Geometrodynamics and the Problem of Motion *

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I. PAST CONTRIBUTIONS AND PRESENT PROBLEMS OF THE THEORY OF MOTION

Importance of the Problem of Motion in the Early Days of General Relativity in Emphasizing the Great Scope of This Theory

GENERAL relativity and the quantum theory of the atom, both born in World War I, and surely destined some day to be married in high state, have grown at very different rates, and raised different degrees of expectation at the several stages of their careers. The quantum principle was overshadowed at first by the drama of relativity; however, it has steadily grown in power and scope and usefulness, until now there is no part of physics that does not acknowledge its suzerainty. Relativity, on the other hand, quickly and dramatically encompassed the description of gravitation, of gravitational waves, and of the three still better known and more easily testable predictions of general relativity. Thereupon productivity in the field began to languish. Only in the last few years has it once more become widely appreciated that in general relativity Einstein gave a kind of master theory of physics out of which many deep meanings and rich physical consequences are still to be read.[1] In the intervening decades the vision had become dimmed of what Riemann had earlier sensed and of what Clifford had still more explicitly in mind[2]: "I hold in fact (1) That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them. (2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave. (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial. (4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity."

Faith in this concept of a *geometrodynamical* universe, disappointed in the long lull that followed the first exciting days of general relativity, was sustained by nothing so much—until recent times—as the idea[3–14] to derive the equations of motion of concentrations of mass-energy from the field equations themselves. No such derivation is possible in a linear theory like electrodynamics, as is well known. The electromagnetic field due to a solitary particle of charge e_1 in motion

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[1] One summary of recent developments in general relativity is to be found in J. A. Wheeler, Nuovo cimento Suppl. 7 (1960); see also C. W. Misner and J. A. Wheeler, Ann. Phys. N. Y. 2, 525 (1957).

[2] W. K. Clifford (1845–1879; creator of the Clifford numbers), Proc. Cambridge Phil. Soc. 2, 157 (1876), (read 1870). See *Mathematical Papers by William Kingdon Clifford*, edited by R. Tucker (Macmillan and Company, Ltd., London, 1882), for this paper and for Clifford on Riemann's famous lecture of 1854.

[3] A. Einstein and J. Grommer, Sitzber. preuss. Akad. Wiss. Physik math. Kl. 2, 235 (1927): motion of concentrations of mass-energy concluded not to be arbitrarily specifiable without violating field equations.

[4] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938); L. Infeld, Phys. Rev. **53**, 836 (1938); A. Einstein and L. Infeld, Ann. Math. **41**, 455 (1940); Can. J. Math. **1**, 209 (1949): equations of motion of pointlike singularities derived from the field equations as an infinite series in powers of the ratio (particle velocity)/(velocity of light).

[5] V. A. Fock, Zhur. Eksptl. i. Teoret. Fiz. **9**, 375 (1939); J. Phys. U.S.S.R. **1**, 81 (1939); *The Theory of Space Time and Gravitation*, translated by N. Kemmer (Pergamon Press, New York, 1959), with minor changes from the Russian edition of 1955; Revs. Modern Phys. **29**, 325 (1957): problem of motion of spherically symmetric nonrotating bodies of finite size. See also N. Petrova, J. Phys. U.S.S.R. **19**, 989 (1949), for second approximation; V. A. Fock, Doklady Akad. Nauk U.S.S.R. **32**, 28 (1941), and Compt. rend. acad. sci. U.R.S.S. **32**, 25 (1941), for first integrals of the equations of motion; and I. G. Fichtenholz, Zhur. Eksptl. i. Teoret. Fiz. **20**, 233 (1950), for Lagrangian form of equations of motion in second approximation.

[6] L. Infeld and A. Schild, Revs. Modern Phys. **21**, 408 (1949): derivation of equations of motion of an infinitesimal test particle in a given background metric.

[7] L. Infeld, Acta Phys. Polon. **13**, 187 (1954); Revs. Modern Phys. **29**, 398 (1957); *Equations of Motion and Gravitational Radiation* (Polska Akademia Nauk, Universytet Warszawski, 1959): treatment of the singularities as δ functions.

[8] See also L. Infeld, Acta Phys. Polon. **10**, 284 (1950); and L. Infeld and A. Scheidigger, Can. J. Math. **3**, 195 (1951), for other aspects of the theory of motion.

[9] A. Papapetrou, Proc. Phys. Soc. (London) **A64**, 57 (1951): another type of derivation of the equations of motions of finite masses from the field equations.

[10] A. Papapetrou and W. Urich, Z. Naturforsch. **10a**, 109 (1955); and V. P. Kashkarov, Zhur. Eksptl. i Teoret. Fiz. **27**, 563 (1954): Derivation of corrections to equations of motion when masses are rotating or endowed with a dipole moment.

[11] E. Corinaldesi in *Jubilee of Relativity Theory*, edited by A. Mercier and M. Kervaire (Birkhäuser Verlag, Basel, 1956), p. 125: derivation of equations of motion of two masses from field equations by methods analogous to those used in the quantum theory of fields.

[12] J. Callaway, Phys. Rev. **92**, 1567 (1953): proof that Einstein's so-called unified field theory leads to the wrong equations of motion for charged concentrations of mass-energy, in the sense that the object moves in extremal electromagnetic fields as if completely uncharged—an argument against that theory.

[13] D. M. Chase, Phys. Rev. **95**, 243 (1954): shows that the standard 1916 general relativity predicts a motion of a charged concentration of mass-energy which agrees with the Lorentz force law.

[14] R. W. Lindquist and J. A. Wheeler, Revs. Modern Phys. **29**, 432 (1957): derivation directly from the field equations of the equation for the expansion and recontraction of a closed universe constituted of a regular lattice of mass centers—a derivation employing the concept of a "Schwarzschild lattice cell" analogous to the Wigner-Seitz approximation in solid-state physics.

$x=x_1(t)$ can be added to the field due to a particle of charge e_2 in motion $x=x_2(t)$, and the sum satisfies the electromagnetic field equations, despite the fact that the particle motions themselves are no longer correct when the interaction between them is allowed for. Not so in general relativity! The nonlinearity of the field equations is such that these equations cannot be satisfied unless the concentrations of mass-energy move in the proper way. In general relativity for the first time one therefore acquired the possibility to derive the equations of motion from the field equations.

Difficult Problem of Models for the Concentrations of Mass-Energy Whose Motion is to be Analyzed

The several techniques developed to derive the equations of motion all deal with bodies endowed with mass, and yet all seek insofar as possible to avoid the issue of the internal constitution of these bodies. That question, it has been supposed, must wait for an explanation of the elementary particles.

Fluid Masses

To secure a provisional model for masses in motion, Fock[5] and Papapetrou[9] therefore considered each mass to be a collection of fluid. The stress-energy density of the fluid $T_{\alpha\beta}$ serves then as source of the gravitational field seen outside of that mass, in accordance with Einstein's field equations,

$$R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R=(8\pi G/c^4)T_{\alpha\beta}. \tag{1}$$

This approach has the virtue of avoiding all singularities in the metric. The internal degrees of freedom of the concentration of mass energy depend in a vital way upon the equation of state assumed for the fluid; and fluid physics is outside the domain of the pure geometrodynamics of Clifford and Einstein. Or as Infeld, another leader in the study of this subject, puts it, "Einstein always thought that to use (1) . . . instead of the source free field equations is somehow in bad taste, because we do not know in (1) what $T_{\alpha\beta}$ is, and we mix a geometrical tensor on the left side with a physical tensor on the right side."

Masses Regarded as Singularities; Difficulties with This Concept

Einstein, Grommer, Infeld, and Hoffmann, on the other hand,[3,4,6–8] keep to pure geometrodynamics, but can do so only at the cost of having to assume that all the moving concentrations of mass-energy are associated with singularities in the metric. The consequences of this assumption for the derived motion are the same—up to the order of approximation so far investigated, the eighth order in v/c—as for the assumption of fluid masses. However, a deep question of principle stands out from this analysis. If singularities are to be admitted at all, what kind of singularities are to be allowed? If every kind of singularity is admitted, then in effect the field equations are abandoned. It is reminder enough on this point to consider the difference in electrostatics between the equations of Laplace, $\nabla^2\varphi=0$, and of Poisson, $\nabla^2\varphi=-4\pi\rho$. As soon as solutions of the former equation are admitted of the type $1/r$, then also solutions are allowed of the form $\sum_k e_k/r_k$, and by extension solutions of the form

$$\varphi_1=\int r_{12}^{-1}\rho_2 d^3x_2. \tag{2}$$

But this function satisfies Poisson's equation, with *arbitrary* $\rho(x)$. In other words, the equation of Laplace has been abandoned. A theory that set out to deal with one entity finds on its hands a second entity. If singularities are admitted, the properties of the sources cannot be discussed adequately entirely within the framework of the theory; or in other words, the theory is no longer complete. If singularities are tolerated in general relativity, then the completeness of this theory is even more thoroughly shattered. There does not exist today even the beginnings of a comprehensive analysis of the kinds of singularities which may arise in solutions of Einstein's field equations.

Difficulty in Giving a Well-Defined Meaning to the Concept of "Locally Spherically Symmetrical Schwarzschildian Type of Metric"

To demand of each singularity that it shall be "of the Schwarzschild character," and "spherically symmetrical," is not a well-defined requirement for two reasons. First, the Schwarzschild metric admits an infinite number of modes of departure from sphericity even when it stands alone.[15] Second, under the influence of other concentrations of mass-energy there are forced departures from sphericity. Similar departures occur in the case of the electric potential

$$\varphi=e_1/r+\sum c_{nm}r^nY_n^{(m)}(\theta,\varphi) \tag{3}$$

arising from a point charge at the origin plus other charges elsewhere. Here there is a meaning to speaking of spherical symmetry; all dipole potentials and potentials of higher multipole order

$$r^{-n-1}Y_m^{(m)}(\theta,\varphi) \tag{4}$$

are excluded. The exclusion of these terms in a well-defined way is possible only because the center of charge can be approached indefinitely closely for investigation. The Schwarzschild metric, on the other hand, has no center to approach. For example, at the value $T=0$ of Schwarzschild's time coordinate there is a smallest possible centered sphere that one can draw; it has the

[15] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957): small first-order departures from the Schwarzschild metric; A. Peres and N. Rosen, Phys. Rev. **115**, 1085 (1959): influence of second-order terms.

proper area

$$\text{area}_{\min}=4\pi r_{\min}^2=4\pi\cdot(2Gm/c^2)^2. \quad (5)$$

Consequently to bring up nearby masses perturbs the metric by an amount which cannot be made negligible in comparison with the effect of the concentration of mass-energy under consideration. This perturbation even deforms the limiting sphere into a new shape.

Even the Mass of One Concentration of Mass-Energy Not under All Conditions a Well-Defined Quantity

There is a final reason to question the existence of any well defined way to speak of "allowable singularities." Consider the situation when two regions of Schwarzschild character move directly towards each other. Then it seems impossible to prevent the amalgamation of the two regions of strongly deformed geometry. But as soon as this possibility is admitted, it would seem necessary to admit that the inverse process can take place: breakup of one concentration of mass-energy into two or more. In other words, it would appear that the concept of "number of singularities" is not even definable. Moreover, there does not seem to exist any justification for tagging concentrations of mass-energy with individual mass values, m_1, m_2, $\cdots$ which are assumed to be constant for all time. Electric charges are another matter. The integral of the electric flux over a surface surrounding two objects is strictly equal to the sum of the integrals taken around the two objects separately, or

$$e=e_1+e_2. \quad (6)$$

The mass, on the other hand, is defined by the rate at which the metric approaches flatness—if it does—as is seen in the factor $(1-2Gm/c^2r)$ in Schwarzschild's expression for the metric. When the region of space under study includes *two* concentrations of mass energy, then the mass as so defined is not the sum of the two masses individually. Instead, it is clear from the simplest analysis of the interaction energy that the total mass have a value of the order

$$m\sim m_1+m_2+\left[\frac{e_1e_2}{r_{12}}-\frac{Gm_1m_2}{r_{12}}+\tfrac{1}{2}(m_1+m_2)^{-1}m_1m_2(\mathbf{v}_2-\mathbf{v}_1)^2\right]\Big/c^2. \quad (7)$$

It is no longer possible to define the mass associated with either object in an unambiguous way.

Mass Always to Some Extent Ambiguous when Surrounding Metric Is Not Asymptotically Flat

The lack of definition in the value of the mass can be stated in more general terms. An object of mass m has linear dimensions at least of the order $L\sim Gm/c^2$. Over dimensions of this order the metric of the surrounding space, with Riemann curvature tensor $R^\alpha{}_\beta{}^\gamma{}_\delta$, deviate from flatness by amounts of the order

$$R^\alpha{}_\beta{}^\gamma{}_\delta L^2. \quad (8)$$

This same quantity governs the fractional ambiguity in the mass in different kinds of observations. This lack of definability of the mass is completely insignificant under any conditions that one now knows how to realize. It is nevertheless crucial as a matter of principle for two objects separated by distances of the order of their own gravitational radii. Then the two objects become so blended that there is no way to give a clear meaning to the mass of any "part" of the system. Then also it is impossible to know what one should *mean* by "equations of motion" of concentrations of mass energy, much less to derive such equations.

II. PURELY GEOMETRODYNAMICAL MODELS FOR MASS

Geons

In view of the difficulties in the theory of motion associated either with the idea of masses of liquid or with the concept of moving singularities in the metric, it is significant that there exists a purely classical and singularity-free model for a concentration of mass energy: a geon.[16–18] Such an object consists of a circulating distribution of electromagnetic wave energy, or of gravitational wave energy, which holds itself together by its own gravitational attraction for a time very long in comparison with travel times through the geon. It is a self-consistent solution of the field equations. The static or effective average gravitational field serves as wave guide to contain the radiation. The radiation—and the radiation alone—in turn serves as the source of this field. The accelerations needed to hold radiation in a circular track of radius R are of the order of c^2/R, whereas the acceleration available from a mass M is of the order GM/R^2. The equality of the two demands a geon mass of the order

$$M\sim(c^2/G)R. \quad (9)$$

When the radiations circulating in the geon all have reduced wavelengths of the same order

$$\lambda\!\!\!^{-}=\lambda/2\pi, \quad (10)$$

and when the radiations are confined to a spherical shell of thickness $\sim\lambda\!\!\!^{-}$, then the rate of leakage of energy from the system is governed by a penetration factor of the

[16] J. A. Wheeler, Phys. Rev. **97**, 511 (1955): structure and modes of transformation of geons in general, and integration of equations for simple spherical geon. See also F. J. Ernst, Jr., Phys. Rev. **105**, 1662 (1957), for variational treatment of the structure of a simple spherical geon.

[17] E. A. Power and J. A. Wheeler, Revs. Modern Phys. **29**, 480 (1957): thermal geons.

[18] F. J. Ernst, Jr., Phys. Rev. **105**, 1665 (1957); Revs. Modern Phys. **29**, 496 (1957): toroidal geons.

Gamow type,

$$\exp(-4.56R/\lambda\!\!\!^{-}). \tag{11}$$

When the wavelength is very short compared to the radius, the decay rate is negligibly small.

Comprehensive Character of General Relativity in View of Geon Concept; Nature of the Description of Motion That It Gives

The existence of geons gives to classical general relativity a comprehensiveness that had not been expected in the beginning. Einstein's theory accounts not only for the fields produced by concentrations of mass-energy, and for the motions of such masses, but also supplies one mechanism for a concentration of energy to hold itself together. To the outside the geon manifests mass, but inside there is nowhere that one can put his finger and say "Here is 'real' mass!" There is in principle no sharp distinction between geons as collections of radiant energy, susceptible to decay and transmutation processes, and the "free" waves that pass through the space between geons, undergoing scattering, absorption, or reemission processes. Legalistically speaking, the state of the universe of classical physics is described by the singularity-free electromagnetic and gravitational magnitudes at every point, and by nothing more.

Geometrodynamics and Its Bearing on the Problem of Motion

The word "geometrodynamics" has been used to describe that formulation of the standard 1916 Einstein-Maxwell theory which excludes from attention all "real" masses and "real" charges; that is to say, all objects and phenomena in which the quantum of action plays an important part. Geons and the motion and transformations of geons belong in this sense to geometrodynamics. These objects must be massive ($>10^{39}$ g) and enormous ($>10^{11}$ cm) if they are to satisfy the conditions for a classical analysis. Therefore they have not the slightest direct connection with the world of elementary-particle physics. This circumstance means that *the problem of motion of classical masses can be reconsidered and in principle for the first time fully analyzed within the closed and logically self-consistent framework of geometrodynamics.* Here no attempt is made at such a complete treatment, a treatment which would lean heavily on the work of Einstein, Infeld, Fock, and others. Instead, this article merely sketches out some points of view and principles which would seem fundamental to any geometrodynamical analysis of the problem of motion.

Concepts of Configuration, Initial Value Data, and Initial Value Equations

Nothing is more central to the geometrodynamical picture than this, that the configuration C (Table I) of the system is completely specified on any spacelike hypersurface σ by the three-dimensional geometry intrinsic to that three-space, and by the values of the electric and magnetic fields throughout that three-space, and by nothing more. Anything stated or derived about motion from geometrodynamics must be stated or derived out of this way of speaking.

To specify the entire dynamics of the system it is enough to specify *nearly* identical intrinsic geometries for two three-spaces, and nearly identical divergence-free magnetic fields within each of these two three-spaces. This is the Lagrangian or sandwich formulation of Sharp, so far established only when the conditions on the two surfaces are nearly identical, but presumed to hold under more general circumstances. Alternatively, a Hamiltonian formulation is available, in which twice as many data—coordinates and momenta—are specified on a single surface. This is the formulation in which Lichnerowicz established the central theorem of geometrodynamics[19]: *regular initial value data* that satisfy the "one-time" initial value conditions of Table I, plus the field equations of Einstein and Maxwell, *allow one to calculate the evolution of the geometry and the electromagnetic field for a finite time* into the past and the future.

Case of Special Simplicity: Time-Symmetric Initial Value Data

It is fortunate that there exists a special type of initial value situation which illustrates the ideas of geometrodynamics with particular simplicity. In this so-called time-symmetric initial value problem[20] it is given that the electric field is zero on the initial three-surface, and it is given in addition that this initial surface, whatever its intrinsic three-geometry, has zero *extrinsic* curvature: ${}^{(3)}K_{ik}=0$; three-surface of time *symmetry.* Under these circumstances several of the initial value conditions that are listed in Table I are satisfied automatically, and the remainder make only two requirements: first,

$$\operatorname{div}\mathbf{H}'=0 \tag{12}$$

[or in geometrized units, with

$$\mathbf{H}'(\text{gauss})=(c^2/G^{\frac{1}{2}})\mathbf{h}'(\text{cm}^{-1}) = 3.49\times10^{24}\,\text{gauss cm}\,\mathbf{h}', \tag{13}$$

demand

$$\operatorname{div}\mathbf{h}'=0]; \tag{14}$$

and second,

$${}^{(3)}R\equiv\left\{\begin{matrix}\text{intrinsic scalar}\\ \text{curvature invariant}\\ \text{of three-surface of}\\ \text{time symmetry}\end{matrix}\right\} = 2(8\pi G/c^4)(\mathbf{H}'^2/8\pi)=2(\mathbf{h}')^2. \tag{15}$$

[19] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Électromagnetisme* (Masson et Cie, Paris, 1955); also *Problèmes Globaux en Méchanique Relativiste* (Hermann, Paris, 1939); J. Math. pure appl. (9) **23**, 37 (1944); Helv. Phys. Acta Suppl. **IV**, 176 (1956).

[20] J. Weber and J. A. Wheeler, Revs. Modern Phys. **29**, 509 (1957); Dieter Brill, Ann. Phys. **7**, 466 (1959); Huzihiro Araki, *ibid.* **7**, 456 (1959).

TABLE I. Concept of configuration C for three kinds of system compared and contrasted. E and H denote electric and magnetic fields; ${}^{(3)}g_{ik}$, the tensor that describes the geometry intrinsic to a three-space; and ${}^{(3)}K_{ik}$, the tensor ("second fundamental form" of differential geometry) that gives the *extrinsic* curvature of a three-space; that is, that tells how this three-space is imbedded—or to be imbedded—in a surrounding four-space.

System	Single particle	Electromagnetic field	Geometrodynamics
Specification of time	Time $t=t'$	Spacelike three-surface σ: $x^\mu = x^\mu(u,v,w)$	"Surface" and "field on surface" specifiable not separately but only in combination: The three-dimension geometry $\mathcal{G}'$ or ${}^{(3)}g_{ik}'(u,v,w)$ and the magnetic field $\mathbf{H}=\mathbf{H}'(u,v,w)$
Specification of configuration C at this time	$x=x'$	Magnetic field in σ: $\mathbf{H}=\mathbf{H}'(u,v,w)$	
Data adequate to forecast dynamics, at least when given for times or three-surfaces not too far removed from each other (Lagrange or "sandwich" formulation)	$x(t')=x'$ and $x(t'')=x''$	$\mathbf{H}$ (on σ') $=\mathbf{H}'$ and $\mathbf{H}$ (on σ'') $=\mathbf{H}''$	$\mathcal{G}'$ and $\mathbf{H}'$ and $\mathcal{G}''$ and $\mathbf{H}''$[c]
Hamiltonian or "one-time" formulation	$x(t')=x'$ and $p(t')=p'$	$\mathbf{H}$ (on σ') $=\mathbf{H}'$ and $\mathbf{E}$ (on σ'') $=\mathbf{E}'$	${}^{(3)}g_{ik}'$ and ${}^{(3)}K_{ik}'$ and $\mathbf{H}'$ and $\mathbf{E}'$
Conditions on specification of sandwich data	No conditions	div $\mathbf{H}'=0$; div $\mathbf{H}''=0$[b]	div $\mathbf{H}'=0$; div $\mathbf{H}''=0$; None on $\mathcal{G}'$ and $\mathcal{G}''$[b]
Conditions on specification of Hamiltonian data ("one-time" initial data)	No conditions	div $\mathbf{H}'=0$; div $\mathbf{E}'=0$	div $\mathbf{H}'=0$; div $\mathbf{E}'=0$; (intrinsic scalar curvature invariant of σ') $-$ (extrinsic curvature invariant of σ') $=$ ($8\pi G/c^4$ times energy density on σ')[d] Three additional conditions on extrinsic curvature set by Poynting flux.
Number of distinct histories[a]	∞^2	$\infty^{4\times 3}$	$\infty^{8\times 3}$

[a] See J. A. Wheeler, Nuovo cimento Suppl. 7 (1960).
[b] In addition, there are conditions of compatibility between $\mathbf{H}'$ and $\mathbf{H}''$ (or between $\mathcal{G}'$ and $\mathcal{G}''$) whose number is of a lower order of infinity than the number of points on σ' or σ''.
[c] Proof given by David Sharp, A.B. senior thesis, Princeton University, 1960 (unpublished).
[d] Proof given by Y. Fourès-Bruhat, Acta Math. **88**, 141 (1952); J. Ratl. Mech. Anal. **4**, 951 (1956).

If these conditions for a time-symmetric initial value problem are satisfied by a regular metric ${}^{(3)}g_{ik}$ and a regular magnetic field $\mathbf{H}'$, then the field equations guarantee the existence of

(1) a uniquely determined four-dimensional history free of singularity for a finite time and

(2) a coordinate system (*not unique*!) such that the metric for this four-space, and the electromagnetic field, satisfy these symmetry requirements:

$$\begin{aligned} g_{00}(T,x,y,z)&=g_{00}(-T,x,y,z);\\ g_{0m}=g_{m0}(T,x,y,z)&=-g_{m0}(-T,x,y,z);\\ g_{mn}(T,x,y,z)&=-g_{mn}(-T,x,y,z);\\ \mathbf{H}(T,x,y,z)&=\mathbf{H}(-T,x,y,z);\\ \mathbf{E}(T,x,y,z)&=-\mathbf{E}(-T,x,y,z). \end{aligned} \tag{16}$$

Here $\mathbf{E}$ and $\mathbf{H}$ are understood to be the electric and magnetic components of the field in a frame of reference of constant T.

Three Examples of Time-Symmetric Initial Value Situations

It may be appropriate to illustrate the machinery of the time-symmetric initial value problem by three simpler examples before looking at a time-symmetric geometrodynamic formulation of the problem of motion. All of these examples assume zero electromagnetic field, so that only a single requirement is imposed on the intrinsic geometry of the three-space of time symmetry: It must have zero intrinsic curvature invariant.

Case of Spherical Symmetry Formulated as an Initial Value Problem Completely Free of Singularities

Example 1. Metric spherically symmetric at the moment of time symmetry. Write

$$\begin{aligned} ds^2&=\psi^4(x,y,z)(dx^2+dy^2+dz^2)\\ &=\psi^4(r,\theta,\varphi)(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2). \end{aligned} \tag{17}$$

Then the condition of vanishing scalar curvature invariant, ${}^{(3)}R=0$, demands

$$\nabla^2\psi=0. \qquad (18)$$

The only solution with spherical symmetry can be written in the form

$$\psi=1+(m^*/2r). \qquad (19)$$

Here an arbitrary constant multiplicative constant has been thrown away because it affects only the scale in which lengths are measured. The constant

$$m^*(\mathrm{cm})=Gm/c^2=(0.74\times10^{-28}\ \mathrm{cm/g})m(g) \qquad (20)$$

is a geometrical measure of mass. *It has to be positive if the intrinsic geometry is to be free of singularity.* The proper circumference of a circle with coordinate value r is

$$\mathrm{circum}=2\pi r\psi^2=2\pi[r+m^*+(m^{*2}/4r)]. \qquad (21)$$

This quantity has a minimum value of $4\pi m^*$ for $r=m^*/2$. It measures the size of a throat which connects two Euclidean spaces.[21] Alternatively and more physically, the throat can be regarded as a *wormhole*[22] connecting together two regions of *one* otherwise Euclidean space—regions which are so far apart that neither per turbs the other. The geometry is everywhere regular. It is enough to write

$$r_1=m^{*2}/4r \qquad (22)$$

to discover that the metric looks the same in terms of r_1 as it does in terms of r. Small values of r are not associated with any singularity at all, but only with very great distances in the other quasi-Euclidean space. The metric in the initial three-space at the moment of time symmetry is seen even better to be regular when r at this time is expressed in terms of a regularizing variable u—most appropriately the variable u of Kruskal[23]:

$$[(r/2m^*)-1]\exp[r/2m^*]=u^2. \qquad (23)$$

This implicit relation provides two values of u for every value of r greater than the minimum value—one lying on the upper extension of the throat, the other on its lower extension—but only a single value of r for each value of u. Let r and f denote those positive functions of u which are implicitly defined by the above relation and by

$$f^2=(32m^{*3}/r)\exp(-r/2m^*). \qquad (24)$$

Then the metric on the initial spacelike surface is

$$ds^2=f^2(u)du^2+r^2(u)(d\theta^2+\sin^2\theta d\varphi^2). \qquad (25)$$

Here u runs from $-\infty$ through the throat at $u=0$ to $u=+\infty$; $f(u)$ runs from small values up to $4m^*$ and then back down to small values; and $r(u)$ runs from large value down to $2m^*$ and back up to large values.

Time Development of this Spherically Symmetric and Initially Nonsingular Metric

The evolution in time of this completely regular geometry is specified in a completely deterministic way by Einstein's equations,

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=0. \qquad (26)$$

This determinism is in no way lessened by the circumstance that the field equations leave quite open the choice of coordinates which are used in describing the resulting four-dimensional space-time continuum. The solution is simplest in Kruskal's coordinates u and v, one spacelike and the other timelike, and the usual polar coordinates θ and φ; the four-metric is

$$ds^2=f^2(u,v)(du^2-dv^2)+r^2(u,v)(d\theta^2+\sin^2\theta d\varphi^2). \qquad (27)$$

Here r is now a function of both u and v, defined by

$$[(r/2m^*)=1]\exp[r/2m^*]=u^2-v^2; \qquad (28)$$

and f is in turn defined as a function of u and v by (24).

[21] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935); C. W. Misner and J. A. Wheeler, Ann. Phys. N. Y. **2**, 525 (1957).

[22] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).

[23] M. D. Kruskal, as reported by J. A. Wheeler, International Conference on Relativistic Theories of Gravitation, Royaumont, June, 1959; Phys. Rev. **119**, 1743 (1960); see also C. Fronsdal, Phys. Rev. **116**, 778 (1959); also C. W. Misner and J. A. Wheeler, footnote 1, pp. 390–395. Misner is the first to have derived the everywhere regular expression for the initial three-geometry directly out of the initial value equations themselves: C. W. Misner, Phys. Rev. **118**, 1110 (1960). This method of finding the geometry, as described in Eqs. (17)–(19) of the present text, is so simple that it appears worthwhile to note its generalization to the case where spherical symmetry is maintained but charge is present in the sense of "flux through the wormhole." Then the geometrized electric field **e** has to satisfy the condition $\mathrm{div}\,\mathbf{e}=0$, or $4\pi(\psi^2 r)^2 e=$ flux constant $=4\pi q^*$, or $e=q^*/\psi^4 r^2$. The other initial value requirement, Eq. (15) (with electric field in place of magnetic field) leads to the equation

$$-8\psi^{-5}\nabla^2\psi=2q^{*2}/\psi^8 r^4.$$

Here the expression on the left for the intrinsic curvature invariant is most quickly recalled—apart from the factor eight—from two circumstances: (1) the intrinsic curvature vanishes when $\nabla^2\psi$ is zero; and (2) this curvature has the dimensions cm^{-2}, or ψ^{-4}. That solution of the differential equation which is asymptotic to unity has only a single constant of integration, the geometrized mass, m^*:

$$\psi=[(1+m^*/2r)^2-(q^*/2r)^2]^{\frac{1}{2}}.$$

Limiting values are

$$\psi=1+m^*/2r$$

for the case of zero charge, $q^*=0$, and

$$\psi=(1+m^*/r)^{\frac{1}{2}}$$

for the case of maximal charge, $q^*=\pm m$. The resulting expression for the metric,

$$ds^2=\psi^4(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2),$$

is a well-known way of rewriting the space part of the Reissner-Nordstrøm generalization of the Schwarzschild solution,

$$ds^2=-(1-2m^*/R+q^{*2}/R^2)dT^2+(1-2m^*/R+q^{*2}/R^2)^{-1}dR^2+R^2(d\theta^2+\sin^2\theta d\varphi^2).$$

The two-coordinate systems are related by the formula $R=\psi^2 r$. The circumference of a circle about the center of symmetry is

$$2\pi R=2\pi\psi^2 r=2\pi[r+m^*+(m^{*2}-q^{*2})/4r].$$

The throat is located at

$$r_{\mathrm{throat}}=(m^{*2}-q^{*2})^{\frac{1}{2}}/2.$$

One mouth of the throat corresponds to $r>r_{\mathrm{throat}}$, the other to $r<r_{\mathrm{throat}}$. The space approaches Euclidean flatness of $r\rightarrow\infty$ and for $r\rightarrow0$, just as it does in the case where there is no charge.

Relation to Schwarzschild Solution

Kruskal shows that the four-metric (27) is equivalent to the usual expression for the Schwarzschild metric,

$$ds^2 = -(1-2m^*/r)dT^2 + (1-2m^*/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (29)$$

in the region where the Schwarzschild expression is defined; but shows also that expression (29) fails to cover a great portion of the four-dimensional continuum where the intrinsic geometry is really regular. For example, (29) covers only one sheet—from the throat on out to great distances—at the moment of time symmetry $T=0$ or $v=0$; and as the time coordinate v advances, expression (29) covers even less than the entire upper sheet out from the connecting throat to great distances.

Shrinkage of the Throat

The throat diminishes in circumference,

$$\begin{bmatrix}\text{proper length}\\ \text{of circumference}\\ \text{of throat}\end{bmatrix} = 2\pi r(u,v)_{u=0} = \begin{cases} 4\pi m^* \text{ at } v=0 \\ \sim 2^{3/2} m^*(1-v^2)^{\frac{1}{2}} \text{ for } v\to\pm 1\end{cases} \quad (30)$$

as time goes on. The curvature invariants become greater and greater and go to infinity as v approaches $+1$ (future) or -1 (past). There is only a finite length of proper cotime,

$$\begin{bmatrix}\text{proper cotime at}\\ \text{throat } (u=0) \text{ from}\\ v=-1 \text{ to } v=+1\end{bmatrix} = \int_{-1}^{1} f(0,v)dv = 2\pi m^*, \quad (31)$$

during which the intrinsic geometry at the throat is free of singularity.

Conclusions from Schwarzschild-Fronsdal-Kruskal Metric

Conclusions. (a) Even simple situations force one to consider geometries whose topology is non-Euclidean.

(b) The initial value problem provides a useful way to speak about the Schwarzschild metric.

(c) An observer at the throat finds that the intrinsic geometry associated with this metric becomes singular after a finite proper time. Therefore the dynamics of the geometry cannot be further analyzed in this particular case without going outside of the framework of classical general relativity. In this respect there is a close analogy with the problem of an electron losing energy by radiation and spiraling in towards a nucleus—*its* final state, too, cannot be properly treated within the framework of classical mechanics. This circumstance is no bar to posing the problem of the time evolution of the initial three-geometry (25); it is only a warning that quantum considerations cannot be escaped[24] when the lapse of proper time at the throat approaches the critical value πm^*.

(d) Until the quantum analysis is pursued to quantitative conclusions, it is difficult to say what eventually happens to a single Schwarzschild throat. Therefore it is still more difficult to follow the evolution for all time of many such throats, as would be required if one were to try to treat the problem of motion of many masses in terms of Schwarzschild-like regions in the metric.

(e) It is really not possible to derive the equations of motion of concentrations of mass energy from the field equations in any rigorous way "representing matter as singularities" for three reasons:

(i) For a finite proper time the Schwarzschild metric is *not* singular at all. Instead, it belongs to a multiply connected topology. It is impossible to hide this topology in any legalistically correct analysis of a space endowed with one or more such centers of mass.

(ii) When the metric *does* become singular after a finite proper time, there is no known way to foretell what happens next. What can be meant by "equations of motion" under these circumstances?

(iii) The dynamics of even a single Schwarzschild concentration of mass-energy cannot even in principle be expressed in the form of equations for the time rate of change of three coordinates alone. The metric field demands for its specification an infinite number of degrees of freedom—degrees of freedom which describe gravitational waves in the field of force of a Schwarzschild throat[25]

Initial Value Problem for Wormhole Metric

Example 2. The Schwarzschild metric can be considered to describe a "wormhole"[22] connecting two remote regions of *one* Euclidean space rather than a "bridge" between *two* Euclidean spaces.[21] When the two mouths of the wormhole, instead of being very far apart in this Euclidean space,[26] have a limited separation, then the curvatures in the space around each mouth reinforce each other in the region between them. If no additional sources of curvature are present, such as free gravi-

[24] For further discussion of this point see footnote 1.

[25] T. Regge and J. A. Wheeler [Phys. Rev. **108**, 1063 (1957)], analyze weak gravitational waves in the Schwarzschild metric. This analysis cannot be carried through in a completely unambiguous way because it is not clear what boundary conditions the metric perturbations must satisfy at the boundary where the intrinsic curvature goes to infinity. Just as the ultimate fate of a Schwarzschild throat cannot be determined from classical geometrodynamics alone, neither can the problem of small departures from sphericity be given a completely definitive treatment entirely within that framework of ideas; however, the *initial value problem* associated with such first-order departures from sphericity can be treated apart from all such issues about singularities and deserves detailed consideration. A. Peres and N. Rosen [Phys. Rev. **115**, 1085 (1959)], discuss some of the corrections which have to be applied to the weak wave analysis when the effective energy content of the wave—a second-order effect—is taken into account.

[26] The distance *through the wormhole* is not the quantity of interest here—it may be very short—but the distance measured *through the* nearly *Euclidean space*.

tational radiation, then the curvature changes with time in this intervening region, and elsewhere, in such a way as to describe the acceleration of the two mouths towards each other.[27] It is convenient to take as starting moment of the analysis the instant when the two mouths have reached a maximum separation and are starting to fall back towards each other. This loose way of speaking can be translated into precise terminology. Let a time-symmetric initial value problem be assumed. In other words, let a definite three-geometry be specified (a) which is everywhere regular, (b) which describes a wormhole with two mouths at finite separation, and (c) which satisfies the initial value requirement on the three-dimensional scalar curvature invariant,

$$^{(3)}R=0. \tag{32}$$

Then the future (and past) evolution of the real geometry—not the coordinate system—is completely determined by Einstein's field equations. The situation is so like that of a Schwarzschild throat with mouths at an infinite separation that one can hardly doubt that the same kind of singularity develops here as there after the lapse of a finite proper time. Therefore no solution of the dynamic problem valid for all time can be expected within the framework of *classical* geometrodynamics. Nevertheless it is of interest to see even for a finite proper time in what way the problem of motion makes sense.

Great Freedom in Specification of Initial Value Problem; Gravitational Waves

To demand of the initial three-geometry the time-symmetric condition $^{(3)}R=0$ and a wormhole topology—and even on top of these to demand axial symmetry about the most direct line between the two mouths—does not uniquely determine the initial three-geometry. For example, a great variety of axially symmetric gravitational waves can be superposed on the two-mouth geometry. Moreover, no unique and intrinsically geometrical (*coordinate free*) prescription[28] has ever been given to sort out gravitational waves from such gross disturbances in the metric. It is quite possible at the present time to believe that *in principle* no way can ever be given to make this discrimination. Then one initial three-geometry is as good as another. In other words, the dynamical system is characterized by an infinite number of degrees of freedom, *no one of which can be distinguished from the others and no one of which can be said to describe "the motion of a mass."*

Misner's Analytic Specification of Initial Value Data for Wormhole Problem

One can, however, introduce purely mathematical restrictions which simplify the initial value problem without in any way making "the motion of the masses" any more definable in terms of a few distinguished coordinates. Misner has given a most interesting treatment of the wormhole initial value problem.[29] He cuts out all but a minimal arbitrariness in the initial geometry by limiting attention to three-metrics representable in the form

$$ds_W{}^2=\Phi^4(\mu,\theta)ds_D{}^2, \tag{33}$$

where

$$ds_D{}^2=d\mu^2+d\theta^2+\sin^2\theta d\varphi^2(-\mu_0\leq\mu<\mu_0). \tag{34}$$

Then the requirement $^{(3)}R=0$ gives a simple linear second order equation for Φ which—with appropriate boundary conditions—leads Misner directly to a solution containing only two adjustable constants:

$$\Phi=a^{\frac{1}{2}}\sum_{n=-\infty}^{+\infty}[\cosh(\mu+2n\mu_0)-\cos\theta]^{-\frac{1}{2}}. \tag{35}$$

These constants, the scale factor a and the ratio factor μ_0, are fixed as follows according to Misner by the demands that (a) the total mass of the system—as determined at a great distance $[(\mu^2+\theta^2)^{\frac{1}{2}}\rightarrow 0]$ by its Schwarzschildean rate of approach to flatness—have a specified value $m(g)$ or $Gm/c^2=m^*$(cm) and (b) the distance from the throat of the metric, out through one mouth, out through the quasi-Euclidean space, into the other mouth, and thus automatically back to the throat of the metric, have a specified value L(cm). These two requirements give two equations for the constants a and μ_0. These equations are most readily formulated by way of a parameter k and its complement $k'=(1-k^2)^{\frac{1}{2}}$ such as is familiar in the theory of the complete elliptic integrals $E(k)$ and $K(k)$; thus,

$$L=(4a/\pi)K(k')E(k); \tag{36}$$

$$m^*=4a\sum_{n=1}^{\infty}(\sinh n\mu_0)^{-1}; \tag{37}$$

$$\mu_0=\pi K(k)/K(k'). \tag{38}$$

The ratio of the prescribed L to the prescribed m^* fixes μ_0 and k; then either L or m^* fixes a.

Evolution in Time of Wormhole

The evolution in time of the three-geometry thus specified can be found in the beginning by power series expansion and thereafter by electronic computation.[30] The intrinsic geometry of the resulting four-space is completely determinate, regardless of the freedom of choice that is open as to the coordinate system to be used to describe that geometry. This geometry contains within itself the story as to the change of the distance L with time and the generation of gravitational waves by

[27] C. W. Misner, Phys. Rev. **118**, 1110 (1960).

[28] The prescription in footnote 27 depends—that article makes clear—upon a particular choice for the canonical variables.

[29] C. W. Misner, footnote 27.

[30] Considerable unpublished work has been done on both methods of approach by Richard Lindquist. To him and to Professor C. W. Misner, appreciation is expressed for numerous discussions of this problem.

the two equal masses as they are accelerated towards each other. From the work of Einstein, Infeld, and Hoffmann, and others, it is clear that the motion accords qualitatively in the beginning with that to be expected from Newton's equations of motion. However, that there is any such thing as a legalistically correct derivation of the equations of motion from the field equations—or even such a thing as a legalistically *defined* equation of motion—is denied in this example by considerations of the same kind as those which came up in the preceding example.

No Unique Way to Exclude Gravitational Radiation

(a) There is no unambiguous way to sort out the relevant degrees of freedom from those associated with gravitational radiation.

No Unique Way to Describe Distance to Extended Geometrodynamical Objects

(b) The distance L of Eq. (36) provides only one of many possible definitions of distance.

Infinite Number of Degrees of Freedom

(c) There are many choices of initial metric compatible with a given L and a given total mass—choices which were not written down only because of a desire for an example mathematically easy to discuss.

No Natural Spherical Symmetry; Throat Nonspherical

(d) The geometry of the throat at the moment of time symmetry [deducible from (33) and (35) by setting $\mu=(2n+1)\mu_0$] is *not* spherically symmetrical; the two mouths interact with each other and distort their common throat. The metric on this throat is

$$ds_{\text{throat}}^2=a^2\{\sum_{n=-\infty}^{+\infty}[\cosh(2n+1)\mu_0-\cos\theta]^{-\frac{1}{2}}\}^4 \times(d\theta^2+\sin^2\theta d\varphi^2)$$

or

$$ds_{\text{throat}}=2a[\Sigma_1-\tfrac{1}{2}\Sigma_3\sin^2(\theta/2)+\cdots]^2 \times(d\theta^2+\sin^2\theta d\varphi^2)^{\frac{1}{2}} \quad (39)$$

with

$$\Sigma_k=\sum_{n=1}^{\infty}1/\sinh^k(n-\tfrac{1}{2})\mu_0.$$

The deformation of the sphere can be examined by calculating its Gaussian curvature as a function of position; that is, as a function of the colatitude θ. For a 2-manifold imbedded in a Euclidean 3-manifold this curvature is governed by the product of the two principal radii of curvature and has the value

$$1/\rho_1\rho_2=-{}^{(2)}R/2. \quad (40)$$

The present throat is *not* imbedded in a Euclidean 3-manifold, but its intrinsic 2-geometry is still well defined, and the right-hand side of (40) can be evaluated in terms of the square bracket in (39):

$$\begin{aligned}-{}^{(2)}R/2 &=-(1/4a^2[\ ]^2\sin\theta)(d/d\theta)[\ ]^{-1}d[\ ]\sin\theta/d\theta \\ &=(1/4a^2\Sigma_1^4)\left[1+\frac{\Sigma_1}{\Sigma_1}+\frac{\Sigma_3^2-\Sigma_1\Sigma_5}{\Sigma_1^2}\tfrac{3}{4}\sin^2\theta+\cdots\right] \\ &=(1/m^*)^2[1+\text{angle independent terms} \\ &\qquad -(3m^{*3}/16L^3)\sin^2\theta+\cdots. \quad (41)\end{aligned}$$

Appreciation is expressed to Mr. Fred Manasse for pointing out that the deformation shows up only in the third and higher orders of m^*/L, as is also to be expected on simple arguments based on the $1/(\text{distance})^3$ dependence of tide producing forces. The curvature is evidently greatest for $\theta=0$ and $\theta=\pi$. In other words, the interaction between the two mouths deforms the throat from a sphere to what is approximately—for not too small separations—a *prolate* or cigar-shaped ellipsoid. This result shows in explicit form that centers of gravitating mass cannot in general be treated as spherically symmetrical.

Again Intrinsic Singularity in Geometry after Finite Proper Time

(e) There is every reason to believe that the metric becomes singular after a finite proper time, thus making it impossible even to speak in classical terms of the further development of the motion.

Example 3. Geons. Does every metric become singular in the course of time and therefore lead *always* to a situation where *classical* geometrodynamics can no longer be applied? The conjecture has recently been put forward that this is always the case in the *closed* universe[1] with the topology of the three-sphere S^3, such as is of the greatest interest in connection with cosmological questions. However, in an asymptotically flat space, time-symmetric examples can be given in which not only is the initial three-geometry completely free of singularity, but also there is every reason to believe that the geometry *remains* free of singularity for all time. The simplest example of this kind is a gravitational wave endowed with axial symmetry which converges from great distances towards a limited region of space, implodes to a configuration of extremal energy concentration, and then spreads out again to infinity. Brill[20] has shown how one can describe such waves at the moment of time symmetry in terms of the metric of Bondi

$$ds^2=\psi^4[e^{2\lambda q_1(z,\rho)}(dz^2+d\rho^2)+\rho^2d\varphi^2]. \quad (42)$$

He has shown that every such wave (a) which has an "energy distribution factor" $q_1(z,\rho)$ limited to a finite region of space

$$q_1(z,\rho)=0 \quad \text{for} \quad (\rho^2+z^2)^{\frac{1}{2}}>a<\infty \quad (43)$$

and (b) which satisfies the initial value equation

$$ {}^{(3)}R=0 \tag{44}$$

or

$$(1/\rho)(\partial/\partial\rho)\rho(\partial\psi/\partial\rho)+(\partial^2\psi/\partial z^2)+\rho^{-2}(\partial^2\psi/\partial\varphi^2)+\lambda\Phi_1(\rho,z)\psi=0, \tag{45}$$

with the "energy distribution source factor" Φ_1 defined by

$$4\Phi_1=\partial^2 q_1/\partial\rho^2+\partial^2 q_1/\partial z^2 \tag{46}$$

and (c) which is everywhere regular, and asymptotically flat in the sense

$$\psi \underset{(r\to\infty)}{=} 1+\frac{m^*}{2r}+\text{terms of order }\frac{1}{r^2}\text{ and higher,} \tag{47}$$

—but not everywhere flat—*necessarily has a positive definite mass:*

$$m^*=(1/2\pi)\int(\nabla\ln\psi)^2\rho d\rho dz d\varphi\geq 0. \tag{48}$$

He and Araki[20] prove, moreover, that there *exist* solutions which satisfy these requirements. From this circumstance it follows that *one can construct masslike concentrations of gravitational energy* without having to appeal to solutions of the Schwarzschild type which become singular after the lapse of a finite proper time. Moreover, geons appear so far to be the *only* purely geometrodynamical entities which go through their entire time evolution without singularity. Therefore these masslike concentrations of energy would seem to provide the only legalistically pure foundation that is available to test what shall be meant by the concept of "equations of motion."

III. BEHAVIOR OF GEONS IS INCONSISTENT WITH THE THEORY OF MOTION IN ITS PRESENT INCOMPLETE FORM

List of Effects Considered

The conventional derivation of the equations of motion from the field equations has already had to be viewed with caution because (1) certain kinds of disturbance in the metric evolve within a finite proper time into a singular state beyond which no purely classical analysis of their further development in time is possible. Geons illustrate four additional respects in which the usual treatment of the equations of motion would seem questionable: (2) the mass is not constant, (3) the mass can accelerate itself by a kind of rocket effect, (4) the structure of certain kinds of geons is sensitive to external fields to an extent for which there is no place in the usual analysis of the equations of motion, (5) all the moments of the distribution of mass-energy in a geon cannot be obtained by simple volume integrations, nor can these objects be treated as δ functions.

Leakage; Imploding Wave vs Proper Geon

The usual analysis of the equations of motion assumes that the mass of the moving object remains constant. However, leakage of energy out of a geon is unavoidable, and leakage changes the mass of the geon. The usual treatment postulates a space that is asymptotically flat. A *classical* geon in such a space ultimately diffuses away its energy to an infinite state of dilution. Only the *rate* of this leakage, not its existence, can be affected by the design of the geon. There is a great variety of geons which are allowed by the field equations. The simplest to discuss are those which possess a moment of time symmetry. The condition of these geons is completely fixed by the specification of the initial three-geometry—and the magnetic field, if any—at the moment of time symmetry. The rate of leakage is highest for gravitational waves whose effective propagation vector is directed most nearly radially. Consider, for example, a single nearly spherical pulse of gravitational radiation. Let it implode to a state of maximum concentration and spread out again. It maintains anything like the state of maximum concentration only for an interval of cotime, $\Delta T=c\Delta t$, of the order of the pulse thickness itself. The concentration of energy has so transitory an existence that it is only as a matter of principle that one might be tempted to name it a "geon." In contrast, the radiation in a proper geon travels around nearly at right angles to the radial direction and under suitable conditions holds together for a time very long in comparison with the time to go once around. Not all geons are characterized by a moment of time symmetry, but for those geons which do possess such symmetry, the equations

$${}^{(3)}R/2=(8\pi G/c^4)(\mathbf{H}^2/8\pi)=\mathbf{h}^2 \tag{49}$$

and

$$\mathbf{E}=0;\quad \mathrm{div}\mathbf{H}=0 \tag{50}$$

provide the only legal requirements on the three-dimensional geometry and the field at the instant of turn about. Up until this moment the geon is slowly sopping up inward directed radiation coming from great distances, and growing in size and mass. After this instant, the geon is leaking energy and gradually dropping in size and mass. This behavior is consistent with the theorem of Lichnerowicz and Papapetrou, that there exist no periodic and singularity-free solutions of the equations of general relativity.

Mass In Equation of Motion Cannot Be Precisely Constant

For all geons, then, time symmetric or not, energized by gravitational radiation or electromagnetic radiation or any mixture of the two, the decay rate is finite. *The mass that appears in the equation of motion is not a constant except when considered at an incomplete level of approximation.* The approximation is very good [cf. Eq. (11)] when the wavelength of the trapped radiation is short compared to the size of the geon. But in principle there is no nonsingular localized and purely geometrodynamical object to which one can point which has for all time a sharply defined mass.

Propulsion by Leakage Radiation

Leakage phenomena have a second consequence for the equations of motion. The escaping radiation may come preferentially from one side of the geon and propel it like a rocket through space at ever increasing velocity.

Construction of Geon with Rocket Property

There are several ways to construct a geon with a rocket property. One of the simplest starts with a large geon in which all the radiant energy is concentrated in a ring or torus of minor radius small in comparison with its major radius. Around one point on this ring is centered a second and much smaller toroidal geon, with its plane perpendicular to the two opposite directions of circulation or radiation in the larger geon. This second geon is not so small that it actually infringes on the active region of the larger geon. For simplicity it is taken to have such a large ratio of $R/\lambda\!\!\!\!-$ that leakage out of it is negligible compared to leakage out of the larger geon. That leakage is governed by the effective refractive index barrier about the larger geon. This primary barrier is cut down significantly in thickness in the neighborhood of the second geon. Moreover, this barrier was already thinnest on the outward curve of the primary geon. Therefore radiation escapes preferentially from one region on the outside of the primary geon. This "bright spot" serves as rocket engine to propel the combined geon system through space

Description in Terms of Radioactive Cascade of Photon Emissions

The oriented emission of radiation can be restated in quantum language. The geon emits photons preferentially into a certain direction. Recoil from these radiative processes drives the geon in the opposite direction. The geon does not follow a simple equation of motion because it so frequently undergoes radioactive decay. Such decay is the normal property of a geon.

No Classical Geon Free of Decay

Can a geon ever settle down to a ground state? Only then can it fully comport with the assumption always tacitly made in trying to derive the equations of motion from the field equation: that disintegration is impossible. The answer to this question is incomplete, but it is enough for the present purpose. No classical geon is ever free of decay. Such geons have masses greater than 10^{39} g, radii greater than 10^{11} cm, and field strengths less than the critical limit, $\mathcal{E}_{\mathrm{crit}}=4.42\times10^{13}$ gauss or es v/cm, at which electron pairs are produced out of the vacuum in great numbers. Geons with lesser dimensions and greater internal fields conceivably exist in principle, but their properties are completely unknown. There is no reason to believe they are stable. Neither is there the slightest evidence that they have any direct connection with elementary particles. Moreover, even if a sufficiently small quantum geon should be stable, its equation of motion would not admit of derivation entirely within the framework of *classical* geometrodynamics. Therefore in any purely *classical* general relativity treatment of the problem of motion it would appear essential as a matter of principle to allow for emission of radiation from the masses under study.

Usual Analysis Neglects Mass Buildup and Mass Decay by Overlooking Radiation of Short Wavelength

At what point is the radiation left out of account in the usual derivation of the equations of motion? In the calculation of surface integrals extended around the centers of mass to evaluate momentum and energy, no account is taken of possible disturbances in the metric of wavelength very *short* in comparison with the characteristic Schwarzschild lengths m_j^*(cm) associated with the masses in question. It is not right to leave such waves out of account because they describe the change in mass of the geon and its rocket propulsion properties.

Sensitivity of Some Geons to Internal Rearrangement and to Disintegration by Even Weak External Fields

There is a third respect in which the usual equations of motion have to be viewed with caution. They purport to describe, among other features, the motion of a concentration of mass-energy in a background field. There is no reason to question the qualitative correctness of the conclusions in the approximation in which geon leakage and geon breakup phenomena can be neglected. However, it is appropriate to note that the probability of leakage and breakup can be greatly affected by the background field itself. It lowers the barrier against leakage in some portions of the geon and raises it in others—the net result generally being to increase the rate of leakage. The effects can be even more extreme when the geon suddenly passes from a region where the metric is flat into a region of strong curvature. Then a large amount of radiation may be spilled out of the system. A thermal geon[17] is particularly susceptible to this kind of breakdown. It is loaded up with radiation to the limit it can hold. Entry of such an object of dimension L into a region of curved space alters the metric components from their previous values by amounts of the order

$$R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}L^2. \tag{51}$$

In consequence, the geon spills out nonuniformly in direction fractions of its entire radiative content of the same order of magnitude. Other kinds of geon are caused to undergo fission into two or more parts.

Influence of an External Field on Leakage of a Simple Spherical Geon

Even the simplest kind of geon, a spherical geon, is susceptible to at least two kinds of internal alteration as a consequence of entry into a region of curved space. First, the rate of leakage is, in general, enhanced—and in a one-sided way, so that a rocket effect results. Second, the internal energy content may be drastically rearranged. Both effects can be discussed in a little more detail.

The leakage effect in a spherical geon involves a Gamow type of penetration factor. In such a geon the metric can be written approximately in the form

$$ds^2 = -e^{\nu}dT^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{52}$$

where

$$e^{\nu} = \begin{cases} 1-2m^*/r & \text{for } r > 2.25m^* + \epsilon \\ \frac{1}{9} & \text{for } r < 2.25m^* - \epsilon \\ \text{rounded transition}^{31} & \text{for } r \text{ very close to } 2.25m^*, \end{cases} \tag{53}$$

$$e^{\lambda} = \begin{cases} (1-2m^*/r)^{-1} & \text{for } r > 2.25m^* + \epsilon \\ 1 & \text{for } r < 2.25m^* - \epsilon \\ \text{abruptly rising function, increasing nearly eightfold}^{31} & \text{for } r \text{ very close to } 2.25m^*. \end{cases} \tag{54}$$

The distance-dependent factor $R(r)$ in the electromagnetic vector potential satisfies the differential equation

$$d^2R/dr^{*2} + [\lambda\!\!\!^{-}{}^{-2} - l^{*2}e^{\nu}/r^2]R = 0, \tag{55}$$

where dr^* is an abbreviation for

$$dr^* = e^{\frac{1}{2}\lambda - \frac{1}{2}\nu}dr. \tag{56}$$

The factor

$$l^{*2} = l(l+1) \tag{57}$$

is connected with the number of nodes l in the angular dependence of the trapped radiation. Very nearly one can write

$$\begin{aligned} l^* &= \begin{pmatrix}\text{wave number of} \\ \text{radiation in active zone}\end{pmatrix}\begin{pmatrix}\text{circumference} \\ \text{of active zone}\end{pmatrix} \\ &= 3\begin{pmatrix}\text{wave number} \\ \text{far away}\end{pmatrix}\begin{pmatrix}2\pi \text{ times} \\ \text{Schwarzschild coordinate } r\end{pmatrix} \\ &= 3r_{\text{active}}/\lambda\!\!\!^{-} \\ &= (27/4)m^*/\lambda\!\!\!^{-}. \end{aligned} \tag{58}$$

The factor in square brackets in (55),

$$[\ ] \doteq \lambda\!\!\!^{-}{}^{-2}[1 - (27m^*/4r)^2(1-2m^*/r)], \tag{59}$$

is negative from just outside the active zone,

$$r_1 = (9m^*/4) \tag{60}$$

out to

$$r_2 = (27m^*/4)[(11/12)^{\frac{1}{2}} - (1/6)]. \tag{61}$$

The amplitude of the electromagnetic potential falls off exponentially in this region. The rate of leakage is governed by the Gamow factor

$$\begin{aligned} e^{-2I} &= \exp - (2/\lambda\!\!\!^{-}) \int_{r_1}^{r_2} [(27m^*/4r)^2(1-2m^*/r) - 1]^{\frac{1}{2}} dr \\ &= \exp(-4.56r_1/\lambda\!\!\!^{-}) \\ &= \exp(-10.2m^*/\lambda\!\!\!^{-}). \end{aligned} \tag{62}$$

When the geon moves into a region of substantial curvature, the metric from the geon on outward deviates from the ideal Schwarzschild character. The disturbed state of the metric is analogous to the state of the electric potential around a point charge when that charge moves into the neighborhood of charges and conductors. The potential in that case changes from (e/r) to

$$(e/r) + \sum_{n,m} c_{nm} r^n Y_n^{(m)}(\theta,\varphi) \tag{63}$$

in the case of a nearly static environment, and to

$$(e/r) + \sum_n c_n r^n P_n(\cos\theta) \tag{64}$$

when that environment has axial symmetry. Similarly here in the case of a nearly static environment of axial symmetry the metric outside the geon is altered[15] to an expression of the form

$$\begin{aligned} ds^2 = &-[(1-2m^*/r)(2m^*/r)^2 \\ &\times \sum_n a_n P_n^{(2)}(1-r/m^*)P_n(\cos\theta)]dT^2 \\ &+[(1-2m^*/r)^{-1} + (2m^*/r)^2(1-2m^*/r)^{-2} \\ &\times \sum_n a_n P_n^{(2)}(1-r/m^*)P_n(\cos\theta)]dr^2 \\ &+[1 + \text{similar terms proportional to the } a_n] \\ &\times r^2(d\theta^2 + \sin^2\theta d\varphi^2) + 2dTd\varphi(r/2m^*)^2 \\ &\times \sum_n b_n F(n+2, 1-n, 4, r/2m^*) \\ &\qquad \times \sin\theta(d/d\theta)P_n(\cos\theta). \end{aligned} \tag{65}$$

Here $P_n^{(2)}(x)$ is the associated Legendre function, with the value $3(1-x^2)$ for $n=2$ and $15x(1-x^2)$ for $n=3$; and F is the hypergeometric function

$$1 + [(n+2)(1-n)/4](r/2m^*) + \cdots. \tag{66}$$

The perturbation in the metric alters the penetration factor (62). The theory of this alteration resembles the theory of alpha decay of deformed nuclei.[32] To a first approximation the alteration in the decay rate is described by an effective penetration factor, evaluated by calculating the Gamow integral straight through the most penetrable part of the deformed barrier to the "bright spot" on the surface where the rocket effect is

[31] Detailed behavior in this transition region has been determined by an electronic computer and is given in footnote 16.

[32] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953) (especially Figs. 30 and 31 on p. 1133); J. O. Rasmussen, University of California Radiation Laboratory Unclassified Rept. UCRL-2431 (1953); J. A. Wheeler in *Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics*, edited by E. H. Bellamy and R. G. Moorhouse (Pergamon Press, New York, 1955), p. 38; R. F. Christy, Phys. Rev. **98**, 1205(A) (1955); J. O. Rasmussen and B. Segall, *ibid.* **103**, 1298 (1956); V. M. Strutinsky, Zhur. Eksptl. i Teoret. Fiz. **30**, 411 (1956); P. O. Fröman, Kgl. Danske Videnskab Selskab. Mat. fys. Medd. **1**, No. 3 (1957); C. J. Gallagher, Jr. and J. O. Rasmussen, J. Inorg. & Nuclear Chem. **3**, 333 (1957); J. O. Rasmussen and E. R. Hansen, Phys. Rev. **109**, 1656 (1958); J. O. Rasmussen, *ibid.* **113**, 1593 (1959); **115**, 1675 (1959); R. K. Chasman and J. O. Rasmussen, *ibid.* **115**, 1257 (1959).

concentrated. The perturbation changes the penetration factor from something of the form $\exp(-2I_0) = \exp(-2I_1/\lambda)$ to something of the form

$$\exp-(1/\lambda)[2I_1+\sum(a_nA_n+b_nB_n)]. \quad (67)$$

Here the coefficients A_n and B_n can be evaluated once and for all, whereas the coefficients a_n and b_n depend upon the strength and form of the external perturbation. The analytical character of (67) has the following consequences for a spherical geon of definite size and mass.

(1) If the wavelength of the trapped radiation is also specified, then there exists a certain critical strength for the coefficients in the external disturbance,

$$a_n \sim a_{n,\mathrm{crit}} \equiv \lambda/A_n, \quad b_n \sim b_{n,\mathrm{crit}} \equiv \lambda/B_n, \quad (68)$$

such that a power series development of the decay rate makes sense for values of these coefficients *less* than the critical values.

(2) If a *sequence* of geons is considered in which λ decreases by a factor 2 from each member in the sequence to the next, then a development of the decay rate in powers of a_n and b_n is *not* possible for the entire sequence of geons, no matter how small the coefficients a_n and b_n in the external perturbation may be. This circumstance shows that a power series development of the equations of motion of geons in powers of the curvature of the background metric is the less justified in principle—though the more justified in practice—the shorter the wavelength of the trapped radiation and the smaller its rate of escape.

Changes in Internal Structure Can Also Be Brought About by Background Field

In addition to altering leakage rates and inducing rocket propulsion, in a nearly spherical geon a background metric can produce drastic changes in internal structure. In a nearly spherical geon the photons course around in all directions which are contained within the thin spherical zone of substantial activity. The photon orbits are unstable[16] with respect to rearrangement in a toroidal geometry. In the rearranged geons part of the radiation goes around the ring one way; the rest, the other way. The extra gravitational attraction between beams of radiation going in opposite directions gives this configuration enhanced stability.

Altered Response of Rearranged Geon to External Fields

The toroidal configuration has an extremal quadrupole moment. In consequence of this moment the response of the geon to external gravitational fields is different from the response of the original nearly spherical geon to the same fields. The external fields also, in the course of time, change the orientation of the quadrupole moment. To date no attempt appears to have been made to derive either effect as a consequence of the gravitational field equations. However, similar phenomena associated with the next lower moment of a concentration of mass-energy have been well studied.[10] The moment of angular momentum is affected in orientation by the appropriate derivatives of the background metric; and the pull of the field gradient on this moment produces a corrective term in the equation for the motion of the center of mass itself.

Moment of Momentum Not Changed in Lowest Approximation

The moment of momentum, lower than the quadrupole moment by one order in powers of the linear extension of the geon, is also simpler than the quadrupole moment in this important respect, that it is less susceptible to being changed by the action of weak slowly varying external fields. The collapse of a nearly spherical geon into a toroidal configuration, promoted by a weak disturbance from outside, does not alter the moment of momentum to the lowest order of approximation. A nonzero value for this approximate integral of the field equations implies a definite orientation for the torus after the rearrangement reaction has taken place, and implies that more radiation goes around in one direction than in the opposite sense. No such approximate conservation law would seem to exist for the higher moments of the system—of which the quadrupole moment has the greatest influence on the response to a slowly varying background metric. In conclusion, then, (a) it appears necessary to suppose that a weak external field can promote a disproportionately great change in the quadrupole moment of the concentration of mass energy; (b) this change in quadrupole moment does alter the effective equation of motion of the mass; and therefore (c) the traditional development of the equations of motion in powers of the external perturbation would seem ill-adapted to describe the motion to all orders of approximation.

Effect of Irradiation by Fields of Short Wavelength, and Leakage of Short-Wave Radiation, on Moments

Not mentioned here is the influence of ingoing radiation of short wavelength, or outgoing leakage radiation, in altering not only the mass of the system—as already discussed—but also the first and second moments of the energy distribution. These effects, quite different from those that have to do with the response of the geon to a slowly varying background field, also find no natural place in the traditional analysis of the equations of motion.

Moments Not Definable by Simple Volume Integrals

Geometrodynamics changes in many ways one's views as to what any derivation of the equations of motion can and should prove. Let one last change be mentioned: a change in what one means by a "moment" of the distribution of mass-energy. In flat or nearly flat space it is natural to define an energy-momentum four-vector

—and thereby a mass—through integrals of the form

$$P_\alpha = \int T_{\alpha\mu} dS^\mu. \tag{69}$$

Here the integration goes over a three-dimensional spacelike hypersurface, and the element of surface has the form

$$\begin{aligned} dS^\mu &= g^{\mu\nu} dS_\nu; \\ dS_\nu &= (-g)^{\frac{1}{2}} [\nu\alpha\beta\gamma] dx^\alpha dx^\beta dx^\gamma. \end{aligned} \tag{70}$$

In this expression the symbol [0 1 2 3] has the value 1 and the general symbol changes sign on interchange of any two indices. Similarly a four-tensor of angular momentum is often defined in a flat space:

$$M_{\mu\nu} = \int (x_\mu T_{\nu\sigma} - x_\nu T_{\mu\sigma}) dS^\sigma; \tag{71}$$

and likewise higher moments. However, a geon is a region of space where the metric departs very greatly from flatness. In the center of an ideal spherical geon the quantity $g_{00} = g_{TT}$ has only one-ninth of its value outside. A very great error can therefore be made in idealizing the metric as flat in any calculation of the mass or higher moments of the distribution of mass-energy. Only because the metric deviates from flatness is it possible for two apparently contradictory expressions[33] for the mass of an ideal spherical electromagnetic geon to agree with each other:

$$Mc^2 = -\int_0^\infty \langle T_T{}^T \rangle 4\pi r^2 dr \tag{72}$$

and

$$Mc^2 = -2\int_0^\infty \langle T_T{}^T \rangle e^{\lambda/2+\nu/2} 4\pi r^2 dr. \tag{73}$$

The metric enters in an important way in these considerations because gravitational forces are so important in holding the geon together.

Deviations from Flatness Essential in Resolving Lorentz-Poincaré Paradox of Structure of Charged Object

How essential it is to allow for departures from flatness in analyzing the energy and momentum of the system shows up even more decisively in another type of geometrodynamical object. The mouth of a wormhole endowed with a flux of electric charge furnishes a classical model[1,16] for a *classical* charged body. Such an object has not the slightest direct connection with the *quantized* charges of the world of elementary-particle physics.[34] However, this classical model resolves for the first time in one self-consistent way the paradox first brought clearly to light by Poincaré and Lorentz.[35] They considered a concentration of electric charge moving through space as a unit. They evaluated the integrated energy and momentum associated with the electromagnetic field of the object. They showed that integrals of *tensor* quantities did not transform under change of reference system as do the components of a four-vector. They noted that these quantities in and by themselves therefore do not supply a proper account of the momentum-energy four-vector of a system endowed with mass. They concluded that the analysis could only then be self-consistent when it took into account the stress-energy-momentum tensor of that field, over and above electromagnetism, which held the charge together. This stabilizing force comes from the metric itself in a wormhole that traps electric lines of force. In this kind of object the electric and gravitational forces are already automatically balanced. The mass is read off directly from the asymptotically Schwarzschildean behavior of the metric far from the mouth of the wormhole.

Volume Integrals Not Appropriate as Means to Calculate Moments

Any attempt to express the mass of the wormhole mouth as an integral of some quantity over the interior would seem to be completely out of place. There is no well-defined "interior." The space is doubly connected. The further such an integration attempts to probe "into one mouth" of the wormhole, the more the region of integration extends out of the other mouth, into a space more and more nearly Euclidean, and of ever greater volume. It would seem equally inappropriate to express the moment of momentum and higher moments of such an object as integrals. They are better defined in terms of the asymptotic character of the metric, just as the moments of a distribution of electric charge are definable in terms of the coefficients c_{nm} in the expansion for the potential outside:

$$\begin{aligned} \phi &= (e/r) + (p_x x + p_y y + p_z z)/r^3 + \cdots \\ &= \textstyle\sum_{n,m} c_{nm} r^{-n-1} Y_n{}^{(m)}(\theta, \varphi). \end{aligned} \tag{74}$$

The same remarks apply to a geon which derives all or part of its mass from its content of gravitational radiation.

IV. CONCLUSIONS

Effects Which Must Be Taken into Account in an Improved Formulation of the Theory of Motion

In conclusion, the standpoint of geometrodynamics brings to light some aspects of the problem of motion which are not usually considered, but which would seem inescapable in any legalistically accurate analysis: (1) evolution of certain kinds of disturbance in the metric (Schwarzschild solutions; wormhole solution) within a

[33] See footnote 17, based on R. C. Tolman, Phys. Rev. **35**, 875 (1930); see also L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), pp. 309 and 323; also C. W. Misner and P. Putnam, Phys. Rev. **116**, 1045 (1960).

[34] Footnote 1; also J. A. Wheeler, Ann. Phys. N. Y. **2**, 604 (1957).

[35] H. Poincaré, Rend. Palermo **21**, 1906; H. A. Lorentz, Verslag. Kon. Akad. Amsterdam **26**, 981 (1917). For a survey of this problem, see A. Pais, *Developments in the Theory of the Electron* (Institute for Advanced Study and Princeton University, Princeton, New Jersey, 1948).

finite time to a state of infinite curvature beyond which no purely classical analysis of the situation is justified; (2) leakage of radiation out of other kinds of geometrical objects—geons, some of which can be free of singularity for all time—which thereby lose the possibility of having masses and moments of momentum which are sharply conserved for all time; (3) radiative propulsion of a kind not normally included in the equations of motion; (4) responses of moments and leakage rates to an external field which are also not envisaged in the usual derivations of the equations of motion from the field equations; and (5) structures which are not adapted to analysis by "δ functions" or by simple volume integrals of a "density of mass energy." In pointing out these effects it is not the intention to question the usefulness of past treatments of the problem of motion. On the contrary, they have had a most inspiring effect in bringing to light the true nature of Einstein's theory. Moreover, it cannot be doubted that their predictions are correct to a high degree of approximation in many cases. However, the point here is this, that some change in method or in principle or both must be required for a deeper analysis of the problem of motion, else effects (1)–(5) would already have found niches waiting for them in the treatment.

Geometrodynamic Foundations for Theory of Motion

It would appear that a reanalysis of the problem of motion within the framework of classical geometrodynamics must have these properties:

(1) It must start with the initial value data on an initial spacelike hypersurface.

(2) It must demand that these data satisfy the initial value equations of Foures and Lichnerowicz.

(3) It must trace out the evolution of the metric and the electromagnetic field in time from the field equations of Maxwell and Einstein.

(4) It must do this in such a way as to be able to speak of concentrations of mass-energy under conditions where this concept makes sense.

(5) The analysis must not be capable of development in power series with indefinitely high accuracy, else to the concentrations of mass-energy would be attributed a sharpness of definition which they do not have and cannot have.

Nature of Series Expansion of Equations of Motion

In connection with the appropriate form of mathematics for analyzing the problem of motion it is most instructive to consider by way of analogy a well known problem in quantum mechanics,[36] the perturbation of the levels of a harmonic oscillator, with potential $V^{(0)}=(m/2)\omega^2x^2$, by a weak supplementary potential of the form

$$V^{(1)}=-\lambda x^3. \tag{75}$$

[36] Max Born and Pascual Jordan, *Elementare Quantenmechanik* (Springer-Verlag, Berlin, 1930), chapter on perturbation theory.

One can apply the familiar apparatus of perturbation theory and develop the typical energy level in a power series in λ,

$$E_n=E_n^{(0)}+\lambda E_n^{(1)}+\lambda^2E_n^{(2)}+\cdots. \tag{76}$$

This procedure makes good sense for many purposes; however, in principle, it is completely nonsensical. For any given $\lambda>0$, no matter how small, one can always find a value of x so great that the potential has passed its peak and descended to a value lower than the energy in question. Consequently, the states in the potential minimum are not stable, but are subject to leakage through the barrier. The legalistically correct machinery of analysis is no longer a countable set of bound state eigenfunctions, but a continuous infinity of wave functions that belong to a continuous spectrum. For narrow ranges of the energy, these wave functions are large in the region of the potential minimum, but this is a distinction of degree, not of kind, from other ranges of the energy. The power series (76), at first convergent, is ultimately divergent—reasonably enough, because it purports to give a value to something that does not even exist! But the first terms in the series are often more useful than the continuum analysis—especially when supplemented by a calculation of an appropriate Gamow-Condon-Gurney leakage factor. By analogy (Table II) this circumstance suggests that the usual power series derivation of the equations of motion from the field equations give reasonable results provided that only the first few terms in the series are taken, and provided that the resulting equations of motion are supplemented by factors and terms correcting for leakage of mass, for rocket propulsion, for internal rearrangement reactions, and for the interaction with the environment of moments, natural and induced. It is not excluded that all these terms can be brought into evidence by some automatic and unthinking type of series development, but the analogy with other problems suggests that each kind of effect requires a separate kind of analysis best to show it forth. Such a situation, if it obtains, is in no way unusual. It is a characteristic of a rich physical theory—as for example Maxwell electrodynamics—to show a variety of effects, many of which require very different mathematical techniques for their elucidation (scattering of light; polarization; reflection, refraction and absorption; etc.). Such a situation is all the more to be expected in general relativity because it is the widest in scope of all physical theories and because it is thoroughly nonlinear in character.

Existence of New Effects in Elementary-Particle Physics Suggested by Present Geometrodynamical Discussion of the Problem of Motion

Unusual about geometrodynamics is the circumstance that only enough is known about its machinery today to trace out its implications at a classical level—at a level of not much direct relevance to experiment. This limitation is not a bar to seeing interesting effects at work at a

TABLE II. Analogies between two mathematically similar problems which have absolutely no direct physical relation to each other.

	Perturbation in quantum mechanics that converts a discrete spectrum into a continuum	Equations of motion of concentrations of mass energy
Power series	Starts as if convergent, then diverges; but first terms are nevertheless very useful.	Series in powers of the interacting masses presumably divergent but nevertheless useful.
Legalistically	Description of decaying state in terms of superposition of eigenstates of the energy in the continuous spectrum; or more generally, in terms of a solution of the time-dependent Schrödinger equation.[a]	Time-dependent solution of the coupled equations of Einstein and Maxwell, completely specified by the initial value data on metric and electromagnetic fields on initial spacelike hypersurface.
Analysis more accurate than power series but less complicated and more useful than fully precise treatment	Complex energy value found from fitting together JWKB approximate wave functions in the several regimes of x, thus leading automatically to a penetration factor of the Gamow type.	Allowance for leakage out of geon—and internal transformations of geon due to external perturbations—by leakage factors of the Gamow type, calculations of polarizability, and other mathematical techniques harmoniously adapted to the physics of the situation.

[a] H. Casimir, Physica **1**, 193 (1934); see also H. A. Bethe, Ann. Physik **4**, 443 (1930); G. Breit, Phys. Rev. **40**, 127 (1932); and especially G. Breit and F. L. Yost, *ibid.* **48**, 203 (1935).

classical level. Some of these classical effects already have well-known analogs in the quantum world of elementary particle physics: (1) the *existence* of concentrations of mass energy; (2) the *existence* of charge; (3) leakage of radiation from a geon, the classical analog of spontaneous radiation from a quantum system; (4) breakup of a geon into two or more parts, the classical analog of the radioactive decay of a particle. Analogs of other classical effects have not so far been seen in elementary particle transformations: (1) one-sided "rocket" propulsion, i.e., emission of a succession of quanta into the same general solid angle; (2) response of decay rate to external fields, particularly inhomogeneous gravitational fields; and (3) spontaneous internal rearrangement catalyzed by presence of an inhomogeneous environment.[37] It is not impossible that one can gain some very distant insight into the reasonableness of these processes for elementary particles by studying further the conditions under which they are significant for geons. Whether or not this is true, the further study of the problem of motion on the basis of geometrodynamics would seem to offer rewards for the future as rich as those that Einstein, Infeld, Fock, and others have won in the past.

SUMMARY

The problem of the motion of concentrations of mass-energy is discussed within the context of geometrodynamics—that formulation of standard general relativity in which attention is restricted to curved empty space, free of all singularities and of all "real masses," which lie outside the framework of classical physics. Mass in classical geometrodynamics derives only from the mass-energy of collections of gravitational radiation and electromagnetic radiation; charge, only from the flux of lines of force through "wormholes" or topological handles in the space. Legalistically speaking, the state of the system is described by giving on two nearby spacelike surfaces σ' and σ'' the intrinsic three-dimensional geometry $^{(3)}\mathcal{G}'$ and $^{(3)}\mathcal{G}''$, and the magnetic field $\mathbf{H}'$ and $\mathbf{H}''$, and nothing more. The whole evolution in time is then determined by the field equations of Maxwell and Einstein. Then only for the sake of convenience is there ever any reason to bring up the question of equations of motion of masses: under circumstances where concentrations of mass-energy hold themselves together for relatively long periods of time. Two models of such masses are considered for the insight they throw on the validity of the usual derivations of the equations of motion. One, Schwarzschild-like concentrations of mass-energy, evolve within a finite proper time into a (1) singular state beyond which no purely classical analysis of their further development in time is possible. This and other circumstances make it doubtful that the concept of "equations of motion" has any well-defined significance for such objects. Of the other model, a geon, there exist many types, some of which are free of singularity for all time. The motion of geons can be described in terms of the concept of "equations of motion" only when that description is corrected for effects which are neglected in the usual derivations of such equations: (2) slow dissipation of mass by leakage; (3) self-acceleration by one-sided emission of leakage radiation; (4) changes in structure or disintegration induced by even weak fields arising from the environment; (5) departure from anything resembling a δ-function structure or from a structure all of whose moments can be calculated by simple volume integration. It is concluded that all five effects must make their appearance in any really correct classical derivation of the equations of motion from the classical field equations. The analysis has nothing whatsoever directly to do with the quantum world of elementary particles; but just as effects (2) and (5) have their analogs in elementary-particle physics, so it is suggested that the quantum analogs of effects (3) and (4) may also reasonably be anticipated for elementary particles under appropriate conditions. It is also suggested that some preliminary insight into the character of such quantum effects may be won by a more detailed analysis of the classical problem. Also discussed is the character of the expansions which appear in the usual theory of motion, and which—it is reasoned—must ultimately diverge and become meaningless. Further accuracy in analysis can then only be achieved by returning to the logical starting point: the initial value data of pure geometrodynamics, and the field equations that govern the time rate of change of this geometry.

[37] For reasons to suspect that nucleons are crushed out of existence and converted into radiation of zero rest mass under conditions where the pressure is far above the value in the nuclear interior, and where the metric changes percentagewise very rapidly in a very small distance, as at the center of a star of critical mass, see B. K. Harrison, M. Wakano, and J. A. Wheeler in "Onzième conseil de physique Solvay," *La structure et l'évolution de l'univers* (Editions Stoops, 78 Coudenberg, Brussels, 1958); also footnote 1.