Matrix Representation of Polarization*

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CONTENTS

I. INTRODUCTION

THE usual descriptions of polarization phenomena require careful consideration of all the angles involved and the general features of polarization sensitive processes are not readily apparent from the complicated equations that result. In the current literature more frequent use is being made of the density matrix for describing polarization. These matrices introduce the Stokes[1] parameters which, when used as a four-vector, allow the ordinary polarization-sensitive cross sections to be written in matrix form. This form is very convenient for the description of polarization phenomena for both electromagnetic radiation and elementary particles and readily shows the similarity between these processes and the customary descriptions of polarized light. The matrix representation of a large number of polarization sensitive interactions is given here. With these matrices the general features of the interactions are readily determined. An introductory account of the density matrix and the development of the Stokes parameters is given in a previous paper.[2]

In quantum mechanics the wave function describing a pure state of polarization can be expanded in a complete set of orthonormal eigenfunctions. For electromagnetic radiation and particles of spin $\frac{1}{2}$, this expansion consists of only two terms,

$$\psi = a_1\psi_1 + a_2\psi_2. \tag{1}$$

The wave functions describing pure states may be chosen in the form

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

On using these, the wave function describing the beam is given by

$$\psi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{3}$$

leading to the following expressions for the expectation values of the unit matrix and the Pauli spin matrices:

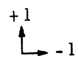
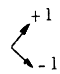
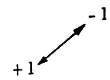
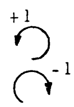
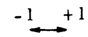
$$\begin{aligned}
I &= \langle 1 \rangle = (a_1^* a_2^*) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a_1 a_1^* + a_2 a_2^*, \\
P_1 &= \langle \sigma_z \rangle = (a_1^* a_2^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 a_1^* - a_2 a_2^*, \\
P_2 &= \langle \sigma_x \rangle = (a_1^* a_2^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 a_2^* + a_2 a_1^*, \\
P_3 &= \langle \sigma_y \rangle = (a_1^* a_2^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = i(a_1 a_2^* - a_2 a_1^*).
\end{aligned} \tag{4}$$

This set of four members, called the "Stokes parameters," represents physically measurable properties and completely characterizes the beam. From the definition of the a_i one sees that I is the total intensity. The choice of the Pauli spin matrices is such that the Stokes

* This work was done under the auspices of the U. S. Atomic Energy Commission.

[1] G. G. Stokes, Trans. Cambridge Phil. Soc. 9, 399 (1852).
[2] W. H. McMaster, Am. J. Phys. 22, 351 (1954).

TABLE I. Comparison of the Stokes parameters.

Stokes parameter		Photon observation		Particle observation
I		Intensity		Intensity
P_1		Plane polarization		Spin in z direction
P_2		Plane polarization at an angle of $\pi/4$ to the right		Spin in x direction
P_3		Left circular polarization		Spin in y direction
		Right circular polarization		

parameter P_1 represents the pure states ψ_1 and ψ_2. For electromagnetic radiation these states are usually chosen to represent two states of orthogonal plane polarization and for particles they refer to polarization in the positive and negative z directions, respectively.

Table I gives a comparison of the meanings of the Stokes parameters for radiation and particles as used here.

In the application of the Stokes parameters to a problem it is convenient to write them in the form of a four-vector:

$$\begin{pmatrix} I \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}. \tag{5}$$

As an example of its form, consider the following simple cases:

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ represents an unpolarized beam.

$\begin{pmatrix} 1 \\ \pm 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}$ represent plane polarization or transverse spin in z and x directions.

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{pmatrix}$ represent circular polarization or spin in y direction.

Since the Stokes parameters are dependent upon the choice of axes, there exists a rotation matrix M which relates the Stokes parameters in one coordinate system to those in another system. On considering a second coordinate system rotated about the direction of propagation at an angle θ to the right of the original coordinate system, then

$$\begin{pmatrix} I' \\ \mathbf{P}' \end{pmatrix} = M \begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}, \tag{6}$$

where

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

for rotations about the axis represented by P_3, with $\omega = 2\theta$ for photons and $\omega = \theta$ for particles. (Derivation and discussion are given in the Appendix.) Thus if in the old system the values are (I, P_1, P_2, P_3), then, in the new system of coordinates, the Stokes parameters of the same beam are

$$\begin{pmatrix} I \\ P_1 \cos\omega + P_2 \sin\omega \\ -P_1 \sin\omega + P_2 \cos\omega \\ P_3 \end{pmatrix}. \tag{8}$$

A rotation in the opposite direction changes the sign of the sine term as expected.

The probability of detecting a photon or a particle characterized by the Stokes parameters $(1,\mathbf{D})$ in an arbitrary beam characterized by the Stokes parameters $(I,\mathbf{P})$ is given by

$$W = \tfrac{1}{2}(1 + \mathbf{P}\cdot\mathbf{D}),$$

which, using vector notation, becomes

$$W = \tfrac{1}{2}(1,\mathbf{D}) \begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}. \tag{9}$$

With a polarization-insensitive detector we are, in effect, making a measurement of each of two orthogonal states and summing. Thus one obtains the formula

$$W = (1\ 0\ 0\ 0) \begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}. \tag{10}$$

When photons or particles undergo an interaction which is sensitive to polarization, then, in general, the

Stokes parameters of the initial beam are transformed into a new set of parameters. The relation between these two Stokes vectors is given by a transformation matrix T characteristic of the interaction. That is,

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = T\begin{pmatrix} I_0 \\ \mathbf{P}_0 \end{pmatrix}, \tag{11}$$

where T is a 4×4 matrix.

When we pass a beam $(I,\mathbf{P})$ through a polarizer T and detect it with an analyzer $(1,\mathbf{D})$ the fractional intensity detected is given by

$$W = \tfrac{1}{2}(1,\mathbf{D})T\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}. \tag{12}$$

II. PHOTON POLARIZATION

A. Polarization of Light

Since polarization is first studied in optics, let us start with some examples in this field.

1. Nicol Prism

A Nicol prism with its transmission axis along $\mathbf{E}_1$, i.e., one which passes only light characterized by the Stokes parameters (1 1 0 0), has the interaction matrix

$$T = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

The zero's of the fourth row and column show that a Nicol prism is insensitive to circular polarization.

This matrix expresses the $\cos^2\phi$ dependence on the orientation of the electric vector as can be seen from the following considerations. For the vector expression

$$\mathbf{E} = \cos\phi\mathbf{E}_1 + \sin\phi\mathbf{E}_2,$$

the Stokes vector is

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = \begin{bmatrix} 1 \\ \cos^2\phi - \sin^2\phi \\ 2\sin\phi\cos\phi \\ 0 \end{bmatrix},$$

and hence

$$T\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = \cos^2\phi \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus a polarization insensitive detector measures

$$W = (1\ 0\ 0\ 0)\cos^2\phi \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \cos^2\phi.$$

For a system of two Nicol prisms, the first acting as a polarizer with interaction matrix T and the second as a detector which accepts polarization characterized by $(1,\mathbf{D})$, the transmitted intensity is given by Eq. (12). For an unpolarized initial beam we have

$$W = \tfrac{1}{2}(1\ 1\ 0\ 0)T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \tfrac{1}{2}(1\ 1\ 0\ 0)\tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \tfrac{1}{2},$$

for parallel transmission axes and

$$W = \tfrac{1}{2}(1\ -1\ 0\ 0)T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0,$$

for crossed prisms.

The next two examples are among those worked out by Perrin.[3]

2. Birefringent Crystal

A birefringent crystal which introduces a phase ϕ between the components of the vibration along two orthogonal axes has the following transformation matrix if we take the slow axis along $\mathbf{E}_1$:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{bmatrix}. \tag{14}$$

In particular, a quarter-wave plate with its slow axis along E_1 is given by $(\phi = \pi/2)$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{15}$$

3. Optical Activity

A crystal exhibiting optical activity which rotates the plane of polarization an angle ϕ to the right has the interaction matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\phi & -\sin 2\phi & 0 \\ 0 & \sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{16}$$

The simplicity of these interaction matrices shows the ease of treating polarized light in the formalism of the Stokes parameters.

B. Polarization of Gamma Rays by Compton Scattering

Next let us consider the polarization of gamma rays by Compton scattering. The Klein-Nishina formula[4] shows that the cross section depends on the directions of polarization of the initial photon, the final photon, the initial electron, and the final electron. The most

[3] F. Perrin, J. Chem. Phys. **10**, 415 (1942).

[4] O. Klein and Y. Nishina, Z. Physik **52**, 853 (1929); Y. Nishina, *ibid.* **52**, 869 (1929).

familiar form of the equation is

$$\frac{d\sigma}{d\Omega}=\tfrac{1}{4}r_0^2\left(\frac{k}{k_0}\right)^2\left[\frac{k_0}{k}+\frac{k}{k_0}-2+4\cos^2\Theta\right], \qquad (17)$$

where $r_0=e^2/mc^2$ is the classical radius of the electron, Θ is the angle between the direction of polarizations of the incident and scattered beams, k_0 and k are the initial and final energies of the gamma in units of mc^2, and $k_0/k=1+k_0(1-\cos\theta)$ expresses the degradation of energy for a scattering angle θ.

This form of the Klein-Nishina formula is applicable only to plane polarizations. The complete equation has been brought into a convenient form by Fano[5] and Tolhoek.[6,7] Here we follow Fano's representation but use Tolhoek's polarization conventions which agree with those we have been developing. This presentation is not complete in that it does not contain terms dependent upon the final state of the electron's spin. A complete set of terms is given in Tolhoek's article.[6]

The matrix of the Klein-Nishina formula used here is a revised form of Fano's original matrix[5] and is given in Eq. (18). In the limiting case $k=k_0\rightarrow 0$, we obtain the matrix for Rayleigh scattering:

$$T=\tfrac{1}{2}r_0^2\left(\frac{k}{k_0}\right)^2\begin{pmatrix} 1+\cos^2\theta+(k_0-k)(1-\cos\theta) & \sin^2\theta & 0 & -(1-\cos\theta)(k_0\cos\theta+\mathbf{k})\cdot\mathbf{S} \\ \sin^2\theta & 1+\cos^2\theta & 0 & (1-\cos\theta)(\mathbf{n}\times\mathbf{n}_0)\cdot(\mathbf{k}_0\times\mathbf{S}) \\ 0 & 0 & 2\cos\theta & (1-\cos\theta)(\mathbf{k}_0\times\mathbf{n})\cdot\mathbf{S} \\ -(1-\cos\theta)(\mathbf{k}\cos\theta+\mathbf{k}_0)\cdot\mathbf{S} & (1-\cos\theta)(\mathbf{n}_0\times\mathbf{n})\cdot(\mathbf{k}\times\mathbf{S}) & (1-\cos\theta)(\mathbf{k}\times\mathbf{n}_0)\cdot\mathbf{S} & 2\cos\theta+(k_0-k)(1-\cos\theta)\cos\theta \end{pmatrix}, \qquad (18)$$

where $P_1=+1$ refers to linear polarization perpendicular to the plane of scattering, $P_2=+1$ refers to linear polarization 45° to the right of P_1, $P_3=+1$ refers to left circular polarization, $\mathbf{S}$ is the spin direction of the initial electron, k_0 and k are the energy of the incident and scattered quanta in units of mc^2, $\mathbf{n}_0$ and $\mathbf{n}$ are unit vectors in the direction of k_0 and k, and θ is the angle of scattering.

The terms involving the electron's spin are contained in the fourth row and column only, which means that the interaction between photons and electron spin occurs only with circularly polarized components of the gamma-ray beam. This is to be expected since both polarizations are associated with angular momentum.

The complete cross section analogous to the Klein-Nishina formula is then given by Eq. (12) which gives a relatively simple procedure for finding polarization-dependent solutions. Several examples are given. For simplicity, whenever only linear polarization is involved we use just the upper left 3×3 submatrix of Eq. (18) which expresses Eq. (17).

1. Compton Scattering of Unpolarized Gamma Rays

As a result of Compton scattering the Stokes parameters of an unpolarized beam undergo the transformation

$$T\begin{pmatrix}1\\0\\0\end{pmatrix}\sim\begin{pmatrix}1+\cos^2\theta+(k_0-k)(1-\cos\theta)\\ \sin^2\theta\\ 0\end{pmatrix}, \qquad (19)$$

from which, since $P_1\sim\sin^2\theta$, as a result of Compton scattering, the beam is partially polarized orthogonal to the plane of scattering.

The degree of polarization as obtained from Eq. (19) is

$$P=\sin^2\theta/[1+\cos^2\theta+(k_0-k)(1-\cos\theta)]. \qquad (20)$$

Sometimes it is convenient to determine the ratio of intensities with opposite polarizations in the scattered beam. This is defined as

$$p=d\sigma_\perp/d\sigma_{\parallel},$$

where

$$d\sigma_\perp=\tfrac{1}{2}(1\ \ 1\ \ 0)T\begin{pmatrix}1\\0\\0\end{pmatrix}=\tfrac{1}{4}\left(\frac{k}{k_0}\right)^2 r_0^2[(k_0-k)(1-\cos\theta)+2]$$

or

$$d\sigma_\perp=\tfrac{1}{4}r_0^2\left(\frac{k}{k_0}\right)^2\left[\frac{k}{k_0}+\frac{k_0}{k}\right],$$

$$d\sigma_{\parallel}=\tfrac{1}{2}(1\ \ -1\ \ 0)T\begin{pmatrix}1\\0\\0\end{pmatrix}=\tfrac{1}{4}r_0^2\left(\frac{k}{k_0}\right)^2 \times[(k_0-k)(1-\cos\theta)+2\cos^2\theta]$$

or

$$d\sigma_{\parallel}=\tfrac{1}{4}r_0^2\left(\frac{k}{k_0}\right)^2\left[\frac{k}{k_0}+\frac{k_0}{k}-2\sin^2\theta\right],$$

yielding the result

$$p=[(k_0-k)(1-\cos\theta)+2]/[(k_0-k)(1-\cos\theta)+2\cos^2\theta], \qquad (21)$$

where the $\perp$ and $\parallel$ refer to plane polarization perpendicular to and parallel to the plane of scattering, respectively. When $p=1$, which occurs for $\theta=0$ and π, the scattered beam is unpolarized. On using the ordinary Klein-Nishina formula, the derivation of this result requires more effort and a careful consideration of the angles involved.

The cross section for a polarization-insensitive detector is found to be

$$\sigma=(1\ \ 0\ \ 0)T\begin{pmatrix}1\\0\\0\end{pmatrix}=\tfrac{1}{2}r_0^2\left(\frac{k}{k_0}\right)^2 \times[(k_0-k)(1-\cos\theta)+1+\cos^2\theta], \qquad (22)$$

$$\sigma=\tfrac{1}{2}r_0^2\left(\frac{k}{k_0}\right)^2\left[\frac{k}{k_0}+\frac{k_0}{k}-\sin^2\theta\right],$$

[5] U. Fano, J. Opt. Soc. Am. **39**, 859 (1949); Revs. Modern Phys. **29**, 74 (1957).
[6] F. W. Lipps and H. A. Tolhoek, Physica **20**, 395 (1954).
[7] H. A. Tolhoek, Revs. Modern Phys. **28**, 277 (1956).

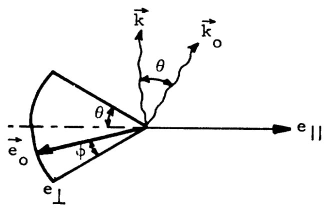


FIG. 1. Compton scattering of a polarized gamma-ray beam. $\mathbf{e}_0$ gives the plane of polarization of the incident beam. $(\mathbf{k},\mathbf{k}_0)$ is the plane of the paper and θ is the Compton scattering angle. *Note.* Letters in figures with overhead arrows correspond to boldface letters in the captions and text.

which is the sum of $d\sigma_\perp$ and $d\sigma_{\parallel}$ as expected. Here we have used (1 0 0) as an analyzer, which is the Stokes vector characterizing an ordinary photon detector, such as a scintillation counter.

2. Compton Scattering of a Polarized Beam

For this discussion we use the geometry shown in Fig. 1. To use Fano's matrix we recall that positive values of P_1 refer to plane polarization perpendicular to the plane of scattering (i.e., along $e_\perp$). The simplest representation of the initial completely polarized beam is by the Stokes vector (1 1 0), but this is in a coordinate system rotated through an angle ϕ to the right of $\mathbf{e}_\perp$ (looking in the direction $-\mathbf{k}_0$); therefore, we must rotate the coordinate system an angle ϕ to the left using the matrix M given by Eq. (7) with the appropriate change of sign. Thus, using the plane of scattering as a reference plane, we have

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = M\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\phi \\ -\sin 2\phi \end{pmatrix} = \begin{pmatrix} 1 \\ \cos^2\phi - \sin^2\phi \\ -2\sin\phi\cos\phi \end{pmatrix}. \quad (23)$$

The cross section for scattering is thus given by

$$\sigma = (1\ 0\ 0)T\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = \tfrac{1}{2}r_0^2\left(\frac{k}{k_0}\right)^2 \times[(k_0-k)(1-\cos\theta)+2\cos^2\phi+2\cos^2\theta\sin^2\phi]$$

or

$$\sigma = \tfrac{1}{2}r_0^2\left(\frac{k}{k_0}\right)^2\left[\frac{k}{k_0}+\frac{k_0}{k}-2\sin^2\theta\sin^2\phi\right]. \quad (24)$$

The probability that the beam is polarized orthogonal to the plane of scattering is given by

$$\sigma_\perp = \tfrac{1}{2}(1\ 1\ 0)T\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = \tfrac{1}{4}r_0^2\left(\frac{k}{k_0}\right)^2 \times[(k_0-k)(1-\cos\theta)+4\cos^2\phi]$$

or

$$\sigma_\perp = \tfrac{1}{4}r_0^2\left(\frac{k}{k_0}\right)^2\left[\frac{k}{k_0}+\frac{k_0}{k}+2-4\sin^2\phi\right], \quad (25)$$

and the probability that the beam is polarized in the plane of scattering is given by

$$\sigma_{\parallel} = \tfrac{1}{2}(1\ -1\ 0)T\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = \tfrac{1}{4}r_0^2\left(\frac{k}{k_0}\right)^2 \times[(k_0-k)(1-\cos\theta)+4\cos^2\theta\sin^2\phi]$$

or

$$\sigma_{\parallel} = \tfrac{1}{4}r_0^2\left(\frac{k}{k_0}\right)^2\left[\frac{k}{k_0}+\frac{k_0}{k}-2+4\cos^2\theta\sin^2\phi\right]. \quad (26)$$

Use is made here of the relation $(k/k_0)+(k_0/k)-2 = (k_0-k)(1-\cos\theta)$.

From Eq. (24) we see that the polarized gamma rays are preferentially scattered in a plane perpendicular to the electric vector. This makes it possible to use Compton scattering as an analyzer in a way analogous to the Nicol prism of optics. Two special cases are of interest, and are sketched in Fig. 2. In case I we have

$$T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{k_0}{k}+\dfrac{k}{k_0} \\ 2 \\ 0 \end{pmatrix},$$

and in case II we have

$$T\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{k}{k_0}+\dfrac{k_0}{k}-2+2\cos^2\theta \\ -2\cos^2\theta \\ 0 \end{pmatrix}.$$

These Stokes vectors show that the beam has been depolarized to some extent in both cases and for the special case of right-angle scattering, $\theta=90°$, the case II beam is completely depolarized.

To use Compton scattering as an analyzer for plane polarization, the two measurements shown in Fig. 3 are made. The partially polarized beam I is Compton scattered and intensity measurements are made for a scattering angle of $\theta=\pi/2$ in the two directions I_2 and I_1, where I_1 is perpendicular to the plane of I and I_2. The ratio of intensities in the two polarization states along P_1 and perpendicular to P_1, p is found from

$$I_1/I_2 = (pR+1)/(p+R), \quad (27)$$

where R is the ratio that would be obtained for a beam

FIG. 2. Special cases of the scattering of a linearly polarized gamma ray.

totally polarized in the (I,I_2) plane. That is, from Fig. 2,

$$R=\frac{\text{case I}}{\text{case II}}=\frac{k_0/k+k/k_0}{k_0/k+k/k_0-2}.$$

Because of the θ dependence of k, the asymmetry ratio R, which is unity for $\theta=0$ and $\theta=\pi$, reaches its maximum value at an angle θ somewhat smaller than $\pi/2$ (e.g., at 1 Mev the maximum value of $R\sim 2.7$ is reached around 78.5°).

A description of a gamma-ray polarimeter using this technique is given by Metzger and Deutsch.[8]

3. Double Compton Scattering of an Unpolarized Beam (Fig. 4)

The cross section for double scattering is given by

$$\sigma=(1\ 0\ 0)T_2T_1\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

The unpolarized incident beam k_0, characterized by (1 0 0), is partially polarized as a result of the first scattering. After this first scattering, the beam $\mathbf{k}_1$ is characterized by Eq. (19), where these Stokes parameters refer to the $(\mathbf{k}_0,\mathbf{k}_1)$ plane. Therefore, in order to use Fano's matrix for the second scattering, we must transform, by the matrix M, these Stokes parameters to refer to the $(\mathbf{k}_1,\mathbf{k}_2)$ plane, which is at an angle ϕ to the $(\mathbf{k}_0,\mathbf{k}_1)$ plane. Therefore

$$\sigma\sim(1\ 0\ 0)T_2\begin{pmatrix}1+\cos^2\theta_1+(k_0-k_1)(1-\cos\theta_1)\\ \sin^2\theta_1(\cos^2\phi-\sin^2\phi)\\ 0\end{pmatrix},$$

which yields

$$\sigma\sim\gamma_{01}\gamma_{12}-\gamma_{01}\sin^2\theta_2-\gamma_{12}\sin^2\theta_1 +2\sin^2\theta_1\sin^2\theta_2\cos^2\phi, \quad (28)$$

where $\gamma_{01}=k_1/k_0+k_0/k_1$ and $\gamma_{12}=k_2/k_1+k_1/k_2$, which is the result given by Wightman[9] using the density matrix. This example of double scattering shows the ease of computation by the use of the Stokes parameters.

4. Magnitude of the Polarization Effect

Figure 5 shows the magnitudes of the polarization that can be obtained by Compton scattering. These curves are for the energy range 0.2–10 Mev.

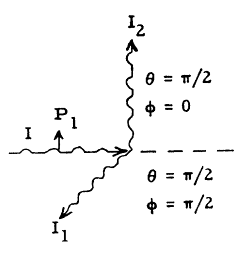


FIG. 3. Compton scattering as an analyzer for plane polarization such that for $P_1=1$ the beam is plane polarized in the (I,I_2) plane. The ratio of polarizations is found from

$$I_1/I_2=(pR+1)/(p+R),$$

where R is the ratio that would be obtained for a totally polarized beam.

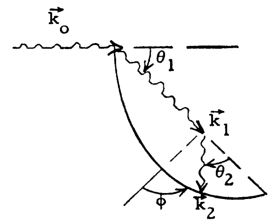


FIG. 4. Double Compton scattering of an unpolarized gamma-ray beam. ϕ is the angle between the $(\mathbf{k}_0,\mathbf{k}_1)$ plane and the $(\mathbf{k}_1,\mathbf{k}_2)$ plane.

The polarization of an unpolarized beam as given by Eq. (20) is shown in Fig. 5(a). As expected, the effect is greatest at low energies and for scattering angles near 90°. The dashed line shows the angle which yields the maximum polarization for various incident energies.

If the incident beam is completely polarized perpendicular to the plane of scattering, the scattered beam is only partially polarized. This was discussed as case I in Fig. 2. The resulting partial polarization is given by

$$P=2\bigg/\left(\frac{k_0}{k}+\frac{k}{k_0}\right) \quad (29)$$

and is shown in Fig. 5(b). By using boths Figs. 5(a) and 5(b) one can determine the amount of polarization resulting from multiple scattering of an incident beam.

In designing polarization analyzers it is necessary to determine the cross sections for the two states of polarization (perpendicular and parallel to the plane

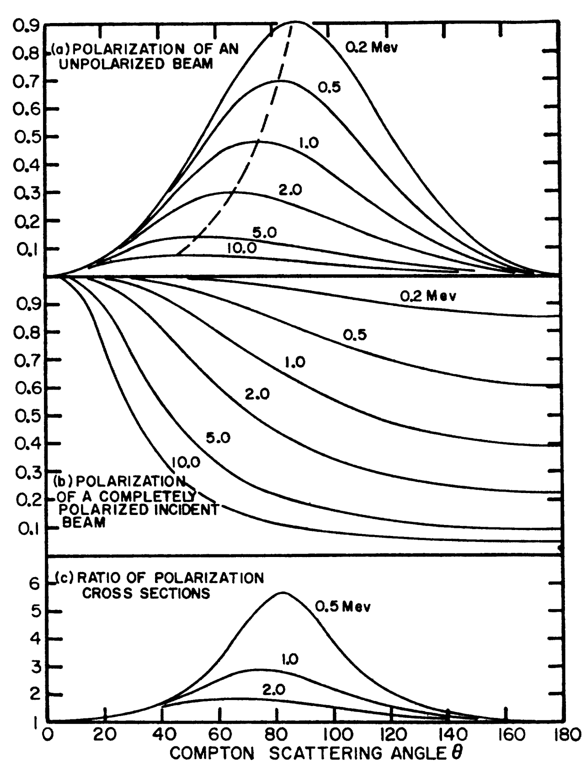


FIG. 5. Polarization by Compton scattering.

[8] F. Metzger and M. Deutsch, Phys. Rev. 78, 551 (1950).
[9] A. Wightman, Phys. Rev. 74, 1813 (1948).

of scattering). These are expressed by the ratio $p=d\sigma_\perp/d\sigma_\parallel$ given by Eq. (21) and shown in Fig. 5(c).

5. *Effects Dependent on Circular Polarization and Electron Spin*

Compton scattering of a polarized gamma beam by polarized electrons is given by

$$\sigma=(1\ 0\ 0\ 0)T\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix},$$

$$\sigma=\tfrac{1}{2}r_0^2\left(\frac{k}{k_0}\right)^2\left[1+\cos^2\theta(k_0-k)(1-\cos\theta)+P_1\sin^2\theta-P_3(1-\cos\theta)(\mathbf{k}_0\cos\theta+\mathbf{k})\cdot\mathbf{S}\right]. \quad (30)$$

This is accomplished experimentally by using magnetized iron in which two of the electrons are aligned with the field. Thus this process can be used as a detector for circularly polarized gammas. The requirement of the additional factor of electron-spin orientation corresponds to the use of a quarter-wave plate in optics.

Compton scattering of an unpolarized gamma beam by polarized electrons yields a beam characterized by the Stokes parameters

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}=T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\sim\begin{bmatrix}1+\cos^2\theta+(k_0-k)(1-\cos\theta)\\ \sin^2\theta \\ 0 \\ -(1-\cos\theta)(\mathbf{k}\cos\theta+\mathbf{k}_0)\cdot\mathbf{S}\end{bmatrix}, \quad (31)$$

which shows that in general the resulting beam is elliptically polarized, and the backscattered beam, $\theta=\pi$, is circularly polarized. The degree of polarization is given by

$$p_c=\frac{|\mathbf{P}|}{I}=\frac{(P_1^2+P_2^2+P_3^2)^{\frac{1}{2}}}{I}$$

$$=(1-\cos\theta)\frac{\{(1+\cos\theta)^2+[(\mathbf{k}\cos\theta+\mathbf{k}_0)\cdot\mathbf{S}]^2\}^{\frac{1}{2}}}{(1+\cos^2\theta)+(k_0-k)(1-\cos\theta)} \quad (32)$$

for the general case and

$$p_c=\frac{2k_0(1+k_0)}{(1+2k_0+2k_0^2)}|\mathbf{n}_0\cdot\mathbf{S}| \quad (33)$$

for backscattering. For high energies, $k_0>1$ (i.e., greater than $\frac{1}{2}$ Mev), the degree of polarization rapidly approaches $|\mathbf{n}_0\cdot\mathbf{S}|$ and we can get a high degree of circular polarization either left or right circularly polarized dependent upon whether the electron spin is parallel or antiparallel with the direction of the incident photon.

Polarized electrons can also be obtained by Compton scattering. The appropriate cross sections and processes are discussed in great detail by Tolhoek.[6,7,10]

[10] H. A. Tolhoek and S. R. Degroot, Physica 20, 85 (1954).

C. Radiation from an Accelerated Electron

For the case of an accelerated electron in an homogeneous magnetic field, illustrated in Fig. 6, the emitted radiation is polarized. We use the formulas of Sokolov and Ternov[11] to illustrate this effect. After integration over the energy spectrum, the Stokes vector of the emitted radiation for the relativistic case is given by

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}=\tfrac{3}{2}w\begin{bmatrix}f_\perp+f_\parallel\\ f_\perp-f_\parallel \\ 0 \\ f_c\end{bmatrix}, \quad (34)$$

where $w=\frac{2}{3}(e^2c/R^2)\epsilon^4$ is the total energy radiated per unit time, ϵ the electron total energy in units of mc^2, $x=\epsilon\cos\theta$, $f_\perp=(5x^2/16)(1+x^2)^{-7/2}$ for polarization $\perp$ to the plane of motion, $f_\parallel=\frac{7}{16}(1+x^2)^{-5/2}$ for polarization in the plane of motion, and $f_c=-(4/\pi\sqrt{3})x(1+x^2)^{-3}$. A measurement to determine the degree polarization of the beam is again given by Eq. (12) and the results of these measurements are shown in Fig. 7.

Figure 7(a) shows that for radiation emitted in the plane of motion ($x=0$) the light is plane polarized in this plane. For $\theta<\pi/2(x>0)$, the light is right circularly polarized, and for $\theta>\pi/2(x<0)$ the light is left circularly polarized, that is, the polarization has the same helicity as the electron's motion.

For the nonrelativistic case, Eq. (34) becomes

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}=\frac{1}{8\pi}\frac{e^2c}{R^2}\beta^4\begin{bmatrix}1+\cos^2\theta\\ -\sin^2\theta \\ 0 \\ -2\cos\theta\end{bmatrix}. \quad (35)$$

The angular distribution of the emitted radiation is given by

$$\frac{d\sigma}{d\Omega}=(1\ 0\ 0\ 0)\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}=\frac{e^2c}{R^2}\frac{\beta^4}{8\pi}(1+\cos^2\theta),$$

and

$$\int\frac{d\sigma}{d\Omega}d\Omega=\frac{2}{3}\frac{e^2c}{R^2}\beta^4,$$

the total radiated energy. The polarization effect expressed by Eq. (35) has been used in the analysis of the Zeeman effect.

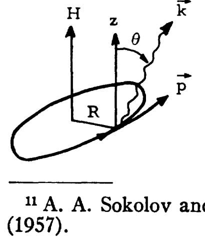


FIG. 6. Radiation of an accelerated electron in an homogeneous magnetic field. The photon is emitted at an angle θ from the direction of the magnetic field.

[11] A. A. Sokolov and I. M. Ternov, Soviet Phys. JETP 4, 396 (1957).

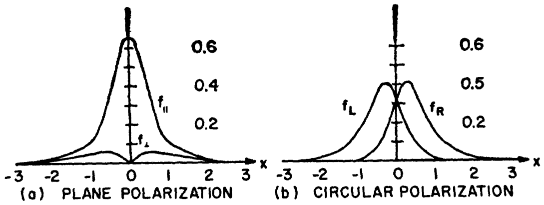


FIG. 7. Relative degrees of polarization of the radiation as a function of $x=\epsilon\cos\theta$. f_L and f_R refer to left and right circular polarization, respectively.

D. Čerenkov Radiation

The polarization effects for Čerenkov radiation from particles of spin $\frac{1}{2}$ are given by Sokolov and Loskutov.[12] The Stokes parameters for the radiation are given by

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = \begin{bmatrix} 2A+B \\ -B \\ 0 \\ -Cs \end{bmatrix}, \tag{36}$$

where

$$A=\frac{e^2}{c^2}\int_0^{\omega_{\max}} \frac{\omega n^2\omega^2\hbar^2}{4c^2p^2}\left(1-\frac{1}{n^2}\right)d\omega,$$

and expresses the intensity polarized perpendicular to the plane of interaction,

$$B=\frac{e^2}{c^2}\int_0^{\omega_{\max}} \omega(1-\cos^2\theta)d\omega,$$

and expresses the intensity polarized in the plane of interaction,

$$C=\frac{c^2}{2c^2}\int_0^{\omega_{\max}} \frac{\hbar n\omega^2}{cp}\left(1-\frac{\cos\theta}{\beta n}\right)d\omega,$$

$s=\pm 1$ represents the longitudinal spin of the electron, and $n=\sqrt{\epsilon}$ is the refractive index. Equation (36) shows that the radiation is polarized in the plane of interaction, i.e., the electric vector is everywhere perpendicular to the surface of the cone of Čerenkov light as shown in Fig. 8. The fourth component of the Stokes vector of Eq. (36) shows that longitudinally polarized electrons produce circularly polarized light. If the electron is polarized along (against) its momentum, $s=1(s=-1)$, then the light is right (left) circularly polarized. That is, the photon has the same helicity as the electron. The degree of circular polarization is given by

$$P_c=\frac{I_L-I_R}{I_L+I_R}=\frac{-Cs}{2A+B},$$

since

$$I_L=\tfrac{1}{2}(1\ 0\ 0\ 1)\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} \text{ and } I_R=\tfrac{1}{2}(1\ 0\ 0\ -1)\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}.$$

[12] A. A. Sokolov and I. M. Loskutov, Soviet Phys. JETP **5**, 523 (1957); **7**, 706 (1958).

III. ELECTRON POLARIZATION

A. Coulomb Scattering of Electrons (Mott Scattering)

Coulomb scattering of electrons is polarization sensitive due to spin-orbit coupling, caused by the interaction of the magnetic moment of the electron with the magnetic field which a moving electron experiences in the electric field of the nucleus. The cross section for Coulomb scattering of electrons was originally derived by Mott.[13] The discussion here follows that of Mendlowitz[14] and Tolhoek.[7,10] The z axis is chosen as the reference axis for both the electron's spin and its initial direction.

The cross section is again given by

$$\sigma=\tfrac{1}{2}(1,\mathbf{D})T\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}$$

with the transformation matrix

$$T=\lambdabar^2\begin{bmatrix} I & 0 & 0 & D \\ 0 & L & -T & 0 \\ 0 & T & L & 0 \\ D & 0 & 0 & I \end{bmatrix}, \tag{37}$$

where $\lambdabar=\hbar/p$ and $2\pi\lambdabar$ is the de Broglie wavelength,

$D=i(fg^*-f^*g)=2q'\csc\theta[FG^*+F^*G]$ and gives the polarization perpendicular to the plane of scattering,

$L=I\cos\theta+(fg^*+f^*g)\sin\theta-2|g|^2\cos\theta$
$=GG^*\sec^2(\theta/2)-q'^2FF^*\csc^2(\theta/2)$ and gives the longitudinal polarization,

$T=-I\sin\theta+(fg^*+f^*g)\cos\theta+2|g|^2\sin\theta$
$=2iq'\csc\theta(FG^*-F^*G)$ and gives the transverse polarization in the plane of scattering.

$q'=(\alpha/\beta)(1-\beta^2)^{\frac{1}{2}}$,

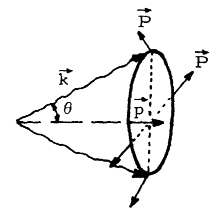
$\alpha=Z/137$;

F and G are functions defined by Mott[13] through the equations

$$f=\lambdabar(iq'F+G),$$
$$g=\lambdabar[iq'F\cot(\theta/2)+G\tan(\theta/2)].$$

The F and G functions have been tabulated by Sherman[15] for a variety of elements and angles.

FIG. 8. Čerenkov light is polarized perpendicular to the surface of the cone of radiation.



[13] N. F. Mott, Proc. Roy. Soc. (London) **A124**, 425 (1929); **A135**, 429 (1932).
[14] H. Mendlowitz, Am. J. Phys. **26**, 17 (1958); H. Mendlowitz and K. M. Case, Phys. Rev. **97**, 33 (1955).
[15] N. Sherman, Phys. Rev. **103**, 1601 (1956).

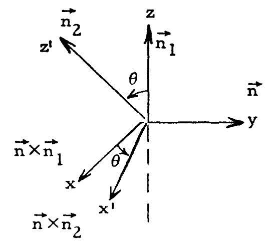


FIG. 9. Schematic diagram showing the angles and unit vectors involved in Mott scattering of electrons.

Sherman's function S is given by

$$S = D/I.$$

In this matrix form it is easy to treat the polarization effects of any case of Mott scattering. The directions of scattering and polarizations implicit in the transformation matrix are shown in Fig. 9. The Stokes parameters of the initial beam refer to the directions

$$\begin{array}{ll} P_1 \to \mathbf{n}_1 & (z \text{ axis}) \\ P_2 \to \mathbf{n}\times\mathbf{n}_1 & (x \text{ axis}) \\ P_3 \to \mathbf{n} = (\mathbf{n}_1\times\mathbf{n}_2)/\sin\theta & (y \text{ axis}) \end{array}$$

and those of the scattered beam refer to the directions

$$\begin{array}{ll} P_1' \to \mathbf{n}_2 & (\text{new } z \text{ axis}) \\ P_2' \to \mathbf{n}\times\mathbf{n}_2 & (\text{new } x \text{ axis}) \\ P_3' \to \mathbf{n} & (y \text{ axis}). \end{array}$$

The unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ represent the initial and scattered beam with a scattering angle of θ and define the (xz) plane.

To illustrate the use of the transformation matrix, several examples are worked out which correspond to the usual experimental cases. Longitudinal polarization cannot be measured by a single scattering experiment alone, just as circular polarization of a beam of light cannot be measured by a Nicol prism alone. The longitudinal polarization must be converted into transverse electron polarization or circularly polarized gammas by means of bremsstrahlung just as circular polarization must be transformed into plane polarization by means of a quarter-wave plate. The longitudinal polarization can be transformed to transverse polarization by single scattering as will be shown, and Tolhoek[16] has pointed out that this can also be accomplished by the use of a transverse electric field, which changes the direction of motion of the electron but leaves its spin orientation in space unchanged. Both methods have been used experimentally.[17,18]

1. Scattering of a Polarized Beam

If the initial beam is polarized perpendicular to the plane of scattering, then

$$\binom{I}{\mathbf{P}} = T\begin{pmatrix} 1 \\ 0 \\ 0 \\ P_3 \end{pmatrix} = \lambda\!\!\!{}^{-2}\begin{pmatrix} I+P_3D \\ 0 \\ 0 \\ D+P_3I \end{pmatrix}$$

and the direction of polarization remains unchanged. The ratio of the intensities in the two opposite directions is given by

$$\frac{I_1}{I_2} = \frac{I+P_3D}{I-P_3D} = \frac{1+P_3S}{1-P_3S}, \tag{38}$$

where

$$I_1 = (1\ 0\ 0\ 0)T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad I_2 = (1\ 0\ 0\ 0)T\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Thus single scattering can be used to determine the degree of polarization from

$$P_3 = (I_1 - I_2)/S(I_1 + I_2). \tag{39}$$

This experiment is shown schematically in Figs. 10(a) and 10(b).

A simple physical picture of the spin dependence of the scattering can be presented. Consider Fig. 10 (b) where the spin angular momentum is down; i.e., the electron's magnetic moment is pointing up. The electron in its rest system sees an effective magnetic field caused by the current of the positive nucleus moving toward it. For those electrons passing to the right of the nucleus (i.e., scattering to the left) the magnetic interaction adds to the electric one; whereas for those that pass to the left the magnetic interaction opposes the electric

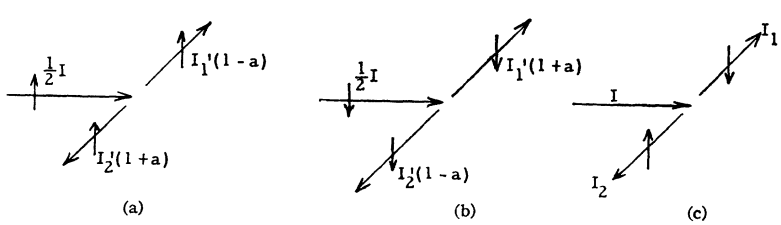


FIG. 10. (a) Spin "up" component; (b) spin "down" component; (c) resulting polarization. In (a) and (b) we have an analysis of an unpolarized beam into two independent components of opposite spin. Each component is scattered with a different intensity so that the resulting beams I_1 and I_2 of the unpolarized beam shown in (c) are partially polarized. The arrows on the beam show the predominant spin directions.

[16] J. A. Tolhoek and S. R. Degroot, Physica **17**, 17 (1951).
[17] A. de Shalit, S. Kuperman, H. J. Lipkin, and T. Rothem, Phys. Rev. **107**, 1459 (1957).
[18] A. I. Alikhanou, G. P. Eliseieu, V. A. Lubimov, and B. V. Ershler, Nuclear Phys. **5**, 588 (1958).

interaction. Thus more electrons and scattered to the left, or into the paper.

2. Scattering of an Unpolarized Beam

We have seen the analyzing properties of electron-scattering experiments from the discussion in Sec. III. A.1; now let us investigate the polarization of an unpolarized electron beam by single scattering. Since scattering is sensitive only to spin states orthogonal to the scattering plane, we expect the scattered beam to be partially polarized orthogonal to the plane of scattering, which is indeed the case as shown earlier, and is now demonstrated by means of the Stokes vector:

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \lambda^2\begin{bmatrix}I\\0\\0\\D\end{bmatrix} = \lambda^2\begin{bmatrix}|f|^2+|g|^2\\0\\0\\i(fg^*-f^*g)\end{bmatrix}. \tag{40}$$

This shows that the beam is partially polarized perpendicular to the plane of scattering with a degree of polarization

$$P=\frac{D}{I}=\frac{i(fg^*-f^*g)}{|f|^2+|g|^2}=S(\theta). \tag{41}$$

Figure 10 illustrates this process. An unpolarized beam of electrons can be regarded as an incoherent superposition of two beams of opposite polarization; hence, the beam can be subdivided into the two components shown in Figs. 10(a) and 10(b). These two components are scattered differently, and thus the scattered beams I_1 and I_2 as shown in Fig. 10(c) are partially polarized with the predominant spin direction as shown. Since Sherman's $S(\theta)$ function is mostly negative, we define

$$a=-S$$

so that the polarization effect is readily apparent.

3. Double Scattering of an Unpolarized Beam

Let us consider the scattering geometry of Fig. 11 for an unpolarized incident beam.

As a result of the first scattering, the beam is characterized by

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \lambda^2\begin{bmatrix}I_1\\0\\0\\D_1\end{bmatrix}.$$

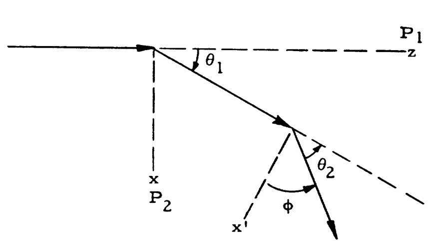


FIG. 11. Schematic diagram of the double-scattering experiment with the xz plane being the plane of the paper.

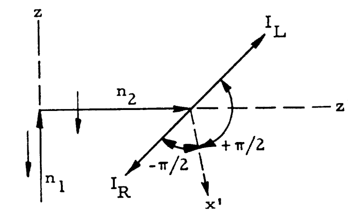


FIG. 12. Double-scattering experiment to measure the longitudinal polarization of β-decay electrons.

For the second scattering, we must rotate our coordinate system an angle about the z' axis. The rotation matrix is

$$M(\phi)=\begin{bmatrix}1&0&0\\0&0&0\\0&\cos\phi&\sin\phi\\0&-\sin\phi&\cos\phi\end{bmatrix}.$$

The intensity of the twice scattered beam is given by

$$\sigma=(1\ 0\ 0\ 0)T_2MT_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix},$$

$$\sigma=\lambda^2(1\ 0\ 0\ 0)T_2\begin{bmatrix}I_1\\0\\D_1\sin\phi\\D_1\cos\phi\end{bmatrix}$$

which yields

$$\sigma=\lambda^4 I_1I_2(1+\delta\cos\phi), \tag{42}$$

where $\delta=S(\theta_1)S(\theta_2)$. Graphs of δ as a function of electron energy are given by Tolhoek[6,7] and others. For electrons, δ has its maximum value for kinetic energies of about 150 kev.

4. Double Scattering of a Longitudinally Polarized Beam

As a result of nonconservation of parity, the electrons emitted in β decay are longitudinally polarized opposite to their direction of motion and the degree of polarization is equal to $\beta=v/c$. One method of measuring this polarization is by the double scattering experiments shown in Fig. 12, where the longitudinal polarization is converted to transverse polarization by Mott scattering. This experiment was performed by de Shalit[17] and the method analyzed by Gürsey and Tassie.[19] This shows how the experiment is treated using the interaction matrix and the Stokes parameters. As a result of the first scattering the beam is represented by the Stokes parameters

$$\begin{pmatrix} I_1 \\ \mathbf{P}_1 \end{pmatrix} = T\begin{bmatrix}1\\P\\0\\0\end{bmatrix} = \lambda^2\begin{bmatrix}I_1\\PL_1\\PT_1\\D_1\end{bmatrix}.$$

[19] F. Gürsey, Phys. Rev. **107**, 1734 (1957); L. J. Tassie, *ibid.* **107**, 1452 (1957).

To describe the second scattering we must rotate the coordinate system by an angle $\phi=\pm\pi/2$ since the second scattering plane is perpendicular to that of the first scatter. Thus

$$\lambda^2\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}\begin{bmatrix}I_1\\PL_1\\PT_1\\D_1\end{bmatrix}=\lambda^2\begin{bmatrix}I_1\\PL_1\\D_1\\-PT_1\end{bmatrix},\quad(\phi=\pi/2)$$

represents the incident beam for scattering into I_L and

$$\lambda^2\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}\begin{bmatrix}I_1\\PL_1\\PT_1\\D_1\end{bmatrix}=\lambda^2\begin{bmatrix}I_1\\PL_1\\-D_1\\PT_1\end{bmatrix},\quad(\phi=-\pi/2)$$

represents the incident beam for scattering into I_R. Thus the left-right asymmetry ratio is given by

$$R=\frac{I_L-I_R}{I_L+I_R}=-\frac{D_2T_1P}{I_2I_1}. \tag{43}$$

For this experiment R is positive since D and T are, in general, negative numbers and $P\approx-\beta\approx-1$. De Shalit used a source of 10 mC of $P^{32}(\beta\sim1)$ and their measured asymmetry ratio was $2R=5.4\times10^{-2}$.

B. Electron-Electron Scattering (Møller Scattering)

Coulomb scattering of electrons was seen to be insensitive to longitudinal polarization; however, longitudinal polarization can be measured by electron-electron scattering. Stehle[20] has given a table of the matrix elements necessary to compute various polarization-sensitive cross sections. These matrix elements have been used to find the interaction matrix for Møller scattering. The matrix presented in Eq. (44) is for the case where the target electron's spin is either parallel or antiparallel with the direction of motion of the incident electron (z axis), and only one of the final electrons is observed. The Stokes parameters of the final electron are given in the new coordinate system rotated through the scattering angle θ about the y axis. The xz plane is chosen as the plane of scattering. With these conventions, the interaction matrix is

$$T=\frac{r_0^2}{4}\frac{1}{\gamma^2(\gamma^2-1)^2\sin^4\theta}\begin{bmatrix}I&\mp A&-B&0\\\pm C&C&\pm D&0\\\pm D&D&\mp E&0\\0&0&0&F\end{bmatrix}, \tag{44}$$

where

$$\begin{aligned}
I&=[(2\gamma^2-1)^2(4-3\sin^2\theta)+(\gamma^2-1)^2(\sin^4\theta+4\sin^2\theta)],\\
A&=[(2\gamma^2-1)(4\gamma^2-3)\sin^2\theta-(\gamma^4-1)\sin^4\theta]\\
B&=2\gamma(\gamma^2-1)\cos\theta\sin^3\theta,\\
C&=2\cos\theta(2\gamma^2-1)(2\gamma^2-1-\gamma^2\sin^2\theta),\\
D&=2\gamma(2\gamma^2-1)\sin\theta\cos^2\theta,\\
E&=2(2\gamma^2-1)(2\gamma^2-1-\sin^2\theta)\cos\theta,\\
F&=2[(2\gamma^2-1)^2-(2\gamma^4-1)\sin^2\theta],
\end{aligned}$$

γ = electron energy in units of mc^2.

In Eq. (44), the upper sign refers to the target electron spin parallel to the incident momentum ($+z$ direction) and the lower sign is for the antiparallel case. The variables γ and θ are in the center of mass system. They are related to the corresponding laboratory variables γ' and θ' by the following formulas[21]:

$$\begin{aligned}
\gamma'&=2\gamma^2-1,\\
\cos\theta&=\frac{2-(\gamma'+3)\sin^2\theta'}{2+(\gamma'-1)\sin^2\theta'},\\
d\Omega&=\frac{8(\gamma'+1)\cos\theta'}{[2+(\gamma'-1)\sin^2\theta']^2}d\Omega'.
\end{aligned} \tag{45}$$

As an example let us consider the case where the incident electron has an arbitrary spin specified by the polar and azimuthal angles χ and ϕ, and the target electron is polarized parallel to the momentum of the incident electron. This case is equivalent to that considered by Ford and Mullin[22] where they chose the incident spin parallel to the incident momentum and let χ and ϕ specify the spin direction of the target electron. In this case the differential scattering cross section is

$$\frac{d\sigma}{d\Omega}=(1\ 0\ 0\ 0)T\begin{bmatrix}1\\\cos\chi\\\sin\chi\cos\phi\\\sin\chi\sin\phi\end{bmatrix} \tag{46}$$

or

$$\frac{d\sigma}{d\Omega}=\frac{r_0^2}{4}\frac{1}{\gamma^2(\gamma^2-1)^2\sin^4\theta}[I-A\cos\chi-B\sin\chi\cos\phi], \tag{47}$$

which is identical with Eq. (7) of Ford and Mullin.

For the case where the target electron is unpolarized, the interaction matrix becomes

$$T=\frac{r_0^2}{4}\frac{1}{\gamma^2(\gamma^2-1)^2\sin^4\theta}\begin{bmatrix}I&0&0&G\\0&C&D&0\\0&D&E&0\\F&0&0&F\end{bmatrix}, \tag{48}$$

[20] P. Stehle, Phys. Rev. **110**, 1458 (1958).

[21] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 255.

[22] G. W. Ford and C. J. Mullin, Phys. Rev. **108**, 477 (1957). In Eq. (7) there should be a factor 4 before $\sin^2\theta$ in the second term. The factor is correct in Eq. (6).

where $G=(\gamma^2-1)^2\sin^4\theta-(4\gamma^2-3)\sin^2\theta$. This matrix has the same nonzero elements as that for Mott scattering. Thus, an unpolarized electron beam is partially polarized perpendicular to the plane of scattering as a result of Møller scattering since

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}=T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\sim\begin{bmatrix}I\\0\\0\\F\end{bmatrix}.$$

From Eq. (48) it is a simple matter to compute the depolarization of an electron beam as a result of electron-electron scattering. For the various pure polarization states we have

$$T\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\sim\begin{bmatrix}I\\C\\D\\F\end{bmatrix};\quad T\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\sim\begin{bmatrix}I\\D\\E\\F\end{bmatrix};$$

and

$$T\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\sim\begin{bmatrix}I+G\\0\\0\\2F\end{bmatrix}. \tag{49}$$

C. Positron-Electron Scattering (Bhabha Scattering)

The interaction matrix for positron-electron scattering can be found by using Stehle's[20] matrix elements. Again we choose the xz plane as the plane of scattering. The interaction matrices are given here for the incident and final positron. If the target electron is unpolarized, then

$$T=\frac{r_0^2}{16\gamma^2(\gamma^2-1)^2}\begin{bmatrix}I&0&0&A\\0&L&J&0\\0&J&F&0\\B&0&0&C\end{bmatrix}. \tag{50}$$

When the target electron is polarized in the positive z direction,

$$T_{+z}=\frac{r_0^2}{16\gamma^2(\gamma^2-1)^2}\begin{bmatrix}I&M&-G&A\\K&L&J&D\\-H&J&F&H\\B&E&G&C\end{bmatrix}, \tag{51}$$

where

$$\begin{aligned}
I&=a^2+\tfrac{1}{4}(b^2+c^2+d^2+e^2),\\
A&=\tfrac{1}{2}(ce+bd)-a^2,\\
B&=\tfrac{1}{2}(bc+de)-a^2,\\
C&=\tfrac{1}{2}(be+dc)+a^2,\\
D&=\tfrac{1}{2}(ce-bd),\\
E&=\tfrac{1}{2}(de-bc),\\
F&=\tfrac{1}{2}(dc-be),\\
G&=\tfrac{1}{2}a(b-c+d-e),\\
H&=\tfrac{1}{2}a(b+c-d-e),\\
J&=\tfrac{1}{2}a(b+c+d+e),\\
K&=\tfrac{1}{4}(-b^2+c^2-d^2+e^2),\\
L&=\tfrac{1}{4}(b^2-c^2-d^2+e^2),\\
M&=\tfrac{1}{4}(-b^2-c^2+d^2+e^2).
\end{aligned}$$

The small letters refer to Stehle's matrix elements

$$\begin{aligned}
a&=\gamma\cot(\theta/2)[1-2\beta^2\sin(\theta/2)],\\
b&=\cot^2(\theta/2)[1+2\beta^2\gamma^2\cos^2(\theta/2)],\\
c&=[-1+2\beta^2\gamma^2\sin^2(\theta/2)],\\
d&=-\gamma^{-2}[1+2\beta^2\gamma^2\cos^2(\theta/2)],\\
e&=-\{\beta^2(2\gamma^2+1)+\cot^2(\theta/2)[(2\gamma^2-1)-2\beta^2\sin^2(\theta/2)]\}.
\end{aligned}$$

The interaction matrix for a target electron whose spin is in the negative z direction can be obtained from Eq. (51) by interchanging

b and e
d and c

in the expressions for the matrix elements.

D. Internal Conversion Electors Following Beta Decay

One of the results of the nonconservation of parity in β decay is that the residual nucleus is left polarized in the direction of the emitted electron. Then, if internal conversion follows the β decay, the conversion electrons have a definite polarization. On using the results of Geshkenbein,[23] we obtain the Stokes vector of the conversion electron in the form

$$\begin{pmatrix} I \\ \mathbf{S} \end{pmatrix}=\begin{bmatrix}A\\(A_1-2A_2)\cos\theta\\(A_1+A_2)\sin\theta\\0\end{bmatrix}, \tag{52}$$

where θ is the angle between the emitted β particle and the conversion electron. The z axis is chosen along the direction of the conversion electron and the xz plane is the plane of interaction.

The transverse polarization of the K-conversion electrons following the β decay of Hg^{203} was measured by Alberghini and Steffen[24] using Mott scattering and they found $(A_1+A_2)=0.35\beta$.

E. Electrons from Mu-Meson Decay

From the two-component neutrino theory of Lee and Yang we find that the muons from π-μ decay are completely polarized along their momentum vector. Consequently, the electrons from the decay of such muons should also be polarized.[25,26] From the cross section of Okum and Shekhter[26] we obtain the interaction matrix

$$T=\frac{\mu w^4}{96\pi^4}(\epsilon^2-u^2)^{\frac{1}{2}}\begin{bmatrix}A&D&0&0\\E&C&0&0\\0&0&B&-F\\0&0&F&B\end{bmatrix}, \tag{53}$$

[23] B. V. Geshkengein, Nuovo cimento **10**, 375 (1958); Soviet Phys. JETP **8**, 865 (1959).
[24] J. E. Alberghini and R. M. Steffen, Nuclear Phys. **14**, 199 (1959).
[25] H. Uberall, Nuovo cimento **6**, 376 (1957).
[26] L. B. Okum and M. Shekhter, Soviet Phys. JETP **7**, 864 (1958).

where

$$\epsilon = E_e/w, \quad u = m_e/w, \quad w = (m_\mu^2 + m_e^2)/2m_\mu,$$

and the z axis has been chosen along the momentum vector of the electron.

Thus the Stokes vector of the emitted electron is represented by

$$\begin{pmatrix} I \\ \mathbf{S} \end{pmatrix} = T\begin{pmatrix} 1 \\ \mathbf{P} \end{pmatrix} \sim \begin{bmatrix} A+DP_1 \\ E+CP_1 \\ BP_2-FP_3 \\ FP_2+BP_3 \end{bmatrix},$$

where P is the polarization vector of the μ meson. The polarization-sensitive cross section is again given by Eq. (12). Thus F represents the transverse polarization perpendicular to the $(\mathbf{P},\mathbf{p}_e)$ plane and B represents the polarization in this plane.

IV. INTERACTIONS BETWEEN PHOTONS AND ELECTRONS

A. General

In this section we describe processes relating photons and electrons. In these interactions the Stokes vector of an electron (or photon) is transformed into the Stokes vector of a photon (or electron) by means of an interaction matrix in the usual manner

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = T\begin{pmatrix} 1 \\ \mathbf{S} \end{pmatrix} \tag{54a}$$

or

$$\begin{pmatrix} I \\ \mathbf{S} \end{pmatrix} = T\begin{pmatrix} 1 \\ \mathbf{P} \end{pmatrix}, \tag{54b}$$

where now we use S_i to indicate the Stokes parameters of electrons and P_i for photons. In this representation the inverse nature of bremsstrahlung and pair production appears in the interaction matrix as an interchange of rows and columns. The similarity of the photoelectric effect and pair production is readily apparent in the form of their interaction matrices.

The meanings of the positive Stokes parameters are

- P_1: linear polarization perpendicular to the plane of the interaction,
- P_2: linear polarization 45° to the right,
- P_3: left circular polarization,
- S_1: spin in the plane of interaction along the z axis.
- S_2: spin in the plane of interaction along the x axis,
- S_3: transverse spin perpendicular to the plane of interaction (y axis).

When the final product is an electron, it is usually necessary to perform a rotation about the y axis by the angle θ of the interaction so that the Stokes vector has the customary meanings; that is, S_1 referring to longitudinal polarization and S_2 to transverse polarization. The angles involved for the matrices are shown in Fig. 13. In the matrix form it is quite simple to arrive at the cross section of a multiple-type experiment. For

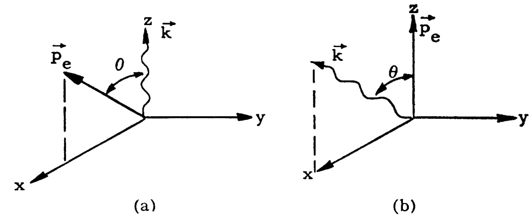


FIG. 13. Schematic diagram for the interaction matrices involving electrons of momentum **p** and photons of momentum **k**. The plane of interaction is always taken as the xz plane. (a) Photons to electrons; (b) electrons to photons.

example, say one would like to measure an electron's longitudinal spin by measuring the asymmetry of the Compton scattering of its bremsstrahlung radiation. In matrix form this experiment is described by

$$d\sigma = \tfrac{1}{2}(1,\mathbf{D})T_cMT_b\begin{pmatrix} 1 \\ \mathbf{S} \end{pmatrix}, \tag{55}$$

where T_b and T_c are the interaction matrices for bremsstrahlung and Compton scattering, M represents a rotation of the coordinate system to bring the Stokes parameters from bremsstrahlung into the coordinate system being used for Compton scattering, and $(1,\mathbf{D})$ represents the detection apparatus.

B. Bremsstrahlung

Bremsstrahlung radiation is in general polarized regardless of the polarization state of the incident electron. Longitudinally polarized electrons give rise to circularly polarized bremsstrahlung which can be detected by spin-dependent Compton scattering. This process has been used experimentally[27] to measure the degree of longitudinal spin that beta particles have as a result of parity nonconservation.[28]

Bremsstrahlung cross sections showing the dependence upon the states of polarization have been worked out by several authors.[29,30] The cross sections in the form developed by Olsen and Maximom[30] are used here to illustrate the use of the matrix representation. The Stokes parameters for the outgoing photon are obtained from Eq. (54a) using the interaction matrix

$$T = \frac{2Z^2}{137} r_0^2 \frac{d\xi}{\epsilon_1} \frac{dk}{k} \begin{bmatrix} I & 0 & 0 & 0 \\ D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -L & -T & 0 \end{bmatrix}, \tag{56}$$

[27] M. Goldhaber, L. Grodsins, and A. W. Sunyar, Phys. Rev. **106**, 826 (1957); S. Galster and H. Schopper, Phys. Rev. Letters **1**, 330 (1958); E. G. Beltrametti and S. Vitale, Nuovo cimento **9**, 289 (1958); and others.

[28] T. D. Lee and C. N. Young, Phys. Rev. **104**, 254 (1956).

[29] R. L. Gluckstern, M. H. Hull, Jr., and G. Greit, Phys. Rev. **90**, 1026 (1953); R. L. Gluckstern and M. H. Hull, Jr., *ibid.* **90**, 1030 (1953); H. Banerjee, *ibid.* **111**, 532 (1958); K. W. McVoy, *ibid.* **106**, 828 (1957); **111**, 1333 (1958).

[30] H. Olsen and L. C. Maximon, Phys. Rev. **110**, 589 (1958); Proc. Phys. Seminar in Trondheim **5** (1958) (to be published in *The Physical Review*). The author is grateful for having received prepublication copies of these two papers and that of footnote 32.

where

$\xi = 1/(1+u^2)$,

u = component of $\mathbf{p}$ perpendicular to $\mathbf{k}$,

ϵ, k = total electron and photon energies in units of mc^2,

$\mathbf{p}$ = electrons initial momentum,

$I = (\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)-2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)$,

$D = 8\epsilon_1\epsilon_2 u^2\xi^2\Gamma$,

$L = k[(\epsilon_1+\epsilon_2)(3+2\Gamma)-2\epsilon_2(1+4u^2\xi^2\Gamma)]$,

$T = 4k\epsilon_2(1-2\xi)\xi\Gamma$,

$\epsilon_2 = \epsilon_1 - k$,

Γ contains the Coulomb and screening effects and $= \ln\alpha - 2 - f(Z)$ for no screening and complete screening; with

$\alpha = 2\epsilon_1\epsilon_2/k = 1/\delta$ for no screening, and

$\alpha = 111\ Z^{-\frac{1}{3}}/\xi$ for complete screening,

$f(Z)$ = Coulomb correction function of Davies, Bethe, and Maximon.[31]

In this representation we are defining the (**pk**) plane as the (xz) plane, and the cross section is for the case where the direction and spin of the final electron are not observed.

From the interaction matrix of Eq. (56), we find that the general Stokes vector for the emitted photon is

$$\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} \approx \begin{bmatrix} I \\ D \\ 0 \\ -S_1L - S_2T \end{bmatrix} \tag{57}$$

which shows that the bremsstrahlung radiation produced by an unpolarized beam of electrons is partially polarized perpendicular to the plane of interaction and that circularly polarized bremsstrahlung can arise only from a polarized electron. Since the linear polarization is not dependent upon the electron's spin, the bremsstrahlung from polarized electrons are, in general, elliptically polarized.

Olsen's cross section shows that the linear polarization is greatest at the low-energy end of the photon spectrum, reaching its maximum value for $\xi = \frac{1}{2}$ at an angle of emission $\theta \approx 1/\epsilon_1$. For example, Gluckstern shows that the low-energy photons ($k < \frac{1}{4}$ Mev) produced in aluminum by 2.5-Mev electrons are about 55% polarized, and for 50-Mev electrons in lead Olsen finds about 45% polarization. For low-energy electrons, Gluckstern and Banerjee's expressions show that at the high-energy end of the photon spectrum there is some polarization in the plane of scattering; however, this effect decreases with increasing electron energy so that for high-energy electrons the high-energy bremsstrahlung is unpolarized.

For the forward bremsstrahlung from longitudinally polarized electrons, Eq. (57) shows that $P_3 = -S_1L$, which means that the sense of circular polarization (right or left) of the bremsstrahlung is the same as that of the electron (parallel or antiparallel to its momentum). The amount of circular polarization increases rapidly with photon energy, and near the upper end of the bremsstrahlung spectrum it is a maximum. This maximum is given by McVoy as being

$$\frac{I_R - I_L}{I_R + I_L} = \left[1 + \frac{(E_0+p_0c)(E_0+mc^2)}{(2E_0-p_0c)(E_0-mc^2)}\right]^{-1}$$

$$\approx \left[1 + \frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right], \tag{58}$$

where I_R and I_L are the intensities of right and left circular polarization. For 2-Mev electrons, this maximum is 97% while at a photon energy of 200 kev, the polarization is down to about 43% and drops rapidly for lower photon energies. For high-energy electrons, Olsen's cross section shows complete circular polarization at the high-energy end of the photon spectrum since, when $k \approx \epsilon_1$, we have

$$\frac{I_R - I_L}{I_R + I_L} = \frac{L}{I} \approx \frac{\epsilon_1^2(3+2\Gamma)}{\epsilon_1^2(3+2\Gamma)} = 1.$$

The circular polarization caused by transverse electron polarization is generally small, i.e., $|L| > |T|$, and $T = 0$ for $\xi = \frac{1}{2}$ and $k = \epsilon_1$, where the polarization due to longitudinal spin is a maximum.

C. Pair Production

Pair production is essentially the inverse process of bremsstrahlung and the interaction matrix is obtained by inverting the rows and columns of the bremsstrahlung interaction matrix and making a change of sign for the ϵ_2 term. In this case the Stokes parameters of the electron is given by Eq. (54b):

$$\begin{pmatrix} I \\ \mathbf{S} \end{pmatrix} = T \begin{pmatrix} I \\ \mathbf{P} \end{pmatrix}$$

with

$$T = \frac{2Z^2}{137} r_0^2 \frac{d\epsilon_1}{k^3} d\xi \begin{bmatrix} I & -D & 0 & 0 \\ 0 & 0 & 0 & -L \\ 0 & 0 & 0 & -T \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{59}$$

where

$I = (\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)+2\epsilon_1\epsilon_2(1+4u^2\xi\Gamma)$,

$D = 8\epsilon_1\epsilon_2 u^2\xi\Gamma$,

$L = k[(\epsilon_1-\epsilon_2)(3+2\Gamma)+2\epsilon_2(1+4u^2\xi^2\Gamma)$,

$T = 4k\epsilon_2\xi(1-2\xi)\Gamma$,

$k = \epsilon_1 + \epsilon_2$.

In this representation the xz plane is the plane of

[31] H. Davies, H. A. Bethe, and L. C. Maximon, Phys. Rev. **93**, 788 (1954).

emission, and positive values of P_1 indicate photon polarization perpendicular to the plane of emission.

For linearly polarized photons the electron is most likely to be emitted in the plane of polarization, since we have the two cases

$$\sigma_{\parallel}=(1\ 0\ 0\ 0)T\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}=I+D$$

and

$$\sigma_{\perp}=(1\ 0\ 0\ 0)T\begin{bmatrix}1\\1\\0\\0\end{bmatrix}=I-D$$

which yield the asymmetry ratio

$$R=(\sigma_{\parallel}-\sigma_{\perp})/(\sigma_{\parallel}+\sigma_{\perp})=D/I.$$

Olsen[30] points out that R is greatest when the electron and positron have about the same energy.

In general, the Stokes parameters of the electron beam are given by

$$\begin{pmatrix}I\\ \mathbf{S}\end{pmatrix}\approx\begin{bmatrix}I-P_1D\\-P_3L\\-P_3T\\0\end{bmatrix}$$

which show that a circularly polarized photon beam results in a polarized electron-positron pair. The degree of longitudinal polarization is just $\pm L/I$ where the $+$ sign arises for right circular photons since $P_3=-1$. The more energetic particle of the pair is polarized in the same sense as the circularly polarized photon.

D. Photoelectric Effect

The similarity between pair production and the photoelectric effect shows up in that the same elements of the interaction matrix are nonzero. The Stokes parameters for the photoelectrons are given by Eq. (54b) and using the interaction matrix derived from Olsen's[32] cross section for the K shell, we find

$$T=\frac{Z^5}{(137)^4}r_0^2\beta^3\frac{\epsilon}{k^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^3}\begin{bmatrix}1+D & -D & 0 & 0\\0 & 0 & 0 & -A\\0 & 0 & 0 & B\\0 & 0 & 0 & 0\end{bmatrix},\tag{60}$$

where

$$D=\frac{1}{k}\left[\frac{2}{k\epsilon(1-\beta\cos\theta)}-1\right],$$

$$A=\frac{\epsilon}{\epsilon+1}\left[\frac{2}{k\epsilon}+\beta\cos+\frac{2}{k\epsilon^2(1-\beta\cos\theta)}\right],$$

$$B=\frac{\epsilon}{\epsilon+1}\beta\sin\theta\left[\frac{2}{k\epsilon(1-\beta\cos\theta)}-1\right].$$

[32] H. Olsen, Kgl. Norske Videnskab. Selskabs Forh. **31**, Nos. 11, 11a (1958).

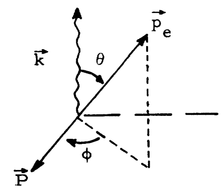


FIG. 14. Schematic diagram for the study of the angular distribution of photoelectrons ejected by linearly polarized photons. **P** represents the direction of polarization and P is the degree of polarization $(0\leqslant P\leqslant 1)$.

From this interaction matrix the general Stokes vector of the photoelectron is

$$\begin{pmatrix}I\\ \mathbf{S}\end{pmatrix}\approx\begin{bmatrix}1+D-P_1D\\-AP_3\\BP_3\\0\end{bmatrix},\tag{61}$$

which shows that the angular distribution is dependent upon the linear polarization of the incident photon and that polarized electrons are produced by circularly polarized gammas.

It is of interest to investigate the angular distribution of photoelectrons ejected by linearly polarized photons, using the geometry shown in Fig. 14. In this geometry the Stokes vector of the incident photon is given by

$$\begin{pmatrix}1\\ \mathbf{P}\end{pmatrix}=\begin{bmatrix}1\\-P\cos2\phi\\-P\sin2\phi\\0\end{bmatrix},\tag{62}$$

where P is the degree of polarization $(0\leq P\leq 1)$. The differential cross section for a completely polarized incident photon beam is then

$$d\sigma=(1\ 0\ 0\ 0)T\begin{bmatrix}1\\-\cos2\phi\\-\sin2\phi\\0\end{bmatrix}\approx 1+D+D\cos2\phi,$$

$$d\sigma\approx 1+2D\cos^2\phi,$$

and upon substitution of those terms involving the angular distribution, we find

$$d\sigma\sim\frac{\sin^2\theta}{(1-\beta\cos\theta)^4}\left\{\frac{k^2}{4}(1-\beta\cos\theta)+\left[\frac{1}{\epsilon}-\frac{k}{2}(1-\beta\cos\theta)\right]\cos^2\theta\right\},\tag{63}$$

which is the same as that given by Sauter.[33] The polarization-dependent term shows some interesting properties. If $\epsilon>5/3$ ($E_{\text{kin}}>0.33$ Mev), there is a range of values of θ and k given by

$$k(k+1)(1-\beta\cos\theta)=2\tag{64}$$

for which the cross section is independent of the polarization of the photon. For photon energies less than this critical value, D is positive and the photo-

[33] F. Sauter, Ann. Physik **11**, 454 (1931); F. Sauter and H. O. Wuster, Z. Physik **141**, 83 (1955).

electrons are ejected predominantly in the plane defined by the incoming photon and its electric vector. For photon energies greater than this critical value, D is negative and the photoelectrons are ejected in the plane of the magnetic vector. This effect can be seen if we define an asymmetry ratio

$$R = d\sigma(0)/d\sigma(\pi/2) = 1+2D, \tag{65}$$

that is, the ratio of the differential cross section for ejection coplanar with, and orthogonal to, the electric vector of the incident photon. This asymmetry ratio is plotted in Fig. 15 as a function of the incident photon energy.

This crossover feature of the photoelectric effect has been verified experimentally by McMaster and Hereford[34] who used the Compton effect as a source of partially polarized photons.

Equation (61) shows that photoelectrons ejected by circularly polarized gammas are polarized. The degree of polarization in the z direction for right circularly polarized photons is seen to be

$$P_z = A$$

and that in the x direction is

$$P_x = -B.$$

There is no polarization in the y direction. Thus the electron is polarized in the plane of emission and the direction of the spin vector is given by

$$\tan\chi = -\frac{B}{A} = -\frac{\beta\sin\theta\left[\dfrac{2}{k\epsilon}-1+\beta\cos\theta\right]}{\left[\left(\dfrac{2}{k\epsilon}+\beta\cos\theta\right)(1-\beta\cos\theta)+\dfrac{2}{k\epsilon^2}\right]}. \tag{66}$$

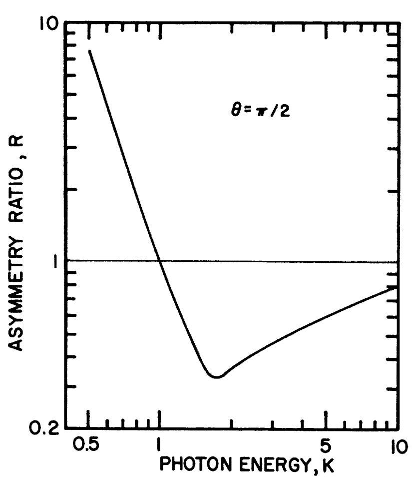


FIG. 15. Asymmetry ratio for photoelectrons ejected from the K shell by linearly polarized photons for $\theta=\pi/2$.

[34] W. H. McMaster and F. L. Hereford, Phys. Rev. **95**, 723 (1954).

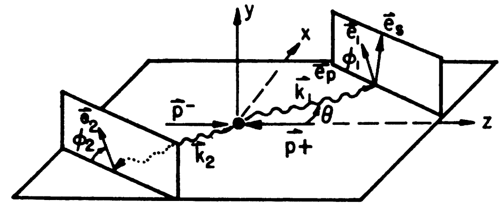


FIG. 16. Center-of-mass system for the two quanta annihilation of positrons. The angles ϕ_1 and ϕ_2 give the angle between the polarization vectors of the two quanta $\mathbf{k}_1$ and $\mathbf{k}_2$ and the plane of interaction (xz plane).

At the cross over energy and angle given by Eq. (64), $\tan\chi=0$, which means that under these conditions the electrons are polarized in the direction of the incident photon with a degree of polarization

$$P=A=1,$$

that is, the photoelectrons are completely polarized. The spin of the electron is parallel to $\mathbf{k}$ for right circularly polarized gammas and antiparallel for left circular polarization.

E. Positron Annihilation-in-Flight

For positron annihilation-in-flight we discuss the center-of-mass geometry shown in Fig. 16. We follow the general form of Page[35] and McMaster.[36] The description of the photons' polarization is simplified by defining a set of functions dependent upon these polarizations as

$$\begin{aligned}
\rho_0 &= e^{i\delta_1}\sin\phi_2\cos\phi_1 - e^{i\delta_2}\cos\phi_2\sin\phi_1,\\
\rho_1 &= e^{i\delta_2}\sin\phi_2\cos\phi_1 - e^{i\delta_1}\cos\phi_2\sin\phi_1,\\
\rho_2 &= \cos\phi_2\cos\phi_1 + e^{i(\delta_1+\delta_2)}\sin\phi_2\sin\phi_1,\\
\rho_3 &= \cos\phi_2\cos\phi_1 - e^{i(\delta_1+\delta_2)}\sin\phi_2\sin\phi_1,
\end{aligned} \tag{67}$$

where the electric vector of the photon is represented by $\mathbf{e}=\cos\phi\mathbf{e}_p+\sin\phi e^{i\delta}\mathbf{e}_s$ with $\mathbf{e}_p$ and $\mathbf{e}_s$ being unit vectors in the plane of interaction and perpendicular to it, respectively, and δ is the phase angle. For linear polarization ($\delta=0$), Eqs. (67) reduce to

$$\rho_0=\sin(\phi_2-\phi_1), \quad \rho_2=\cos(\phi_2-\phi_1),$$
$$\rho_1=\sin(\phi_2+\phi_1), \quad \rho_3=\cos(\phi_2+\phi_1).$$

For a complete description of the polarizations of the

TABLE II. Photon polarization states.

State	Linear				Circular			
	pp	ss	ps	sp	RR	LL	RL	LR
ρ_0	0	0	1	−1	$-i$	i	0	0
ρ_1	0	0	1	1	0	0	i	$-i$
ρ_2	1	1	0	0	1	1	0	0
ρ_3	1	−1	0	0	0	0	1	1

[35] L. A. Page, Phys. Rev. **106**, 394 (1957).
[36] W. H. McMaster, Nuovo cimento **7**, 395 (1960).

two photons $\mathbf{k}_1$ and $\mathbf{k}_2$ we can specify the ρ's in either a set of plane-polarization states or circular polarization states. For the plane-polarization description, let p and s refer to polarization in and perpendicular to the plane of interaction, respectively. For the circular polarization description, let R and L indicate right and left circular polarization, respectively.[37] With these conventions, the values of the ρ's for each set are given in Table II.

The cross section for positron annihilation-in-flight can now be written in matrix form as

$$\frac{d\sigma}{d\Omega}=(1,\mathbf{S})T\begin{pmatrix}1\\ \mathbf{P}\end{pmatrix}, \tag{68}$$

where $\mathbf{S}$ and $\mathbf{P}$ refer to the electron and positron, respectively. The interaction matrices have the form

$$T_c=\frac{r_0^2}{8\beta\epsilon^2(1-\beta^2\cos^2\theta)^2}\begin{bmatrix} H_0^2+H^2 & 2i(H_xH_y-H_0H_z) & 2iH_0H_x & 0\\ 2i(H_0H_z+H_xH_y) & H^2-2H_z^2-2H_0^2 & 0 & 0\\ 2iH_0H_x & 0 & H_0^2-H^2+2H_x^2 & 0\\ 0 & 0 & 0 & H_0^2-H^2+2H_y^2\end{bmatrix} \tag{69}$$

for the description of circular polarization states of the annihilation quanta, and

$$T_p=\frac{r_0^2}{8\beta\epsilon^2(1-\beta^2\cos^2\theta)^2}\begin{bmatrix} H_0^2+H^2 & 0 & 0 & 0\\ 0 & H^2-2H_z^2-H_0^2 & 2H_xH_z & 0\\ 0 & 2H_xH_z & H_0^2-H^2+2H_x^2 & 0\\ 0 & 0 & 0 & H_0^2-H^2+2H_y^2\end{bmatrix} \tag{70}$$

for the description of plane-polarization states of the annihilation quanta. For a transformation to the laboratory system one utilizes the standard formulas

$$E=2\epsilon^2-1,\quad \beta=\left(\frac{E-1}{E+1}\right)^{\frac{1}{2}},$$

and

$$\theta_{\text{lab}}=\tan^{-1}\left[\frac{\sin\theta(1-\beta^2)^{\frac{1}{2}}}{\beta+\cos\theta}\right],$$

where β is given in the c.m. system; E and ϵ are the total energies in units of mc^2 in the lab and c.m. system, respectively.

The matrix elements of Eqs. (69) and (70) are given by

$$\begin{aligned} &H^2=H_x^2+H_y^2+H_z^2,\\ &H_0=(\beta/\epsilon)(\rho_2+\rho_3\sin^2\theta),\\ &H_x=-\rho_1\beta\sin\theta,\\ &H_y=\rho_3\beta\sin\theta\cos\theta,\\ &H_z=\rho_0/\epsilon.\end{aligned} \tag{71}$$

1. Unpolarized Electrons and Positrons

The differential cross section for unpolarized initial and final spins is obtained from

$$\frac{d\sigma}{d\Omega}=(1\,0\,0\,0)T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\sim H_0^2+H^2 \tag{72}$$

for both the linear and circular polarization descriptions.

On substituting from Eqs. (71), one obtains

$$\frac{d\sigma}{d\Omega}=\frac{r_0^2}{8\beta\epsilon^2}\frac{1}{(1-\beta^2\cos^2\theta)^2}\left\{\frac{1}{\epsilon^2}[\rho_0^2+\beta^2(\rho_2+\rho_3\sin^2\theta)^2]+\beta^2\sin^2\theta(\rho_1^2+\rho_3^2\cos^2\theta)\right\}. \tag{73}$$

This equation shows that in the low-energy limit ($\beta\to 0$) the photons tend to be cross-polarized (ps and sp). In the circular polarization description, the photons are RR and LL, i.e., there is no net angular momentum.

If the polarizations of the two photons are not observed, then one obtains the differential cross section by summing over all directions of photon polarization as given in Table II:

$$\frac{d\sigma}{d\Omega}=\frac{r_0^2}{4\beta\epsilon^2}\left[\frac{1+2\beta^2\sin^2\theta-\beta^4-\beta^4\sin^4\theta}{(1-\beta^2\cos^2\theta)^2}\right]. \tag{74}$$

2. Longitudinal Spins

For the plane polarization description we have

$$\frac{d\sigma}{d\Omega}=(1\,S_z\,0\,0)T_p\begin{bmatrix}1\\P_z\\0\\0\end{bmatrix}\sim H_0^2+H^2+P_zS_z(H^2-2H_z^2-H_0^2), \tag{75}$$

which leads to the two cases of parallel and antiparallel spins

$$(d\sigma/d\Omega)_{\text{par}}\sim 2(H_x^2+H_y^2)$$

and

$$(d\sigma/d\Omega)_{\text{opp}}\sim 2(H_0^2+H_z^2).$$

These equations show that at low energies the annihilation takes place with spins opposed, while at high

[37] In this representation, e.g., right circular polarization of both photons is specified by the vectors

$$\mathbf{e}_1=\tfrac{1}{2}(\mathbf{e}_p-i\mathbf{e}_s)\quad\text{and}\quad \mathbf{e}_2=\tfrac{1}{2}(\mathbf{e}_p+i\mathbf{e}_s),$$

thus each photon advances as a right-hand screw.

energies annihilation with parallel spins predominates. In both cases the gammas tend to be cross-polarized; however, for $\theta=0$, $(d\sigma/d\Omega)_{\text{par}}=0$. This feature has been used experimentally to measure the longitudinal polarization of positrons in β decay.[38]

For the circular polarization description, we have

$$d\sigma/d\Omega \sim H_0^2+H^2+2iP_z(H_xH_y-H_0H_z) +2iS_z(H_0H_z+H_xH_y) +S_zP_z(H^2-2H_z^2-H_0^2), \quad (76)$$

which leads to the two cases of parallel and antiparallel spins

$$(d\sigma/d\Omega)_{\text{par}} \sim 2(H_x^2+H_y^2 \pm 2iH_xH_y),$$

and

$$(d\sigma/d\Omega)_{\text{opp}} \sim 2(H_0^2+H_z^2 \pm 2iH_0H_z),$$

where the upper sign refers to positron spin in the positive Z direction. For positrons polarized in the positive Z direction, the ratio of right circular polarization (RCP) to left circular polarization (LCP) for these two cases in

$$\left(\frac{RCP}{LCP}\right)_{\text{par}} = \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^2$$

and

$$\left(\frac{RCP}{LCP}\right)_{\text{opp}} = \frac{(1-\beta)^2+\beta^2\sin^4\theta}{(1+\beta)^2+\beta^2\sin^4\theta}$$

These results are illustrated in Fig. 17 where we see that the photon **k** tends to pick up the helicity of the particle whose momentum is nearly parallel to **k**.

Many times an experiment is conducted with just one of the particles polarized. For the discussion which follows we limit observation to the higher-energy photon in the direction of the positron's momentum.

For positrons polarized in the negative Z direction

(a) Antiparallel with positive positron spin. (b) Antiparallel with negative positron spin.

(c) Parallel with positive positron spin. (d) Parallel with negative positron spin.

FIG. 17. Circular polarization correlations at high energy for longitudinal spin. Two values of θ are illustrated, $\theta_1<\pi/2$ and $\theta_2>\pi/2$. The solid arrows designate the directions of the electron's and positron's spin. The electron is shown on the left.

[38] See, for example, S. Frankel, P. G. Hansen, O. Nathan, and G. M. Temmer, Phys. Rev. **108**, 1099(L) (1957).

(a) Polarized positrons, unpolarized target. (b) Unpolarized positrons, polarized target.

FIG. 18. Circular polarization produced by high-energy positron annihilation-in-flight when one of the particles is unpolarized. The electron target is on the left and the positron is incident from the right.

incident on unpolarized electrons, we have

$$\frac{d\sigma}{d\Omega}=(1\ 0\ 0\ 0)T\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} \sim (H_0^2+H^2+2iH_0H_z-2iH_xH_y). \quad (77)$$

From this equation one finds that at all energies the observed photons are predominantly RCP.

When the target electrons are polarized in the positive Z direction and the incident positrons are unpolarized, the cross section is given by

$$\frac{d\sigma}{d\Omega}=(1\,1\,0\,0)T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \sim H_0^2+H^2+2iH_0H_z+2iH_xH_y. \quad (78)$$

This equation shows that at $\beta=0$ the photons are unpolarized, at high energies they are LCP, and for an intermediate range they are RCP. These two cases are shown in Fig. 18.

3. *Transverse Spins*

The polarization correlations for transverse spins are readily obtained from Eqs. (69) and (70). Let us consider the case where one of the particles is polarized in the plane of annihilation and the electron is unpolarized. Then

$$\frac{d\sigma}{d\Omega}=(1\,0\,0\,0)T\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad\text{or}\quad \frac{d\sigma}{d\Omega}=(1\,0\,1\,0)T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

Thus $d\sigma/d\Omega \sim H_0^2+H^2$ for the plane-polarization description which states immediately that there is no polarization correlation; however, if we consider circular polarization, we have

$$d\sigma/d\Omega \sim H_0^2+H^2+2iH_0H_x. \quad (79)$$

The maximum polarization effect occurs for $\theta=\pi/2$, and if we define the degree of polarization as

$$P=(d\sigma_R-d\sigma_L)/(d\sigma_R+d\sigma_L),$$

where $d\sigma_R=\frac{1}{2}(d\sigma_{RR}+d\sigma_{RL})$ and $d\sigma_L=\frac{1}{2}(d\sigma_{LL}+d\sigma_{LR})$, then one finds

$$P=2\beta/\epsilon[1+2\beta^2(1-\beta^2)], \quad (80)$$

which states that the gamma ray is predominantly *RCP* and thus partakes in the helicity of the polarized particle. On the other hand, if one of the particles is polarized perpendicular to the plane of annihilation, no polarization correlation exists since

$$\frac{d\sigma}{d\Omega}=(1\,0\,0\,1)T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=(1\,0\,0\,0)T\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\sim H_0^2+H^2.$$

V. NEUTRON POLARIZATION

A. General

The presentation in this section follows the development of Lepore.[39] In solving neutron scattering problems, the wave function solution of Schrödinger's equation has the asymptotic form

$$\psi=e^{i\mathbf{k}_0\cdot\mathbf{r}}\chi+(e^{ikr}/r)f(\theta)\chi,$$

where χ is the spin function of the incident nucleon, and $f(\theta)$ is the amplitude of the scattered wave at infinity which can be written in the form

$$f(\theta)=A(\theta)+B(\theta)\boldsymbol{\sigma}\cdot\mathbf{n} \tag{81}$$

for a spin zero nucleus, where $\mathbf{n}$ is a vector perpendicular to the plane of scattering and is defined by

$$\mathbf{k}\times\mathbf{k}_0=\mathbf{n}k^2\sin\theta.$$

The intensity of the scattered beam is given by

$$\begin{aligned}\sigma(\theta)&=|f(\theta)|^2=(A^*+B^*\boldsymbol{\sigma}\cdot\mathbf{n})(A+B\boldsymbol{\sigma}\cdot\mathbf{n}),\\ \sigma(\theta)&=(A^*A+B^*B)+(A^*B+B^*A)\mathbf{P}\cdot\mathbf{n},\end{aligned} \tag{82a}$$

where $\mathbf{P}$ is the polarization vector and is in the $\boldsymbol{\sigma}$ direction. The polarization of the scattered beam is given by

$$\mathbf{P}=[f(\theta)^*\boldsymbol{\sigma}f(\theta)]/|f(\theta)|^2.$$

The numerator on expansion becomes

$$N=(A^*+B^*\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\sigma}(A+B\boldsymbol{\sigma}\cdot\mathbf{n}),$$
$$N=AA^*\boldsymbol{\sigma}+AB^*(\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\sigma}+BA^*\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\mathbf{n})+BB^*(\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\mathbf{n}).$$

Now

$$\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\mathbf{n})=\mathbf{n}-i(\boldsymbol{\sigma}\times\mathbf{n})$$
$$(\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\sigma}=\mathbf{n}+i(\boldsymbol{\sigma}\times\mathbf{n})$$
$$(\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\mathbf{n})=2(\boldsymbol{\sigma}\cdot\mathbf{n})\mathbf{n}-\boldsymbol{\sigma}.$$

On replacing $\boldsymbol{\sigma}$ by $\mathbf{P}$, the numerator becomes

$$N=AA^*\mathbf{P}+(AB^*+BA^*)\mathbf{n}+i(AB^*-BA^*)(\mathbf{P}\times\mathbf{n})+BB^*2[(\mathbf{P}\cdot\mathbf{n})\mathbf{n}-\mathbf{P}]. \tag{82b}$$

Equations (82a) and (82b) can be rewritten in matrix form as

$$\sigma(\theta)=\tfrac{1}{2}(1,\mathbf{D})T\begin{pmatrix}1\\\mathbf{P}\end{pmatrix},$$

where the interaction matrix is given by

$$T=\frac{1}{4\pi}\begin{bmatrix}I&0&0&D\\0&L&-T&0\\0&T&L&0\\D&0&0&I\end{bmatrix}, \tag{83}$$

which is the same form as that derived for electron scattering. The matrix elements for Eq. (83) are given by

$$I=AA^*+BB^*,$$
$$L=AA^*-BB^*,$$
$$T=i(AB^*-BA^*),$$
$$D=AB^*+BA^*.$$

This shows that the scattering of an unpolarized incident beam results in a beam partially polarized orthogonal to the plane of scattering with the Stokes parameters being given by

$$\begin{pmatrix}I\\\mathbf{P}\end{pmatrix}=\frac{1}{4\pi}\begin{bmatrix}I\\0\\0\\D\end{bmatrix}. \tag{84}$$

The positive Stokes parameters in this representation have the values

P_1: longitudinal polarization
P_2: transverse polarization in the plane of scattering
P_3: transverse polarization orthogonal to the plane of scattering.

The resulting degree of polarization is

$$P=\frac{D}{I}=\frac{AB^*+BA^*}{AA^*+BB^*}. \tag{85}$$

To observe this polarization one must perform a double-scattering experiment. This experiment is illustrated in Fig. 19. The first scattering results in the Stokes parameters of Eq. (84). The plane of the second scattering is at an angle ϕ to the first plane, so that it is

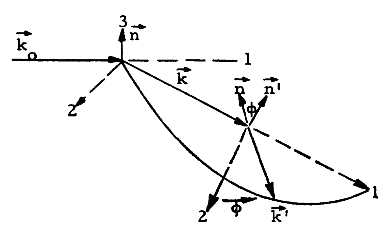


FIG. 19. Double-scattering experiment. The second scattering forms a plane $(\mathbf{k},\mathbf{k}')$ at an angle ϕ to the first scattering plane $(\mathbf{k}_0,\mathbf{k})$.

[39] J. V. Lepore, Phys. Rev. **79**, 137 (1950); see also, K. B. Mather and P. Swan, *Nuclear Scattering* (Cambridge University Press, New York, 1958), p. 269.

necessary to perform a rotation about the 1 axis. Thus the Stokes parameters for the incident beam in the second scattering is given by

$$M\begin{pmatrix} I \\ \mathbf{P} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} I_1 \\ 0 \\ 0 \\ D_1 \end{bmatrix} \begin{bmatrix} I_1 \\ 0 \\ D_1\sin\phi \\ D_1\cos\phi \end{bmatrix} = \begin{pmatrix} I \\ \mathbf{P}' \end{pmatrix}.$$

The intensity of the second scattered beam given by

$$\sigma = (1\,0\,0\,0)T_2 M T_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (86)$$

is just

$$\sigma = I_1 I_2 + D_1 D_2 \cos\phi = I_1 I_2 [1 + P(\theta_1)P(\theta_2)\cos\phi]. \qquad (87)$$

The asymmetry of the second scattered beam is generally determined when both scatterings take place in the same plane, i.e., $\phi = 0$. Then if θ_2 is taken in both directions, the asymmetry ratio of the scattered intensities is

$$R = [1 + P(\theta_1)P(\theta_2)]/[1 - P(\theta_1)P(\theta_2)].$$

B. Small-Angle Scattering

In the small-angle scattering of neutrons in the Mev range, Schwinger[40] shows that the resulting neutrons should be partially polarized due to the spin-orbit interaction arising from the motion of the neutron's magnetic moment in the nuclear Coulomb field. For Eq. (81) Schwinger uses

$$f(\theta) = f_0(\theta) + i\gamma \cot(\theta/2)(\boldsymbol{\sigma}\cdot\mathbf{n}),$$

which yields the matrix elements

$$I = \sigma G + 4\pi\gamma^2 \cot^2(\theta/2),$$
$$L = \sigma G - 4\pi\gamma^2 \cot^2(\theta/2),$$
$$T = -8\pi \,\mathrm{Re}\, f_0(\theta),$$
$$D = 2\gamma k\sigma \cot(\theta/2),$$

where G is the ratio of the actual forward scattered intensity to that of an isotropic scatterer,

$$\gamma = \tfrac{1}{2}\mu n \left(\frac{\hbar}{Mc}\right)\left(\frac{e^2}{\hbar c}\right) Z = 1.46\times 10^{-16} Z \text{ cm}.$$

$$k = p/\hbar.$$

For an incident unpolarized beam the degree of polarization is given by Eq. (85) as

$$P = [2\gamma k\sigma \cot(\theta/2)]/[\sigma G + 4\pi\gamma \cot(\theta/2)]. \qquad (88)$$

To estimate the order of magnitude of this polarization, Schwinger uses the approximations

$$\sigma = 2\pi R^2, \quad G = \tfrac{1}{2}(kR)^2, \quad R = 1.5\times 10^{-13} A^{\frac{1}{3}}.$$

[40] J. Schwinger, Phys. Rev. 73, 407 (1948).

The angle of scattering which gives the maximum polarization is given by

$$\tan(\theta/2) = (\gamma/R)(2/kR) \quad \text{for } kR \gg 1,$$

and at this angle neutrons in the low Mev range are about 100% polarized.

Sample[41] has also performed small-angle scattering calculations and his expression for the differential cross section differs from Schwinger's only in that the polarization dependent term is about 25% greater.

VI. CONCLUSION

By the introduction of the Stokes parameters for the description of polarization it is possible to express the cross sections of polarization-sensitive interactions in matrix form. These matrices reduce the mathematical description of polarization to a very simple form which readily points out the general features of the process. This method has the advantage that it shows the direct analogy of the polarization of particles and photons to the polarization of light with which one is usually familiar. The similarity among the interaction matrices points out the related phenomena in polarization processes.

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APPENDIX. TRANSFORMATION OF THE STOKES PARAMETERS UNDER A ROTATION OF THE COORDINATE SYSTEM

Under a rotation of the coordinate axes the Stokes parameters are also transformed. This is accomplished by the transformation matrix M, which expanded yields the four equations

$$I' = m_{11}I + m_{12}P_1 + m_{13}P_2 + m_{14}P_3,$$
$$P_1' = m_{21}I + m_{22}P_1 + m_{23}P_2 + m_{24}P_3,$$
$$P_2' = m_{31}I + m_{32}P_1 + m_{33}P_2 + m_{34}P_3,$$
$$P_3' = m_{41}I + m_{42}P_1 + m_{43}P_2 + m_{44}P_3.$$

Let us consider a rotation about the axis represented by P_3. For photons this corresponds to a rotation in a plane orthogonal to the direction of propagation and does not effect the degree of circular polarization. For particles, the polarization in the y direction is unaffected. Thus the matrix must have the form

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we consider ordinary space rotations as indicated in Fig. 20 and use the identity $\psi = a_1\psi_1 + a_2\psi_2 = b_1\psi_1' + b_2\psi_2'$,

[41] J. T. Sample, Can. J. Phys. 34, 36 (1956).

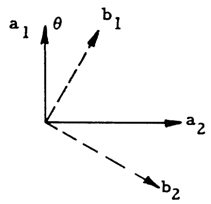

FIG. 20. $b_1=a_1\cos\theta+a_2\sin\theta$; $b_2=-a_1\sin\theta+a_2\cos\theta$.

then one finds as the orientation coefficients

$$I'=b_1b_1^*+b_2b_2^*=a_1a_1^*+a_2a_2^*=I,$$
$$P_1'=b_1b_1^*-b_2b_2^*=P_2\sin2\theta+P_1\cos2\theta,$$
$$P_3'=b_1b_2^*+b_2b_1^*=-P_1\sin2\theta+P_2\cos2\theta,$$
$$P_3'=i(b_1b_2^*-b_2b_1^*)=i(a_1a_2^*-a_2a_1^*)=P_3.$$

Thus, for radiation (whose orthogonal state vectors correspond to orthogonal space vectors) we have for rotations about the axis corresponding to P_3

$$M_3=\begin{bmatrix}1&0&0&0\\0&\cos2\theta&\sin2\theta&0\\0&-\sin2\theta&\cos2\theta&0\\0&0&0&1\end{bmatrix}.$$

For particles, the orthogonal state vectors do not correspond to the space vectors (ψ_1 and ψ_2 representing spins in the $\pm z$ direction are orthogonal state vectors but the two space directions $\pm z$ are not orthogonal), so we cannot use the same expansion of b_1 and b_2; however, considering the meanings of P_1 and P_2, we see that the transformation matrix M is suitable if 2θ is replaced by θ; that is, a 90° rotation performs the transformations

$$P_1'=P_2,\quad P_2'=-P_1.$$

For particles we are mainly interested in rotations about the z axis, which is generally chosen as the direction of motion. The rotation matrix for this case can be obtained from that first given by a cyclic rotation of the rows and columns. Thus

$$M_1=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\theta&\sin\theta\\0&0&-\sin\theta&\cos\theta\end{bmatrix}.$$

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Approximate Methods in the Quantum Theory of Many-Fermion Systems

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I. INTRODUCTION

THIS paper considers the Hartree-Fock approximation from two complementary points of view. In the first, the method is a convenient first step towards an *ab initio* solution of the many-particle Schrödinger equation for a system of fermions. It can be shown that the Hartree-Fock wave function satisfies mathematical conditions which ensure that a large class of matrix elements, in the perturbation solution of Schrödinger's equation, should vanish.[1,2] For this reason it is a useful zeroth-order wave function in a perturbation calculation. From this point of view, modifications to the Hartree-Fock method which simplify the details of calculations are desirable, if their effect on the perturbation calculation can easily be evaluated.

From the second point of view, the Hartree-Fock approximation is the last hand-hold for elementary physical intuition before it is forced to work directly in terms of the superposition of wave amplitudes that depend on large numbers of independent variables. It can be argued that any attempt to think in terms of physical models, rather than pure mathematics, past this point is necessarily deceptive. In the Hartree-Fock approximation (or at least in the unrestricted Hartree-Fock approximation which is discussed in the following) there is a one-to-one correspondence between particles and one-particle wave functions (orbitals) which justifies the loose physical language used in talking about "an electron in an outer shell," for example, or "an electron moving through a lattice." Furthermore, in the Hartree-Fock approximation one is free to make up wave packets from the orbitals, and to localize them both conceptually and mathematically, so that it is not completely fallacious to talk about the force between two particles when mathematically this is described by the potential energy integral between density distributions made from the probability amplitudes denoted by two localized orbitals.

A modification of the Hartree-Fock method which

[1] L. Brillouin, *Actualités sci. et ind.* No. 159 (1934); R. Lefebvre, Compt. rend. **237**, 1158 (1953).
[2] R. K. Nesbet, Proc. Roy. Soc. (London) **A230**, 312 (1955).