\mathbf{K}=\mathbf{R}+\mathbf{M}=(1/2\sqrt{3})(\lambda_1+\lambda_2+2,\ \lambda_2+1).$$

Then, $\xi(\lambda_1,\lambda_2)$ may be written

$$\begin{aligned}\xi(\lambda_1,\lambda_2)&\\ =\{&\exp[i(\lambda_1+\lambda_2+2)\varphi_1/2\sqrt{3}]\\ &-\exp[-i(\lambda_1+\lambda_2+2)\varphi_1/2\sqrt{3}]\}\\ &\times\{\exp[i(\lambda_2+1)\varphi_2/2\sqrt{3}]-\exp[-i(\lambda_2+1)\varphi_2/2\sqrt{3}]\}\\ &-\{\exp[i(\lambda_2+1)\varphi_1/2\sqrt{3}]-\exp[-i(\lambda_2+1)\varphi_1/2\sqrt{3}]\}\\ &\times\{\exp[i(\lambda_1+\lambda_2+2)\varphi_2/2\sqrt{3}]\\ &\qquad-\exp[-i(\lambda_1+\lambda_2+2)\varphi_2/2\sqrt{3}]\}.\end{aligned}$$

The dimensions of the irreducible representations, $N=\chi(\lambda_1,\lambda_2,\ \varphi_1=\varphi_2=0)$, are

$$N=(1+\lambda_1)(1+\lambda_2)[1+\tfrac{1}{2}(\lambda_1+\lambda_2)][1+\tfrac{1}{3}(\lambda_1+2\lambda_2)].$$

We note that $\chi^*=\chi$, so that representations related by complex conjugation are always equivalent.

In Fig. 4 we have drawn the weight diagrams for the 4, 5, and 10 dimensional representations of C_2. The solid lines with arrows denote the weight vectors while the dotted lines, which are perpendicular to the roots, represent the planes of reflection that leave the weight diagram unchanged (the set of reflections S defined above). The 4 and 5 dimensional representations, $D^{(4)}(1,0)$ and $D^{(5)}(0,1)$, are the fundamental irreducible representations of B_2, while $D^{(10)}(2,0)$ is the regular representation.[47]

E. Synthesis of Representations of Lie Algebras

For physical application, it is imperative to have explicit matrix representations of the low dimensional Lie algebras. As has been implied in the preceding paragraph, the straightforward generalization of the favorite method of constructing the matrix representation of a rank one group is somewhat awkward for higher rank groups. Of the several alternative methods which offer promise, we choose one which has useful by-products. In particular, the generalized Clebsch-Gordan coefficients[48] will materialize as part of the fallout of results.

As a warming-up exercise, we recall certain facts about the group SU_2. Let the basis for an irreducible representation $D(J)$, uniquely characterized by the total angular momentum $J(J+1)$, be labeled as $\psi_M{}^J$ where J is an integer or a half-integer and M runs from J to $-J$ in integral steps. In particular, select the spin $\frac{1}{2}$ representation $D(\frac{1}{2})$ whose highest weight is the fundamental dominant weight of SU_2. Then the representation is given in terms of Pauli matrices[49]:

$$\hat{H}_1=\tfrac{1}{2}\sigma_3; \qquad \hat{T}_+=(1/2\sqrt{2})(\sigma_1+i\sigma_2); \qquad \hat{T}_-=(1/2\sqrt{2})(\sigma_1-i\sigma_2)$$
$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix};\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix};\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad \text{(III.7)}$$

and the basis is $\psi_m{}^{\frac{1}{2}}$, $m=\frac{1}{2}, -\frac{1}{2}$. It is possible to arrive at a new representation inequivalent to $D(\frac{1}{2})$ by forming the direct product representation in the space spanned by the $\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}$. The action of $\hat{H}_1$ and $\hat{T}_\pm$ on the product basis is, of course,

$$\hat{T}_A\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}=(\hat{T}_A\psi_m{}^{\frac{1}{2}})\psi_{m'}{}^{\frac{1}{2}}+\psi_m{}^{\frac{1}{2}}(\hat{T}_A\psi_{m'}{}^{\frac{1}{2}})$$
$$\hat{T}_A\psi_m{}^{\frac{1}{2}}=\textstyle\sum_{m'}(\hat{T}_A)_m{}^{m'}\psi_{m'}{}^{\frac{1}{2}}, \qquad \text{(III.8)}$$

where $\hat{T}_A$ is $\hat{T}_z=H_1$, $\hat{T}_+$, or $\hat{T}_-$. The product representation is, in general, reducible; for example,

$$\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}=\sum_{M,J}(JM|\tfrac{1}{2}m,\tfrac{1}{2}m')\psi_M{}^J, \qquad \text{(III.9)}$$

where $(JM|\frac{1}{2}m,\frac{1}{2}m')$ are the Clebsch-Gordan coefficients which reduce the representation. To accomplish the reduction, we note that $\hat{T}_+$ and $\hat{T}_-$ commute with $\hat{T}^2$, and, since the eigenvalue of $\hat{T}^2$ uniquely characterizes an irreducible representation, they cannot lead out of an irreducible representation when applied in any order and any number of times to a single basis vector. The highest weight M in the product representation $\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}$, namely $M=\frac{1}{2}+\frac{1}{2}$, belongs to an irreducible representation and hence the space spanned by the vectors generated by application of $\hat{T}_+$ and $\hat{T}_-$ to $\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}}$ is irreducible under SU_2. Thus the orthonormal vectors

$$\psi_1{}^1\equiv\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}}$$
$$\psi_0{}^1\equiv\hat{T}_-(\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}})=(1/\sqrt{2})(\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}}+\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}) \qquad \text{(III.10a)}$$
$$\psi_{-1}{}^1\equiv2(\hat{T}_-)^2(\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}})=\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}$$

are a basis for an irreducible representation $D(1)$ of SU_2 and the remaining linear independent vector in the direct product space $\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}$ is

$$\psi_0{}^0\equiv(1/\sqrt{2})(\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}-\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}}). \qquad \text{(III.10b)}$$

This $\psi_0{}^0$ generates $D(0)$ for SU_2. The Clebsch-Gordan coefficients are read off from Eqs. (III.10a) and (III.10b) while the irreducible Lie algebra follows by computing the $\hat{T}_A$ matrix elements by using Eq. (III.8).

To find an arbitrary irreducible representation $D(J)$, it is only necessary to split off the highest irreducible representation of the direct product space

$$\psi_{m(1)}{}^{\frac{1}{2}}\psi_{m(2)}{}^{\frac{1}{2}}\cdots\psi_{m(2J)}{}^{\frac{1}{2}}\equiv(\psi_m{}^{\frac{1}{2}})^{2J}. \qquad \text{(III.11)}$$

The orthonormal basis which results is[50]:

$$\psi_M{}^J=N(J,M)(\hat{T}_-)^{J-M}(\psi_{\frac{1}{2}}{}^{\frac{1}{2}})^{2J}, \qquad M=J,\cdots,-J,$$
$$N(J,M)=\left[\frac{(J+M)!\,2^{J-M}}{(J-M)!\,(2J)!}\right]^{\frac{1}{2}} \qquad \text{(III.12)}$$

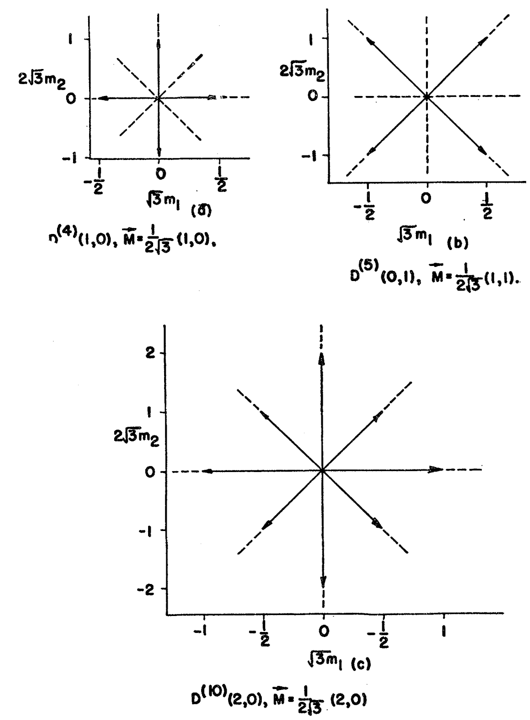

FIG. 4. Weight diagrams for C_2. Solid lines denote weight vectors; dotted lines represent the reflection planes.

[48] We refer here to the coefficients prescribing the linear combinations of direct product states relative to which the representation reduces.

[49] The operators $\hat{T}_\pm$ are usually defined without the factor $1/\sqrt{2}$. Throughout this paper, we shall adopt the sign convention of Condon and Shortley (reference 29) for isotopic spin. This implies that all the signs of $L_\pm$ matrix elements are positive although the physical particles are sometimes identified as the negative of the bases defining this representation.

[50] To derive $N(J,M)$, use the identity
$$\hat{T}_+\hat{T}_-=\tfrac{1}{2}(\hat{T}_+\hat{T}_-+\hat{T}_-\hat{T}_++\hat{T}_z)=\tfrac{1}{2}(\hat{T}^2-\hat{T}_z{}^2-\hat{T}_z)$$
to obtain a recursion relation.

and the operators $\hat{T}_z$, $\hat{T}_+$, $\hat{T}_-$ enjoy the properties:

$$\hat{T}_-\psi_M{}^J=[N(J,M)/N(J,M-1)]\psi_{M-1}{}^J$$
$$=[\sqrt{\tfrac{1}{2}}(J+M)(J-M+1)]^{\frac{1}{2}}\psi_{M-1}{}^J \quad \text{(III.13)}$$
$$\hat{T}_+\psi_M{}^J=[\tfrac{1}{2}(J-M)(J+M+1)]^{\frac{1}{2}}\psi_{M+1}{}^J$$
$$\hat{T}_z\psi_M{}^J=M\psi_M{}^J$$

which then gives the constitution of the $D(J)$ representation. We now develop the generalization of the foregoing conclusions to simple groups of higher rank.

To construct the irreducible representations of Lie algebras of rank two and higher, we show that all that is required is:

(a) The l fundamental irreducible representations whose highest weights are characterized by one of the l fundamental dominant weights.

(b) a reduction procedure for direct product representations.

Before deriving the theorems needed to synthesize representations, a few words on the characterization of the representation space are in order. In order to specify the representation of the algebra, it is sufficient to give the representations of the basis elements $\hat{H}_i$ and $\hat{E}_\alpha$. We define the representation by prescribing the action of $\hat{H}_i$ and $\hat{E}_\alpha$ on an orthonormal complete set of ket vectors $|\{\lambda_1\cdots\lambda_l\},\nu\rangle$ spanning the N dimensional representation space ($\nu=1,\cdots,N$). When no ambiguities arise, the ket $|\{\lambda_1\cdots\lambda_l\},\nu\rangle$ will often be abbreviated as $|\{N\},\nu\rangle$ and even $|\nu\rangle$. Since the H_i intercommute, they can be simultaneously diagonalized, and, since they are taken to be Hermitian, their eigenvalues are real. We choose a representation in which the H_i are diagonal. Thus the label ν in $|\{N\},\nu\rangle$ stands for a fixed eigenvalue of each of the H_i (the weight $\mathbf{m}$) in addition to other discriminating labels (g) which are needed in the case of multiple weights. Furthermore, the matrices E_α satisfy the relation: $(E_\alpha)^\dagger=E_{-\alpha}$.

If $|\{N\},\nu\rangle$ is the basis for one representation of a Lie algebra and $|\{N'\},\nu'\rangle$ a basis for a second representation, the direct product space spanned by the basis $|\{N\},\nu;\{N'\},\nu'\rangle$ is again a representation of the Lie algebra whose elements L_A act upon the kets $|\{N\},\nu;\{N'\},\nu'\rangle$ in the following manner:

$$L_A|\{N\},\nu;\{N'\},\nu'\rangle$$
$$=L_A{}^{(N)}\otimes 1^{(N')}|\{N\},\nu;\{N'\},\nu'\rangle$$
$$+1^{(N)}\otimes L_A{}^{(N')}|\{N\},\nu;\{N'\},\nu'\rangle. \quad \text{(III.14)}$$

Here $L_A{}^{(N)}$, $1^{(N)}$ and $L_A{}^{(N')}$, $1^{(N')}$ act only on the N and N' dimensional representations, respectively. The direct product representation defined by Eq. (III.14) is, in general, reducible in a way which is shown below.

Given the abstract Lie algebra as presented in Sec. II, we now seek to construct in a systematic way the matrix sets representing the algebra. The method is essentially predicated upon four theorems:

Theorem I. If $H_i|\mathbf{m},g\rangle=m_i|\mathbf{m},g\rangle$, then $H_iE_{-\alpha}|\mathbf{m},g\rangle=[m_i-r_i(\alpha)]E_{-\alpha}|\mathbf{m},g\rangle$.

Proof: $[H_i,E_{-\alpha}]=-r_i(\alpha)E_{-\alpha}$ by Eq. (II.6).

Therefore

$$H_iE_{-\alpha}|\mathbf{m},g\rangle=E_{-\alpha}H_i|\mathbf{m},g\rangle-r_i(\alpha)E_{-\alpha}|\mathbf{m},g\rangle$$
$$=[m_i-r_i(\alpha)]E_{-\alpha}|\mathbf{m},g\rangle.$$

We seek the value of a such that the ket $aE_{-\alpha}|\mathbf{m},g\rangle$ is of unit length. Note, incidentally, that $aE_{-\alpha}|\mathbf{m},g\rangle$ is orthogonal to $|\mathbf{m},g\rangle$ since the H_i eigenvalues of these two states differ.

Theorem II. If $E_\alpha|\mathbf{m},g\rangle=0$, then the normalization constant a is $a=[\mathbf{r}(\alpha)\cdot\mathbf{m}]^{\frac{1}{2}}$.

Proof: $[E_\alpha,E_{-\alpha}]=\mathbf{r}(\alpha)\cdot\mathbf{H}$ by Eq. (II.7).

Therefore

$$\langle\mathbf{m},g|[E_\alpha,E_{-\alpha}]|\mathbf{m},g\rangle$$
$$=\langle\mathbf{m},g|E_\alpha E_{-\alpha}|\mathbf{m},g\rangle=|a|^{-2}$$
$$=\langle\mathbf{m},g|\mathbf{r}(\alpha)\cdot\mathbf{H}|\mathbf{m},g\rangle=\mathbf{r}(\alpha)\cdot\mathbf{m}\ \text{Q.E.D.}$$

In a direct product representation, the greatest dominant weight $\mathbf{M}$ is the sum of the greatest dominant weights $\mathbf{M}^{(N)}$ and $\mathbf{M}^{(N')}$ of the constituent N and N' dimensional representations.

Theorem III. The space spanned by the basis vectors generated by application of $\hat{H}_i$ and $\hat{E}_\alpha$, in any order and any number of times, to $|\mathbf{M}\rangle$ is irreducible under the Lie algebra.

Proof: $|\mathbf{M}\rangle$ is a basis vector of an irreducible representation. Hence, the space spanned by application of $\hat{H}_i$ and $\hat{E}_\alpha$ to $|\mathbf{M}\rangle$ provides an irreducible representation for the algebra by the very definition of irreducibility.

The number of orthonormal vectors which span the reduced direct product space generated in the above manner is the dimension of the resulting representation.

To construct the irreducible representations contained in a direct product representation, we proceed as follows:

(a) Select the ket in the direct product space with the highest weight $|\mathbf{M}\rangle$.

(b) Apply the operators E_α, $E_\alpha E_\beta$, $\cdots$, to $|\mathbf{M}\rangle$. Orthonormalize by the Schmidt process all resulting kets. The orthonormalization is carried out by using the orthonormal properties of the constituent representations, i.e.,

$$\langle\{N\},\nu;\{N'\},\nu'|\{N\},\nu'';\{N'\},\nu'''\rangle$$
$$=\delta_{\nu\nu''}\delta_{\nu'\nu'''}. \quad \text{(III.15)}$$

Kets having different weights will automatically come out orthogonal to each other. The dimensions of the irreducible representations of the algebra have been evaluated in a previous section from the character of the associated group so that this information can be used to predict the number of linearly independent vectors.

(c) Next, in the subspace orthogonal to that generated from $|\mathbf{M}\rangle$, select the ket $|\mathbf{M}'\rangle$ with the highest weight. Generate from $|\mathbf{M}'\rangle$ another space irreducible under the Lie algebra in the same way as an irreducible space was generated from $|\mathbf{M}\rangle$.

(d) The action of the elements of the Lie algebra on the orthonormal vector basis thus generated is readily ascertained by noting the action of $\hat{H}_i$ and $\hat{E}_\alpha$ on the spaces from which the direct product was constructed [Eq. (III.14)].

Given the l explicit representations characterized by each of the l fundamental dominant weights, every irreducible representation of the algebra can be generated by reducing a suitably chosen direct product. Let $\mathfrak{A}^{(s)}$ be the matrix algebra $D(0,0,\cdots,1,\cdots 0)$, where the 1 is in the sth position, whose highest weight is the fundamental dominant weight:

$$\mathbf{M}^{(s)}=(M_1^{(s)},M_2^{(s)},\cdots,M_l^{(s)}).$$

The highest weight of an arbitrary irreducible representation is $\mathbf{M}=\sum_s \lambda_s \mathbf{M}^{(s)}$.

Theorem IV. The irreducible representation of the Lie algebra characterized by the highest weight $\mathbf{M}=\sum_s \lambda_s \mathbf{M}^{(s)}$ is the first irreducible representation obtained by reduction of the product algebra

$$\underbrace{\mathfrak{A}^{(1)}\times\cdots\times\mathfrak{A}^{(1)}}_{\lambda_1 \text{ times}}\times\underbrace{\mathfrak{A}^{(2)}\times\cdots\times\mathfrak{A}^{(2)}}_{\lambda_2\text{ times}}\times\cdots\times\underbrace{\mathfrak{A}^{(l)}\times\cdots\times\mathfrak{A}^{(l)}}_{\lambda_l\text{ times}}.$$

Proof: The highest weight in the product algebra is $\mathbf{M}=\sum_s \lambda_s \mathbf{M}^{(s)}$. By generating a space irreducible under the Lie algebra from the ket $|\mathbf{M}\rangle$, by a generalization of the procedure illustrated above for the direct product of two spaces, an irreducible representation results.

We now go on to use the above method to construct some irreducible representations of SU_3, C_2, and G_2. In particular, all the fundamental representations which go into making the direct product representations will be generated.

F. Matrix Representations of SU_3

The fundamental representations are $D^{(3)}(1,0)$ and $D^{(3)}(0,1)$. Besides constructing these representations we also reduce the regular representation $D^{(8)}(1,1)$ out of the product $D^{(3)}(1,0)\otimes D^{(3)}(0,1)$.

$D^{(3)}(1,0)$. The weight diagram was given in Fig. 2(a); the highest weight is the fundamental dominant weight

$$\mathbf{M}^{(1)}=\tfrac{1}{6}(\sqrt{3},1) \qquad \text{(III.16)}$$

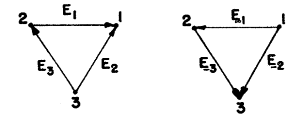

FIG. 5. Action of E_α on $D^{(3)}(1,0)$ of SU_3.

We may write $|\{3\},a\rangle$, $a=1$, 2, 3 or simply $|a\rangle$ for the three states, and use the labeling of Fig. 5. Then the H_i are the diagonal matrices whose eigenvalues are the respective components m_i of the weights. That is[51]

$$H_1=\sum_a m_1(a)\,|a\rangle\langle a|=\frac{1}{2\sqrt{3}}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix},$$
$$H_2=\sum_a m_2(a)\,|a\rangle\langle a|=\frac{1}{6}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}. \qquad \text{(III.17)}$$

According to theorem I (Sec. III E), when $\hat{E}_{-\alpha}$ operates on a state with weight $\mathbf{m}$, it creates a state with weight $\mathbf{m}-\mathbf{r}(\alpha)$. This is symbolized in Fig. 5. Clearly, if $\mathbf{m}-\mathbf{r}(\alpha)$ is not a weight, then $\hat{E}_{-\alpha}|\mathbf{m}\rangle=0$. Therefore, in this simple case, all the constants of proportionality are given by theorem II (Sec. III E) to be $\pm[\mathbf{r}(\alpha)\cdot\mathbf{m}]^{\frac{1}{2}}$. Hence

$$\begin{aligned}E_{-1}|\{3\},1\rangle&=[\mathbf{r}(1)\cdot\mathbf{m}(1)]^{\frac{1}{2}}|\{3\},2\rangle\\&=6^{-\frac{1}{2}}|\{3\},2\rangle,\\E_{-2}|\{3\},1\rangle&=[\mathbf{r}(2)\cdot\mathbf{m}(1)]^{\frac{1}{2}}|\{3\},3\rangle\\&=6^{-\frac{1}{2}}|\{3\},3\rangle,\\E_{-3}|\{3\},2\rangle&=[\mathbf{r}(3)\cdot\mathbf{m}(2)]^{\frac{1}{2}}|\{3\},3\rangle\\&=6^{-\frac{1}{2}}|\{3\},3\rangle.\end{aligned} \qquad \text{(III.18)}$$

The phases of E_{-1} and E_{-2} are arbitrary, but once they have been selected, the phase of E_{-3} is determined by the convention (II.17), since

$$[E_{-1},E_2]=N_{-1,2}E_3=6^{-\frac{1}{2}}E_3. \qquad \text{(III.19)}$$

In the form of matrices, (III.18) becomes

$$E_{-1}=6^{-\frac{1}{2}}\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}=6^{-\frac{1}{2}}|2\rangle\langle 1|,$$
$$E_{-2}=6^{-\frac{1}{2}}\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}=6^{-\frac{1}{2}}|3\rangle\langle 1|, \qquad \text{(III.20)}$$
$$E_{-3}=6^{-\frac{1}{2}}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}=6^{-\frac{1}{2}}|3\rangle\langle 2|,\quad E_\alpha=E_{-\alpha}{}^\dagger.$$

$D^{(3)}(0,1)$. If $\mathfrak{U}$ is a unitary matrix representation of the group SU_3, then $\mathfrak{U}^*$, the complex conjugate matrices, are also a representation. Let $\mathfrak{U}$ be of the

[51] Here $m_i(a)$ is the ith component of the weight of the ath state.

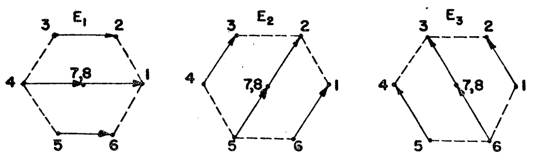

FIG. 6. Action of E_α on $D^{(8)}(1,1)$ of SU_3.

form $\mathcal{U}=\exp(i\epsilon^A L_A)$, then $\mathcal{U}^*=\exp(-i\epsilon^A \tilde{L}_A)$ since $\mathcal{U}^+=\mathcal{U}^{-1}$. Hence the "contragredient" representation of the Lie algebra is $L_A'=-\tilde{L}_A$ where L_A are given in Eq. (20). In view of the reality of these L_A, we find

$$H_i'=-\tilde{H}_i=-H_i,\qquad E_\alpha'=-\tilde{E}_\alpha=-E_\alpha{}^\dagger=-E_{-\alpha}. \tag{III.21}$$

The first of Eq. (21) shows that the weight diagrams for contragrediently related representations are transformed into each other by reflection through the origin. Thus we get the weight diagram of Fig. 2(b). Equation (21) would not hold with a different labeling. From (17), (20), and (21):

$$H_1'=\frac{1}{2\sqrt{3}}\begin{pmatrix}-1&&\\&1&\\&&0\end{pmatrix},\quad H_2'=\frac{1}{6}\begin{pmatrix}-1&&\\&-1&\\&&2\end{pmatrix},$$
$$E_1'=-6^{-\frac{1}{2}}|2\rangle\langle 1|,\qquad E_2'=-6^{-\frac{1}{2}}|3\rangle\langle 1|,$$
$$E_3'=-6^{-\frac{1}{2}}|3\rangle\langle 2|,\qquad E_{-\alpha}'=E_\alpha'^\dagger. \tag{III.22}$$

Thus

$$|\{3^*\},2\rangle=-6^{\frac{1}{2}}E_1'|\{3^*\},1\rangle,$$
$$|\{3^*\},3\rangle=-6^{\frac{1}{2}}E_2'|\{3^*\},1\rangle,$$
$$|\{3^*\},3\rangle=-6^{\frac{1}{2}}E_3'|\{3^*\},2\rangle.$$

This representation is inequivalent to $D^{(3)}(1,0)$ because the set of eigenvalues of H_i' is different from that of H_i. (See also Secs. III B and V.)

$D^{(8)}(1,1)$. The highest weight $\mathbf{M}$ of this representation is $\mathbf{M}^{(1)}+\mathbf{M}^{(2)}$, where $\mathbf{M}^{(1)}=\frac{1}{6}(\sqrt{3},1)$ and $\mathbf{M}^{(2)}=\frac{1}{6}(\sqrt{3},-1)$ are the fundamental dominant weights of $D(1,0)$ and $D(0,1)$, respectively. Hence $D(1,1)$ is contained in $D(1,0)\otimes D(0,1)$. The weight diagram is given in Fig. 2(e); we shall label the states as in Fig. 6, writing $|\{8\},A\rangle$ for the Ath state.

Each product state $|\{3\},a;\{3^*\},b\rangle$ has a unique weight equal to the sum of the weights of $|\{3\},a\rangle$ and $|\{3^*\},b\rangle$. Conversely, for $A=1,\cdots,6$ there is only one product state with the weight of $|\{8\},A\rangle$. Hence, with a choice of phases that turns out to be convenient later:

$$\begin{aligned}|\{8\},1\rangle&=|\{3\},1;\{3^*\},2\rangle,\\|\{8\},2\rangle&=|\{3\},1;\{3^*\},3\rangle,\\|\{8\},3\rangle&=|\{3\},2;\{3^*\},3\rangle,\\|\{8\},4\rangle&=-|\{3\},2;\{3^*\},1\rangle,\\|\{8\},5\rangle&=-|\{3\},3;\{3^*\},1\rangle,\\|\{8\},6\rangle&=|\{3\},3;\{3^*\},2\rangle.\end{aligned} \tag{III.23}$$

The states with weight zero are of the form

$$\sum_{a=1}^{3}\rho_a|\{3\},a;\{3^*\},a\rangle, \tag{III.24}$$

with real coefficients ρ_a. The transformations of the states by the $\hat{E}_\alpha$ is given by the action of $\hat{E}_\alpha$ on $|\{3\},a\rangle$ and on $|\{3^*\},a\rangle$, and is symbolized in Fig. 6. When Eqs. (23) are operated on by the $\hat{E}_\alpha$ or by products of the $\hat{E}_\alpha$, it is easy to see from (21) that the states (III.24) always occur in such linear combinations that

$$\sum\rho_a=0.$$

Hence, only two linearly independent combinations (III.24) occur in $D^{(8)}(1,1)$, as is in fact obvious from the fact that $D^{(8)}(1,1)$ is eight-dimensional. A possible choice of two orthonormal states is

$$\begin{aligned}&|\{8\},7\rangle\\&\quad=(1/\sqrt{2})[|\{3\},2;\{3^*\},2\rangle-|\{3\},1;\{3^*\},1\rangle],\\&|\{8\},8\rangle\\&\quad=6^{-\frac{1}{2}}[-|\{3\},2;\{3^*\},2\rangle-|\{3\},1;\{3^*\},1\rangle\\&\qquad\qquad+2|\{3\},3;\{3^*\},3\rangle].\end{aligned} \tag{III.25}$$

Here the state $|\{8\},7\rangle$ has been chosen, in anticipation of future convenience, as the state obtained by applying $\hat{E}_{-1}$ to $|\{8\},1\rangle$. Once $|\{8\},7\rangle$ has been chosen, $|\{8\},8\rangle$ is unique. The E_α are given by their effect on each of the product states, for example, using (20) and (22):

$$E_2|\{8\},6\rangle=E_2|\{3\},3;\{3^*\},2\rangle=6^{-\frac{1}{2}}|\{3\},1;\{3^*\},2\rangle=6^{-\frac{1}{2}}|\{8\},1\rangle.$$

In this way we get

$$\begin{aligned}6^{\frac{1}{2}}E_1&=\sqrt{2}|1\rangle\langle 7|+\sqrt{2}|7\rangle\langle 4|+|2\rangle\langle 3|+|6\rangle\langle 5|,\\6^{\frac{1}{2}}E_2&=\tfrac{1}{2}\sqrt{2}|2\rangle(\langle 7|+\sqrt{3}\langle 8|)\\&\quad+\tfrac{1}{2}\sqrt{2}(|7\rangle+\sqrt{3}|8\rangle)\langle 5|+|3\rangle\langle 4|+|1\rangle\langle 6|,\\6^{\frac{1}{2}}E_3&=-\tfrac{1}{2}\sqrt{2}|3\rangle(\langle 7|-\sqrt{3}\langle 8|)\\&\quad+\tfrac{1}{2}\sqrt{2}(|7\rangle-\sqrt{3}|8\rangle)\langle 6|+|4\rangle\langle 5|-|2\rangle\langle 1|,\\E_{-\alpha}&=E_\alpha{}^\dagger.\end{aligned} \tag{III.26}$$

The H_i, are the diagonal matrices

$$H_i=\sum_{A=1}^{8}m_i(A)|A\rangle\langle A|, \tag{III.27}$$

where $\mathbf{m}(A)$ is the weight of the Ath state. The phases here are consequences of the phases in (23).

We found that $D^{(8)}(1,1)$ contains only the two linear combination (25) of the three states (24). The third linear combination, orthogonal to (25) and normalized, is

$$|\{1\}1\rangle=(1/\sqrt{3})\textstyle\sum_a|\{3\}a;\{3^*\}a\rangle. \tag{III.28}$$

This is an invariant, $E_\alpha=H_i=0$. Thus the decomposition of $D(1,0)\otimes D(0,1)$ is

$$D^{(3)}(1,0)\otimes D^{(3)}(0,1)=D^{(8)}(1,1)\oplus D^{(1)}(0,0). \tag{III.29}$$

An easier way of finding $D^{(8)}(1,1)$ uses the fact that this is the regular representation, as we shall show. The regular representation[47] is that in which L_A is represented by the matrices $-(C_A)_B{}^D$ whose components are the structure constants $-C_{AB}{}^D$. When the commutation relations are in the standard form (II.12), the capital latin index $A=1, \cdots, 8$ is replaced by $i=1, 2$ and $\alpha=\pm1, \pm2, \pm3$. Thus, referring to (II.12), the $\hat{H}_i$ are represented by $-(C_i)_A{}^B$, whose nonvanishing matrix elements are $-C_{i\alpha}{}^\alpha=-r_i(\alpha)$; the $\hat{E}_\alpha$ are represented by $-(C_\alpha)_A{}^B$, whose nonvanishing matrix elements are $-C_{\alpha i}{}^\alpha=+r_i(\alpha)$, $-C_{\alpha-\alpha}{}^i=-r_i(\alpha)$, and $-C_{\alpha\beta}{}^\gamma=-N_{\alpha\beta}$. Summarizing

$$H_i=-C_i=-\sum_\alpha r_i(\alpha)|\alpha\rangle\langle\alpha|, \quad \text{(III.30)}$$

$$E_\alpha=-C_\alpha=+\sum_i r_i(\alpha)|i\rangle\langle\alpha| -\sum_i r_i(\alpha)|-\alpha\rangle\langle i|-\sum_\beta N_{\alpha\beta}|\beta\rangle\langle\gamma|. \quad \text{(III.31)}$$

Here, as in (II.12), $|\gamma\rangle$ is the state whose root is $\mathbf{r}(\alpha)+\mathbf{r}(\beta)$.

Comparing (30) and (31) with (27) and (26), and taking $r_i(\alpha)$ from (II.18), we find complete agreement with the following identifications

$$\begin{aligned} |\alpha\rangle\to|A\rangle:\quad & |-1\rangle\to|1\rangle, \quad |-2\rangle\to|2\rangle, \\ & \qquad\qquad\quad\; |-3\rangle\to|3\rangle, \\ & |+1\rangle\to-|4\rangle, \quad |+2\rangle\to-|5\rangle, \\ & \qquad\qquad\quad\; |+3\rangle\to|6\rangle, \\ |i\rangle\to|A\rangle:\quad & |1\rangle\to-|7\rangle, \\ & |2\rangle\to-|8\rangle. \end{aligned} \quad \text{(III.32)}$$

The complex conjugate of $D^{(8)}(1,1)$ is related to it by reflection through the origin of the weight diagram. This gives the same diagram with a different labeling. The operator reflecting through the origin is[52]

$$C=-|1\rangle\langle4|\pm|2\rangle\langle5|\pm|3\rangle\langle6|\pm|4\rangle\langle1| \pm|5\rangle\langle2|\pm|6\rangle\langle3|\pm|7\rangle\langle7|\pm|8\rangle\langle8|.$$

The signs are determined by

$$CL_AC^{-1}=L_A'=-\tilde{L}_A, \quad \text{(III.33)}$$

where L_A are the matrices (26), (27). The solution is

$$\begin{aligned} C=&-|1\rangle\langle4|-|4\rangle\langle1|+|3\rangle\langle6|+|6\rangle\langle3| \\ &-|5\rangle\langle2|-|2\rangle\langle5|+|7\rangle\langle7|+|8\rangle\langle8| \\ =&\tilde{C}=C^{-1}. \end{aligned} \quad \text{(III.34)}$$

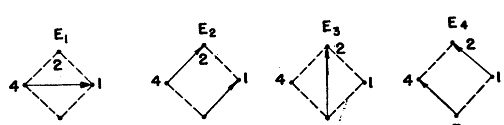

FIG. 7. Action of E_α on $D^{(4)}(1,0)$ of C_2.

[52] This operator is the same as will be introduced later as a "metric tensor." It could also have been defined by the property that

$$\sum_{A,B} C_{AB}|\{8\},A;\{8\},B\rangle=|\{1\}\rangle$$

is an invariant.

The existence of C means that $D^{(8)}(1,1)$ is equivalent to its contragredient representation. This is both displayed and proved in (33).

G. Matrix Representations of C_2

The fundamental representations are $D^{(4)}(1,0)$ and $D^{(5)}(0,1)$ and the regular representation is $D^{(10)}(2,0)$.

$D^{(4)}$*(1,0)*: The weight diagram was given in Fig. 4(a), we label the states as in Fig. 7, $|\{4\},a\rangle$, $a=1, 2, 3, 4$. The actions of E_α, $\alpha=1, \cdots, 4$ are summarized in Fig. 7; the action of $E_{-\alpha}$ are the same with the arrows reversed. As in the case of SU_3, Theorem II is sufficient to allow one to write down the explicit forms of E_α almost immediately. Thus, the analogs of (17) and (18) are

$$H_i=\sum_{a=1}^{4} m_i(a)|a\rangle\langle a| \quad \text{(III.35)}$$

$$\begin{aligned} E_1&=6^{-\frac{1}{2}}|1\rangle\langle4|, \\ E_2&=(1/2\sqrt{3})(|1\rangle\langle3|-|2\rangle\langle4|), \\ E_3&=-6^{-\frac{1}{2}}|2\rangle\langle3|, \\ E_4&=(1/2\sqrt{3})(|2\rangle\langle1|+|4\rangle\langle3|), \\ E_{-\alpha}&=(E_\alpha)^\dagger. \end{aligned} \quad \text{(III.36)}$$

The weight diagram for the contragredient representation is obtained by reflection through the origin. In this case we get the same diagram with a different labeling. Hence, the operator reflecting through the origin is

$$C=-|1\rangle\langle4|\pm|4\rangle\langle1|\pm|2\rangle\langle3|\pm|3\rangle\langle2|.$$

The phases must be chosen to agree with (21), that is

$$CL_AC^{-1}=L_A'=-\tilde{L}_A. \quad \text{(III.37)}$$

The solution is

$$\begin{aligned} C&=-|1\rangle\langle4|+|4\rangle\langle1|+|2\rangle\langle3|-|3\rangle\langle2| \\ &=-\tilde{C}=-C^{-1}=-C^\dagger. \end{aligned} \quad \text{(III.38)}$$

The existence of C means that $D^{(4)}(1,0)$ is equivalent to its contragredient representation. This is both displayed and proved in (35).

$D^{(5)}$*(0,1)*. The weight diagram is that of Fig. 4(b); we use the labeling of Fig. 8. The action of E_α is also symbolized in Fig. 8. The matrices are obtained

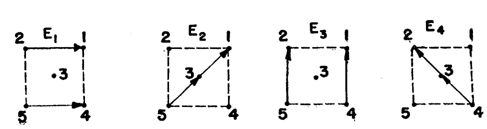

FIG. 8. Action of E_α on $D^{(5)}(0,1)$ of C_2.

exactly as before, namely:

$$H_i=\sum_{k=1}^{5} m_i(k)|k\rangle\langle k|, \tag{III.39}$$

$$\begin{aligned} E_1&=6^{-\frac{1}{2}}(|1\rangle\langle 2|+|4\rangle\langle 5|),\\ E_2&=6^{-\frac{1}{2}}(|1\rangle\langle 3|-|3\rangle\langle 5|),\\ E_3&=6^{-\frac{1}{2}}(|1\rangle\langle 4|+|2\rangle\langle 5|),\\ E_4&=6^{-\frac{1}{2}}(|2\rangle\langle 3|+|3\rangle\langle 4|),\\ E_{-\alpha}&=(E_\alpha)^\dagger. \end{aligned} \tag{III.40}$$

Again the contragredient representation is equivalent. The matrix C in this case is

$$C=|5\rangle\langle 1|+|1\rangle\langle 5|-|2\rangle\langle 4|-|4\rangle\langle 2|+|3\rangle\langle 3| =\tilde{C}. \tag{III.41}$$

$D^{(10)}(2,0)$. The weight diagram was given in Fig. 4(c). The highest weight is exactly twice the highest weight of $D^{(4)}(1,0)$, and $D^{(10)}(2,0)$ is contained in $D^{(4)}(1,0)\otimes D^{(4)}(1,0)$. We begin by calling $|\{10\},1\rangle$ the state $|\{4\},1;\{4\},1\rangle$. Since the E_α operate in the same way on the two factors, it is evident that $E_\alpha|\{10\},1\rangle$, $E_\alpha E_\beta|\{10\},1\rangle$, etc., are all symmetric in the two factors. Hence we have, with a convenient set of phases,

$$\begin{aligned}
|\{10\},1\rangle&=|\{4\},1;\{4\},1\rangle =-|-1\rangle,\\
|\{10\},2\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},1;\{4\},2\rangle +\tfrac{1}{2}\sqrt{2}|\{4\},2;\{4\},1\rangle=+|-2\rangle,\\
|\{10\},3\rangle&=|\{4\},2;\{4\},2\rangle =-|-3\rangle,\\
|\{10\},4\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},2;\{4\},4\rangle +\tfrac{1}{2}\sqrt{2}|\{4\},4;\{4\},2\rangle=+|-4\rangle,\\
|\{10\},5\rangle&=|\{4\},4;\{4\},4\rangle =+|+1\rangle,\\
|\{10\},6\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},4;\{4\},3\rangle +\tfrac{1}{2}\sqrt{2}|\{4\},3;\{4\},4\rangle=+|+2\rangle,\\
|\{10\},7\rangle&=|\{4\},3;\{4\},3\rangle =+|+3\rangle,\\
|\{10\},8\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},1;\{4\},3\rangle +\tfrac{1}{2}\sqrt{2}|\{4\},3;\{4\},1\rangle=-|+4\rangle,\\
|\{10\},9\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},1;\{4\},4\rangle +\tfrac{1}{2}\sqrt{2}|\{4\},4;\{4\},1\rangle=+|1\rangle,\\
|\{10\},10\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},2;\{4\},3\rangle +\tfrac{1}{2}\sqrt{2}|\{4\},3;\{4\},2\rangle=-|2\rangle.
\end{aligned} \tag{III.42}$$

The labeling on the right-hand side is the one that allows us to use Eqs. (30) and (31) directly. The simplest derivation is by means of [cf. (V.5)]

$$6^{\frac{1}{2}}\sum_{a,b,d} C_{ab}(E_{\pm\alpha})_{bd}|ad\rangle=|\pm\alpha\rangle;$$

$$6^{\frac{1}{2}}\sum_{a,b,d} C_{ab}(H_i)_{bd}|ad\rangle=|i\rangle.$$

H. Matrix Representations of G_2

The fundamental representations are $D^{(7)}(1,0)$ and $D^{(14)}(0,1)$, the latter being the regular representation.

$D^{(7)}(1,0)$. The weight diagram is that of Fig. 3(a), and we use the labeling indicated there, thus $|\{7\},k\rangle$, $k=1, \cdots, 7$. As in the other examples, Theorem II (Sec. III E) suffices to determine the matrix elements of E_α. The result is

$$\begin{aligned}
H_i&=\sum_{k=1}^{7} m_i(k)|k\rangle\langle k|,\\
E_1&=(1/2\sqrt{3})(\tfrac{1}{2}\sqrt{2}|1\rangle\langle 2|+\tfrac{1}{2}\sqrt{2}|3\rangle\langle 4|+|5\rangle\langle 6|+|6\rangle\langle 7|),\\
E_2&=(1/2\sqrt{2})(|5\rangle\langle 4|+|1\rangle\langle 7|),\\
E_3&=(1/2\sqrt{3})(\tfrac{1}{2}\sqrt{2}|5\rangle\langle 3|-|6\rangle\langle 4|+|1\rangle\langle 6|-\tfrac{1}{2}\sqrt{2}|2\rangle\langle 7|),\\
E_4&=(1/2\sqrt{2})(|1\rangle\langle 3|+|2\rangle\langle 4|),\\
E_5&=(1/2\sqrt{3})(|6\rangle\langle 3|-\tfrac{1}{2}\sqrt{2}|7\rangle\langle 4|-\tfrac{1}{2}\sqrt{2}|1\rangle\langle 5|+|2\rangle\langle 6|),\\
E_6&=(1/2\sqrt{2})(-|2\rangle\langle 5|+|7\rangle\langle 3|);\ E_{-\alpha}=(E_\alpha)^\dagger.
\end{aligned} \tag{III.43}$$

The C operator which changes $D^{(7)}$ into its complex conjugate and reflects the weight-diagram through the origin is defined by

$$CL_AC^{-1}=-\tilde{L}_A.$$

The solution is

$$C=|5\rangle\langle 7|+|7\rangle\langle 5|-|1\rangle\langle 4|-|4\rangle\langle 1| +|2\rangle\langle 3|+|3\rangle\langle 2|-|6\rangle\langle 6|. \tag{III.44}$$

$D^{(14)}(0,1)$. Since this is the regular representation, the matrices H_i and E_α are given by (30) and (31). The weight diagram is that of Fig. 3(b).

IV. COMPOSITION AND DECOMPOSITION OF LIE ALGEBRA REPRESENTATIONS

The basis of the vector space affording a representation of a simple group may be characterized by the simultaneous eigenvalues of the maximum number of mutually commuting Lie algebra operators, designated by the symbols $H_1, H_2, \cdots, H_l$ where l is the rank of the group. However, the characterization of the representation space basis is not complete if only the H_i eigenvalues are assigned to the basis vectors because the same set of eigenvalues of the H_i, the weight $\{m_1\cdots m_l\}\equiv\mathbf{m}$, can occur more than once in a specific representation, i.e., weights other than the dominant weight $\{M_1\cdots M_l\}\equiv\mathbf{M}$ are, in general, not simple. The goals of this section are (a) to find the set of weights and their multiplicities in every representation, and (b) to reduce the direct product of irreducible representations into a direct sum of irreducible representations. The course which we pursue is a purely geometric one, and represents an extension of the classical method.

A. Geometric Characterization of a Representation

Let us restrict the considerations to the groups of rank two. The seven-dimensional representation of G_2 can be characterized by plotting the array of points (m_1,m_2) whose coordinates are the weights of the representation.[53] Figure 9(a) shows the resulting array of points. In this specific example, the multiplicity of each of the weights is one and so each weight is associated with one and only one point. When the multiplicity of a weight is greater than one, this will be indicated. Such is the case in the eight-dimensional representation of SU_3, for which the associated point set is given in Fig. 9(b).

Before proceeding with the task of composing and reducing representations, we introduce the formal operations on sets of points which are utilized in the subsequent sections.

B. Algebra of Sets of Points

To illustrate the algebraic manipulations to which sets of points can be subjected, consider first sets of collinear points. A set of points on a line with a center * and with signed multiplicities attached to each point will be associated with a function which is a sum of powers of a single variable x, as follows:

(a) Each point is associated with a term in the function; the latter has as many terms as there are points,
(b) The coordinate of each point relative to the set center * represents the power of x in the relevant term,
(c) The numerical coefficient of the term is the attached signed multiplicity.

Thus the set of points in Fig. 10(a) represents the algebraic expression $0.3x^{-4}-x^{-1}+2x^2$.

In what follows, only integral multiplicities come into consideration and if a single point without indicated multiplicity but with an attached sign occurs, the associated term in the algebraic expression is assigned a coefficient ± 1 depending on the indicated sign. A final liberty with the above conventions is to assume that, in the absence of an indicated center of a point set, this coincides with the geometric center of the point set.

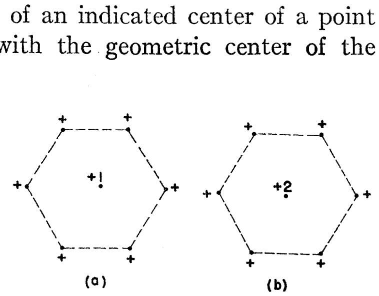

FIG. 9. (a) Representation set for G_2, (b) representation set for SU_3.

[53] These arrays are nothing but the weight diagrams of Sec. III, with the multiplicities added.

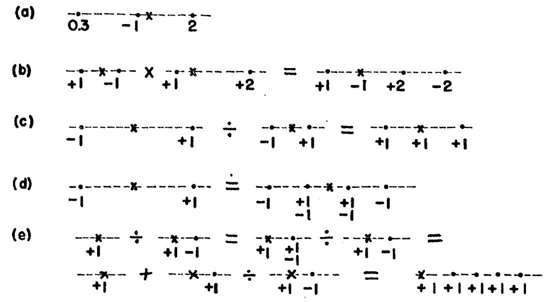

FIG. 10. Algebraic processes on linear point sets.

Addition of two sets of points (ζ and ζ') with a common center is defined to be the union of the two sets; $\zeta+\zeta'\equiv\zeta\cup\zeta'$, the multiplicities adding algebraically. *Subtraction* of two sets of points ζ and ζ' is defined to be the addition of ζ to the set $-\zeta'$ obtained from ζ by changing the signs of all multiplicities.

To *multiply* one set of points ζ by another set ζ', the center of the set ζ' is placed on each of the points of the set ζ and each term of ζ' is multiplied by the multiplicity of the point of ζ upon which its center sits. The new set of points obtained in such a manner is defined to be the product set $\zeta\times\zeta'$. For example, Fig. 10(b) is the geometric equivalent of $(x^{-1}-x)\times(x^{-1}+2x^3)=(x^{-2}-1+2x^2-2x^4)$.

Division is defined to be the inverse of multiplication. The most trivial case of division is the case in which the two sets of points ζ and ζ' are identical. The result of the division $\zeta\div\zeta'$ is simply a single point at the common center of ζ and ζ'. In general, one set of points ζ' exactly divides a congruent set ζ if the multiplicities of every point of ζ is a fixed multiple Z of its image point in ζ'. The result of this division operation is a point of multiplicity Z which sits where the center of ζ' falls when superimposed on ζ. If the set ζ' is not congruent to the set ζ, it is possible to create a subset of ζ, denoted by ζ'' and exactly divisible by ζ', by adding and subtracting points, of the same multiplicity at appropriate positions in the set ζ. After dividing such a subset ζ'' away, we are left with the problem of dividing the residual set $\zeta-\zeta''$ by ζ'. By continuing this process, we may ultimately arrive at a residual set itself exactly divisible by ζ' without modification. As an example, consider the problem illustrated in Fig. 10(c) whose algebraic analog is $(x^3-x^{-3})/(x-x^{-1})$. By adding and subtracting a point of multiplicity $+1$ at each of the positions -1 and $+1$ [Fig. 10(d)], the exact division can be effected. If two sets of points are not exactly divisible, division can still be carried out by adding and subtracting points to the dividend set *ad infinitum*. Figure 10(e) illustrates the geometric method of carrying out the expansion $1/1-x=1+x+x^2+\cdots$. In what follows, we use only exactly divisible point sets.

All of the above manipulations are quite trivial for linear sets of points. However, it is possible to generalize

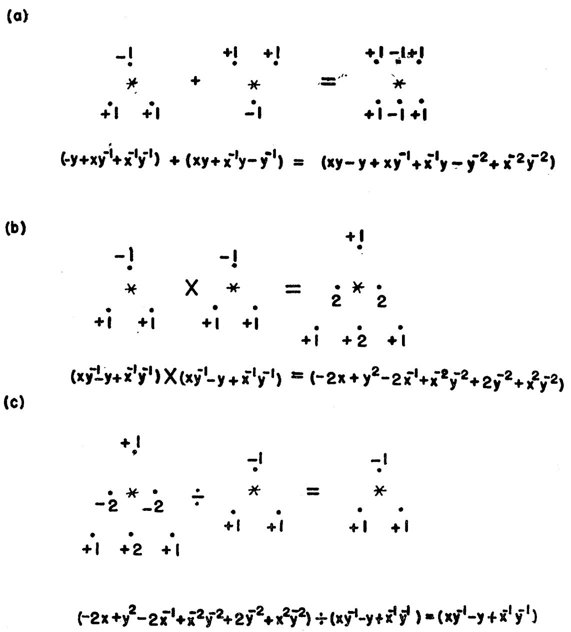

FIG. 11. Algebraic processes on two dimensional sets of points. (a) Addition; (b) multiplication; (c) division.

to an algebra of sets of points in an n-dimensional space, each point being characterized by a coordinate $\mathbf{m}=(m_1m_2\cdots m_l)$ and an assigned multiplicity and the total set being provided with a center. Every such point is again associated with a term in an algebraic expression in n variables. For example, the point at $\mathbf{m}=(m_1m_2\cdots m_l)$ with multiplicity μ_{m} is the geometric representation of $\mu_{\mathrm{m}}x_1^{m_1}x_2^{m_2}\cdots x_l^{m_l}$. All algebraic processes on algebraic expressions in n variables of the form $\sum_{\mathrm{m}}\mu_{\mathrm{m}}x_1^{m_1}\cdots x_l^{m_l}$ can now be given a geometric analog.

Since our concern is with functions in two variables, we illustrate in Fig. 11 some algebraic processes carried out on sets of points in two dimensions. It is to be remarked that the operations on the sets of points are completely isomorphic to the corresponding algebraic processes and as such are, for example, associative and commutative.

C. Construction of Weights and Multiplicities of Irreducible Representations

Our goal in this section is to assign to every irreducible representation of a group a set of points

TABLE I. Coordinates of points in the set $\xi(\lambda_1\lambda_2)$ for SU_3.

$(6/\sqrt{3})x$	$6y$	Multiplicity
$(\lambda_1+\lambda_2+2)$	$(\lambda_1-\lambda_2)$	$+1$
(λ_1+1)	$(\lambda_1+2\lambda_2+3)$	-1
$-(\lambda_1+1)$	$(\lambda_1+2\lambda_2+3)$	$+1$
$-(\lambda_1+\lambda_2+2)$	$(\lambda_1-\lambda_2)$	-1
$-(\lambda_2+1)$	$-(2\lambda_1+\lambda_2+3)$	$+1$
(λ_2+1)	$-(2\lambda_1+\lambda_2+3)$	-1

(called the representation set from now on) and derive the admissible sets of points constituting a representation. The fundamental observation is the following: The character of the representation is the algebraic expression associated with the representation set.[53] For a group of rank l the algebraic variables associated with the representation set may be selected as $x_i=e^{i\varphi_i}$. Recall now that every representation of a rank two group is characterized by two integers λ_1 and λ_2 where λ_1, λ_2 run over all non-negative integers. The general expressions for the characters of all the groups which interest us have been given by Weyl.[8] Letting $\chi(\lambda_1,\lambda_2)$ denote the set of points constituting the representation, the general expression for $\chi(\lambda_1,\lambda_2)$ is

$$\chi(\lambda_1,\lambda_2)=\xi(\lambda_1,\lambda_2)/\xi(0,0), \qquad \text{(IV.1)}$$

where the algebraic expressions $\xi(\lambda_1,\lambda_2)$ were given in Secs. III A and III B. The set of points $\xi(\lambda_1,\lambda_2)$ is called the *girdle* of points uniquely characterizing a

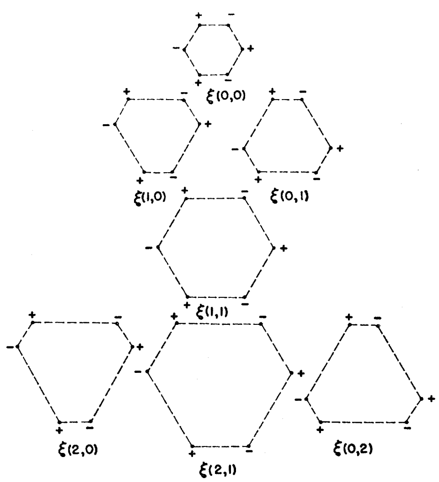

FIG. 12. Some girdles of SU_3.

representation. We thus see that to generate the representation set, the girdle $\xi(\lambda_1,\lambda_2)$ must be divided by the girdle $\xi(0,0)$. Since the $\chi(\lambda_1,\lambda_2)$ form a finite set of points, $\xi(\lambda_1,\lambda_2)$ must be exactly divisible by $\xi(0,0)$.

To illustrate the detailed mechanics of generating representation sets, we turn to the groups SU_3, C_2, and G_2.

SU_3. The coordinates of the six points making up $\xi(\lambda_1,\lambda_2)$ are given in Table I. They are the values of the components of $(S\mathbf{K})$ of Eq. (III.4). For SU_3 the girdle $\xi(\lambda_1,\lambda_2)$ forms the vertices of a hexagon which has the following properties:

(a) Every other side is of the same length, either $\frac{1}{3}\sqrt{3}(\lambda_1+1)$ or $\frac{1}{3}\sqrt{3}(\lambda_2+1)$,
(b) The hexagons are always symmetric about the y axis,
(c) A hexagon is symmetric about the x axis if and only if $\lambda_1=\lambda_2$. In this case, the hexagon is regular (all sides being equal).

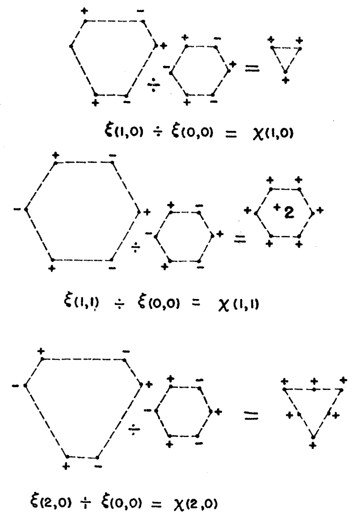

FIG. 13. Some characters of SU_3 obtained by the division process.

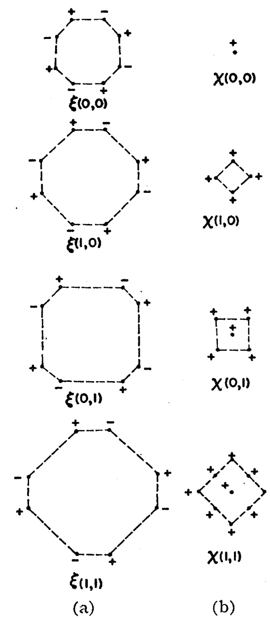

FIG. 14. Some girdles and characters of C_2.

If $\chi(\lambda_1,\lambda_2)$ is a representation set, then the complex conjugate representation set $\chi^*(\lambda_1,\lambda_2)=\chi(\lambda_2,\lambda_1)$ (for SU_3 only) is obtained by inverting the $\chi(\lambda_1,\lambda_2)$ hexagon through the origin, and changing the signs of the multiplicities. An equivalent procedure is to reflect $\chi(\lambda_1,\lambda_2)$ in the x axis and leave the multiplicities unchanged. Thus, the necessary and sufficient condition for equivalence of $D(\lambda_1,\lambda_2)$ and $D^*(\lambda_1,\lambda_2)$ is that the $\xi(\lambda_1,\lambda_2)$ hexagon be regular.

Figure 12 illustrates the girdles of some low-dimensional representations of SU_3. Triangular graph paper is admirably suited for the plot.

The construction of the weights and multiplicities of a representation is now effected by dividing $\xi(\lambda_1,\lambda_2)$ by $\xi(0,0)$ and identifying the quotient points as the representation set. In Fig. 13, we carry out some representative divisions.

C_2. With the use of Table II any girdle can be found; in particular, those illustrated in Fig. 14(a). The points of $\xi(\lambda_1,\lambda_2)$ define the vertices of an octagon symmetric about the x and y axes. Every representation is therefore equivalent to its complex conjugate representation. The sides of the octagon alternate in length between $\frac{1}{3}\sqrt{3}(\lambda_2+1)$ and $(\frac{2}{3})^{\frac{1}{2}}(\lambda_1+1)$.

TABLE II. Coordinates of the point s in the set $\xi(\lambda_1,\lambda_2)$ for C_2.

$2\sqrt{3}\,x$	$2\sqrt{3}\,y$	Multiplicity
$(\lambda_1+\lambda_2+2)$	(λ_2+1)	$+1$
(λ_2+1)	$(\lambda_1+\lambda_2+2)$	-1
$-(\lambda_2+1)$	$(\lambda_1+\lambda_2+2)$	$+1$
$-(\lambda_1+\lambda_2+2)$	(λ_2+1)	-1
$-(\lambda_1+\lambda_2+2)$	$-(\lambda_2+1)$	$+1$
$-(\lambda_2+1)$	$-(\lambda_1+\lambda_2+2)$	-1
(λ_2+1)	$-(\lambda_1+\lambda_2+2)$	$+1$
$(\lambda_1+\lambda_2+2)$	$-(\lambda_2+1)$	-1

Figure 14(b) gives the result of dividing the $\xi(\lambda_1,\lambda_2)$ of Fig. 14(a) by $\xi(0,0)$.[54]

G_2. Table III specifies the sets $\xi(\lambda_1,\lambda_2)$ as dodecahedrons symmetric about the x and y axis. Thus the complex conjugate representations are equivalent. As in C_2 and SU_3, the sides of the $\xi(\lambda_1,\lambda_2)$ polygon alternate in length, in this case between $\frac{1}{2}(\lambda_2+1)$ and $\frac{1}{6}\sqrt{3}(\lambda_1+1)$. Figure 16 contains the representation sets, $\chi(1,0)$ and $\chi(0,1)$, while Fig. 15 illustrates some girdles.

D. Reduction of Direct Products of Representations

In the previous section, we have shown how to derive all the representation point sets including the multiplicity assignments. However, for the purpose of reducing the direct product of representations, only

TABLE III. Coordinates of points in the set $\xi(\lambda_1,\lambda_2)$ for G_2.

$4\sqrt{3}\,x$	$4y$	Multiplicity
$(2\lambda_1+3\lambda_2+5)$	(λ_2+1)	$+1$
$(\lambda_1+3\lambda_2+4)$	$(\lambda_1+\lambda_2+2)$	-1
(λ_1+1)	$(\lambda_1+2\lambda_2+3)$	$+1$
$-(\lambda_1+1)$	$(\lambda_1+2\lambda_2+3)$	-1
$-(\lambda_1+3\lambda_2+4)$	$(\lambda_1+\lambda_2+2)$	$+1$
$-(2\lambda_1+3\lambda_2+5)$	(λ_2+1)	-1
$-(2\lambda_1+3\lambda_2+5)$	$-(\lambda_2+1)$	$+1$
$-(\lambda_1+3\lambda_2+4)$	$-(\lambda_1+\lambda_2+2)$	-1
$-(\lambda_1+1)$	$-(\lambda_1+2\lambda_2+3)$	$+1$
(λ_1+1)	$-(\lambda_1+2\lambda_2+3)$	-1
$(\lambda_1+3\lambda_2+4)$	$-(\lambda_1+\lambda_2+2)$	$+1$
$(2\lambda_1+3\lambda_2+5)$	$-(\lambda_2+1)$	-1

[54] The method of dividing point sets by point sets turns out to be quite powerful. Further details will be found in a paper by two of the authors (J.D. and C.F.).

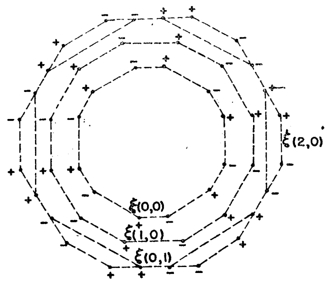

FIG. 15. Some girdles of G_2.

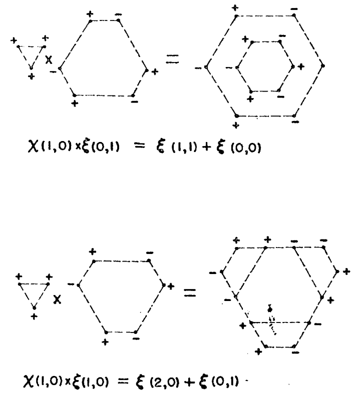

FIG. 17. Geometric derivation of girdles in direct product representation of SU_3.

the girdle $\xi(\lambda_1,\lambda_2)$ associated with the representation is needed, as we now prove.

The direct product of two representations of a simple group reduces completely and uniquely into a sum of irreducible representations some of which may occur more than once. Letting $\nu(\mu_1,\mu_2)$ designate the number of times a specific representation $\chi(\mu_1,\mu_2)$ occurs in the reduction of a direct product of irreducible representations, we have the following equality between point sets

$$\chi(\lambda_1,\lambda_2)\otimes\chi(\lambda_1',\lambda_2')=\sum_{\mu_1\mu_2}\nu(\mu_1,\mu_2)\chi(\mu_1,\mu_2). \quad \text{(IV.2)}$$

If we use the fundamental relation Eq. (IV.1), Eq. (IV.2) reduces to

$$\frac{\xi(\lambda_1,\lambda_2)\times\xi(\lambda_1',\lambda_2')}{\xi(0,0)}=\sum_{\mu_1\mu_2}\nu(\mu_1,\mu_2)\xi(\mu_1,\mu_2), \quad \text{(IV.3)}$$

where we have multiplied both sides of Eq. (IV.2) by $\xi(0,0)$. Because only the girdles of the irreducible representations $\chi(\mu_1,\mu_2)$ occur on the right-hand side of Eq. (IV.3), we need only carry out the point set process $\{\xi(\lambda_1,\lambda_2)\times\xi(\lambda_1',\lambda_2')\div\xi(0,0)\}$, and then identify the girdles and their multiplicities $\nu(\mu_1,\mu_2)$ in the resulting set to reduce completely the product representations. Use of one of the several alternative forms of $\xi(\lambda_1,\lambda_2)\times\xi(\lambda_1',\lambda_2')\div\xi(0,0)$, namely

$$\chi(\lambda_1,\lambda_2)\times\xi(\lambda_1',\lambda_2')=\xi(\lambda_1,\lambda_2)\chi(\lambda_1',\lambda_2')$$
$$=\xi(0,0)\chi(\lambda_1,\lambda_2)\chi(\lambda_1',\lambda_2')$$

will simplify the computations in some cases.

As examples of the reduction process, we carry out

(a) $\chi(1,0)\times\chi(1,0)$ and $\chi(1,0)\times\chi(0,1)$ in SU_3

(b) $\chi(1,0)\times\chi(1,0)$ for C_2

(c) $\chi(1,0)\times\chi(1,0)$ for G_2.

Figures 17 and 18 illustrate the reduction processes for SU_3 and C_2, respectively. The superimposed girdle diagram of Fig. 15 is the product of $\xi(0,0)\times\chi(1,0)\times\chi(1,0)$ for G_2.

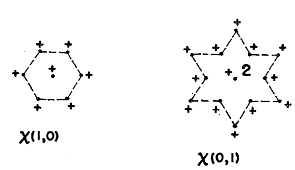

FIG. 16. Some characters of G_2.

V. TENSOR ANALYSIS OF SIMPLE LIE GROUPS

In this section we present some results by an alternative, purely algebraic method, which to a certain extent is complementary to the geometric method. The specific advantages of the algebraic method is that it deals directly with the bases of the representation space (the "wave functions"), and that it gives directly the explicit form of invariants, product representations and transformation matrices.

Let m be the dimensionality of any representation of some simple Lie group. The matrix algebra of that representation consists of Hermitian traceless matrices. Since the matrix algebra of an m-dimensional representation of SU_m is the set of *all* Hermitian traceless matrices, it follows that the group in question is a subgroup of SU_m. For example, C_2 and G_2 are subgroups of SU_4 and SU_7, respectively. Therefore the reduction of a product of several m-dimensional representations is a refinement of the reduction according to SU_m. It is very helpful, then, to begin with a discussion of SU_m for arbitrary m.

A. Group SU_m

Let ψ_a, $a=1,\cdots,m$, be a basis for an m-dimensional representation of SU_m. The matrices representing a basis for the Lie algebra are any set of m^2-1 independent Hermitian traceless matrices. The *contragredient* representation ψ^a is defined by[55]

$$\psi_a\rightarrow(\delta_a{}^b+i\epsilon^A L_{Aa}{}^b)\psi_b,\quad \psi^a\rightarrow\psi^b(\delta_b{}^a-i\epsilon^A L_{Ab}{}^a). \quad \text{(V.1)}$$

[For $m=3$, these representations are those labeled $D^{(3)}(1,0)$ and $D^{(3)}(0,1)$ in Secs. III and IV. The weight diagrams are those of Fig. 2(a) and Fig. 2(b).]

Next consider the "tensors" $\psi_{abc\cdots}{}^{ef\cdots}$. These are quantities transforming in the same way as products of the representations ψ_a and ψ^a. Thus ψ_{ab} has m^2 components which transform among themselves like the

[55] This will be recognized as agreeing with the definitions of Eqs. (I.1) and (III.21).

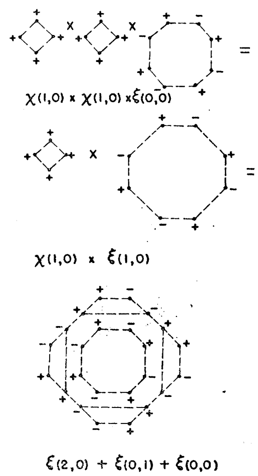

FIG. 18. Geometric derivation of girdles in the direct product representation of C_2.

m^2 quantities $\psi_a\psi_b$, $\psi_a{}^b$ transforms like $\psi_a\psi^b$, etc. The tensors form bases for representations called *product representations*; the present definition agrees with that of Sec. III.

Product representations are usually[56] reducible. The reduction of second-rank tensors according to SU_m is entirely elementary. The tensor $\psi_a{}^b$, for example, transforms according to (1), as follows[57]:

$$\psi_a{}^b \to (\delta_a{}^c+i\epsilon^A L_{Aa}{}^c)(\delta_d{}^b-i\epsilon^B L_{Bd}{}^b)\psi_c{}^d$$
$$=\psi_a{}^b+i\epsilon^A(L_{Aa}{}^c\delta_d{}^b-L_{Ad}{}^b\delta_a{}^c)\psi_c{}^d. \quad \text{(V.2)}$$

In particular, if we put $a=b$ and sum, we find

$$\psi_a{}^a \to \psi_a{}^a.$$

Thus the trace is invariant, meaning that the m^2 dimensional representation $\psi_a{}^b$ may be reduced into a one-dimensional representation and the m^2-1 dimensional representation whose basis is the traceless tensor

$$\psi_a{}^b-\frac{1}{m}\delta_a{}^b\psi_c{}^c=P_a{}^b{}_c{}^d\psi_d{}^c. \quad \text{(V.3)}$$

Here $P_a{}^b{}_c{}^d$ is the projection operator

$$P_a{}^b{}_c{}^d=\delta_a{}^d\delta_c{}^b-\frac{1}{m}\delta_a{}^b\delta_c{}^d, \quad \text{(V.4)}$$

whose rows are labeled by a, b and whose columns are labeled by c, d.

The proof that (3) is the basis of an irreducible representation is instructive. First, we show that (3) is the *regular representation*[47] for SU_m, and that it *contains* the regular representation for any subgroup of SU_m. Let r be the order of the subgroup, and consider the r linearly-independent combinations.

$$\varphi_A=L_{Ab}{}^a\psi_a{}^b=L_{Ab}{}^a(\delta_a{}^c\delta_d{}^b-\frac{1}{m}\delta_a{}^b\delta_d{}^c)\psi_c{}^d. \quad \text{(V.5)}$$

The second equality is a result of the fact that the matrices $L_{Aa}{}^b$ are traceless, and shows that φ_A depend on the traceless tensor (3) only. From (2) and (5) we get

$$\varphi_A\to\varphi_A+i\epsilon^B(L_{Ad}{}^a L_{Ba}{}^c-L_{Bd}{}^b L_{Ab}{}^c)\psi_c{}^d$$
$$=\varphi_A+i\epsilon^B C_{AB}{}^D L_{Dd}{}^c\psi_c{}^d$$
$$=\varphi_A+i\epsilon^B C_{AB}{}^D\varphi_D. \quad \text{(V.6)}$$

Hence, the φ_A are the basis of that representation of the r-parameter subgroup in which the operators L_B are represented by the structure constants $C_{AB}{}^D$, and that is the regular representation. Equation (5) shows that this representation is contained in the traceless $\psi_a{}^b$. In the special case of SU_m, $r=m^2-1$, and $L_{Aa}{}^b$ is the set of *all* Hermitian traceless matrices. Hence, in that case the regular representation φ_A is equivalent to the representation whose basis is the traceless $\psi_a{}^b$. Since the former is irreducible[47] (for any simple group), so is the latter.

With the proof that (3) is irreducible, the reduction of $\psi_a{}^b$ has been completed. We can also prove that ψ_a and ψ^a are inequivalent. For suppose that they are equivalent. Then there exists a nonsingular form invariant matrix A^{ab} such that $\psi^a=A^{ab}\psi_b$. This could be used to prove that $\psi_a{}^b$ and ψ^{ab} were equivalent, which is impossible since ψ^{ab} reduces quite differently, as we shall see immediately. Hence, no matrix exists for raising and lowering indices.

The reduction problem for tensors of arbitrary rank, but with all indices either upstairs or downstairs, has a complete and beautiful solution in terms of Young tableaux.[58] We do not present the general theory here, since it is only of marginal interest, and thus do not prove that the representations obtained are irreducible. However, whenever appropriate, we indicate the connection between the representations and the tableaux. The complete reduction of the second-rank tensor ψ_{ab} is given by

$$\psi_{ab}=\psi_{ab,}+\psi_{a,b},$$

where

$$\psi_{ab,}\equiv\tfrac{1}{2}(\psi_{ab}+\psi_{ba}),\quad \psi_{a,b}\equiv\tfrac{1}{2}(\psi_{ab}-\psi_{ba}).$$

The symmetric part $\psi_{ab,}$ has $\frac{1}{2}m(m+1)$ components and corresponds to the Young tableau of Fig. 19(a). The skew part $\psi_{a,b}$, has $\frac{1}{2}m(m-1)$ components and the Young tableau is that of Fig. 19(b).

Roughly, indices appearing in the same row in a Young diagram are subject to symmetrization, while indices appearing in the same column are subject to

[56] The only exception is the case when one of the factors is the identity representation.

[57] The tensor $\psi_a{}^b$ would be written $|\{m\},a;\ \{m\},b\rangle$ in the notation of Sec. III E. Equation (V.2) is an application of Eq. (III.14).

[58] A readable exposition is given in D. Rutherford; *Substitutional Analysis* (Edinburgh University Press, Edinburgh, Scotland, 1948).

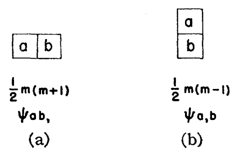

FIG. 19. The Young tableaux related to the reduction of the second-rank tensor in m dimensions.

antisymmetrization. The notation is the following: A comma between the indices separate those of the first row from those of the second row, a second comma separates the indices in the second row from those of the third, and so on. The completely symmetric tensor $\psi_{a\cdots d,}$ is furnished with a comma to distinguish it from the general nonsymmetrized tensor $\psi_{a\cdots d}$.

Corresponding to the reduction of the third-rank tensor there are the four Young tableaux of Fig. 20. The irreducible bases, as well as the dimensionalities, are indicated; the latter, of course, add up to m^3. Whereas $\psi_{abc,}$ and $\psi_{a,b,c}$ are uniquely defined as the completely symmetric and the completely skew parts, respectively, the other two parts have mixed symmetry and their definition is slightly ambiguous.[59] This is due to the fact that they are a pair of equivalent representations of SU_m. A possible choice is:

$$\psi_{ab,c}=\tfrac{1}{4}(\psi_{abc}-\psi_{acb}+\psi_{bac}-\psi_{bca}),$$
$$\psi_{ac,b}=\tfrac{1}{4}(\psi_{acb}-\psi_{bca}+\psi_{cab}-\psi_{bac}).$$

With this choice the four parts are orthogonal. This summarizes the complete reduction of ψ_{abc}.

We have seen how covariant tensors are reduced according to their symmetry, and how the mixed tensor $\psi_a{}^b$ reduces by separating the trace. For a general mixed tensor, judicious use of both operations gives the complete reduction into irreducible representations of SU_m. The theorem that is needed is that a mixed tensor is irreducible if and only if; (1) the symmetry of the lower indices is that of a single Young tableau, (2) the symmetry of the upper indices is that of a single Young tableau, and (3) contraction with respect to one upper and one lower index gives zero. The tensor $\psi_{bc}{}^a$ is easily reduced into the following four parts; the two m-dimensional representations $\psi_b=\psi_{ba}{}^a$ and $\psi_b'=\psi_{ab}{}^a$, the traceless symmetric part

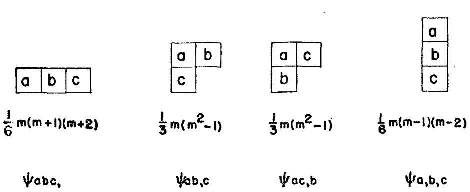

FIG. 20. The Young tableaux related to the reduction of the third-rank tensor in m dimensions.

[59] For tensors of higher rank, the ambiguity is much greater. T. Yamanouchi has prescribed a general procedure which always leads to orthogonal wave functions in Proc. Phys. Math. Soc. Japan, **18**, 623 (1936); **19**, 436 (1937).

having the $\frac{1}{2}m^2(m+1)-m$ components

$$\psi_{bc,}{}^a-\frac{1}{m+1}(\delta_b{}^a\psi_{dc,}{}^d+\delta_c{}^a\psi_{db,}{}^d), \qquad \text{(V.7)}$$

and the traceless skew part

$$\psi_{b,c}{}^a-\frac{1}{m-1}(\delta_b{}^a\psi_{d,c}{}^d-\delta_c{}^a\psi_{d,b}{}^d) \qquad \text{(V.8)}$$

which has $\frac{1}{2}m^2(m-1)-m^2$ components.

B. Group SU_3

We have seen how tensors of rank 2 or 3 reduce under SU_m. A significant simplification occurs in the case $m=3$, because the Levi-Civita tensors ϵ_{abc} and ϵ^{abc}, which equal $+1$ (-1) if abc is an even (odd) permutation of 123 and zero otherwise, have only three indices.

The relation between the above reduction of second-rank tensors and the labeling of representations introduced earlier is (more information in Table IV):

$$\begin{array}{ccc} \psi_a & \psi^a & \psi_{a,b} \\ D^{(3)}(1,0) & D^{(3)}(0,1) & D^{(3)}(0,1) \\ \text{(a)} & \text{(b)} & \text{(c)} \\ \psi^{a,b} & \psi_{ab,} & \psi^{ab,} \\ D^{(3)}(1,0) & D^{(6)}(2,0) & D^{(6)}(0,2) \\ \text{(d)} & \text{(e)} & \text{(f)} \end{array} \qquad \text{(V.9)}$$

as we now prove. The first relation, identifying ψ_a as the basis for $D^{(3)}(1,0)$, is essentially a definition. Then (9b) follows from the fact that ψ^a is contragredient to ψ_a, and $D^{(3)}(0,1)$ is contragredient to $D^{(3)}(1,0)$. Next consider (9c), according to which $\psi_{a,b}$ is equivalent to ψ^a. This equivalence is exhibited and proved by the relation $\psi^a=\epsilon^{abc}\psi_{b,c}$, which expresses the three components of ψ^a in terms of the three linearly-independent components of $\psi_{b,c}$. In general, the operation of converting two lower indices on a tensor into one upper index by means of ϵ^{abc}, is nonsingular if and only if the tensor is skew in the two lower indices. This follows from the relation

$$\epsilon_{abc}\epsilon^{ade}=\delta_b{}^d\delta_c{}^e-\delta_b{}^e\delta_c{}^d. \qquad \text{(V.10)}$$

Finally, relation (9e) follows from the fact that $\psi_{ab,}$ is (the highest dimensional) part of ψ_{ab}.

In terms of outer products of representations, (9) shows that[60]

$$D^{(3)}(1,0)\otimes D^{(3)}(1,0)=D^{(6)}(2,0)\oplus D^{(3)}(0,1),$$
$$\psi_a\otimes\psi_b\sim\psi_{ab,}\oplus\psi_{a,b}. \qquad \text{(V.11)}$$

A second relation follows from

$$[D^{(3)}(1,0)]^*=D^{(3)}(0,1),$$
$$\psi_a{}^*\sim\psi^a. \qquad \text{(V.12)}$$

[60] The symbol $\sim$ reads "transforms like."

TABLE IV. Representations of SU_3. All mixed tensors are supposed to be traceless, e.g., $\psi_{ac}{}^{a}=0$. The missing representation "64" is $D^{64}(3,3)$ with the basis $\psi_{def}{}^{abc}$ and the isotopic content $0, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, \frac{5}{2}, \frac{5}{2}, 3$. The dimension of $D(\lambda_1,\lambda_2)$ is $\frac{1}{2}(\lambda_1+1)\times(\lambda_2+1)(\lambda_1+\lambda_2+2)$. The regular representation is $D^8(1,1)$.

Complete designation	Abbr. design	Highest weight	Fig. no.	Isotopic content	Basic	$\otimes D^3(1,0)$	$\otimes D^6(2,0)$	$\otimes D^8(1,1)$	$\otimes D^{10}(3,0)$
$D^1(0,0)$	1	$(0,0)$		0	ψ	3	6	8	10
$D^3(1,0)$	3	$\frac{1}{6}(\sqrt{3},1)$	2(a)	$0,\frac{1}{2}$	ψ_a	6+3*	10+8	15+6*+3	15′+15
$D^3(0,1)$	3*	$\frac{1}{6}(\sqrt{3},-1)$	2(b)	$0,\frac{1}{2}$	ψ^a	8+1	15+3	15*+6+3*	24+6
$D^6(2,0)$	6	$\frac{1}{3}(\sqrt{3},1)$	2(c)	$0,\frac{1}{2},1$	ψ_{ab}	10+8	15′+15+6*	24+15*+6+3*	24+21+15*
$D^6(0,2)$	6*	$\frac{1}{3}(\sqrt{3},-1)$	2(d)	$0,\frac{1}{2},1$	ψ^{ab}	15*+3*	27+8+1	24*+15+6*+3	42+15+3
$D^8(1,1)$	8	$\frac{1}{3}(\sqrt{3},0)$	2(e)	$0,\frac{1}{2},\frac{1}{2},1$	$\psi_a{}^b$, χ_A	15+6*+3	24+15*+6+3*	27+10+10*+8+8+1	35+27+10+8
$D^{10}(3,0)$	10	$\frac{1}{2}(\sqrt{3},1)$	22	$0,\frac{1}{2},1,\frac{3}{2}$	ψ_{abc}	15′+15	24+21+15*	35+27+10+8	35+28+27+10
$D^{10}(0,3)$	10*	$\frac{1}{2}(\sqrt{3},-1)$		$0,\frac{1}{2},1,\frac{3}{2}$	ψ^{abc}	24*+6*	42*+15*+3*	35*+27+10*+8	64+27+8+1
$D^{15}(2,1)$	15	$\frac{1}{2}(\sqrt{3},+\frac{1}{3})$		$0,\frac{1}{2},\frac{1}{2},1,1,\frac{3}{2}$	$\psi_{bc}{}^a$				
$D^{15}(1,2)$	15*	$-\frac{1}{3})$							
$D^{15}(4,0)$	15′	$\frac{2}{3}(\sqrt{3},+1)$		$0,\frac{1}{2},1,\frac{3}{2},2$	ψ_{abcd}				
$D^{15}(0,4)$	15′*	$-1)$							
$D^{21}(5,0)$	21	$\frac{5}{6}(\sqrt{3},+1)$		$0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2}$	ψ_{abcde}				
$D^{21}(0,5)$	21*	$-1)$							
$D^{24}(3,1)$	24	$\frac{2}{3}(\sqrt{3},+\frac{1}{2})$		$0,\frac{1}{2},\frac{1}{2},1,1,\frac{3}{2},\frac{3}{2},2$	$\psi_{bcd}{}^a$				
$D^{24}(1,3)$	24*	$-\frac{1}{2})$							
$D^{27}(2,2)$	27	$\frac{2}{3}(\sqrt{3},0)$		$0,\frac{1}{2},\frac{1}{2},1,1,1,\frac{3}{2},\frac{3}{2},2$	$\psi_{cd}{}^{ab}$				
$D^{28}(6,0)$	28	$(\sqrt{3},+1)$		$0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3$	ψ_{abcdef}				
$D^{28}(0,6)$	28*	$-1)$							
$D^{35}(4,1)$	35	$\frac{5}{6}(\sqrt{3},+3)$		$0,\frac{1}{2},\frac{1}{2},1,1,\frac{3}{2},\frac{3}{2},2,2,\frac{5}{2}$	$\psi_{bcde}{}^a$				
$D^{35}(1,4)$	35*	$-3)$							
$D^{36}(7,0)$	36	$\frac{7}{6}(\sqrt{3},+1)$		$0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2}$	$\psi_{abcdefg}$				
$D^{36}(0,7)$	36*	$-1)$							
$D^{42}(3,2)$	42	$\frac{5}{6}(\sqrt{3},+1)$		$0,\frac{1}{2},\frac{1}{2},1,1,1,\frac{3}{2},\frac{3}{2},\frac{3}{2},2,2,\frac{5}{2}$	$\psi_{cde}{}^{ab}$				
$D^{42}(2,3)$	42*	$-1)$							
$D^{45}(8,0)$	45	$\frac{4}{3}(\sqrt{3},+1)$		$0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2},4$	$\psi_{abcdefgh}$				
$D^{45}(0,8)$	45*	$-1)$							
$D^{48}(5,1)$	48	$(\sqrt{3},+\frac{2}{3})$		$0,\frac{1}{2},\frac{1}{2},1,1,\frac{3}{2},\frac{3}{2},2,2,\frac{5}{2},\frac{5}{2},3$	$\psi_{bcdef}{}^a$				
$D^{48}(1,5)$	48*	$-\frac{2}{3})$							

The reduction of $\psi_a{}^b$ was discussed in detail. For $m=3$,

$$D^{(3)}(1,0)\otimes D^{(3)}(0,1)=D^{(8)}(1,1)\oplus D^{(1)}(0,0),\qquad \psi_a\otimes\psi^b\sim(\psi_a{}^b-\tfrac{1}{3}\delta_a{}^b\psi_c{}^c)\oplus\delta_a{}^b\psi_c{}^c. \tag{V.13}$$

The analogs of (11) and (13) for third-rank tensors are

$$D^{(3)}(1,0)\otimes D^{(3)}(1,0)\otimes D^{(3)}(1,0) =D^{(10)}(3,0)\oplus D^{(8)}(1,1)\oplus D^{(8)}(1,1)\oplus D^{(1)}(0,0), \qquad \psi_a\otimes\psi_b\otimes\psi_c\sim\psi_{abc},\oplus\psi_{ab,c}\oplus\psi_{ac,b}\oplus\psi_{a,b,c}, \tag{V.14}$$

and

$$D^{(3)}(1,0)\otimes D^{(3)}(1,0)\otimes D^{(3)}(0,1) =D^{(15)}(2,1)\oplus D^{(6)}(0,2)\oplus D^{(3)}(1,0)\oplus D^{(3)}(1,0) \qquad \psi_a\otimes\psi_b\otimes\psi^c\sim\psi_{ab,}{}^c\oplus\psi_{a,b}{}^c\oplus\psi_{ac}{}^c\oplus\psi_{cb}{}^c. \tag{V.15}$$

The equivalence of $\psi_{ab,c}$ with $D^{(8)}(1,1)$ is exhibited by $\psi_a{}^d=\epsilon^{bcd}\psi_{ab,c}$ (obviously $\psi_a{}^d$ is traceless). In (15), by $\psi_{ab,}{}^c$ and $\psi_{a,b}{}^c$, we mean the traceless parts (7) and (8). The equivalence of the latter to $D^{(6)}(0,2)$ is displayed by $\epsilon^{bce}[\psi_{b,c}{}^a-\frac{1}{2}(\delta_b{}^a\psi_{d,c}{}^d-\delta_c{}^a\psi_{d,b}{}^d)]=\psi^{ea,}$. We might also argue as follows. Since the traceless part of $\psi_{b,c}{}^a$ is irreducible, and raising of the lower indices by means of ϵ^{bce} is a similarity transformation, the result must be one of the irreducible parts of ψ^{ea}. Since the dimension is $\frac{1}{2}m^2(m-1)-m=6$ the irreducible part in question must be the six-dimensional symmetric part $\psi^{ea,}$.

It is clearly possible to convert, in the manner just illustrated by several examples, any tensor of mixed symmetry into tensors of lower rank, symmetric in all upstairs indices and symmetric in all downstairs indices. For the latter, the reduction is completed by separating out the traceless part. Hence, a complete set of irreducible representations is given by the set of traceless symmetrized tensors $\psi_{ab\cdots,}{}^{cd\cdots}$. If λ_1 is the number of lower indices, and λ_2 is the number of upper indices, the irreducible representations may be labeled $D(\lambda_1,\lambda_2)$. Since this is the highest[61] representation contained in the product of λ_1 factors of $D(1,0)$ and λ_2 factors of $D(0,1)$, the present labeling agrees exactly with that of Sec. III.

Alternatively, all indices may be lowered, converting each upper index into two lower ones. Starting with a symmetrized traceless mixed tensor with λ_1 lower and λ_2 upper indices, this process must give an irreducible representation, i.e., a tensor with the symmetry of a particular Young tableau. It is easily verified that the table in question has two rows, with $\lambda_1+\lambda_2$ boxes in the first row and λ_2 boxes in the second row. The reason why no tableaux with three rows are obtained is that adding a column with three rows means multiplying with the representation $\psi_{a,b,c}$, which is an invariant.

The dimension of $\psi_{ab\cdots,}$, symmetric in λ_1 indices, is $\frac{1}{2}(\lambda_1+1)(\lambda_1+2)$. Hence $\psi_{a\cdots,}{}^{b\cdots}$, symmetric in λ_1 lower and λ_2 upper indices, has $\frac{1}{4}(\lambda_1+1)(\lambda_1+2)(\lambda_2+1)\times(\lambda_2+2)$ components. The tensor obtained by contracting one upper and one lower index has $\frac{1}{4}\lambda_1(\lambda_1+1)\lambda_2\times(\lambda_2+1)$ components. Hence the traceless part has $\frac{1}{2}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)$ components, and this is therefore the dimensionality of $D(\lambda_1,\lambda_2)$. The same result was obtained in Sec. IV by the geometric method,[54] which is more suited to that kind of calculation.

The reduction of the product of any two representations is easily calculated by the above methods. The results of Table IV have been obtained by this method as well as independently by the geometric method. In Table IV may also be found the "wave functions" for any representation of SU_3 with dimension less than 50. The projection operators, which effect the symmetrization and subtracts out the trace, are easily written down as in (3) and (4), and allows us to obtain the transformation matrices explicitly. One example may be sufficient to illustrate this. The transformation of the basis (3), obtained from (2) and (4), is given by the representation

$$L_A \rightarrow P_e{}^f{}_c{}^b(L_{Ab}{}^d\delta_a{}^c - L_{Aa}{}^c\delta_b{}^d). \qquad \text{(V.16)}$$

C. Group $C_2(B_2)$

This is the group of 4×4 matrices that leaves a nondegenerate skew form h^{ab} invariant.[62] This is evidently a subgroup of SU_4, and the reduction of product representations is merely a refinement of that carried out for SU_m, with $m=4$. The fact that the form-invariant h^{ab} exists, and may be used as a raising and lowering operator if we define h_{ab} by[63]

$$h_{ab}h^{bc} = \delta_a{}^c,$$

means that the two representations ψ_a and ψ^a are equivalent. The equivalence is exhibited and proved by noting that $h^{ab}\psi_b$ transforms like ψ^a. Both ψ_a and ψ^a are (different and equivalent) bases for the representation denoted $D^{(4)}(1,0)$ in a previous section. Clearly a tensor of arbitrary mixed rank can be converted into a tensor with all the indices downstairs. The reduction problem then consists of two steps: First reduce according to SU_4 (that is, split the tensor into its various possible symmetry classes, or Young tables), then separate out the "traces" formed with h^{ab}. Remembering that h^{ab} is skew, so that taking the trace on a pair of symmetrized indices gives zero, we easily find the results of Table V. [The method of the last section is even easier, and for higher representations, it is the only practical one.] As in the case of SU_3, the low dimensionality (4 in this case) allows a simplification. Thus the completely skew tensor $\psi_{a,b,c}$ is equivalent to $\psi^d = \epsilon^{abcd}\psi_{a,b,c}$, where ϵ^{abcd} is the Levi-Civita symbol.

Let $L_{Aa}{}^b$ be the infinitesimal generators of the fundamental representation $D^{(4)}(1,0)$ of C_2. The form invariance of h^{ab} means that

$$h^{ab} \rightarrow h^{ab} - i\epsilon^A(L_{Ac}{}^a h^{cb} + L_{Ac}{}^b h^{ac}) = h^{ab}.$$

Writing $h^{ac}L_{Ac}{}^b \equiv L_A{}^{ab}$, we get

$$L_A{}^{ab} = L_A{}^{ba}.$$

Hence the infinitesimal generators, with the lower index raised, are symmetric. Hence the number of linearly independent $L_A{}^{ab}$ is 10 which is the order of C_2. In order to obtain a complete set of 16 independent matrices we introduce 5 linearly-independent skew matrices $\sigma_i{}^{ab}$, $i=1, 2, 3, 4, 5$ and choose them so that

$$\sigma_i{}^{ab}h_{ab} = 0.$$

We are now able to understand the reduction of ψ_{ab} and the higher tensors in greater detail. We have already noted that $\psi_{a,b}$ contains the invariant $h^{ab}\psi_{a,b}$. The five-dimensional representation, which is the traceless part of the skew part, can now conveniently be written

$$\varphi_i \equiv \sigma_i{}^{ab}\psi_{ab}, \quad i=1, \cdots, 5. \qquad \text{(V.17)}$$

The proof of this statement follows. The six skew components of $\psi_{a,b}$ form a basis for a representation,

[61] That is, the one with the highest weight.

[62] Any skew metric may be transformed into the form

$$h^{ab} = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{bmatrix}.$$

This is the choice we have made in Eq. (III.36).

[63] If h^{ab} is as in reference 62, then

$$h_{ab} = \begin{bmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}.$$

TABLE V. Representations of $C_2[B_2]$. The bases satisfy the "subsidiary conditions" $h^{ab}\psi_{a,b}=h^{ac}\psi_{ab,c}=0$, $\sigma^i{}_a{}^b\varphi_{bi}=\sigma^i{}_a{}^b\varphi_{ij,}{}^a=g^{ij}\varphi_{ij,}=g^{ij}\varphi_{ijk,}=g^{ij}\varphi_{ij,k}=g^{ij}\varphi_{ij,}{}^a=0$.

Complete designation	Abbr. design.	Highest weight	Fig. no.	Isotopic content	Basis	$\otimes D^4(1,0)$	$\otimes D^5(0,1)$	$\otimes D^{10}(2,0)$	$\otimes D^{14}(0,2)$
$D^1(0,0)$	1	$(0,0)$		0	ψ	4	5	10	14
$D^4(1,0)$	4	$\frac{1}{2\sqrt{3}}(1,0)$	4(a)	$0,0,\frac{1}{2}[\frac{1}{2},\frac{1}{2}]$	ψ_a	10+5+1	16+4	20+16+4	40+16
$D^5(0,1)$	5	$\frac{1}{2\sqrt{3}}(1,1)$	4(b)	$0,\frac{1}{2},\frac{1}{2}[0,0,1]$	$\psi_{a,b},\varphi_i$	16+4	14+10+1	35′+10+5	35′+30+5
$D^{10}(2,0)$	10	$\frac{1}{2\sqrt{3}}(2,0)$	4(c)	$0,0,0,\frac{1}{2},\frac{1}{2},1$ $[0,1,1,1]$	$\psi_{ab,},\varphi_{i,j},\chi_A$	20+16+4	35′+10+5	35+35′+14+10+5+1	
$D^{14}(0,2)$	14	$\frac{1}{2\sqrt{3}}(2,2)$		$0,\frac{1}{2},\frac{1}{2},1,1,1$ $[0,0,0,1,1,2]$	$\varphi_{ij,}$	40+16	35+30+5		
$D^{16}(1,1)$	16	$\frac{1}{2\sqrt{3}}(2,1)$		$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1,1$ $[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2}]$	$\varphi_{ai};\psi_{ab,c}$	35+14+10+5	40+20+16+4		
$D^{20}(3,0)$	20	$\frac{1}{2\sqrt{3}}(3,0)$		$0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},$ $1,1,\frac{3}{2}[\frac{1}{2},\frac{1}{2},\frac{3}{2},$ $\frac{3}{2},\frac{3}{2},\frac{3}{2}]$	$\psi_{abc,}$				
$D^{30}(0,3)$	30	$\frac{1}{2\sqrt{3}}(3,3)$		$0,\frac{1}{2},\frac{1}{2},1,1,1,\frac{3}{2},\frac{3}{2},$ $\frac{3}{2},\frac{3}{2}[0,0,0,0,$ $1,1,1,2,2,3]$	$\varphi_{ijk,}$				
$D^{35}(4,0)$	35	$\frac{1}{2\sqrt{3}}(4,4)$		$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},$ $1,1,1,\frac{3}{2},\frac{3}{2},2[0,1,$ $1,1,2,2,2,2,2]$	$\psi_{abcd,}$				
$D^{35}(2,1)$	35′	$\frac{1}{2\sqrt{3}}(3,1)$		$0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},$ $\frac{1}{2},\frac{1}{2},1,1,1,1,\frac{3}{2},\frac{3}{2}$ $[0,0,1,1,1,1,1,$ $1,2,2,2]$	$\varphi_{ij,k}$				
$D^{40}(1,2)$	40	$\frac{1}{2\sqrt{3}}(3,2)$		$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1,1,$ $1,1,1,1,\frac{3}{2},\frac{3}{2},\frac{3}{2}$ $[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},$ $\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{5}{2}]$	$\varphi_{ij,}{}^a$				

that is, they transform among themselves. Therefore the six linearly-independent combinations φ_i, $i=1, 2, \cdots, 5$ and $h^{ab}\psi_{ab}$ transform among themselves. But $h^{ab}\psi_{ab}$ is invariant and orthogonal to φ_i. Therefore, the φ_i transform among themselves; that is, the φ_i form the basis for a five-dimensional representation. We do not prove here that this representation is irreducible, but it can easily be seen to be the representation $D^{(5)}(0,1)$ discussed in preceding sections. The way that the φ_i transform among themselves is given by

$$\varphi_i=\sigma_i{}^{ab}\psi_{ab}\longrightarrow\sigma_i{}^{ab}(\delta_a{}^c+i\epsilon^A L_{Aa}{}^c)(\delta_b{}^d+i\epsilon^B L_{Bb}{}^d)\psi_{cd} \equiv(\delta_i{}^j+i\epsilon^A L_{Ai}{}^j)\sigma_j{}^{ab}\psi_{ab}. \quad \text{(V.18)}$$

As is the usual treatment of the Pauli σ matrices, we interpret $\sigma_i{}^{ab}$ as a constant tensor. This nomenclature is justified by noting that the above definition of $L_{Ai}{}^j$ gives

$$\sigma_i{}^{ab}\longrightarrow(\delta_c{}^a-i\epsilon^A L_{Ac}{}^a)(\delta_d{}^b-i\epsilon^B L_{Bd}{}^b) \times(\delta_i{}^j+i\epsilon^c L_{ci}{}^j)\sigma_j{}^{cd}=\sigma_i{}^{ab}.$$

That is, $\sigma_i{}^{ab}$ is form invariant.

This representation $D^{(5)}(0,1)$ may appropriately be called the *vector representation*. The form

$$g_{ij}=\sigma_i{}^{ab}\sigma_{jba} \quad \text{(V.19)}$$

is clearly symmetric, nonsingular and constant (form invariant). It may be used to raise and lower vector indices. For example, we have from (18):

$$L_{Ai}{}^j=\sigma_i{}^{ab}(L_{Aa}{}^c\delta_b{}^d+\delta_a{}^c L_{Ab}{}^d)\sigma^j{}_{cd}.$$

Clearly $L_A{}^{ij}$ are the 10 skew 5×5 matrices, and their skewness is equivalent to the form invariance of g_{ij}. Hence, this representation of C_2 is B_2, the orthogonal group in five dimensions. (The isomorphism between C_2 and B_2 was pointed out by Cartan.)

To complete this discussion of the reduction of ψ_{ab}, we note that the ten-dimensional representation $D(2,0)$, which is the symmetric part of ψ_{ab}, is just the regular representation:

$$\chi_A=L_A{}^{ab}\psi_{ab}.$$

The $\sigma_i{}^{ab}$ play the same role here as in ordinary spinor analysis, providing the link between the "spinor" indices a, b, $\cdots$, and the "vector indices," i, j, $\cdots$. For tensors of higher rank, it may be convenient to employ a mixed notation. Thus, the basis ψ_{ab}, is equivalent to $\varphi_{i,j}$ and the basis for $D(1,1)$ is either

$$\psi_{ab,c} \quad \text{with} \quad h^{bc}\psi_{ab,c}=0,$$

or

$$\varphi_{i,j}{}^{a} \quad \text{with} \quad \sigma^{i}{}_{ab}\varphi_{i,j}{}^{a}=0.$$

D. Group G_2

Because G_2 is a subgroup[64] of O_7 it is helpful first to discuss the latter group. The spinor-representation of O_7 is eight-dimensional. Let g^{ab} be a symmetric nonsingular matrix and g_{ab} its inverse:

$$g^{ab}g_{bc}=\delta_{c}{}^{a}.$$

We use this tensor to raise or lower indices in spinor space (i.e., latin indices a, b, c). Let $\gamma_{ia}{}^{b}$ be a set of seven 8×8 matrices that satisfy

$$\begin{aligned} g^{ac}\gamma_{ic}{}^{b}&=\gamma_{i}{}^{ab}=-\gamma_{i}{}^{ba}, \\ (\gamma_i\gamma_j+\gamma_j\gamma_i)_a{}^b&=-2g_{ij}\delta_a{}^b. \end{aligned} \tag{V.20}$$

The numbers g_{ij} are defined by these equations, once a fixed g^{ab} and a fixed set of γ matrices have been chosen. The latter should be taken to be linearly independent; then g_{ij} is a nonsingular quadratic form with an inverse defined by

$$g^{ij}g_{jk}=\delta_{k}{}^{i}.$$

These matrices are used to raise and lower latin indices i, j, k, henceforth called vector indices. Hence g^{ab} and g^{ij} are the metric tensors in spinor space and in vector space, respectively.

The γ matrices may be used to construct a complete set of matrices (64 independent ones) in spinor space. The 21 independent matrices

$$G_{ija}{}^{b}\equiv\tfrac{1}{4}[\gamma_i,\gamma_j]_a{}^b$$

are of particular interest. First we note that $G_{ij}{}^{ab}$ are skew and independent of $\gamma_i{}^{ab}$:

$$G_{ij}{}^{ab}=-G_{ij}{}^{ba},\quad G_{ij}{}^{ab}\gamma_{kab}=0.$$

Therefore, the 21 $G_{ij}{}^{ab}$ and the 7 $\gamma_i{}^{ab}$ form a complete set of 28 linearly independent skew matrices. Next, defining the 35 matrices

$$G_{ijka}{}^{b}\equiv\tfrac{1}{6}[\gamma_i\gamma_j\gamma_k]_a{}^b,$$

where $[\gamma_i\gamma_j\gamma_k]$ is the antisymmetrized product of γ_i, γ_j, γ_k, we note that $G_{ijk}{}^{ab}$ are symmetric and independent of g^{ab}:

$$G_{ijk}{}^{ab}=G_{ijk}{}^{ba},\quad G_{ijk}{}^{ab}g_{ab}=0.$$

Hence, the $G_{ijk}{}^{ab}$ and g^{ab} form a complete set of 36 independent symmetric matrices.

As a simple consequence of the "anticommutativity" of the γ matrices, we find

$$G_{ij},G_{kl}=g_{il}G_{kj}-g_{ik}G_{lj}+g_{lj}G_{ik}-g_{kj}G_{il},$$

which are the correct commutation relations for the group of rotations in seven dimensions.[65] Therefore the $(G_{ij})_a{}^b$ are the infinitesimal generators of that group.

The group G_2 may now be obtained as a subgroup of O_7 in the following way.[66] Let η^a be a constant spinor. Of course O_7 does not admit such an object, and η^a is not constant under O_7. However, there exists a subgroup of O_7 that does leave η^a invariant, and it turns out that this subgroup is G_2. Hence η^a is constant with respect to G_2 only. The subspace of spinor space which is normal to η^a is seven dimensional, and there exists a very convenient way of labeling the seven components of ψ_a which span this subspace. For, let $\eta_i{}^a$ be defined by

$$\eta_i{}^a\equiv-\gamma_{ib}{}^a\eta^b,\quad \eta_{ia}=\gamma_{iab}\eta^b=\eta_i{}^b g_{ba}.$$

Then clearly $\eta_i{}^a\eta_a=0$. Hence the seven components $\eta_i{}^a\psi_a$ of ψ_a are the basis for a representation of that subgroup of O_7 that leaves η_a invariant.

In order to find the matrices of this group, let us define

$$\Gamma_{ijk}=\gamma_{iab}\eta_j{}^a\eta_k{}^b.$$

It is not at first obvious how this can be solved for γ_{iab}, since the $\eta_i{}^a$ are singular. It is clear, however, that γ_{iab} is of the form

$$\gamma_{iab}=A\Gamma_{ijk}\eta_a{}^j\eta_b{}^k+B(\eta_{ia}\eta_b-\eta_{ib}\eta_a).$$

From the commutation relations we find, with the normalization

$$\eta^a\eta_a=1,\quad \text{that}\quad \eta_i{}^a\eta_{ja}=g_{ij},$$

and this immediately gives $A=B=1$, or

$$\gamma_{iab}=\Gamma_{ijk}\eta^{aj}\eta^{bk}+\eta_{ia}\eta_b-\eta_a\eta_{ib}.$$

Using this formula in the anticommutation relations, we find that the necessary and sufficient conditions for the Γ_{ijk} to yield $\gamma_{ia}{}^b$ with the defining properties (a) that the Γ_{ijk} be totally skew, and (b) that

$$\Gamma_i{}^{ln}\Gamma_{jn}{}^k+\Gamma_j{}^{ln}\Gamma_{in}{}^k=\delta_i{}^l\delta_j{}^k+\delta_j{}^l\delta_i{}^k-2g_{ij}g^{lk}. \tag{V.21}$$

Some simple consequences are

$$\Gamma_{il}{}^l=0,\quad \Gamma_{ikl}\Gamma^{jkl}=6\delta_i{}^j.$$

Although not obvious, it is nevertheless true that the above properties suffice to reduce the product of any three Γ matrices to a sum of terms that are linear in

[64] The first physical application of this fact appears to have been made by G. Racah, Phys. Rev. 76, 1352 (1949).

[65] The differential operators $(l/i)(\chi_i\partial_j-\chi_j\partial_i)$ are a realization of these commutation relations.

[66] The following development was suggested by the calculations of reference 19.

Γ's. The formula is

$$\Gamma_{mn}{}^{s}\Gamma_{s}{}^{li}\Gamma_{l}{}^{jk} = -\delta_m{}^i\Gamma_n{}^{jk} - \delta_m{}^j\Gamma_n{}^{ki} - \delta_m{}^k\Gamma_n{}^{ij} - g^{ik}\Gamma_{mn}{}^j + \delta_n{}^i\Gamma_m{}^{jk} + \delta_n{}^j\Gamma_m{}^{ki} + \delta_n{}^k\Gamma_m{}^{ij} + g^{ij}\Gamma_{mn}{}^k.$$

The generators of G_2 are those linear combinations

$$S^{ij}G_{ija}{}^b,$$

that satisfy

$$S^{ij}G_{ija}{}^b\eta_b = 0.$$

This is easily reduced to

$$S^{ij}\Gamma_{ijk} = 0.$$

The general solution (taking $S^{ij} = -S^{ji}$) is a linear combination of the following matrices

$$P(14)_{(mn)}{}^{ij} \equiv \tfrac{1}{2}(\delta_m{}^i\delta_n{}^j - \delta_m{}^j\delta_n{}^i) - \tfrac{1}{6}\Gamma_{kmn}\Gamma^{kij},$$

of which 14 are linearly independent. Hence G_2 has 14 parameters, and the generators are

$$L_{(mn)a}{}^b \equiv P(14)_{(mn)}{}^{ij}G_{ija}{}^b.$$

In vector space this becomes

$$L_{(mn)k}{}^l \equiv L_{(mn)a}{}^b\eta_b{}^l\eta_k{}^a.$$

Hence, the generators of G_2 is the set of skew matrices orthogonal to Γ_{ijk}:

$$L_{(mn)}{}^{ij}\Gamma_{ijk} = 0.$$

The reduction problem for G_2 can now be solved. Let $D^{(7)}(1,0)$ stand for the representation ψ_i. The second-rank tensor is first split into the symmetric and the skew part. The symmetric part $\psi_{ij,}$ contains the invariant $g^{ij}\psi_{ij}$ and the remaining 27 components form an irreducible representation that we label $D^{(27)}(2,0)$. The 21 skew components $\psi_{i,j}$ break up into the $D(1,0)$ $\Gamma^{ijk}\psi_{ij} \equiv \varphi^k$ and a remainder with 14 components. The latter make up the regular representation $L_{(mn)}{}^{ij}\psi_{ij}$, which we call $D^{(14)}(0,1)$. (These labels agree with those of the previous sections.)

The reduction of the third-rank tensor ψ_{ijk} is nontrivial. First write down all the operators that exist for reducing the number of indices, that is, all the form-invariant matrices with 3 to 5 indices:

$$A = \Gamma^{ijk}, \quad B_m = g^{ij}\delta_m{}^k,$$
$$A_m = \tfrac{1}{3}(\Gamma^{ijl}\Gamma_{lm}{}^k + \Gamma^{jkl}\Gamma_{lm}{}^i + \Gamma^{kil}\Gamma_{lm}{}^j),$$
$$A_{mn,} = \tfrac{1}{2}(\Gamma_m{}^{ij}\delta_n{}^k + \Gamma_n{}^{ij}\delta_m{}^k) - 1/7 g_{mn}\Gamma^{ijk},$$
$$A_{m,n} = \tfrac{1}{2}(\Gamma_m{}^{ij}\delta_n{}^k - \Gamma_n{}^{ij}\delta_m{}^k) - \tfrac{1}{6}\Gamma_{mn}{}^l\Gamma_l{}^{0k}\Gamma_0{}^{ij}.$$

Here A_m has been made completely skew in i, j, k, since any symmetric part would reduce to B_m by use of (21). We have subtracted the trace A from $A_{mn,}$ and the part $A_m \sim \Gamma_m{}^{nl}A_{nl}$ from $A_{m,n}$. In this way we are assured that $A_{mn,}{}^{ijk}\psi_{ijk}$ and $A_{m,n}{}^{ijk}\psi_{ijk}$ are irreducible. These operators are then applied to each of the symmetry classes of Fig. 20. We start with the skew part $\psi_{i,j,k}$ which has 35 components. Applying B_m gives zero trivially; $A_{m,n}$ also yields zero after some calculation. Thus we are left with

$$\psi_{i,j,k} = A_{mn,}{}^{ijk}\psi_{i,j,k} \oplus A_m{}^{ijk}\psi_{ijk} \oplus A^{ijk}\psi_{ijk}, \quad \therefore \quad D^{(27)}(2,0) \oplus D^{(7)}(1,0) \oplus D^{(1)}(0,0). \tag{V.22}$$

To each of the two parts $\psi_{ij,k}$ and $\psi_{ik,j}$ with mixed symmetry (having 112 components each), B_m, $A_{mn,}$ and $A_{m,n}$ gives $D^{(7)}$, $D^{(27)}$, and $D^{(14)}$, respectively. The remaining components, of which there are $112-7-27-17=64$, are irreducible. Since this is the highest representation[61] in $D(1,0)\otimes D(0,1)$ it must be $D(1,1)$. Thus

$$\psi_{ij,k} = B_m{}^{ijk}\psi_{ij,k} \oplus A_{mn,}{}^{ikj}\psi_{ij,k} \oplus A_{m,n}{}^{ikj}\psi_{ij,k} \oplus \text{“remainder”}, \quad \therefore \quad D^{(7)}(1,0) \oplus D^{(27)}(2,0) \oplus D^{(14)}(0,1) \oplus D^{(64)}(1,1). \tag{V.23}$$

The "remainder" is the tensor $\psi_{ij,k}$, that satisfies the "subsidiary conditions"

$$B_m{}^{ijk}\psi_{ij,k} = A_{mn,}{}^{ikj}\psi_{ij,k} = A_{m,n}{}^{ikj}\psi_{ij,k} = 0.$$

The result for $\psi_{ik,j}$ is, of course, exactly similar. The completely symmetric part $\psi_{ijk,}$, can be contracted with B_m only. Therefore, the remaining 77 components are irreducible. Since this is the highest representation in $[D(1,0)]^3$, it must be $D^{(77)}(3,0)$. Thus

$$\psi_{ijk,} = B_m{}^{ijk}\psi_{ijk,} \oplus \text{“remainder”} \quad \therefore \quad D^{(7)}(1,0) \oplus D^{(77)}(3,0).$$

The complete results for ψ_{ijk} are listed in Table VI.

It may be helpful, to support our claim that our method supplies explicit matrices of the transformations for each representation, to write them down for some of the representations that are listed in Table VI.

We found the transformation matrices $L_{(mn)i}{}^j$ for the representation $D^{(7)}(1,0)$. These are given explicitly in terms of the Γ^{ijk}. For the present purpose the tensor character of the label (mn) on $L_{(mn)i}{}^j$ is irrelevant, and it is perhaps less confusing to replace it by a single index A running from 1 to 14. Then the transformations of the tensors before symmetrization are

$$\psi_{ij} \to \psi_{ij} + \epsilon^A(L_{Ai}{}^k\delta_j{}^l + L_{Aj}{}^l\delta_i{}^k)\psi_{kl},$$
$$\psi_{ijk} \to \psi_{ijk} + \epsilon^A(L_{Ai}{}^l\delta_j{}^m\delta_k{}^n + L_{Aj}{}^m\delta_i{}^l\delta_k{}^n + L_{Ak}{}^n\delta_i{}^l\delta_j{}^m)\psi_{lmn}.$$

The representation $D^{(27)}(2,0)$ is obtained from ψ_{ij} by the projection operator that symmetrizes and makes traceless, namely,

$$P(27)_{ij}{}^{kl} = \tfrac{1}{2}(\delta_i{}^k\delta_j{}^l + \delta_i{}^l\delta_j{}^k) - (1/7)g_{ij}g^{kl}. \tag{V.24}$$

Hence the matrices for $D^{(27)}(2,0)$ are simply,

$$L_A \to P(27)_{ij}{}^{kl}(L_{Ak}{}^m\delta_l{}^n + L_{Al}{}^n\delta_k{}^m).$$

We already found the projection operator $P(14)_{ij}{}^{kl}$ similarly:

$$P(1)_{ij}{}^{kl} = (1/7)g_{ij}g^{kl}, \quad P(7)_{ij}{}^{kl} = \tfrac{1}{6}\Gamma_{ijm}\Gamma^{klm}. \tag{V.25}$$

TABLE VI. Representations of G_2. The bases satisfy $\Gamma^{ijk}\psi_{i,j}=g^{ij}\psi_{ij,}=g^{ij}\psi_{ij,k}=\Gamma^{jkl}\psi_{ij,k}=g^{ij}\psi_{ijk,}=g^{ij}\psi_{ij,k,l}=\Gamma^{jkm}\psi_{ij,kl}=0$.

Complete designation	Abbr. design.	Highest weight	Fig. no.	Isotopic content	Basis	$\otimes D^7(1,0)$	$\otimes D^{14}(0,1)$	$\otimes D^{27}(2,0)$
$D^7(1,0)$	7		3(a)	$\frac{1}{2},\frac{1}{2},1$	ψ_i	27+14+7+1	64+27+7	77+64+27+14+7
$D^{14}(0,1)$	14		3(b)	$0,0,0,1,\frac{3}{2}$	$\psi_{i,j},\chi_A$	64+27+7	77+77′+27+14+1	
$D^{27}(2,0)$	27			$0,\frac{1}{2},\frac{1}{2},1,1,1,\frac{3}{2},\frac{3}{2},2$	$\psi_{ij,}$	77+64+27+14+7		
$D^{64}(1,1)$	64			$\begin{cases}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\\ 2,2,2,2,\frac{5}{2},\frac{5}{2}\end{cases}$	$\psi_{ij,k}$			
$D^{77}(3,0)$	77			$\begin{cases}0,0,0,\frac{1}{2},\frac{1}{2},1,1,1,1,\frac{3}{2},\\ \frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},2,2,2,\frac{5}{2},\frac{5}{2}\end{cases}$	$\psi_{ijk,}$			
$D^{77}(0,2)$	77′			$\begin{cases}0,0,0,0,0,0,1,1,1,\frac{3}{2},\frac{3}{2},\\ \frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},2,\frac{5}{2},\frac{5}{2},3,3,3\end{cases}$	$\psi_{ij,k,l}$			

It is easily verified that these operators are indeed projection operators, and that they add up to $\delta_i{}^k\delta_j{}^l$. The reader is now able to write down the matrices for any one of the representations in Table VI.

VI. APPLICATIONS

A. Introductory Remarks

We have presented the tools that are needed to construct a physical theory of the strongly interacting particles. The important question now at hand is to select the group which is appropriate, make the proper identification between the basis of the representations and the physical states (particles, resonances, etc.) and determine the experimental predictions that ensue.

The myriad of schemes that may be constructed is limited only by the imagination of the inventor. Consequently, we have been unable to find a course between the Scylla of being too abstract and the Charybdis of leaving out many logical possibilities. It would seem then, that our purpose is best served by giving illustrative *examples*, from which the general pattern of procedure may be gleaned. It cannot be emphasized too strongly, however, that *all of the remainder of this section is offered for illustration only*.

We identify the components of the basis of an irreducible representation of a group with a set of physical states (particles, resonances, scattering states, etc.). Since we have only considered groups of "charge"-space transformations, which commute with every space-time transformation, each of the physical states within one irreducible multiplet must have the same space-time properties, i.e., spin, relative parity, baryon number, etc. In particular, the square of the total four momentum, or the mass, of each of these states must be the same. But a cursory examination of the mass spectrum of the known baryons and mesons tends to preclude the possibility that these particles could form, in any meaningful way, the components of the basis of an irreducible representation of any group larger than the four-parameter semi-simple group of isotopic spin and hypercharge conservation. Namely, the only apparent approximate multiplet structure seems to be that associated with isotopic spin. This is the basis for one possible point of view.

Another point of view can be based on an analogy, which, if fruitful, would allow us to consider meaningfully the baryons as members of a "supermultiplet." That the analogy may be misleading, in whole or in part, is understood; we cite it only as one possible flight of fancy.

Let us consider again the concept of isotopic spin. As commonly conceived, the particle interactions break into an isotopic invariant part and a much weaker symmetry-breaking part, most likely due to the electromagnetic field. In the absence of the latter, the neutron and proton are identified as the degenerate members of an isotopic spin doublet, a spin 1/2 basis for an irreducible representation of the isotopic spin rotation group. Being members of such a multiplet, all their space-time properties are the same, including their masses. The main effect of the symmetry-breaking interaction, in this case, is to remove the degeneracy in the masses and interactions of the proton and neutron. This is so, because electromagnetic interactions conserves parity and baryon number, and is Lorentz invariant, and hence does not change the other space-time properties of the states, such as spin, relative parity, and baryon number. That these statements are independent of the symmetry-breaking coupling strength is obvious. It is conceivable that the interaction could change the number of states by giving rise to some new resonant states. In the isotopic spin case, it would seem that these new states either do not exist or are far removed in mass from the perturbed doublet. Thus, it has been found, since the proton and neutron are related states even in the presence of the symmetry-breaking interaction, that to some extent it is still meaningful to consider them as members of a doublet.

One may now carry the analogy over to the case of some higher symmetry. That is, one might speculate that the particle interactions split into a symmetry-preserving and a symmetry-destroying part, the latter involving some "fields" not contained in the former.

One such possibility is the global symmetry scheme[67] where the bosons involved in the symmetry-preserving part are only pions, while in the violating part, they are only K mesons.[68] Alternatively, the symmetric interaction could involve both K and π mesons, while the symmetry-breaking interaction could be due to some other, as yet unknown, field. It is this latter interaction, then, which would be responsible for the observed mass splittings and other mischief. If the analogy is not misleading, we may, even in the presence of this interaction, still speak in a meaningful way of the "supermultiplets," the components of the basis of an irreducible representation. Namely, we are able to say that all the components must have the same spin, relative parity, and baryon number. In addition, as before, we assume that if any new states arise as a result of the symmetry-breaking interaction, they are separated by a large mass from the presently known resonances.[69] This guarantees that the number of states will be preserved in the presence of the interaction. Thus, by completely describing one component of a multiplet, we have completely specified all the other components, except as to their masses and widths. If, in addition, we adopt the viewpoint of Lee and Yang[17] with regard to relating the widths of the various components, we are left with the masses as the main quantum number perturbed by the symmetry-breaking interaction.

According to the methods developed in the preceding sections, the components of the basis of an irreducible representation are identified by the weights $\mathbf{m}$ to which they belong.

For representations of simple group of rank two, it turns out that at most two linear combinations of H_1 and H_2 can be interpreted as I_3. This is because the spectrum of I_3 must be symmetric about $I_3=0$. For both SU_3 and G_2 only one of the two possibilities, namely $I_3\propto H_1$, are considered; but for $C_2(B_2)$ either choice gives rise to reasonable physical models (see below). In Tables IV–VI the *isotopic content* of many representations are recorded. This is the number of isotopic spin singlets, doublets, triplets, etc., contained in a representation. It may most easily be read off the weight diagram. The number of times that the total isospin I' is contained is equal to the number of states with $I_3=I'$ minus the number of states with $I_3=I'+1$.

B. Analysis of Invariant Amplitudes

In most cases, an attempt at a physical theory will begin with associating a particular representation D_B of some group G with a set of particles called the "fundamental baryons." Which baryons are fundamental depends on the model; it is not even necessary that the "fundamental baryons" be stable baryons. To fix the ideas, however, we assume that this is the case, and refer unambigously to "the baryons." Let ψ_a be the wave function for these baryons, that is, the basis for D_B, and let $\bar{\psi}^a$ be the wave function for the "antibaryons." Clearly $\bar{\psi}^a$ is contragredient to ψ_a and is the basis for $D_{B^*}=(D_B)^*$. The first experiment that may be discussed, even before introducing the bosons, is baryon-baryon or baryon-antibaryon scattering. The relevant four point function is of the form (suppressing the space-time variables)

$$\mathcal{A}=A_{ac}{}^{bd}\langle T(\psi^{\dagger a}\psi_b\psi^{\dagger c}\psi_d)\rangle, \qquad \text{(VI.1)}$$

where the coefficients $A_{ac}{}^{bd}$ must be chosen so as to make $\mathcal{A}$ invariant.

Consider first the four-point function for a specified set of baryons, one of the terms in the sum $\mathcal{A}$:

$$\langle T(\psi^{\dagger a}\psi_b\psi^{\dagger c}\psi_d)\rangle.$$

A knowledge of the effect of the E_α on a basis, ψ_a, of an irreducible representation can be used to find relations among the four-point functions for different processes. First of all, we insert the operator form of the commutation relations $[\hat{E}_\alpha,\hat{E}_{-\alpha}]=r^i(\alpha)\hat{H}_i$ into the four-point function above to obtain

$$\langle T(\psi^{\dagger a}\psi_b[E_\alpha,E_{-\alpha}]\psi^{\dagger c}\psi_d)\rangle=r^i(\alpha)\langle T(\psi^{\dagger a}\psi_b\hat{H}_i\psi^{\dagger c}\psi_d)\rangle.$$

The $\hat{E}_\alpha$ and the $\hat{H}_i$ are linear operators acting on the product basis as in (III.14). By remembering that when they act on the vacuum they give zero and that $E_\alpha{}^\dagger=E_{-\alpha}$, we find

$$\begin{aligned}&\langle T\{[(E_{-\alpha}\psi)^{\dagger a}\psi_b-\psi^{\dagger a}(E_\alpha\psi)_b]\\ &\qquad\times[-(E_\alpha\psi)^{\dagger c}\psi_d+\psi^{\dagger c}(E_{-\alpha}\psi)_d]\}\rangle\\ &\qquad-\langle T\{[(E_\alpha\psi)^{\dagger a}\psi_b-\psi^{\dagger a}(E_{-\alpha}\psi)_b]\\ &\qquad\times[-(E_{-\alpha}\psi)^{\dagger c}\psi_d+\psi^{\dagger c}(E_\alpha\psi)_d]\}\rangle\\ &\quad=r^i(\alpha)[-m_i(c)+m_i(d)]\langle T(\psi^{\dagger a}\psi_b\psi^{\dagger c}\psi_d)\rangle, \qquad \text{(VI.2)}\end{aligned}$$

where $m_i(c)$ and $m_i(d)$ are the weights of ψ^c and ψ_d, respectively. Thus, by knowning the effect of the E_α on the basis of the baryon representation, we can determine an equality between the four-point functions of two different processes.

As an obvious example of the relationship just found, consider the scattering $p+\bar{n}\rightarrow p+\bar{n}$ and let the $E_{-\alpha}$ be the isotopic spin lowering operator. The above relation then becomes

$$\begin{aligned}&\langle T(\psi_p{}^\dagger\psi_n\psi_n{}^\dagger\psi_p)\rangle\\ &\qquad=\tfrac{1}{2}\langle T[(\psi_p{}^\dagger\psi_p-\psi_n{}^\dagger\psi_n)(\psi_p{}^\dagger\psi_p-\psi_n{}^\dagger\psi_n)]\rangle, \qquad \text{(VI.3)}\end{aligned}$$

which states that the $I=1$, $I_3=1$ four-point function is equal to the $I=1$, $I_3=0$ function, i.e., that the four-point function depends only upon I and not upon I_3, a well-known result. The remaining equalities can obviously be obtained by repeating the procedure of inserting these commutation relations into the newly formed four-point functions; this procedure clearly

[67] See, M. Gell-Mann, reference 2.
[68] Or vice versa, see J. Schwinger, reference 2.
[69] This assumption, or a similar one, is a *sine qua non* of any theory of higher symmetries.

terminates when we reach the $I=1$, $I_3=1$ state in this example. In the general case, when the E_α are not just restricted to the isotopic spin operators, we proceed in the same manner. Namely, with repeated use of the general relation we can generate a string of equal four-point functions. From the example above, it is clear that this string will terminate after a finite number of steps since there can only be a finite number of independent relations. This procedure determines all the four-point functions which are equal to the one we started with. Similar statements, of course, can be made for the n-point function.

If we choose our original four-point function such that $\psi^{\dagger c}\psi_d$ is a component of the basis of an irreducible representation (as in the above example), then all the related four-point functions may be completely characterized by this irreducible representation, which in turn is characterized by its highest weight. They will be independent of the other weights (in much the same way as in the above example, they were characterized by I and independent of I_3). If $\psi^{\dagger c}\psi_d$ and $\psi^{\dagger c'}\psi_{d'}$ belong to two different irreducible representations, then the four-point functions in which they appear are, of course, unrelated (just as the $I=0$ amplitude is unrelated to the three $I=1$ amplitudes). We now show that it is possible to gain a much deeper insight into the structure and interrelations of four-point functions after we have found the most general matrix $A_{ac}{}^{bd}$ that makes (1) invariant.

To find all possible solutions of this problem is the same as determining the one-dimensional representations contained in $D_{B^*}\otimes D_{B^*}\otimes D_B\otimes D_B$. It is both convenient and traditional to do this in two steps. For example, for baryon-antibaryon scattering, one first decomposes $D_{B^*}\otimes D_B$:

$$D_{B^*}\otimes D_B=\sum\oplus\ \nu_\sigma D_\sigma, \qquad \text{(VI.4)}$$

where the sum is over inequivalent irreducible representations, and the ν_σ are integers. The invariants in (1) are then the invariants in

$$\sum\oplus\ \nu_\sigma{}^2(D_\sigma\otimes D_\sigma{}^*), \qquad \text{(VI.5)}$$

where each $D_\sigma\otimes D_\sigma{}^*$ contains exactly one invariant. Techniques for finding the ν_σ in (2) were amply discussed in Secs. IV and V, and many examples were listed in Tables IV–VI. Although the ν_σ contain some information that is quite important in applications to follow, we need a more explicit form of the reduction for the present purpose.

Suppose that a particular N_1-dimensional, irreducible representation D_1, whose basis we label by the letters $\mu, \nu, \rho, \cdots$, is contained in the product $D_{B^*}\otimes D_B$ or $\bar{\psi}^c\psi_d$. This means that there exists linear combinations

$$(\Omega_\mu{}^{(1)})_c{}^d\bar{\psi}^c\psi_d, \quad \mu=1, 2, \cdots, N_1, \qquad \text{(VI.6)}$$

which transform among themselves according to D_1. The numbers $(\Omega_\mu{}^{(1)})_c{}^d$ may be regarded as the components of a constant (=form invariant) tensor, and will be called, after proper normalization, an *isometry*. Although the name may be new, the concept is well known, and several examples have already appeared in previous sections: (1) The Pauli $\sigma_{ia}{}^b$ matrices connect the product of two spinors to a vector ($\psi^\dagger\sigma_i\psi$), (2) The matrices 1, γ_μ, $\frac{1}{4}\sqrt{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$, $\gamma_5\gamma_\mu$, γ_5 used for writing down Lorentz invariant couplings connect the product of two four-spinors to tensors, (3) The matrices of any representation D of a Lie algebra connects the product $D\otimes D^*$ to the regular representation (as was emphasized in Sec. V), and (4) Matrices $\sigma_{ia}{}^b$ and Γ_{ijk} were introduced in Sec. V.

The normalization that qualifies these operators for the title isometry is

$$(\Omega_\mu{}^{(1)})_c{}^d(\Omega^{(1)\nu})_d{}^c=\delta_\mu{}^\nu,$$
$$(\Omega_\mu{}^{(1)})_c{}^d(\Omega^{(1)\mu})_e{}^f=P(1)_{ce}{}^{df}. \qquad \text{(VI.7)}$$

Here $(\Omega^{(1)\nu})_c{}^d\equiv g^{\nu\mu}(\Omega_\mu{}^{(1)})_c{}^d$ when $g^{\nu\mu}$ exists; in general it is the isometry of the representation $D_1{}^*$ contragredient to D_1. [It may be proved that $D_1\otimes D_1{}^*$ contains $D_1{}^*$ if it contains D_1.] The matrix $P(1)_{ce}{}^{df}$, in which c, d labels the rows and e, f labels the columns, is the projection operator associated with D_1. Several examples of (7) are well known:

(1). $\frac{1}{2}\sqrt{2}\epsilon_{abc}\frac{1}{2}\sqrt{2}\epsilon^{bcd}=\delta_a{}^d,$
$\frac{1}{2}\sqrt{2}\epsilon_{abc}\frac{1}{2}\sqrt{2}\epsilon^{ade}=\frac{1}{2}(\delta_b{}^d\delta_c{}^e-\delta_b{}^e\delta_c{}^d)=P_{bc}{}^{de}$

where $P_{bc}{}^{de}$ is the antisymmetrization operator;

(2). $\frac{1}{2}\sqrt{2}(\sigma_i)_a{}^b\frac{1}{2}\sqrt{2}(\sigma^j)_b{}^a=\delta_i{}^j,$
$\frac{1}{2}\sqrt{2}(\sigma_i)_a{}^b\frac{1}{2}\sqrt{2}(\sigma^i)_c{}^d=\delta_a{}^d\delta_c{}^b-\frac{1}{2}\delta_a{}^b\delta_c{}^d=P_a{}^b{}_c{}^d$

where the σ_i are the Pauli matrices and $P_a{}^b{}_c{}^d$ is projection operator that separates out the trace;

(3). In Sec. V, Eq. (V.19) we found that

$$(\tfrac{1}{6})^{\frac{1}{2}}\Gamma_{ijk}(\tfrac{1}{6})^{\frac{1}{2}}\Gamma^{ljk}=\delta_i{}^l,$$
$$(\tfrac{1}{6})^{\frac{1}{2}}\Gamma_{ijk}(\tfrac{1}{6})^{\frac{1}{2}}\Gamma^{ilm}=P(7)_{jk}{}^{lm}$$

where $P(7)_{jk}{}^{lm}$ is the projection operator (V.25) that projects out the 7-dimensional representation of G_2 from $D^{(7)}\otimes D^{(7)}$.

As in (4), let σ label the inequivalent irreducible representations, and write $\Omega^{(\sigma,\kappa)}$, $\kappa=1, \cdots, \nu_\sigma$ for the ν_σ isometries associated with each of the ν_σ equivalent representations D_σ. Then the equivalence between $(\Omega_\mu{}^{(\sigma,1)})_c{}^d$ and $(\Omega_\mu{}^{(\sigma,2)})_c{}^d$ means that there exists a nonsingular matrix $(P^{(\sigma,1,2)})_c{}^d{}_e{}^f$ such that

$$(P^{(\sigma,1,2)})_c{}^d{}_e{}^f(\Omega_\nu{}^{(\sigma,2)})_e{}^f=(\Omega_\mu{}^{(\sigma,1)})_c{}^d.$$

Using (7) we get

$$(P^{(\sigma,1,2)})_c{}^d{}_e{}^f=(\Omega_\mu{}^{(\sigma,1)})_c{}^d(\Omega^{(\sigma,2)\mu})_e{}^f. \qquad \text{(VI.8)}$$

From this we see that $P^{(\sigma,1,2)}$ is an isometry. In particular, if the indices are the same as in $P^{(\sigma,1,1)}$ we get back the projection operators. Thus we label the projection operators associated with one of the D_σ by $P^{(\sigma,1,1)}$. Then the properties of the isometries, and in particular the projection operators, may be summarized by

$$(\Omega_\mu{}^{(\sigma,\kappa)})_c{}^d(\Omega^{(\sigma',\kappa')\nu})_d{}^c=\delta_{\sigma\sigma'}\delta_{\kappa\kappa'}\delta_\mu{}^\nu, \qquad \text{(VI.9)}$$

$$(\Omega_\mu{}^{(\sigma,\kappa)})_c{}^d(\Omega^{(\sigma,\kappa')\mu})_e^f=(P^{(\sigma,\kappa,\kappa')})_c{}^d{}_e{}^f, \qquad \text{(VI.10)}$$

$$(P^{(\sigma,\kappa,\kappa')})_c{}^d{}_e{}^f(P^{(\sigma',\kappa'',\kappa''')})_{fg}{}^{eh}=\delta_{\sigma\sigma'}\delta_{\kappa'\kappa''}(P^{(\sigma,\kappa,\kappa''')})_c{}^d{}_g{}^h. \qquad \text{(VI.11)}$$

A direct result of Schur's lemma[70] is that the most general form of $A_{ac}{}^{bd}$ that makes (1) invariant is given by

$$A_{ac}{}^{bd}=\sum_{\sigma,\kappa,\kappa'}F^{\sigma,\kappa,\kappa'}(P^{(\sigma,\kappa,\kappa')})_a{}^b{}_c{}^d,$$
$$\mathcal{A}=\sum_{\sigma,\kappa,\kappa'}F^{\sigma,\kappa,\kappa'}(\bar\psi^a\psi_b)(P^{(\sigma,\kappa,\kappa')})_a{}^b{}_c{}^d(\bar\psi^c\psi_d). \qquad \text{(VI.12)}$$

where $F^{\sigma,\kappa,\kappa'}$ are arbitrary and include all references to space-time coordinates or transformation properties. Using (10):

$$\mathcal{A}=\sum_{\sigma,\kappa',\kappa,\mu}F^{\sigma,\kappa,\kappa'}(\bar\psi\Omega_\mu{}^{(\sigma,\kappa)}\psi)(\bar\psi\Omega^{(\sigma,\kappa')\mu}\psi). \qquad \text{(VI.13)}$$

This is the explicit realization of (5). The number of terms with the same σ is $\nu_\sigma{}^2$.

The number of terms in (13) is $\sum\nu_\sigma{}^2$ and depends, of course, on the choice of the group G and the representation D_B. The procedure that we have outlined is a direct generalization of the well known treatment of isotopic spin. In that case, the index κ is superfluous, since the ν_σ in (4) are always zero or one. Thus, the summation over σ, κ, κ' reduces to a sum over I, the total isotopic spin. If all the ψ_a have isotopic spin 1/2, (13) reduces to

$$\mathcal{A}=F^1(\bar\psi\tfrac{1}{2}\sqrt{2}\sigma_i\psi)(\bar\psi\tfrac{1}{2}\sqrt{2}\sigma^i\psi)+F^0[\bar\psi\tfrac{1}{2}\sqrt{2}(\delta_i{}^j-\tfrac{1}{3}\sigma_i\sigma^j)\psi]\times[\bar\psi\tfrac{1}{2}\sqrt{2}(\delta_j{}^i-\tfrac{1}{3}\sigma_j\sigma^i)\psi]. \qquad \text{(VI.14)}$$

The process of applying the generator $\hat E_\alpha$ and $\hat H_i$ to a basis $\bar\psi^c\psi_d$ of an irreducible representation in (12) or (13) clearly can lead to any other basis of the same irreducible representation, but cannot lead out of that representation. Thus the method that was outlined following (1) relates four-point functions within each term of the σ, κ', κ sum in (13). In fact, that method is simply a way of calculating the isometries. For example, the relation (3) expresses the fact that the right-hand side and the left-hand side occur with equal weight F^1 in (14).

C. Resonances and Mesons

Scattering in one or more states of σ, κ, κ' may exhibit resonances. The resonant states are then multiplets transforming according to D_σ. In order to determine the possible resonance multiplets and their transformation properties, it is sufficient to know the Clebsch-Gordan Series (4). For simple groups of rank two, and low-dimensional representations, this information is contained in Tables IV–VI.

Nothing in our development thus far distinguishes between stable and unstable resonant states. Therefore, it is impossible to make any definite predictions about the number of mesons in a given model. However, in the limit in which the invariance is exact, the various resonance states within one multiplet will have the same mass, width, etc. This might lead one to expect that if one member of a multiplet is stable, so are all the other members of that multiplet. If this is true, the number of mesons will be related to the dimensionalities of the representations occurring in the decomposition (2).[22]

If one likes to write an unrenormalized Lagrangian involving Yukawa couplings, it is necessary to find the trilinear invariants involving ψ_a, $\bar\psi^b$, and the meson field. If stable mesons are indeed possible intermediary states in $B-\bar B$ scattering, then these same trilinear forms are needed to write the vertex function. This remains true even if the mesons are regarded as bound states of the $B-\bar B$ system. From a mathematical point of view, these trilinear couplings are already known. All that is needed is to reinterpret the quantities $(\bar\psi^a\psi_b)$ appearing in (12) as the components of the meson field. For practical purposes, however, it is convenient to label the mesons by a single index, as for example φ^μ, such that each component corresponds to one physical meson. Let D_M be the representation for which φ^μ is the basis. In order for a trilinear invariant to exist, D_M must be equivalent to one of the terms in (4). That is, an isometry $(\Omega_\mu{}^{(M)})_a{}^b$ must exist such that φ^μ transforms contragrediently to $(\bar\psi\Omega_\mu{}^{(M)}\psi)$. Then the trilinear invariants are of the desired form, namely

$$(\bar\psi\Omega_\mu{}^{(M)}\psi)\varphi^\mu. \qquad \text{(VI.15)}$$

In the manner of Eq. (1), consider the three-point function for a specific set of two baryons and a meson, one component of the general invariant three-point function (15),

$$\langle T(\psi^{\dagger a}\psi_b\varphi^\mu)\rangle.$$

Again, insert the operator commutation relation to obtain

$$\langle T(\psi^{\dagger a}\psi_b[\hat E_\alpha,\hat E_{-\alpha}]\varphi^\mu)\rangle=r^i(\alpha)\langle T(\psi^{\dagger a}\psi_b\hat H_i\varphi^\mu)\rangle.$$

Proceeding as previously, we find

$$\langle T\{[(E_{-\alpha}\psi)^{\dagger a}\psi_b-\psi^{\dagger a}(E_\alpha\psi_b)][-(E_{-\alpha}\varphi^\mu)]\rangle-\langle T\{[(E_\alpha\psi)^{\dagger a}\psi_b-\psi^{\dagger a}(E_{-\alpha}\psi_b)][-(E_\alpha\varphi^\mu)]\rangle=-r^i(\alpha)m_i(\mu)\langle T(\psi^{\dagger a}\psi_b\varphi^\mu)\rangle.$$

A trivial example is afforded by the pion-nucleon

[70] I. Schur, Sitzber. preuss. Akad. Wiss., Physik.-math. Kl. 1905, p. 406.

vertex. Consider

$$\langle T(\psi_p{}^\dagger\psi_n\varphi_{\pi^+})\rangle$$

and $\hat{E}_\alpha$ as the isotopic spin raising operator. The well-known result follows

$$\langle T(\psi_p{}^\dagger\psi_n\varphi_{\pi^+})\rangle=\frac{1}{\sqrt{2}}\langle T[(\psi_p{}^\dagger\psi_p-\psi_n{}^\dagger\psi_n)\varphi_{\pi^0}]\rangle.$$

Since we have demonstrated the method both in the case of the three-point and four-point functions, it should be obvious that this method can be generalized to n-point functions involving both mesons and baryons.

Let us now proceed to the specific cases of SU_3, B_2, C_2, and G_2. In the examples contrived for SU_3 and G_2, we follow a line of reasoning according to which the eight known baryons are more fundamental physical states than are the baryon resonances, (or baryon excited states). Specifically, no resonance state or unobserved baryon is to appear in the same multiplet with any of the eight observed baryons. Such a distinction is quite unfounded, even though it seems to be the most fashionable procedure at present. We remove this restriction in our examples of theories built on B_2 and C_2.

D. Model Built on SU_3

If we assume that the eight baryons can form the bases for one or more representations, then the dimensionality of these representations must add up to eight. An inspection of Table IV for SU_3 shows that there is only one possibility with the correct isotopic content; the eight-dimensional representation $D^{(8)}(1,1)$. This implies that all the baryons must have the same space-time properties. If we assume that there are only the seven known mesons, it is impossible to assign the correct isotopic content under SU_3. In addition, if we require that the meson-baryon vertex function does not vanish, which incidentally corresponds to the existence of pole terms in dispersion relations, the dimensionality of the meson representations must be either 1, 8, 10, or 27. This follows from the fact that the Kronecker product of two eight-dimensional representations of baryons contains representations of only those dimensions (Table IV). One possible way out of the dilemma is to postulate the existence of an eighth meson which has not been experimentally detected as yet.[71] This is the approach of Gell-Mann,[18] which

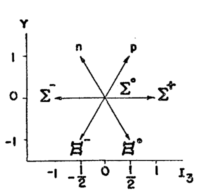

FIG. 21. Weight diagram for D^8 (1,1) of SU_3 with bases associated with baryons. For meson bases, the substitution, $(p,n,\Xi^0,\Xi^-,\Sigma^+,\Sigma^0,\Sigma^-,\Lambda)\rightarrow(K^+,K^0,-\bar{K}^0,\bar{K}^+,\pi^+,\pi^0,\pi^-,\pi^{00})$, should be made.

[71] See, for example, M. Gettner and W. Selove, Phys. Rev. **120**, 593 (1960); J. Poirer and M. Pripstein, Phys. Rev. **122**, 1917 (1961).

we follow here. It then follows that the meson representation is also eight-dimensional, and that all 8 mesons have the same space-time properties. [For example, Σ and Λ have the same parities, and the parity of $(K\Sigma)$ is the same as that of (πN).]

Because $D^{(8)}$ occurs twice in the product $D^{(8)}\otimes D^{(8)}$, there are two of the isometries in (15). To find them is to make a very slight extension of the tensor analysis developed for SU_3 in Sec. V. The baryon wave function is written ψ_A, in keeping with our convention to use capital Latin indices for the regular representation. The antibaryons are labeled $\bar{\psi}^B$. Clearly the structure constants $C_{BD}{}^A$ supply one of the two isometries. The normalization is fixed by the usual definition

$$g_{DE}=C_{BD}{}^AC_{AE}{}^B. \qquad \text{(VI.16)}$$

From the commutation relations (II.3) we find

$$C_{BD}{}^A=\text{trace}[L_BL_DL^A-L_DL_BL^A]. \qquad \text{(VI.17)}$$

We can define the second isometry by

$$C'_{BD}{}^A=\text{trace}[L_BL_DL^A+L_DL_BL^A]. \qquad \text{(VI.18)}$$

Although these relations are true regardless of which representation L_A occurs on the right, the most convenient choice is $D^{(3)}(1,0)$, given in (III.20). Both g_{DE} and $C_{BD}{}^A$ were calculated in Sec. III F.

The most general three-point function is

$$\langle F^1(\bar{\psi}^BC_{BD}{}^A\psi_A)\varphi^D+F^2(\bar{\psi}^BC'_{BD}{}^A\psi_A)\varphi^D\rangle, \qquad \text{(VI.19)}$$

where $\varphi_E=g_{DE}\varphi^D$ is the meson field.

In Fig. 21 we have furnished the weight diagram for $D^{(8)}(1,1)$ with the appropriate baryon symbols. We associate I_3 with $\sqrt{3}m_1$, and Y with $2m_2$, and summarize the relations between the four different labels that we have used:

$$\begin{aligned}
|A\rangle&: |1\rangle,\ |2\rangle,\ |3\rangle,\ |4\rangle,\ |5\rangle,\ |6\rangle,\ |7\rangle,\ |8\rangle;\\
|\alpha\rangle&: |-1\rangle,\ |-2\rangle,\ |-3\rangle,\ -|+1\rangle,\\
|i\rangle&: \quad -|+2\rangle,\ |+3\rangle,\ -|1\rangle,\ -|2\rangle;\\
\text{Baryons}&: -|\Sigma^+\rangle,\ |p\rangle,\ |n\rangle,\ |\Sigma^-\rangle,\ |\Xi^-\rangle,\ |\Xi^0\rangle,\ |\Sigma^0\rangle,\ |\Lambda\rangle;\\
\text{Mesons}&: -|\pi^+\rangle,\ |K^+\rangle,\ |K^0\rangle,\ |\pi^-\rangle,\ |K^-\rangle,\ -|\bar{K}^0\rangle,\ |\pi^0\rangle,\ |\pi^{00}\rangle.
\end{aligned} \qquad \text{(VI.20)}$$

The action of the operators $\hat{H}_i$ and $\hat{E}_\alpha$ was given both in (III.26, 27) and in (III.30, 31). Using the dictionary (20), this is easily translated. The result for baryons is given in Table VII.

We are now in a position to make the predictions of the theory. Consider the scattering of a meson M and a baryon B, $M+B\rightarrow M'+B'$. The pertinent four-point function (suppressing the space-time variables) is

$$\langle T(\psi_{B'}{}^\dagger\psi_{M'}{}^\dagger\psi_B\psi_M)\rangle.$$

The combination $\psi_B\psi_M$ is the Kronecker product of

TABLE VII. Action of E_α on baryons for $D^{(8)}(1,1)$ in SU_3. Table for bosons is obtained by substitution $(p,n,\Xi^0,\Xi^-,\Sigma^+,\Sigma^0,\Sigma^-,\Lambda) \rightarrow (K^+,K^0,-\bar{K}^0,\bar{K}^+,\pi^+,\pi^0,\pi^-,\pi^{00})$.

$E_\alpha \backslash \psi$	p	n	Ξ^0	Ξ^-	Σ^+	Σ^0	Σ^-	Λ
$6^{\frac{1}{2}}E_1$		p		Ξ^0		$-\sqrt{2}\Sigma^+$	$\sqrt{2}\Sigma^0$	
$6^{\frac{1}{2}}E_{-1}$	n		Ξ^-		$-\sqrt{2}\Sigma^0$	$\sqrt{2}\Sigma^-$		
$2\sqrt{3}E_2$			$-\sqrt{2}\Sigma^+$	$\Sigma^0 +\sqrt{3}\Lambda$		p	$\sqrt{2}n$	$\sqrt{3}p$
$2\sqrt{3}E_{-2}$	$\Sigma^0 +\sqrt{3}\Lambda$	$\sqrt{2}\Sigma^-$			$-\sqrt{2}\Xi^0$	Ξ^-		$\sqrt{3}\Xi^-$
$2\sqrt{3}E_3$			$\Sigma^0 -\sqrt{3}\Lambda$	$\sqrt{2}\Sigma^-$	$+\sqrt{2}p$	$-n$		$\sqrt{3}n$
$2\sqrt{3}E_{-3}$	$+\sqrt{2}\Sigma^+$	$-\Sigma^0 +\sqrt{3}\Lambda$				Ξ^0	$\sqrt{2}\Xi^-$	$-\sqrt{3}\Xi^0$
$\sqrt{3}H_1$	$\frac{1}{2}p$	$-\frac{1}{2}n$	$\frac{1}{2}\Xi^0$	$-\frac{1}{2}\Xi^-$	Σ^+		$-\Sigma^-$	
$2H_2$	p	n	$-\Xi^0$	$-\Xi^-$				

two eight-dimensional representations, one for the meson and one for the baryon. This reduces, according to Table IV, as follows:

$$8\otimes 8 = 1\oplus 8\oplus 8\oplus 10\oplus 10^*\oplus 27.$$

There are eight $(=1^2+2^2+1^2+1^2+1^2)$ different four-point invariants, or equivalently 8 independent amplitudes.[72]

So far we have considered the representations for the known baryons and particles. It is conceivable that the other representations of this group might also be realized, not for stable particles, but perhaps for what we might call unstable particles, i.e., the resonances, excited isobaric states, or whatever. In particular, we concentrate on the well-known (3,3) resonance in pion-nucleon scattering and its possible analogs in other baryon-meson scattering processes. We have emphasized before the limitations of such a procedure (see the general discussion of this Section). We note again that the product representation of one baryon and one meson decomposes into irreducible representations of dimensions 1, 8, 8, 10, 10*, and 27. The weight (m_1,m_2) of the compound state π^+p which is a member of the (3,3) resonance, is $\frac{1}{2}(\sqrt{3},1)$. This is the highest weight of the 10-dimensional representation and one of the weights in the 27-dimensional representation. We assume that the (3,3) isobar states are members of the 10-dimensional multiplet.

The weight diagram for the 10-dimensional representation is shown in Fig. 22. Besides the $T=3/2$, $Y=1$, multiplet, which we identify as the (3,3) isobar states (N^*), we have a $T=1$, $Y=0$ triplet, a $T=1/2$, $Y=-1$ doublet, and a $T=0$, $Y=-2$ singlet. The triplet $T=1$, $Y=0$ has the same charge quantum numbers as the excited states Y^* of the $\Lambda\pi$ system.[73] It is very attractive to consider the Y^* as an analog of the N^*. In order for them to belong to the same supermultiplet, these two multiplets must have the same space-time quantum numbers. We therefore assume Y^* to have spin 3/2 and negative orbital parity.

In order to compare these two states and make certain predictions which can be verified by experiments, we must assume certain features of the symmetry-breaking forces. We may assume, after Lee and Yang,[17] that the symmetry-breaking forces are short-range in character and that long-range phenomena are relatively insensitive to them, even though they must be strong enough to account for the mass splittings. Then the same cause that splits the baryon masses is responsible for the difference of the energy levels of N^* and Y^*, while the resonance widths should be predictable from the symmetry. This is because the width of a resonance is proportional to the overlap of the resonance-state wave function and the initial- (or final-) state wave function at the "channel entrance," as we know from nuclear physics,[13,74] so that the relative widths are essentially independent of short range effects.

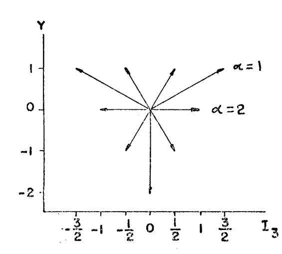

FIG. 22. Weight diagram for $D^{(10)}(3,0)$ for SU_3. The weight labeled as $\alpha=1$ corresponds to the isobar state $(N^*)^{++}$; $\alpha=2$ to $(Y^*)^+$.

[72] It is possible to distinguish between the two equivalent 8-dimensional representations by adding a discrete element (reflection) to the group. Invariance under this operation would prohibit transitions between the two octets and reduce the number of invariant amplitudes to six. See M. Gell-Mann, reference 18.

[73] M. Alston, L. Alvarez, P. Eberhard, M. Good, W. Graziano, H. Ticho, and S. Wojcichi, Phys. Rev. Letters **5**, 520 (1960).

[74] R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1953), Chap. 10.

The state corresponding to the highest weight $(I_3,Y)=(\frac{3}{2},1)$ of the 10-dimensional representation is that linear combination

$$\alpha|p\pi^+\rangle+\beta|\Sigma^+K^+\rangle,$$

which is annihilated by E_1, E_2, E_3, and E_{-3}, as discussed in Sec. III E. Therefore, the normalized state $|\{10\},1\rangle$ can be chosen to be

$$|\{10\},1\rangle=\tfrac{1}{2}\sqrt{2}[|p\pi^+\rangle-|\Sigma^+K^+\rangle]. \quad \text{(VI.21)}$$

The state of interest, consisting of $\Lambda\pi^+$, $\Sigma^+\pi^0$, $\cdots$, can be obtained by operating with E_{-2} on $|\{10\},1\rangle$, i.e.,

$$|\{10\},2\rangle=\tfrac{1}{6}\sqrt{3}[\sqrt{3}|\Lambda\pi^+\rangle+|\Sigma^0\pi^+\rangle+\sqrt{2}|p\bar{K}^0\rangle +\sqrt{2}|\Xi^0K^+\rangle-|\Sigma^+\pi^0\rangle-\sqrt{3}|\Sigma^+\pi^{00}\rangle]. \quad \text{(VI.22)}$$

The partial width for the transition from a $|\{10\},\alpha\rangle$ multiplet to a $|BM\rangle$ state is given by

$$\Gamma_{BM}=\left(q^3\frac{E_B}{E_B+E_M}\right)|C|^2|\langle\{10\},\alpha|BM\rangle|^2, \quad \text{(VI.23)}$$

where q=c.m. momentum, E_B=baryon energy in c.m., E_M=meson energy in c.m., and the first bracket on the right is the kinematical factor arising from the phase-space and the centrifugal barrier for the p-wave; and C is a quantity independent of the "magnetic quantum number" α. The generalized Clebsch-Gordan coefficient $\langle\{10\},\alpha|BM\rangle$ can be read off directly from the foregoing expressions. We list in Table VIII the relative partial widths predicted by the SU_3 symmetry. It is interesting to note that, if the mass of π^{00} is near that of the π, the decay process of the Y^* can produce π^{00} copiously, since the branching ratio of $Y^*\rightarrow\Lambda+\pi^+$ to $Y^*\rightarrow\Sigma^++\pi^{00}$ is approximately unity. This does not seem to agree with experimental findings, however.

E. Model Built on C_2

For this example we discard the assumption that different components of the same basis of an irreducible representation must be identified with the baryons only or with the resonances only. This allows us a good deal more flexibility in making an identification of the particles with a basis. For the purpose of illustration, we have chosen one of the many schemes which might be devised.

Upon inspection of the lower dimensional weight diagrams for C_2 in Fig. 4, we see that the N, Λ, and Ξ can be identified as the basis of the five-dimensional "vector" representation, $D^{(5)}(0,1)$, where $I_3=\sqrt{3}m_1$, and $Y=2\sqrt{3}m_2$. By making the association from Sec. III, $(1,2,3,4,5)\rightarrow(p,n,\Lambda,\Xi^0,\Xi^-)$, (compare Figs. 8 and 23), we can use Eqs. (III.37, 38) to construct Table IX. With this assignment, the Σ must be components of a basis for another irreducible representation and as such could have space-time quantum numbers which differ from those assigned to the $N-\Lambda-\Xi$ set. Specifically, this scheme would admit an odd relative $\Sigma\Lambda$ parity and an odd $K\Sigma$ parity relative to πN.[75] From the weight diagrams (Fig. 4), we see that the lowest dimensional representation in which the isotopic spin and hypercharge content allows both the π and K mesons is $D^{(10)}$ (Fig. 24). This is a representation which admits the existence of an invariant effective Yukawa interaction, because, as may be seen from Table V, $D^{(5)}\otimes D^{(5)}=D^{(1)}\oplus D^{(10)}\oplus D^{(14)}$.

In addition to the K and π, however, $D^{(10)}$ requires three isotopic spin zero mesons, D, with $Y=2$, 0, -2 (charge, $Q=1$, 0, -1, respectively). Of the three, the existence of the charged ones, $D^\pm$, and the consequences thereof, have been discussed by Yamanouchi.[76] The prediction of the existence of a neutral particle, D^0, is a novel feature of the C_2 scheme. Although it is a neutral isotopic scalar meson, it differs from the π^{00} of SU_3 in that it is a member of a hypercharge rotation triplet. If the mass of the D^0 were near that of the $D^\pm$, about 730 Mev as suggested by Yamanouchi,[76] it would have sufficient energy to decay into either 2π or 3π via the strong interactions. The 2π mode, however, can be shown to be forbidden because of parity while the 3π mode is allowed only insofar as the symmetry of C_2 is broken (for such a low-energy process, one would expect the symmetry to be violated

Table VIII. Comparison of relative partial widths of the N^* and Y^* resonances. In computing the relative partial widths $q^3E_B/(E_B+E_M)$ in Eq. (VI.23) are taken from reference 17.

Isobar	Resonance energy (experimental) in Mev	Disintegration products, BM	$\lvert\langle\{10\},\alpha\lvert BM\rangle\rvert^2$ $\alpha=1$	$\alpha=2$	Relative partial width
$(N^*)^{++}$	1237	$p\pi^+$	1/2		1
		$\Lambda\pi^+$		1/4	0.38
$(Y^*)^+$	1385	$\Sigma^+\pi^0$		1/12	0.03
		$\Sigma^0\pi^+$		1/12	0.03
		$\Sigma^+\pi^{00}$		1/4	?

Table IX. Action of E_α on baryons for $D^{(5)}(0,1)$ in C_2.

$E_\alpha \backslash \psi$	p	n	Λ	Ξ^0	Ξ^-
$6^{\frac{1}{2}}E_1$		p			Ξ^0
$6^{\frac{1}{2}}E_{-1}$	n			Ξ^-	
$6^{\frac{1}{2}}E_2$			p		$-\Lambda$
$6^{\frac{1}{2}}E_{-2}$	Λ		$-\Xi^-$		
$6^{\frac{1}{2}}E_3$				p	n
$6^{\frac{1}{2}}E_{-3}$	Ξ^0	Ξ^-			
$6^{\frac{1}{2}}E_4$			n	Λ	
$6^{\frac{1}{2}}E_{-4}$		Λ	Ξ^0		
$\sqrt{3}H_1$	$\frac{1}{2}p$	$-\frac{1}{2}n$		$\frac{1}{2}\Xi^0$	$-\frac{1}{2}\Xi^-$
$2\sqrt{3}H_2$	p	n		$-\Xi^0$	$-\Xi^-$

[75] S. Barshay, Phys. Rev. Letters **1**, 97 (1958). Recent experimental evidence is compared with this conjecture in Y. Nambu and J. J. Sakurai, Phys. Rev. Letters **6**, 377 (1961).
[76] T. Yamanouchi, Phys. Rev. Letters **3**, 480 (1959).

to a rather large extent). If it were energetically possible for the D^0 to decay into $K+\bar{K}$, such a mode would again be ruled out by parity conservation.

So far, we have not assigned the Σ's to an irreducible representation. The lowest dimensional representation that can contain them is easily seen to be $D^{(10)}$. This implies the existence of baryon resonances, associated in the same irreducible representation with the Σ's, which have the same space-time properties, e.g., $J=1/2$, and the following isotopic spin and hypercharge assignments: $I=1/2$, $Y=1$; $I=1/2$, $Y=-1$; and $I=0$, $Y=2$, 0, -2. The first isotopic doublet would appear as a nucleon-pion resonance, the second as a $\Xi\pi$ resonance, in the $J=1/2$ state. The hypercharge triplet would appear as a resonance in the NK, the $N\bar{K}$ and ΞK, and the $\Xi\bar{K}$ scattering states. As pointed out before, the masses of such states remain theoretically unknown.

For demonstration purposes, let us use a combination of the techniques developed in Secs. III and V to analyze the product representation $\psi_i{}^j \equiv \psi_i \psi^j$, where ψ_i is the basis of the five-dimensional representation. We choose its components as $(p,n,\Lambda,\Xi^0,\Xi^-)$. According to Sec. V, there exists a symmetric metric, g^{ij}, which relates ψ^i with ψ_i. In order to determine the form of g^{ij} we first formally form the invariant

$$\chi = g^{ij}\psi_i\psi_j.$$

By remembering that this invariant must have a weight (0,0), it must be a linear combination

$$\chi = a\Xi^- p + b\Xi^0 n + c\Lambda\Lambda + dp\Xi^- + en\Xi^0.$$

In order to determine the coefficients a, b, $\cdots$, we use the fact that $E_\alpha \chi = 0$ for any E_α. The immediate result is

$$\chi = a(\Xi^- p - \Xi^0 n + \Lambda\Lambda + p\Xi^- - n\Xi^0).$$

With a normalization such that $g^2=1$, g^{ij} may now be written as

$$g_{ij} = g^{ij} = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}, \qquad \text{(VI.24)}$$

so that

$$(\psi_i) = \begin{pmatrix} p \\ n \\ \Lambda \\ \Xi^0 \\ \Xi^- \end{pmatrix}, \quad (\psi^i) = \begin{pmatrix} \Xi^- \\ -\Xi^0 \\ \Lambda \\ -n \\ p \end{pmatrix},$$
$$(\bar{\psi}^i) = \begin{pmatrix} \bar{p} \\ \bar{n} \\ \bar{\Lambda} \\ \bar{\Xi}^0 \\ \bar{\Xi}^- \end{pmatrix}, \quad (\bar{\psi}_i) = \begin{pmatrix} \bar{\Xi}^- \\ -\bar{\Xi}^0 \\ \bar{\Lambda} \\ -\bar{n} \\ \bar{p} \end{pmatrix}. \qquad \text{(VI.25)}$$

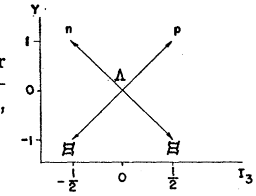

FIG. 23. Weight diagram for $D^{(5)}(0,1)$ of C_2 with bases associated with baryons (p, n, Λ, Ξ^0, and Ξ^-).

This matrix g_{ij} is the same as that introduced under the pseudonym C in Eq. (III.39). It is now possible to construct the bilinear forms $\psi_i{}^j$ for the 10- and 14-dimensional representations:

$$\psi(10)_i{}^j = \bar{\psi}^j\psi_i - \bar{\psi}_i\psi^j;$$
$$\psi(14)_i{}^j = \bar{\psi}^j\psi_i + \bar{\psi}_i\psi^j - \tfrac{2}{5}\delta_i{}^j\bar{\psi}^k\psi_k. \qquad \text{(VI.26)}$$

Specifically, for the 10-dimensional representation, in terms of $B\bar{B}$,

$$\chi_1 = -\psi_1{}^2 = -\bar{n}p - \bar{\Xi}^-\Xi^0,$$
$$\chi_2 = \frac{1}{\sqrt{2}}(\psi_1{}^1 - \psi_2{}^2) = \frac{1}{\sqrt{2}}(\bar{p}p - \bar{n}n + \bar{\Xi}^0\Xi^0 - \bar{\Xi}^-\Xi^-),$$
$$\chi_3 = \psi_2{}^1 = \bar{p}n + \bar{\Xi}^0\Xi^-, \qquad \chi_4 = \psi_1{}^4 = \bar{\Xi}^0 p + \bar{\Xi}^- n,$$
$$\chi_5 = \frac{1}{\sqrt{2}}(\psi_1{}^1 + \psi_2{}^2) = \frac{1}{\sqrt{2}}(\bar{p}p + \bar{n}n - \bar{\Xi}^0\Xi^0 - \bar{\Xi}^-\Xi^-),$$
$$\chi_6 = \psi_4{}^1 = \bar{p}\Xi^0 + \bar{n}\Xi^-, \qquad \chi_7 = \psi_3{}^1 = \bar{p}\Lambda - \bar{\Lambda}\Xi^-,$$
$$\chi_8 = -\psi_3{}^2 = -\bar{n}\Lambda - \bar{\Lambda}\Xi^0, \qquad \chi_9 = \psi_1{}^3 = \bar{\Lambda}n + \bar{\Xi}^0\Lambda,$$
$$\chi_{10} = \psi_2{}^3 = \bar{\Lambda}p - \bar{\Xi}^-\Lambda.$$

Since the 10-dimensional representation is the regular representation, these χ_A, if assigned the space-time properties of a four-vector (i.e., by inserting a γ_μ into each term, e.g., $\bar{p}n \to \bar{p}\gamma_\mu n$), form the baryon part of the current which is conserved due to the group C_2. If the spin zero mesons, K, π, etc., were considered compound baryon-antibaryon systems, these χ_A would, of course, be the complete conserved currents in the interaction representation.[77] In order to avoid being quoted as not having considered strongly-interacting

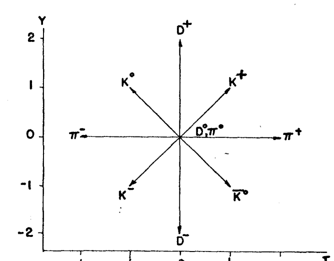

FIG. 24. Weight diagram for $D^{(10)}(2,0)$ of C_2 with bases associated with mesons.

[77] The currents can easily be written down in interaction representation. The transformation to Heisenberg representation will introduce extra terms in the current, if there are derivatives of the fields in the interaction Lagrangian.

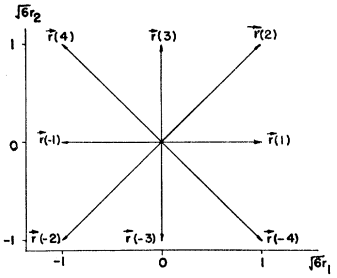

FIG. 25. Root diagram for B_2.

intermediate vector mesons,[78,79] we point out here that by the extension to space-time dependent transformations, these ten currents would be coupled to ten such mesons; this technique is trivially extended to the other groups.

On the other hand, if the χ_A are given the space-time property of a pseudoscalar, the effective Yukawa coupling between the baryons (N,Λ,Ξ) and the pseudoscalar mesons can be written down. Writing the ten mesons as a 10-component $M_A=(-\pi^+,\pi^0,\pi^-,D^+,D^0,D^-,\bar{K}^+,-\bar{K}^0,K^0,K^+)$, the coupling becomes[80]

$$I=g\chi_A M^A, \tag{VI.27}$$

where

$$M^A=(-\pi^-,\pi^0,\pi^+,D^-,D^0,D^+,K^+,\ -K^0,\bar{K}^0,\bar{K}^+),$$
$$M^A=g^{AB}M_B.$$

A simpler method than the above exists for finding the χ_A. Since they form the basis of the regular representation, they are given by $\chi_A=\bar{\psi}^a L_{Aa}{}^b\psi_b$ where the $L_{Aa}{}^b$ can be read directly from Table IX. The advantage of the method described above is that we can now also immediately write down the 14-dimensional basis.

Model Built on B_2

Another possible scheme based on the symmetry of C_2 $(=B_2)$ is obtained by rotating the coordinates of the root diagram, Fig. 1(b) of Sec. II by 45°. We recapitulate the procedure of constructing the Lie algebra, using the root diagram, Fig. 25. In this basis of the algebra

$$\begin{aligned}&[\hat{H}_1,\hat{E}_1]=6^{-\frac{1}{2}}\hat{E}_1, \quad [\hat{H}_1,\hat{E}_2]=6^{-\frac{1}{2}}\hat{E}_2, \cdots,\\ &[\hat{H}_2,\hat{E}_1]=0, \qquad\quad [\hat{H}_2,\hat{E}_2]=6^{-\frac{1}{2}}\hat{E}_2, \cdots,\end{aligned} \tag{VI.28}$$

and we choose the $N_{\alpha\beta}$ to be

$$N_{1,4}=N_{-1,3}=N_{3,-4}=N_{1,3}=N_{-1,2}=N_{2,-3}=6^{-\frac{1}{2}}. \tag{VI.29}$$

The highest weight of the representation (λ_1,λ_2) is, in this case,

$$\mathbf{M}=\lambda_2/6^{-\frac{1}{2}}(1,0)+\lambda_1/6^{-\frac{1}{2}}(\tfrac{1}{2},\tfrac{1}{2}). \tag{VI.30}$$

The dimensionality is given by

$$N=(1+\lambda_1)(1+\lambda_2)[1+\tfrac{1}{2}(\lambda_1+\lambda_2)]\times[1+\tfrac{1}{3}(\lambda_1+2\lambda_2)]. \tag{VI.31}$$

The dimensionalities of the representations $D(\lambda_1,\lambda_2)$ $=D(0,0)$, $D(1,0)$, $D(0,1)$, $D(2,0)$, $D(0.2)$, $\cdots$, are 1, 4, 5, 10, 14, $\cdots$, just as before.

We can identify the Λ particle as the basis of the one-dimensional representation. Inspection of the weight diagram, Fig. 26(a), shows that (p,n,Ξ^0,Ξ^-) and $(K^+,K^0,-\bar{K}^0,+K^-)$ can be chosen as the bases of the four-dimensional representation.

The isotopic content of the five dimensional representation of Fig. 26(b) requires, in addition to the isotopic triplet with $Y=0$, which we identify with $\Sigma^+,\Sigma^0,\Sigma^-$ (π^+,π^0,π^-), two more charged baryons $X^\pm$ ($D^\pm$ for bosons) with $T_3=0$, $Y=0$.

We now illustrate the tensor analysis of Sec. V on the basis of this model. Since we have identified the H_i differently than in the previous case (Sec. III G; Sec. VI E), the matrices derived below are not the same as before. The ten operators may be represented by 4×4 traceless matrices:

$$\begin{aligned}
&(H_1)_a{}^b=2(6)^{\frac{1}{2}}\begin{bmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{bmatrix}, \quad (H_2)_a{}^b=2(6)^{\frac{1}{2}}\begin{bmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{bmatrix},\\
&(E_1)_a{}^b=2(3)^{\frac{1}{2}}\begin{bmatrix}0&1&&\\&0&&\\&&0&1\\&&&0\end{bmatrix}, \quad (E_3)_a{}^b=2(3)^{\frac{1}{2}}\begin{bmatrix}0&&-1&\\&0&&1\\&&0&\\&&&0\end{bmatrix},\\
&(E_2)_a{}^b=6^{-\frac{1}{2}}\begin{bmatrix}0&&&1\\&0&&\\&&0&\\&&&0\end{bmatrix}, \quad (E_4)_a{}^b=6^{-\frac{1}{2}}\begin{bmatrix}0&&&\\&0&-1&\\&&0&\\&&&0\end{bmatrix}, \quad (\psi_a)=\begin{bmatrix}f\\n\\\Xi^0\\\Xi^-\end{bmatrix}\\
&(E_{-\alpha})_a{}^b=(E_\alpha{}^+)_a{}^b=(E_\alpha)_b{}^a.
\end{aligned} \tag{VI.32}$$

[78] J. J. Sakurai, Ann. Phys. **11**, 1 (1960).
[79] R. Utiyama, Phys. Rev. **101**, 1597 (1956); S. L. Glashow and M. Gell-Mann (to be published). The latter authors have independently suggested the B_2 and C_2 models discussed in this section.
[80] The relation between M_A and $M^A=g^{AB}M_B$ is most easily determining by requiring $\chi^A\chi_A=g^{AB}\chi_A\chi_B$ to be invariant.

These matrices may be derived by the method developed in Sec. IV. The metric h^{ab} is defined to be the skew matrix that makes $h^{ab}\psi_a\psi_b$ form invariant. One can easily verify that

$$\Xi^- p - \Xi^0 n + n\Xi^0 - \Xi^- p$$

is form invariant since the $\hat{E}_\alpha$ operating on it annihilate it. Therefore, we choose h^{ab} to be

$$h^{ab}=\begin{bmatrix} & & & -1\\ & & 1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}=-h_{ab},\quad h_{ac}h^{cb}=\delta_a{}^b. \qquad \text{(VI.33)}$$

Note that $h^{ac}(L_i)_c{}^b=(L_i)_c{}^a h^{bc}$, i.e., $L_i{}^{ab}=L_i{}^{ba}$. The contragredient bases are

$$\psi^a=h^{ac}\psi_c=\begin{bmatrix}-\Xi^-\\ \Xi^0\\ -n\\ p\end{bmatrix},\quad \bar{\psi}^a=\begin{bmatrix}\bar{p}\\ \bar{n}\\ \bar{\Xi}^0\\ \bar{\Xi}^-\end{bmatrix}. \qquad \text{(VI.34)}$$

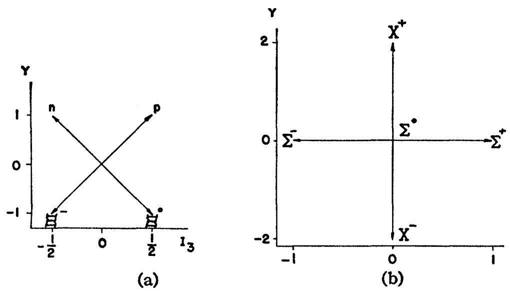

FIG. 26. (a) Weight diagram for $D(1,0)$ of B_2. For meson bases, the substitution, $(p,n,\Xi^0,\Xi^-)\rightarrow(K^+,K^0,-\bar{K}^0,\bar{K}^+)$, should be made. (b) Weight diagram for $D(0,1)$ of C_2. For meson bases, the substitution, $(\Sigma^+,\Sigma^0,\Sigma^-,X^+,X^-)\rightarrow(\pi^+,\pi^0,\pi^-,D^+,D^-)$, should be made.

We take the five skew 4×4 matrices $\sigma_i{}^{ab}$ just introduced in Eq. (V.17), satisfying $h_{ab}\sigma_i{}^{ba}=\text{trace}\,h\sigma_i=0$, to be

$$\sigma_1{}^{ab}=\tfrac{1}{2}\begin{bmatrix} & & & -1\\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix},\quad \sigma_2{}^{ab}=1/\sqrt{2}\begin{bmatrix}0 & 0 & -1 & 0\\ 0 & & & \\ 1 & & & \\ 0 & & & \end{bmatrix},\quad \sigma_3{}^{ab}=1/\sqrt{2}\begin{bmatrix} & & & 0\\ & & & -1\\ & & & 0\\ 0 & 1 & 0 & 0\end{bmatrix},$$
$$\sigma_4{}^{ab}=1/\sqrt{2}\begin{bmatrix}0 & -1 & 0 & 0\\ 1 & & & \\ 0 & & & \\ 0 & & & \end{bmatrix},\quad \sigma_5{}^{ab}=1/\sqrt{2}\begin{bmatrix} & & & 0\\ & & & 0\\ & & & 1\\ 0 & 0 & -1 & 0\end{bmatrix}. \qquad \text{(VI.35)}$$

These matrices are chosen such that $\chi_i\equiv\bar{\psi}^a(\sigma_i)_a{}^b\psi_b=\bar{\psi}^a h_{ac}(\sigma_i)^{cb}\psi_b$ are normalized bases of the five-dimensional representation that transform as the

$$(\psi_i)=(\Sigma^0,\ -\Sigma^+,\Sigma^-,\ -X^+,X^-)$$

or the

$$(M_i)=(\pi^0,\ -\pi^+,\pi^-,\ -D^+,D^-);$$

$$\chi_1=\tfrac{1}{2}(\bar{p}p-\bar{n}n-\bar{\Xi}^0\Xi^0+\bar{\Xi}^-\Xi^-),$$

$$\chi_2=\frac{1}{\sqrt{2}}(-\bar{n}p+\bar{\Xi}^-\Xi^0),\quad \chi_4=\frac{1}{\sqrt{2}}(\bar{\Xi}^-n+\bar{\Xi}^0p),$$

$$\chi_3=\frac{1}{\sqrt{2}}(\bar{p}n-\bar{\Xi}^0\Xi^-),\qquad \chi_5=\frac{1}{\sqrt{2}}(-\bar{n}\Xi^--\bar{p}\Xi^0).$$

The symmetric five-dimensional metric g_{ij} is defined as in Eq. (V.19):

$$g_{ij}=\sigma_i{}^{ab}\sigma_{jba}=\text{trace }\sigma_i\sigma_j$$

$$=\begin{bmatrix}1 & & & & \\ & 0 & -1 & & \\ & -1 & 0 & & \\ & & & 0 & -1\\ & & & -1 & 0\end{bmatrix}=g^{ij};\ g^{il}g_{lj}=\delta^i{}_j. \qquad \text{(VI.36)}$$

The contragredient bases M^i of the five-dimensional representation are obtained by

$$M^i=g^{ij}M_j=\begin{bmatrix}\pi^0\\ -\pi^-\\ \pi^+\\ -D^-\\ D^+\end{bmatrix}. \qquad \text{(VI.37)}$$

The explicit form of the L_A in the five-dimensional representation is obtained from $(L_A)_i{}^j=2\sigma_i{}^{ab}(L_A)_b{}^c\sigma^j{}_{ca}$. We can go on to construct explicit forms of tensors *ad infinitum.* The above examples suffice to illustrate the method.

Let us now turn back to physics. As an example, let us consider the invariant Yukawa type coupling of the (π,D) to the (N,Ξ). It is clear that, by construction, the $\sigma_{ia}{}^b$ are just the $(\Omega_\mu{}^{(\tau)})_a{}^b$ discussed in the early part of this section where τ refers to the five-dimensional representation and $i=\mu$. The invariant coupling is, therefore,

$$I=\bar{\psi}^a\sigma g_{ia}{}^b\psi_b M^i,$$
$$=\frac{g}{2}\{(\bar{p}\gamma_5p-\bar{n}\gamma_5n-\bar{\Xi}^0\gamma_5\Xi^0+\bar{\Xi}^-\gamma_5\Xi^-)\pi^0$$
$$+\sqrt{2}[(\bar{n}\gamma_5p-\bar{\Xi}^-\gamma_5\Xi^0)\pi^-$$
$$-(\bar{\Xi}^0\gamma_5p+\bar{\Xi}^-\gamma_5n)D^-+\text{h.c.}]\}. \qquad \text{(VI.38)}$$

In this case, the number of independent coupling constants required is one, because the product repre-

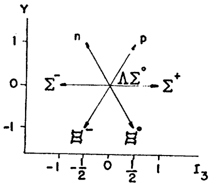

FIG. 27. Weight diagram for $D^{(7)}(1,0)$ with bases associated with baryons. For meson bases, substitute

$$(K^+,K^0,-\bar{K}^0,\bar{K}^+,\pi^+,\pi^0,\pi^-)$$

for

$$(p,n,\Xi^0,\Xi^-,\Sigma^+,\Sigma^0,\Sigma^-).$$

sentation $D^{(4)*}\otimes D^{(4)}$ contains the irreducible representation $D^{(5)}$ only once.

There are three independent amplitudes for scattering of (N,Ξ) and $(\bar{N},\bar{\Xi})$, corresponding to the decomposition (Clebsch-Gordan series) $D^{(4)*}\otimes D^{(4)}=D^{(1)}\oplus D^{(5)}\oplus D^{(10)}$. Let us consider the process

$$a+\bar{b}\rightarrow a'+\bar{b}',$$

where a, b, a', and b' label members of the (N,Ξ) multiplet. The T matrix for this process can be written

$$\begin{aligned}(a'\bar{b}'|T|a\bar{b})=&F'\delta_{a'}{}^{b'}\delta_b{}^a\\&+F^2\sigma_{ia'}{}^{b'}\sigma^i{}_b{}^a\\&+F^3(\Omega_A{}^{(10)})_{a'}{}^{b'}(\Omega_A{}^{(10)})_b{}^a,\end{aligned}\quad\text{(VI.39)}$$

where the isometric operators $(\Omega_A{}^{(10)})_b{}^a$ are proportional to $L_{Ab}{}^a$:

$$(\Omega_A{}^{(10)})_b{}^a=\pm\frac{10}{(\text{trace }L_BL^B)^{\frac{1}{2}}}L_{Ab}{}^a.$$

Finally, let us discuss the Y^* isobar states in this scheme. Since the five-dimensional representation is the lowest dimensional one with the right isotopic spin and hypercharge content to accommodate the triplet, $(Y^*)^+$, $(Y^*)^0$, and $(Y^*)^-$, we propose to identify the Y^* with components of a basis of $D^{(5)}(0,1)$. We have already assigned the Σ and $X^\pm$ to a five-dimensional representation. Therefore, the Y^* and the Σ would have to have the same transformation properties in charge space, while their differences would be described by space-time quantum numbers. An interesting feature of this model is that the decay process $Y^*\rightarrow\Sigma+\pi$ is forbidden by symmetry while $Y^*\rightarrow\Lambda+\pi$ is allowed,[81] independent of spin of the Y^* or the relative $\Lambda\Sigma$ parity. The reason is as follows. The π and the Σ are members of five dimensional representations. Since the product representation $D^{(5)}\otimes D^{(5)}$ does not contain any irreducible $D^{(5)}$, it is impossible for a Y^* to decay into a π and a Σ. On the other hand, since the Λ particle is a basis of the one dimensional representation, the product representation of the Λ and the π multiplet naturally gives rise to a five dimensional representation.

G. Model Built on G_2

If we assume that there are eight baryons which can form the bases for one or more representations, then the dimensionality of these representations must add up to eight. An inspection of Table VI for G_2 shows that the only possibility involves two representations, $D^{(1)}(0,0)$ and $D^{(7)}(1,0)$. This implies that seven of the baryons must have the same space-time quantum numbers; the eighth baryon may have a different set of these quantum numbers. In contrast with SU_3, we see that there is a seven-dimensional representation which allows the possibility of using only the seven known mesons, i.e., $D^{(7)}(1,0)$.[82] In this scheme, of course, it would also be possible to accommodate an eighth meson, the π^{00} which would then correspond to the one-dimensional representation, $D^{(1)}(0,0)$. Until such time as this meson is experimentally detected, we shall consider only the known particles. It is again clear that these seven mesons must have the same space-time quantum numbers.

In Fig. 27 we have drawn the weight diagram for the seven-dimensional representation of G_2. From this, it is clear that if we associate $2\sqrt{3}\hat{H}_1$ with the operator for I_3 and $4\hat{H}_2$ with the operator for Y (hypercharge), then each of the baryons has a specific weight associated with it. There remains only the question of the Σ^0 and Λ which both have zero eigenvalues for these two operators. Since we want charge independence to hold for the strong interactions (this implies the existence of the isotopic spin lowering operator as one of the E_α), the Σ^0 must belong to the seven-dimensional representation and that the Λ is the basis for the one-dimensional representation. Because the Σ and Λ belong to different representations, we see that G_2 can accommodate opposite parities for the Σ and Λ. It predicts specifically, however, that the ΣK parity must be the same as that of $N\pi$

In order to give the usual isotopic spin and hypercharge assignment to the N, Σ, and Ξ, in accordance with the association of H_1 and H_2 with I_3 and Y given above, we must make the following connection between the states given in Sec. III B and the particles

$\|A\rangle$:	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$	$\|4\rangle$	$\|5\rangle$	$\|6\rangle$	$\|7\rangle$	
Baryons:	p	n	Ξ^0	Ξ^-	$-\Sigma^+$	Σ^0	Σ^-	(VI.40)
Mesons:	K^+	K^0	$-\bar{K}^0$	$\bar{K}^+$	$-\pi^+$	π^0	π^-.	

With the aid of this dictionary, it is easy to construct Table X for the particles from the results of Sec. III H.

We proceed now to analyze the scattering $B+M\rightarrow B'+M'$ in the same manner as described in the general part of this section. The pertinent four-point function is

$$\langle T[\psi_{B'}{}^\dagger\psi_{M'}{}^\dagger\psi_B\psi_M]\rangle.$$

The combination $\psi_B\psi_M$ is the Kronecker product of two seven-dimensional representations, one for the meson and one for the baryon. This reduces, following Table VI, according to $7\otimes7=1\otimes7\otimes14\otimes27$. In the manner described previously, we conclude that there are only four different four-point functions or amplitudes for the scattering of the seven baryons by the bosons.

[81] This is in accord with experimental observation (see reference 73).

[82] The model built on G_2 was first suggested by Behrends and Sirlin (reference 19) and, independently, by another of the authors (C.F.) (unpublished).

For further physical predictions, it is necessary to find the metric tensor. This can be done by noting that the invariant part of the compound representation $\chi = g^{ij}\psi_i\varphi_j$ must be a linear combination of the form

$$\chi = ap\bar{K}^+ + bn\bar{K}^0 + c\Sigma^+\pi^- + d\Sigma^-\pi^+ + e\Sigma^0\pi^0 + f\Xi^-K^+ + g\Xi^0K^0.$$

By use of the fact that $E_\alpha\chi = 0$, we can readily find the coefficients a, b, $c\cdots$, and thereby find

$$(g^{ij}) = (g_{ij}) = \begin{bmatrix} 0 & & & 1 & & & \\ & 0 & -1 & & & & \\ & -1 & 0 & & & & \\ 1 & & & 0 & & & \\ & & & & 0 & & -1 \\ & & & & & 1 & \\ & & & & -1 & & 0 \end{bmatrix}, \qquad \text{(VI.41)}$$

where

$$(\psi_i) = (p, n, \Xi^0, \Xi^-, -\Sigma^+, \Sigma^0, \Sigma^-).$$

It follows that

$$(\psi^i) = (\Xi^-, -\Xi^0, -n, p, -\Sigma^-, \Sigma^0, +\Sigma^+).$$

[This matrix was introduced in Eq. (III.42).]

We further need to determine the isometries $6^{-\frac{1}{2}}\Gamma^{ijk}$. This is most easily done by noting that the basis for the seven-dimensional compound representation is given by $\chi^i = \Gamma^{ijk}\psi_j\varphi_k$. We proceed by noting that the χ^i with the fundamental dominant weight $\mathbf{M}^{(1)}$ is a linear combination of the form:

$$-\chi_5 = \chi^7 = ap\bar{K}^0 + b\Xi^0K^+ + c\Sigma^+\pi^0 + d\Sigma^0\pi^+.$$

Since $E_1\chi_5$ and $E_3\chi_5$ would have weights that cannot belong to $D^{(7)}$, they must vanish. These conditions are sufficient to determine the constants a, b, c, and d. The result is

$$\chi^7 = \sqrt{2}(p\bar{K}^0 + \Xi^0K^+) - \Sigma^+\pi^0 + \Sigma^0\pi^+.$$

The matrix $\Gamma^{7jk} = -\Gamma_5{}^{jk}$ may readily be found from this expression. By operating on χ_5 with E_5, we find, [remembering that

$$E_5\psi_5 = -E_5(\Sigma^+) = -\tfrac{1}{4}\sqrt{\tfrac{2}{3}}p = -\tfrac{1}{4}\sqrt{\tfrac{2}{3}}\psi_4]$$

$$p\pi^0 + \sqrt{2}n\pi^+ - \Sigma^0K^+ - \sqrt{2}\Sigma^+K^0 = \chi_1 = \chi^4.$$

Proceeding successively in this manner, we can determine all the χ^i, and thereby all the (Γ^{ijk}). For convenience, we list these matrices

$$\Gamma^1 = \Gamma_4 = \left[\begin{array}{cccc|ccc} & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & \sqrt{2} \\ & & & & 0 & -1 & 0 \\ \hline & & & & 0 & & \\ & & & 1 & & & \\ & & -\sqrt{2} & 0 & & & \end{array}\right]; \quad \Gamma^2 = -\Gamma_3 = \left[\begin{array}{cccc|ccc} & & & & & & \\ & & & & & & \\ & & & & & -1 & \\ & & & & \sqrt{2} & 0 & \\ \hline & & & -\sqrt{2} & & & \\ & & 1 & 0 & & & \\ & & & 0 & & & \end{array}\right],$$

$$\Gamma^3 = -\Gamma_2 = \left[\begin{array}{cccc|ccc} & & & & & & -\sqrt{2} \\ & & & & & 1 & \\ & & & & & & \\ & & & & & & \\ \hline & & & & & & \\ & -1 & & & & & \\ \sqrt{2} & & & & & & \end{array}\right]; \quad \Gamma^4 = \Gamma_1 = \left[\begin{array}{cccc|ccc} & & & & 0 & 1 & 0 \\ & & & & -\sqrt{2} & & \\ & & & & & & \\ & & & & & & \\ \hline 0 & \sqrt{2} & & & & & \\ -1 & & & & & & \\ 0 & & & & & & \end{array}\right],$$

$$\Gamma^5 = -\Gamma_7 = \left[\begin{array}{cccc|ccc} & & & 0 & & & \\ & & & \sqrt{2} & & & \\ & & & & & & \\ 0 & -\sqrt{2} & & & & & \\ \hline & & & & & & \\ & & & & & & 1 \\ & & & & 0 & -1 & 0 \end{array}\right]; \quad \Gamma^6 = \Gamma_6 = \left[\begin{array}{cccc|ccc} & & & -1 & & & \\ & & -1 & & & & \\ & 1 & & & & & \\ 1 & & & & & & \\ \hline & & & & & & -1 \\ & & & & & 0 & \\ & & & & 1 & & \end{array}\right]; \qquad \text{(VI.42)}$$

$$\Gamma^7 = -\Gamma_5 = \left[\begin{array}{cccc|ccc} & & +\sqrt{2} & 0 & & & \\ & & & & & & \\ -\sqrt{2} & & & & & & \\ 0 & & & & & & \\ \hline & & & & 0 & -1 & \\ & & & & +1 & 0 & \\ & & & & & & 0 \end{array}\right].$$

TABLE X. Action of E_α on baryons for $D^{(7)}(1,0)$ in G_2. Table for bosons is obtained by substitution $(p,n,\Xi^0,\Xi^-,\Sigma^+,\Sigma^0,\Sigma^-) \to (K^+,K^0,-\bar{K}^0,\bar{K}^+,\pi^+,\pi^0,\pi^-)$.

$E_\alpha \backslash \psi$	p	n	Ξ^0	Ξ^-	Σ^+	Σ^0	Σ^-
$2(6)^{\frac{1}{2}}E_1$		p		Ξ^0		$-\sqrt{2}\Sigma^+$	$\sqrt{2}\Sigma^0$
$2(6)^{\frac{1}{2}}E_{-1}$	n		Ξ^-		$-\sqrt{2}\Sigma^0$	$\sqrt{2}\Sigma^-$	
$2\sqrt{2}E_2$				$-\Sigma^+$			p
$2\sqrt{2}E_{-2}$	Σ^-				$-\Xi^-$		
$2(6)^{\frac{1}{2}}E_3$			$-\Sigma^+$	$-\sqrt{2}\Sigma^0$		$\sqrt{2}p$	$-n$
$2(6)^{\frac{1}{2}}E_{-3}$	$\sqrt{2}\Sigma^0$	$-\Sigma^-$			$-\Xi^0$	$-\sqrt{2}\Xi^-$	
$2\sqrt{2}E_4$			p	n			
$2\sqrt{2}E_{-4}$	Ξ^0	Ξ^-					
$2(6)^{\frac{1}{2}}E_5$			$\sqrt{2}\Sigma^0$	$-\Sigma^-$	$+p$	$\sqrt{2}n$	
$2(6)^{\frac{1}{2}}E_{-5}$	$+\Sigma^+$	$\sqrt{2}\Sigma^0$				$\sqrt{2}\Xi^0$	$-\Xi^-$
$2\sqrt{2}E_6$			Σ^-		$+n$		
$2\sqrt{2}E_{-6}$		$-\Sigma^+$					Ξ^0
$2\sqrt{3}H_1$	$\frac{1}{2}p$	$-\frac{1}{2}n$	$\frac{1}{2}\Xi^0$	$-\frac{1}{2}\Xi^-$	Σ^+		$-\Sigma^-$
$4H_2$	p	n	$-\Xi^0$	$-\Xi^-$			

It is now a trivial matter to list various amplitudes in a compact notation. For example, the invariant three-point function is

$$\Gamma^{ijk}\langle T(\bar{\psi}_i\psi_j\varphi_k)\rangle. \quad \text{(VI.43)}$$

Another simple example is afforded by Λ production mesons on baryons, $M+B\to M'+\Lambda$. The four-point function is

$$\Gamma^{ijk}\langle T(\bar{\psi}_\Lambda\psi_i\varphi_j\varphi_k)\rangle. \quad \text{(VI.44)}$$

With regard to G_2, it might be interesting to play again the game of finding the processes which might have resonances corresponding to the (3,3) pion-nucleon resonance. At this point, we re-emphasize, the limitations of this game (see the general discussion above). First, the product representation of one baryon and one meson decomposes into representations with dimensionalities of 1, 7, 14, and 27. But the weight of the π^+p state, say, which is a member of the (3,3) resonance, is $(1/4\sqrt{3})(3,\sqrt{3})$. This is just the highest weight for the 14-dimensional representation $D(0,1)$ and it is one of the weights for the 27-dimensional representation. Thus the (3,3) resonance must belong to either the 14 or 27 dimensional representation.

Again, as an example, we have drawn the weight diagram for the 14-dimensional representation in Fig. 3(b). From this, it is clear that besides the $I=\frac{3}{2}$, $Y=1$ multiplet, which we might identify as the 3,3 resonance, the isotopic content includes an $I=\frac{3}{2}$, $Y=-1$ multiplet, an $I=1$, $Y=0$ multiplet, and three singlets, $I=0$, $Y=2$, 0, -2. All of these multiplets must have $J=\frac{3}{2}$. The actual product representation written in terms of the product MB may be found in the manner illustrated above. Namely, the basis for the highest weight of the 14-dimensional representation must be of the form $ap\pi^++b\Sigma^+K^+$. But E_α, for a positive root $r(\alpha)$, acting on this basis must be zero. Specifically, application of E_{-5} gives $a=-b$, so that the basis for the highest weight is $\frac{1}{2}\sqrt{2}(p\pi^+-\Sigma^+K^+)$. The bases for the other weights can be obtained by repeated use of all the E_α. In contrast with SU_3, the $\pi\Lambda$ resonance cannot be associated with the (3,3) pion nucleon resonance, since the $\pi\Lambda$ resonance must be 7-dimensional which does not contain an $I=\frac{3}{2}$ multiplet.

If the (3,3) resonance were identified with the 27-dimensional representation, we would proceed in the same manner. The result would be that the (3,3) resonance would be associated with a different set of isotopic spin multiplets.