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Emission, Absorption, and Conductivity of a Fully Ionized Gas at Radio Frequencies*

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CONTENTS

1. INTRODUCTION

THE interaction between radiation and a fully ionized gas is phenomenologically described by the coefficients of emission, absorption, and electrical conductivity. The alternate use of these coefficients tends to obscure the fact that all of them are manifestations of the same atomic mechanisms and therefore only one coefficient is sufficient to specify each of the others. The mechanism pre-eminent in the radio frequency range is free-free transitions, commonly referred to as bremsstrahlung by physicists.

Unfortunately there is no unanimity as to what is the correct and complete mathematical description of this mechanism. Moreover, authors differ widely in assessing the physical vs mathematical character of the approximations conventionally introduced. In the midst of such controversy it may be useful to attempt a comprehensive review of the problems of free-free transitions which is attracting attention of an ever growing number of specialists in many branches of physics, and to reduce the results to a form useful for practical application, e.g., in radioastronomy, microwave physics, and thermofusion studies.

We begin with a description of the several *classical* methods: derivation of the emission coefficient by means of a Fourier analysis of either the dipole moment, the velocity, the acceleration of the particle, or of the potential acting on the particle. We then take up two approximations, viz., the straightline approximation to the hyperbolic path of the electron in the field of the ion, and a frequency condition making certain integrals solvable by quadrature. If care is taken to avoid inconsistent approximations during successive stages of the analysis, we shall find no differences between the several classical variants.

We then examine whether it is adequate to treat the free-free transitions as two-body collisions and arrive at the important result that, contrary to prevailing opinion, the shielding of the ion's potential by free electrons can be neglected.

Turning to the corresponding quantum mechanical calculations, we state the general result applying to the high energy range as well as the rf spectrum. For the latter we give coefficients remaining correct under conditions where classical calculations break down.

So far, we have considered a single electron of prescribed velocity. The summary effect given by an assembly of particles with a spread in velocities is then found by integrating over the distribution function. This is carried out in detail (the quantum corrections included) for a Maxwellian distribution which is, of course, the case of greatest interest in practice.

In the concluding section we analyze the relation between emission and absorption, assuming local thermodynamic equilibrium (LTE). Without this restriction it would be necessary to reformulate many radiation laws, like Kirchoff's law, and to discuss the general aspects of steady nonequilibrium states which are beyond the scope of this article.

Certain other restrictions are adhered to throughout this review. Firstly, the refractive index is always assumed to be unity because the inclusion of arbitrary values of the refractive index would require an elaborate discussion of the basic radiation laws. To our knowledge, this problem has not yet been solved in an approxi-

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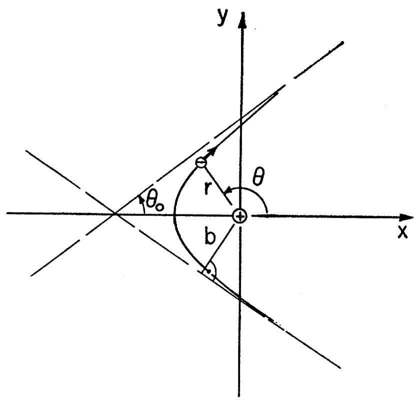

FIG. 1. Geometry of an electron-ion encounter.

mation commensurate with the level aimed at in this paper. Laboratory physicists as well as radioastronomers should not find this restriction too severe, since it affects a rather narrow band around the plasma frequency.

Secondly, external magnetic fields are ignored. In a magnetic field, cyclotron radiation would be emitted and absorbed, and, unless the magnetic field is excessive or particle energies are relativistic, cyclotron radiation can simply be added to free-free radiation in the same way a spectral line is superimposed on a continuum. Accordingly, outside the range of cyclotron resonance our discussion is adequate.

Thirdly, the ionic potential is of the Coulomb type with an arbitrary nuclear charge and therefore only applies to such ions as would have hydrogen-like spectra. Hence, this treatment is inapplicable to an astrophysically important case of free-free radiation, namely, the free-free transition of the negative hydrogen ion, i.e., the transitions of a free electron in the field of a neutral hydrogen atom. However our scheme of approximations should hold for the free-free transitions of, say, single-ionized helium.

Finally, the particle energies are supposed to be nonrelativistic. Under relativistic conditions the quantum mechanical approach, also, must be drastically altered. Relativistic effects require discussion in the context of a few astrophysical problems (Crab Nebula, for instance) and for certain thermofusion devices, which are not considered here. Consequently, the term "classical" is used as a disjunction to "quantum mechanical."

2. CLASSICAL EMISSION COEFFICIENT DERIVED FROM A FOURIER ANALYSIS OF THE ACCELERATION

If the energy loss due to radiation is negligible,[1] the electron describes a hyperbolic orbit in a reference frame attached to the ion. Let the orbital plane be the x-y plane of a Cartesian system or of the corresponding system r, θ. These geometrical conditions are illustrated in Fig. 1.

This description assumes implicitly infinite mass for the ion, so that the center of gravity coincides with the ion's position. Numerically the assumption is of no consequence.

The polar equation of the hyperbola is

$$1/r=(1-\epsilon\cos\theta)/(b-\tan\theta_0). \tag{1}$$

The geometrical meaning of the collision parameter b is evident from Fig. 1. The eccentricity ϵ and the asymptotic angle θ_0 are related by

$$\epsilon=\sec\theta_0. \tag{2}$$

From Coulomb's law

$$|\ddot{r}|=Ze^2/mr^2, \tag{3}$$

we find for the asymptotic angle θ_0 (Rutherford's scattering formula)

$$\tan\theta_0=mbv_0^2/Ze^2=(\epsilon^2-1)^{\frac{1}{2}}. \tag{4}$$

Here, $\ddot{r}$ is the acceleration, m the mass, $-e$ the charge of the electron, $+Ze$ the charge of the ion, and v_0 the initial velocity, i.e., the electron's velocity at time $t=-\infty$.

From conservation of angular momentum, we have

$$d\theta=v_0 b\,dt/r^2. \tag{5}$$

The integration of Eq. (5) is elementary and leads to the well-known result

$$t=-\left(\frac{b}{v_0}\right)\frac{2\epsilon\tan(\theta/2)}{(\epsilon-1)+(\epsilon+1)\tan^2(\theta/2)}-(\epsilon^2-1)^{-\frac{1}{2}} \cdot\ln\left\{\frac{(\epsilon-1)^{\frac{1}{2}}+(\epsilon+1)^{\frac{1}{2}}\tan(\theta/2)}{(\epsilon-1)^{\frac{1}{2}}-(\epsilon+1)^{\frac{1}{2}}\tan(\theta/2)}\right\}. \tag{6}$$

The expression for the emission in all frequencies, and into the solid angle 4π per encounter, is according to Hertz's classical formula:

$$Q=\frac{2e^2}{3c^3}\int_{-\infty}^{+\infty}(\ddot{r})^2dt=\frac{2e^2}{3c^3}\int_{-\infty}^{+\infty}[\ddot{x}^2+\ddot{y}^2]dt. \tag{7}$$

In order to obtain the frequency distribution of Q, a Fourier analysis of the components $\ddot{x}$ and $\ddot{y}$ of the acceleration is required. Their Fourier components are defined by the relations

$$\ddot{x}(\omega)=\pi^{-1}\int_{-\infty}^{+\infty}\ddot{x}(t)\cos(\omega t)\,dt, \tag{8}$$

$$\ddot{y}(\omega)=\pi^{-1}\int_{-\infty}^{+\infty}\ddot{y}(t)\sin(\omega t)\,dt, \tag{9}$$

where the components of the acceleration as functions of r and θ follow from Eq. (3):

$$\ddot{x}(t)=-(Ze^2/mr^2)\cos\theta,\ \ddot{y}(t)=-(Ze^2/mr^2)\sin\theta. \tag{10}$$

[1] For justification see Sec. 9.

By Parseval's theorem the relation between the spectral distribution Q_ω and the total emission Q is

$$Q=\int_{-\infty}^{+\infty}Q(t)\,dt=\pi\int_0^\infty Q_\omega d\omega. \tag{11}$$

Note that the physical dimension of Q is that of an energy, whereas $Q(t)$ is a rate of energy (energy per unit time interval). The quantity Q_ω represents a spectrally resolved energy (energy per unit interval of angular frequency). Hence, $Q_\omega d\omega$ has the same physical dimension as Q.

Substituting the polar angle θ for the time coordinate in Eqs. (8) and (9), we obtain

$$\ddot{x}(\omega)=\frac{Ze^2}{\pi m v_0 b}\int_{\theta_0}^{2\pi-\theta_0}\cos(\omega t)\cos\theta d\theta, \tag{12}$$

$$\ddot{y}(\omega)=\frac{Ze^2}{\pi m v_0 b}\int_{\theta_0}^{2\pi-\theta_0}\sin(\omega t)\sin\theta d\theta, \tag{13}$$

where, for sake of brevity, the time has not been explicitly removed from the trigonometric functions. These equations have a limited use in connection with the straight-line approximation to be discussed in Sec. 4.

Proceeding with the general development, we introduce a parametric representation of r and t which allows us to reduce the integrals in Eqs. (12) and (13) to Hankel functions.

We define a parameter ξ by

$$\sinh\xi=(\epsilon^2-1)^{\frac{1}{2}}\sin\theta/(1-\epsilon\cos\theta), \tag{14}$$

and find instead of Eqs. (1) and (6) the parametric expressions for r and t

$$r=b_0(\epsilon\cosh\xi-1) \tag{15}$$

and

$$t=b_0/v_0(\epsilon\sinh\xi-\xi)/v_0, \tag{16}$$

with the following abbreviations

$$b_0=Ze^2/mv_0^2,\ \tan\theta_0=b/b_0. \tag{17}$$

This representation is well-known from celestial mechanics and has been used by Landau and Lifshitz (1951)[2] in their treatment of bremsstrahlung. The physical meaning of the quantity b_0 is that of a "collision parameter" applying to a deflection by 90°.

The rectilinear coordinates

$$x=r\cos\theta,\ y=r\sin\theta \tag{18}$$

transform into

$$x(\xi)=b_0(\cosh\xi-\epsilon) \tag{19}$$

and

$$y(\xi)=b_0\sinh\xi. \tag{20}$$

The components of the acceleration, according to Eq. (16), are

$$\ddot{x}(\xi)=\frac{v_0^2}{b_0}\frac{\epsilon-\cosh\xi}{(\epsilon\cosh\xi-1)^3} \tag{21}$$

and

$$\ddot{y}(\xi)=-\frac{v_0^2}{b_0}(\epsilon^2-1)^{\frac{1}{2}}\frac{\sinh\xi}{(\epsilon\cosh\xi-1)^3}. \tag{22}$$

Substitution of these expressions in Eqs. (8) and (9) yields

$$\ddot{x}(\omega)=-\frac{v_0}{\pi}\int_{-\infty}^{+\infty}\frac{\epsilon-\cosh\xi}{(\epsilon\cosh\xi-1)^2}\cos[\Omega(\epsilon\sinh\xi-\xi)]d\xi \tag{23}$$

and

$$\ddot{y}(\omega)=+\frac{v_0}{\pi}(\epsilon^2-1)^{\frac{1}{2}}\int_{-\infty}^{+\infty}\frac{\sinh\xi}{(\epsilon\cosh\xi-1)^2}\cdot\sin[\Omega(\epsilon\sinh\xi-\xi)]d\xi, \tag{24}$$

where

$$\Omega=\omega b_0/v_0. \tag{25}$$

Integration by parts of Eqs. (23) and (24) leads to

$$\ddot{x}(\omega)=-\frac{v_0\Omega}{\pi}\int_{-\infty}^{+\infty}\sinh\xi\sin[\Omega(\epsilon\sinh\xi-\xi)]d\xi \tag{26}$$

and

$$\ddot{y}(\omega)=-(v_0\Omega/\pi\epsilon)(\epsilon^2-1)^{\frac{1}{2}}\int_{-\infty}^{+\infty}\cos[\Omega(\epsilon\sinh\xi-\xi)]d\xi. \tag{27}$$

For details see Appendix A.

The integrals in Eqs. (26) and (27) can be expressed in terms of Bessel functions of the third kind (Hankel functions of imaginary argument and order):

$$\ddot{x}(\omega)=-v_0\Omega(d/du)H_{i\Omega}^{(1)}(u)|_{u=i\Omega\epsilon} \tag{28}$$

and

$$\ddot{y}(\omega)=-iv_0(\Omega/\epsilon)(\epsilon^2-1)^{\frac{1}{2}}H_{i\Omega}^{(1)}(i\Omega\epsilon). \tag{29}$$

Details are given in Appendix B.

On combining Eqs. (7), (11), (28), and (29), we obtain for the spectral distribution

$$\pi Q_\omega d\omega=\frac{2\pi e^2}{3c^3}\left(\frac{Ze^2}{mv_0^2}\right)^2\left\{\left[\frac{dH_{i\Omega}^{(1)}(u)}{du}\right]^2_{u=i\Omega\epsilon}-\frac{\epsilon^2-1}{\epsilon^2}[H_{i\Omega}^{(1)}(i\Omega\epsilon)]^2\right\}d\omega. \tag{30}$$

This equation is the definitive classical description of the radiation emitted during a single electron-ion encounter. Before evaluating the cumulative effects of all encounters in an assembly of ions and electrons, we shall compare Eq. (30) with previous work and examine the consequences of certain approximations to Eqs. (12) and (13).

For motion of the electron in a closed orbit (ellipse or circle), trigonometric functions take the place of the hyperbolic functions under the integrals in Eqs.

[2] References are given in alphabetical order in the Bibliography.

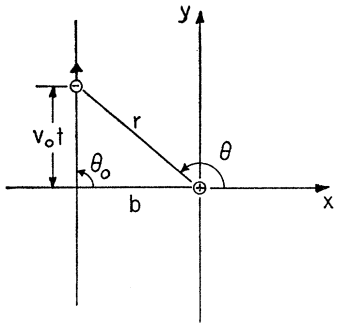

FIG. 2. Straight-line approximation.

(23) and (24), and in that case the integrals can be expressed by Bessel functions of the first kind. A generalization to nonperiodic orbits, corresponding to a finite range of integration in Eqs. (23) and (24), was given by Oster (1960a).

3. DIPOLE MOMENT AND EMISSION

The derivation of Eq. (30) was patterned after the mathematical procedure of Landau and Lifshitz (1951). Their point of departure is the same classical formula for the total radiation [Eq. (7)], and their final result does agree with Eq. (30), but they make intermediate use of the dipole moment

$$\mathbf{d}=-e\mathbf{r} \tag{31}$$

rather than the acceleration. The Fourier components of $\mathbf{d}$ and $\mathbf{r}$ are related to the Fourier components of the acceleration $\ddot{\mathbf{r}}$ and the velocity $\dot{\mathbf{r}}$ by

$$\ddot{\mathbf{r}}(\omega)=-i\omega\dot{\mathbf{r}}(\omega)=(\omega^2/e)\mathbf{d}(\omega). \tag{32}$$

Landau and Lifshitz find it convenient to subject to Fourier analysis the components of the *velocity* obtained by differentiation of Eqs. (19) and (20) with respect to the time, expressed in terms of the parameter ξ, as in Eq. (16).

Admittedly our derivation in Sec. 2 does not differ in physical substance from the approach Landau and Lifshitz have chosen. The variant here preferred facilitates intercomparison of various treatments employing successive approximations whose interaction is difficult to disentangle. Equation (30) already appears in a footnote to Kramers' classical paper [Kramers (1923)].

4. STRAIGHT-LINE APPROXIMATION

According to Eq. (7), the total emission is proportional to the square of the acceleration, while the acceleration itself, according to Coulomb's law, is inversely proportional to the square of the electron-ion distance. Consequently the high energy photons will be emitted primarily during the very close encounters, the rf quanta primarily during distant encounters.

In the rf region to be considered in this article, the distant encounters correspond to hyperbolic orbits that are almost straight lines, i.e.,

$$\epsilon\gg1, \tag{33}$$

Obviously the bulk of the emission is centered on frequencies near the reciprocal of the "duration of an encounter." The natural time interval characterizing a single encounter is the ratio of velocity and collision parameter, i.e.,

$$1/\omega\approx b/v_0. \tag{34}$$

A typical value for the rf range is a wave length of 10 cm, or $\omega\approx2\cdot10^{10}$ cps. Taking $v_0=2\cdot10^8$ cm/sec which is about the average velocity at 100 000°, we find an eccentricity of the order 10^6. This argument provisionally justifies the straight-line approximation in dealing with hyperbolic orbits. In discussing the radiation from an assembly of particles with a spread in velocities, we must verify, of course, that this approximation is legitimate for all particles except a negligible number requiring a more complete treatment.

If the hyperbola is replaced by a straight line, it is unnecessary to introduce the parameter ξ [cf. Eqs. (15) and (16)], and we can immediately subject Eqs. (12) and (13) to Fourier analysis. Introducing $\tan\theta$ as the new variable, we find from Eq. (6)

$$u\equiv\tan\theta=-v_0t/b. \tag{35}$$

Equation (35) also can be recovered from Eq. (16), since for $\epsilon\gg1$

$$\sinh\xi\approx-\tan\theta,\quad\text{with}\quad\tan\theta_0=b/b_0\approx\epsilon. \tag{36}$$

For later use we express the radius vector r also as function of the variable u:

$$r=b_0\epsilon(1+u^2)^{\frac{1}{2}}. \tag{37}$$

The geometrical conditions are illustrated by Fig. 2.

The Fourier components of the acceleration, Eqs. (12) and (13), then read

$$\ddot{x}(\omega)=\frac{Ze^2}{\pi mv_0b}\int_{-\infty}^{+\infty}\cos(\Omega\epsilon u)\frac{du}{(1+u^2)^{\frac{3}{2}}} \tag{38}$$

and

$$\ddot{y}(\omega)=-\frac{Ze^2}{\pi mv_0b}\int_{-\infty}^{+\infty}\sin(\Omega\epsilon u)\frac{du}{(1+u^2)^{\frac{3}{2}}}. \tag{39}$$

Equations (38) and (39) were previously derived and numerically evaluated by Oster (1959). A more convenient form of Eq. (39) is found by an integration by parts:

$$\ddot{y}(\omega)=-\frac{Ze^2}{\pi mv_0b}\Omega\epsilon\int_{-\infty}^{+\infty}\cos(\Omega\epsilon u)\frac{du}{(1+u^2)^{\frac{1}{2}}}. \tag{40}$$

The integrals from Eqs. (38) and (40) are readily expressed in terms of Bessel functions (cf. Appendix C), so that the total emission becomes

$$\pi Q_\omega d\omega=\frac{2\pi e^2}{3c^3}\omega^2\left[\frac{Ze^2}{mv_0^2}\right]^2\left\{\left[\frac{d}{d(i\Omega\epsilon)}H_0^{(1)}(i\Omega\epsilon)\right]^2-\left[H_0^{(1)}(i\Omega\epsilon)\right]^2\right\}d\omega. \tag{41}$$

We now verify directly that in the limiting case $\epsilon \gg 1$ the general expression for the spectrum [Eq. (30)] tends to Eq. (41). By the straight-line condition of Eq. (33) the coefficient of the second Hankel function goes to unity. Rewriting Eq. (33) in terms of collision parameter and particle velocity, we find from Eq. (4)

$$\epsilon \propto bv_0^2 \to \infty \tag{42}$$

The order of the Bessel function, in fact, tends to zero, since according to Eqs. (25) and (34)

$$\Omega \propto \omega v_0^{-3} \propto b^{-1} v_0^{-2} \to 0, \tag{43}$$

while the combination

$$\Omega\epsilon \propto bv_0^2 \cdot \omega v_0^{-3} = \omega b v_0^{-1}, \tag{44}$$

the argument of the Hankel functions, may have any value. However, we will show in the next section that this value is small compared with unity in the rf region.

We now see that the validity of the straight-line approximation depends on a combination of collision parameter and particle velocity. Either particle velocity or collision parameter, or both, ought to be high enough to ensure both conditions (42) and (43).

5. LOW-FREQUENCY LIMIT

Both the original literature and textbooks on radioastronomy, frequently use still another assumption which we shall show to be inappropriate. If the frequencies are so low that over the "time interval during which the radiation is effectively emitted" the trigonometric functions in Eqs. (8) and (9) are approximately constant, the Fourier integrals can be found by quadrature. Hence, under the geometrical conditions outlined in Fig. 1, the sine term is zero, the cosine term unity:

$$\ddot{x}(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \ddot{x}(t)\,dt,\ \ddot{y}(\omega) = 0, \tag{45}$$

which leads to

$$\ddot{x}(\omega) = -\frac{Ze^2}{\pi m v_0 b}\int_{\theta_0}^{2\pi-\theta_0} \cos\theta d\theta = -\frac{2v_0}{\pi\epsilon}. \tag{46}$$

The emission per encounter becomes

$$\pi Q_\omega d\omega = (8e^2/3\pi c^3)(v_0^2/\epsilon^2)\,d\omega. \tag{47}$$

The independence of the emitted radiation of the frequency is often mentioned by radioastronomers. As noted by Scheuer (1960), the derivation just given corresponds to the Fourier analysis of an acceleration of vanishing duration, mathematically represented by a Delta function.

In order to establish the upper limit for ω which would ensure validity of Eq. (45), we investigate the limit $\omega \to 0$ of the complete expression, Eq. (30). The theory of Bessel functions leads to the following expansion [Appendix D] for the Hankel functions in the limit $\Omega \to 0$:

$$H_0^{(1)}(i\Omega\epsilon) = -(2/\pi i)[\ln(\tfrac{1}{2}\Omega\epsilon) + \gamma^*], \tag{48}$$

where Euler's constant

$$\gamma^* = 0.577\cdots,\quad \gamma = e^{\gamma^*} = 1.78\cdots. \tag{49}$$

The next following term in the expansion reads

$$-(2/\pi i)(\tfrac{1}{2}\Omega\epsilon)^2[\ln(\tfrac{1}{2}\Omega\epsilon) + \gamma^* - 1]. \tag{50}$$

From Eq. (48) follows for the derivative

$$dH_0^{(1)}/d(i\Omega\epsilon) = +2/\pi\Omega\epsilon \tag{51}$$

with the next term of the order $o(\Omega\epsilon)$.

Comparing Eqs. (48) and (51), we find that the contribution from the logarithmic term can be neglected in the limit of low frequencies, because

$$(\Omega\epsilon)^{-1} \gg \ln(\Omega\epsilon). \tag{52}$$

Hence, we recover the previous result, Eq. (47). The important new information, however, is the exact form of Eq. (52) which gives the precise condition of validity, an alternate form of which is analogous to Eqs. (42) and (44) and reads

$$\Omega\epsilon \propto \omega b v_0^{-1} \to 0 \tag{53}$$

This expression comprises the weaker condition

$$\Omega \propto \omega v_0^{-3} \to 0, \tag{54}$$

Eq. (43), which is automatically fulfilled whenever the straight-line approximation holds.

The fact that the expression of Eq. (54) is identical for the straight-line approximation and the present frequency condition is the reason why Hankel functions of order 0 turn up in either case. The second condition for the validity of the straight-line approximation, [Eq. (42)], is not fulfilled automatically, however: Frequency condition and straight-line approximation are completely independent in this respect. This can be seen by choosing a small value for ω and also small values for b and v_0; then, Eq. (53) can be fulfilled, while Eq. (42) is clearly violated.

A comment may be added on the two limiting cases of Eq. (47) as far as ϵ is concerned, namely, the straight-line approximation and the parabolic orbit. As shown in Sec. 4, the former leads to the condition, Eq. (33), which can be written as

$$\epsilon \approx \tan\theta_0 = mbv_0^2/Ze^2. \tag{55}$$

The spectrum in the low-frequency limit becomes

$$\pi Q_\omega d\omega = \frac{8Ze^6}{3\pi c^3 m^2 b^2 v_0^2} d\omega. \tag{56}$$

On the other hand, very close encounters, which are of major importance in the x-ray region, can be described with sufficient accuracy by a nearly parabolic orbit whose characteristics are

$$\epsilon = \sec\theta_0 \approx 1. \qquad (57)$$

The resulting spectrum now reads

$$\pi Q_\omega d\omega = (8e^2 v_0^2 / 3\pi c^3) d\omega. \qquad (58)$$

Eqs. (56) and (58) have been widely used and were discussed, among others, by Smerd and Westfold (1949).

In particular, Eq. (58) is the limiting case $\omega \to 0$ of a formula derived by Kramers (1923). He followed the same procedure as we did in Sec. 4, but started from a nearly parabolic orbit instead of the straight-line approximation. He then ended up with Hankel functions of order $\frac{1}{3}$ and $\frac{2}{3}$, which can be shown to be the appropriate limiting case for $\epsilon \to 1$ of the Hankel functions of the more general type derived in Sec. 2, Eq. (30).

Although this result concerns primarily the x-ray region and is of no special interest for the rf problem, it is mentioned here because it will be used to examine the relations between classical and quantum mechanical calculations in Sec. 10.

6. INTEGRATION OVER THE COLLISION PARAMETER

The calculations presented so far deal with a single electron of prescribed initial velocity passing a given ion at a fixed distance, i.e., with a fixed collision parameter. The next step in calculating the emission coefficient per unit volume consists of averaging over the collision parameters.

The number of encounters between one electron and N_i ions in the parameter range b, $b+db$, according to the well-known target consideration, is

$$N_i v_0 2\pi b db. \qquad (59)$$

Let N_i be normalized to unit volume and v_0 be, as before, the electron's initial velocity.

The emission of a single electron/sec and in the frequency range ω, $\omega+d\omega$ is then

$$2\pi N_i v_0 \pi \int_0^\infty Q_\omega b db (d\omega) = 2\pi N_i v_0^2 (2\pi e^2/3c^3) \omega^2 (Ze^2/mv_0^2)^2$$
$$\cdot \int_0^\infty \left\{ \left[\frac{dH_{i\Omega}^{(1)}(u)}{du} \right]^2_{u=i\Omega\epsilon} - [(\epsilon^2-1)/\epsilon^2][H_{i\Omega}^{(1)}(i\Omega\epsilon)]^2 \right\}$$
$$\cdot b db d\omega, \qquad (60)$$

where the unabridged expression for the spectrum from Eq. (30) has been used.

We transform the integration over b into an integration over ϵ by making use of Eq. (4):

$$b db = b_0^2 \epsilon d\epsilon. \qquad (61)$$

The new limits for the integration are 1 and ∞. Choosing finally

$$u \equiv i\Omega\epsilon \qquad (62)$$

as variable, we find for the integral in Eq. (60):

$$-\frac{b_0^2}{\Omega^2} \int_{i\Omega}^{i\Omega\infty} \left\{ \left[\frac{dH_{i\Omega}^{(1)}(u)}{du} \right]^2 - \frac{u^2+\Omega^2}{u^2} [H_{i\Omega}^{(1)}(u)]^2 \right\} u du. \qquad (63)$$

The bracket in Eq. (63) can be transformed into a total differential which is readily integrated (Appendix E):

$$-\frac{b_0^2}{\Omega^2} \int_{i\Omega}^{i\Omega\infty} \frac{d}{du} \left[u H_{i\Omega}^{(1)}(u) \frac{d}{du} H_{i\Omega}^{(1)}(u) \right] du$$
$$= +(b_0^2/\Omega^2) \{ i\Omega H_{i\Omega}^{(1)}(i\Omega) (d/du) H_{i\Omega}^{(1)}(u)|_{u=i\Omega} \}. \qquad (64)$$

Then, Eq. (60) assumes the form (for details see Appendix F):

$$\pi \langle Q_\omega \rangle_b d\omega = N_i (4\pi^2 e^2/3c^3) v_0^3 (Ze^2/mv_0^2)^2 i\Omega H_{i\Omega}^{(1)}(i\Omega)$$
$$\cdot (d/du) H_{i\Omega}^{(1)}(u)|_{u=i\Omega} d\omega. \qquad (65)$$

The same type of integration has been effected by Landau and Lifshitz [(1951), p. 200] in deriving expressions for the radiation from a parallel beam of electrons.

We now turn to the comparison after integration over the collision parameter of the two major approximations, i.e., the straight-line orbit and the low-frequency limit, with the complete Eq. (65).

The straight-line approximation is found from Eq. (41), making use of the formulas from Appendexes E and F for the functions of order zero:

$$\pi \langle Q_\omega \rangle_b d_\omega = N_i (4\pi^2 e^2/3c^3) v_0^3 (Ze^2/mv_0^2)^2 i\Omega H_0^{(1)}(i\Omega)$$
$$\cdot (d/du) H_0^{(1)}(u)|_{u=i\Omega} d\omega. \qquad (66)$$

An inspection of Eq. (E2) shows that Eq. (66) is valid if the order ν is small with respect to the argument:

$$\nu^2/u^2 \ll 1, \qquad (67)$$

i.e., $\epsilon \gg 1$.

The derivation of Eq. (66) has assumed that the major contributions to the radiation at frequencies ω come from encounters with $\epsilon \gg 1$, so that the integration over ϵ can be extended to the limit $\epsilon = 1$, although strictly the integrand has been derived under the assumption that $\epsilon \gg 1$. This procedure will be justified from the mathematical point of view in Appendix G.

For low frequencies, we expand the Bessel functions in Eq. (65) with the aid of Eqs. (48) and (51) and of the formulas of Appendix D and obtain

$$\pi \langle Q_\omega \rangle_b d\omega = N_i \frac{16 e^2}{3c^3} v_0^3 \left(\frac{Ze^2}{mv_0^2} \right)^2 \ln\left(\frac{2mv_0^3}{\gamma Z e^2 \omega} \right) d\omega. \qquad (68)$$

This result is highly significant in two respects. In the first place it means that the emission is bounded for all nonvanishing frequencies, but diverges logarithmically for $\omega \to 0$. This divergence does not result from integration over an infinite range of collision parameters. Secondly, even in the limit of vanishing frequency, the emission still depends on the frequency by way of the logarithm. Either point contradicts current opinion. [cf. Smerd and Westfold (1949), Unsöld (1955), and Oster (1959).] It may be useful to elucidate the reasons for this misunderstanding.

In Sec. 5, Eq. (47), we had found that in the low-frequency limit the spectrum for a prescribed collision parameter, represented by ϵ, is independent of ω and is proportional to ϵ^{-2} on the physical assumption that the emission of radiation occurs during an infinitesimal time interval. The mathematical correlate of this argument was [cf. Eq. (53)] that $\Omega\epsilon\rightarrow 0$. Starting from Eq. (47) one has to evaluate the integral

$$\pi\langle Q_\omega\rangle_b d\omega = 2\pi N_i b_0^2 \frac{8e^2 v_0^2}{3\pi c^3}\int_1^\infty \frac{d\epsilon}{\epsilon}(d\omega). \qquad (69)$$

This integral obviously diverges. The mishap suggests checking the derivation of Eq. (69) as to a tacit violation of antecedent assumptions. Reflection will show that the condition $\Omega\epsilon\rightarrow 0$, or more precisely

$$(\Omega\epsilon)^{-1} \gg \ln(\Omega\epsilon), \qquad (70)$$

as discussed in Sec. 5, is not valid for large values of ϵ. Hence the assumption of an infinitesimal time interval for the emission of radiation is not satisfied if the electron encounters an ion along a path with large collision parameter.

7. CRITIQUE OF A POTENTIAL CUTOFF

To escape the difficulty with the diverging integral (69), it has been argued on *physical* grounds that the integration over ϵ ought to be cut off at a *finite* distance from the scattering ion. This reasoning, in itself, is correct in the sense that the mobility of the free electrons in a plasma tends to establish electrical neutrality everywhere. The distance beyond which the ions' potential can, on the average, be taken as shielded by the electrons, is the Debye-distance (rather than the mean ionic distance chosen by Smerd and Westfold). A mathematically more sophisticated formulation results from replacing the ions' Coulomb potential by a "Debye-Hückel-potential," which is essentially Coulombian out to the Debye-distance and falls off exponentially beyond this distance. This procedure was adopted for conductivity problems in the stellar interior by Persico (1926), whose work elaborates earlier computations by Eddington (1925) and the classical paper by Debye and Hückel (1923).

It is perhaps understandable that no effort was made to mitigate the divergence problem encountered in the last section, which we have now shown to be spurious. For in related work on dc conductivities,[3] the cutoff procedure is indeed necessary in order to avoid a logarithmic divergence of the conductivity coefficients. This divergence occurs solely in the limit $\omega\rightarrow 0$ and is wholly irrelevant to the integration over the collision parameters.

The real problem can now be stated as follows: What is the numerical influence on the emissivity, at a given frequency, of the potential cutoff at, say, the Debye length

$$b_m = (KT)^{\frac{1}{2}}/2\pi^{\frac{1}{2}} e N_e^{\frac{1}{2}}? \qquad (70)$$

Here, K is the Boltzmann constant, T the kinetic temperature, and N_e the number density of electrons per unit volume.

At this point we remark on the accuracy of the proposed treatment. Firstly, cutting off the ions' Coulomb potential at a distance given by Eq. (70) is only qualitatively correct. Substitution of an exponential decrease outside of b_m, instead of this step potential, is from the physical point of view not more accurate, although it may be expedient mathematically.

Secondly, whereas the shielding corrections of the type of Eq. (70) express a property of an *assembly* of electrons which is characterized by its mean kinetic energy, our analysis of the *individual encounters* has been carried out with complete generality and therefore lends itself to a rigorous evaluations of the mean square fluctuations of radiant energy. Introducing subsequently a cut-off at a *mean* distance in a sense violates the spirit of the entire preceding analysis. To give a self-consistent theory of the Debye shielding is beyond the scope of this review. Fortunately, we shall find that the shielding effects are numerically of minor importance in the rf range.[4]

The argument just completed, in fact, precludes the direct computation of the *absorption* coefficient from Lorentz's theory which takes its departure from equations representing an *average* particle. Since here the concept of an average particle is introduced from the very beginning and not merely as a second-order approximation, the direct computation of absorption coefficients in this manner is but a crude approximation.

Finally, the cutoff of the potential ought to be used not only while integrating over the collision parameters, but even in calculating the Fourier components in Eqs. (8) and (9), in which the integration over time should be restricted to a finite range, namely, the time it takes a particle to cross the Debye sphere. The need for such a correction has not received sufficient recognition, although the common omission of this correction can be justified only after it has been established that the radiation emitted at orbital points outside of the Debye sphere is negligible. It is undoubtedly incorrect to omit this correction while

[3] cf. Cohen, Spitzer, and Routly (1950), Spitzer and Härm (1953). Landshoff (1949, 1951) calculated the higher-order approximations to the Chapman-Enskog method of solving Boltzmann's transport equation [Chapman and Cowling (1955)].

[4] The exact form of Eq. (70) requires further explanation. The customary derivation of the Debye distance b_m includes a dependence on the nuclear charge Z of the form $(1-Z)^{-\frac{1}{2}}$, which adds in the case of protons a factor $2^{\frac{1}{2}}$ to the denominator. [cf. Spitzer (1956).] The purely qualitative argument given in the body of our review makes this numerical factor rather arbitrary. We have therefore omitted this insignificant factor and obtain as reward the simple form of Eq. (76).

retaining the shielding argument in order to avoid a logarithmic divergence.

We hope that this rather lengthy discussion will serve to clear up some of the misunderstandings besetting the literature.

8. SHIELDING CORRECTIONS TO THE EMISSIVITY

As stated before, the main problem to be solved in connection with the shielding correction is the *numerical* importance of the term which arises from the finite upper limit b_m in the integration over collision parameters in Eq. (60). Associated with b_m is an eccentricity

$$\epsilon_m = (mv_0^2/Ze^2)\, b_m \tag{71}$$

and [cf. Eq. (62)] a parameter

$$u_m = i\Omega\epsilon_m = i\frac{\omega}{v_0}b_m = i\frac{\omega}{v_0}\frac{(KT)^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}eN_e^{\frac{1}{2}}}. \tag{72}$$

To be consistent, the velocity v_0 likewise ought to be replaced by an average value. We choose

$$\langle v_0 \rangle = (KT/m)^{\frac{1}{2}}, \tag{73}$$

putting unity for the numerical factor that distinguishes rms, most probable, etc., velocities from each other. This choice is no less arbitrary than the remainder of the whole shielding argument. We obtain

$$u_m = i\omega m^{\frac{1}{2}}/2\pi^{\frac{1}{2}}eN_e^{\frac{1}{2}}. \tag{74}$$

Introducing the plasma frequency ω_p,

$$\omega_p^2 = (4\pi e^2/m)\, N_e, \tag{75}$$

we can write instead of Eq. (74),

$$i\Omega\epsilon_m = i\omega/\omega_p. \tag{76}$$

The atomistic counterpart of the plasma frequency is the reciprocal of the time an electron takes to cross (half) the Debye sphere. This result looks rather strange. It is well known that no radiation of frequencies smaller than ω_p can be transmitted through a plasma, since for $\omega=\omega_p$ the refractive index goes through zero, the plasma exhibiting for frequencies $\omega<\omega_p$ properties which resemble closely the phenomenon of total reflection in normal optics. [Cf. for details Oster (1960b).] The conventional derivation of this plasma property does not lay claim to greater accuracy than that of Eq. (76), i.e., either derivation revolves around the concept of an average particle and therefore retains an inherent indeterminacy by a factor of order one.

Since we have excluded from our review (cf. Sec. 1) refractive indices differing from unity, i.e., values of

$$n^2 \approx 1-\omega_p^2/\omega^2 < 1, \tag{77}$$

we are restricted to treating cases where

$$\Omega\epsilon_m = \omega/\omega_p \gg 1. \tag{78}$$

This restriction must be borne in mind when estimating the numerical values of the correction term

$$i(\omega/\omega_p)H_{i\Omega}^{(1)}[i(\omega/\omega_p)](d/du)H_{i\Omega}^{(1)}(u)|_{u=i\omega/\omega_p} \tag{79}$$

to be introduced into Eq. (65).

The arguments of the Hankel functions retained in Eq. (65) can be given a simple physical meaning. For this purpose, we rewrite Ω in terms of ω and ω_0 [cf. Eq. (25)]:

$$i\Omega = i\omega/\omega_0. \tag{80}$$

The reciprocal of ω_0 is the time a particle of velocity v_0 takes to cross a distance b_0, which, it will be recalled, is the collision parameter corresponding to a deflection of 90°:

$$\omega_0 = mv_0^3/Ze^2. \tag{81}$$

Substituting now for v_0 the average velocity, in the scheme of approximation discussed in the preceding section, we write instead of Eq. (81), specializing for a nuclear charge $Z=1$,

$$\omega_0 = (KT)^{\frac{3}{2}}/m^{\frac{1}{2}}e^2. \tag{82}$$

Then, the term from which Eq. (79) must be subtracted is

$$i(\omega/\omega_0)H_{i\Omega}^{(1)}[i(\omega/\omega_0)](d/du)H_{i\Omega}^{(1)}(u)|_{u=i\omega/\omega_0}. \tag{83}$$

A numerical comparison will be found in Appendix H.

In the meantime we give a qualitative justification for having neglected the shielding effects represented by Eq. (79). At the beginning of Sec. 4 we advanced the argument that the bulk of radiation will be centered on frequencies near the reciprocal of the duration of an encounter, for which it is reasonable to substitute the time a particle takes to move through a distance equal to the collision parameter. Typical radio frequencies correspond to eccentricities far in excess of unity, whereas ω_0 corresponds to an eccentricity $\epsilon\approx 1$; accordingly, $\omega \ll \omega_0$ in the rf range. Since the Hankel functions decrease rapidly for increasing argument, the contribution from Eq. (83) to the emission will exceed greatly the contribution from Eq. (79).

9. FOURIER ANALYSIS OF THE ION'S POTENTIAL

We discuss at some length the work of Scheuer (1960) because he carried out a novel computation of the emissivity consisting of a Fourier analysis of the *potential* any given electron is exposed to at a given, fixed time. Hence the Fourier analysis is carried out in terms of wave numbers rather than frequencies. Scheuer's expression for the potential of an ion at $x=b$, $y=z=0$, i.e.,

$$V = Ze(b^2+y^2)^{-\frac{1}{2}} = \frac{2Ze}{\pi}\int_0^\infty K_0(\xi b)\cos(\xi x)\,d\xi, \tag{84}$$

tacitly assumes that the particle moves along a straight line. Therefore his analysis corresponds precisely to the special case discussed in Sec. 4 of this review, but this restriction is never mentioned by Scheuer. Accordingly, the parameter b in Eq. (84) is the very collision parameter.[5]

From Eq. (84) Scheuer evaluates the field components and transforms the result to an arbitrary reference frame. The crucial step of his calculation is the summation over the contributions from all the ions to the mean square of the field components. In particular, he assumes that the ions' positions are uncorrelated and moreover, that the electron always is affected by every ion irrespective of its distance. These two assumptions amount precisely to *neglecting* the shielding and automatically restrict the discussion to two-body collisions.

In the abstract and in the introduction to his paper, Scheuer claims to have legitimized the description of free-free radiation by two-body encounters, but in fact he has not given a comparison of a many-body analysis with a two-body analysis,[6] as our reformulation of his postulates should have made clear.

It is nevertheless worthwhile to review the remainder of his computation. First, he converts the spatial Fourier components into frequencies, arguing that the field components with a wave number ξ cause periodic accelerations with frequency

$$\omega=\xi v_0. \tag{85}$$

Carrying out the integration over the infinite range of collision parameters, he then finds exactly our result in the limit of large eccentricities [Eq. (66)].

Although Scheuer cuts off the integration over the collision parameters at an arbitrary *lower* limit, say, b_s, this difference between Scheuer's calculation and ours does not affect the verdict reached before regarding two-body collisions. The cut-off at b_s was introduced by Scheuer in order to establish an arbitrary boundary between distant and close encounters. For collision parameters between 0 and b_s he adds contributions that were derived under the set of approximations outlined in our Sec. 5. By splitting the range of integration at b_s, Scheuer has not enlarged the precision of his computation. The differences between his two methods of calculation in the respective ranges of collision parameters are merely of mathematical nature and without losing physical accuracy he could as well have used his former method alone with a suitable lower limit due to the straight-line approximation.

10. QUANTUM MECHANICAL TREATMENT

All the arguments and calculations reviewed so far rely exclusively on classical radiation theory. Consequently, Planck's constant $2\pi\hbar=6.67\times10^{-27}$ erg sec never appeared, because implicitly it was discarded when we postulated (with Kramers) that the energy loss of the electron due to the emission of radiation is negligible or, in other words,

$$\hbar\omega\ll mv_0^2/2. \tag{86}$$

A comprehensive quantum mechanical analysis of free-free transitions is found in Sommerfeld's book (1939) on atomic spectra, which summarizes Sommerfeld's own work and that of several collaborators. A complete reproduction of this derivation perhaps is unnecessary as we shall find no important differences between classical and quantum mechanical calculations in the rf range. For this reason, we have presented the classical approach in detail, which gives simple physical correlates to the several mathematical approximations found in the literature. However, it should be emphasized that the quantum mechanical approach is exhaustive in its own right and, as to the range of applicability outside the rf range, even superior.

We epitomize Sommerfeld's treatment as follows. The emissivities are derived from the matrix elements

$$\mathbf{M}=\int\psi_1\mathbf{r}\psi_2{}^*d\tau, \tag{87}$$

where ψ_1 and ψ_2 ($\psi_2{}^*$ is as usual the complex conjugate) are the wave functions of the impinging and receding electron, computed under the assumption that the atom is a bare nucleus, i.e., that the use of hydrogen eigenfunctions is legitimate. The classical counterpart of this restriction is the use of Coulomb's law in Sec. 2. In this approximation, retardation effects are neglected, which certainly is legitimate for nonrelativistic plasmas and the rf range. [For details, compare Heitler (1955), Sec. 25.]

The wave functions ψ_1 and ψ_2 are [Sommerfeld's Eqs. VII, (2.4)–(2.6)] functions of the space coordinates and of the electron's velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e., before and after the encounter with the ion, or alternately, functions of the corresponding wave numbers k_1 and k_2, viz.,

$$k_1=m\mid v_1\mid/\hbar,\quad k_2=m\mid v_2\mid/\hbar. \tag{88}$$

Hence, in general, the energy loss of the electron due to emission of radiation is taken into account, whereas the restriction expressed by Eq. (86) corresponds to the classical neglect of the energy loss. In the quantum mechanical treatment the collision parameter does not appear explicitly. In other words, the quantum mechanical results cannot be compared with the classical ones before the latter have been integrated over all collision parameters.

The next step of Sommerfeld's procedure is the integration over the volume, as defined in Eq. (87). After some lengthy mathematics Sommerfeld obtains an expression [his Eqs. VII, (2.26)–(2.27)] for the matrix elements still containing as parameters the angle of the

[5] Scheuer uses a slightly different notation for the Bessel functions of imaginary argument, following Heaviside and Jeffreys: Their $Kh_n(z)$ equals Watson's $2K_n(z)/\pi$ adopted in this article. Moreover, we have changed from Scheuer's orientation of axes to the one previously used.

[6] Scheuer recognizes this deficiency only in passing, p. 238, first paragraph. The result of Scheuer's calculation which he believes to be an original discovery was already in 1923 obtained by Kramers and is even recognizable in the work of Schott (1912).

incoming and the outgoing electron waves with respect to a certain arbitrary direction of observation.

In order to find the total emission during all encounters producing photons of prescribed energy, i.e., encounters with a constant deficit $|v_1-v_2|$, one has to integrate the sum of the squares of the matrix elements over all angles, for the outgoing electron wave as well as for the incoming one. This step is analogous to the integration over the collision parameters in the classical picture. Sommerfeld's result [Eq. VII, (4.12)] is

$$\pi\langle Q_\omega\rangle_b d\omega=\frac{e^2\hbar^2}{4\pi^2m^2c^3}[\exp(2\pi|n_1|)-1]^{-1}\cdot[1-\exp(-2\pi|n_2|)]^{-1}\frac{4n_1n_2}{(n_1-n_2)^2}d\omega\cdot\frac{d}{dx}\{{}_2F_1(-n_1,-n_2,1;x)\}^2{}_{x=x_0}, \tag{89}$$

with

$$x_0=-4n_1n_2(n_1-n_2)^{-2}. \tag{90}$$

In Eq. (89), ${}_2F_1$ denotes the generalized hypergeometric functions of argument x and $2+1$ parameters in the customary notation. [As adopted by Watson (1958), p. 100.] The parameters n_1 and n_2 are defined by the relations

$$n_1=\frac{Zme^2}{ik_1\hbar^2}=\frac{Ze^2}{iv_1\hbar};\quad n_2=\frac{Ze^2}{iv_2\hbar}. \tag{91}$$

Equation (89) has been widely used for the derivation of the bremsstrahlung spectrum of x-ray tubes and, more recently, of thermofusion devices. Greene (1959) has investigated in detail the various approximations to Eq. (89), in particular, the Born approximation.

It remains to be shown that in the rf region the classical computations are a valid approximation, i.e., that the correspondance principle can safely be used in most cases of interest in laboratory work and in astrophysics. We expect this procedure to be legitimate, because the rf spectrum is at the low-energy tail of the total bremsstrahlung spectrum. However, at high temperatures quantum corrections are indeed required. It may be useful, therefore, to go into the details of the low-energy approximations to the all-embracing quantum mechanical formula of Eq. (89).

The comparison between classical and quantum mechanical calculations can be made in several ways. Formally, of course, the classical expression must result from the mathematical limiting process $\hbar\rightarrow0$ which implies

$$|n_1|\rightarrow\infty,\quad|n_2|\rightarrow\infty,\quad|x_0|\rightarrow\infty. \tag{92}$$

We delay a discussion under what physical conditions this mathematical limit is a valid approximation. Later in this section we shall see that for very high particle velocities the limit (92) is not approached, even in the rf range.

In the limit (92) the hypergeometric function can be expanded (cf. Sommerfeld, Appendix 16D) and expressed as an integral over a closed path in the complex plane

$${}_2F_1(\alpha,\beta,\gamma;x)=\text{const}\oint u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}du. \tag{93}$$

The integration is carried out by the method of steepest descent.

At this point Sommerfeld introduces a physical assumption, namely, that the electron moves on a nearly parabolic orbit corresponding to a deflection angle of 180°. With this approximation, Sommerfeld's expansion of the hypergeometric function leads to Hankel functions of order $\frac{1}{3}$ and $\frac{2}{3}$ (the latter for the derivative). This is the same as Kramer's classical result in the limit of parabolic orbits (cf. Sec. 5).[7]

Bethe and Heitler (1934) and Sauter (1934) made use of the Born approximation which is particularly relevant to the rf problem. The classical limit of the Born approximation corresponds to the straight-line approximation to the hyperbolic orbit. In fact Greene (1959) finds the spectrum to be proportional to

$$K_0(\hbar\omega/2KT)\propto H_0^{(1)}[i(\hbar\omega/2KT)] \tag{94}$$

on the Born approximation. Recalling that for small arguments the derivative of the Hankel function of order zero is proportional to the reciprocal of the argument, Eq. (94) reproduces Eq. (66). In particular, the argument of the Hankel function in the classical limit goes to $i\Omega$ as defined in Eq. (25).

So far, we have verified that in the limit (92) we recover the classical results. We now turn to the task of defining under what physical conditions the limit (92) is a legitimate approximation to Eq. (89).

Following the discussion by Elwert (1939), (1948), we consider the asymptotic behavior of the hypergeometric function in Eq. (89). Using the well-known relation

$$k_1^2-k_2^2=(2m/\hbar^2)\hbar\omega, \tag{95}$$

Elwert defines a quantity which in our notation reads

$$\omega/\omega_g=1-(k_2/k_1)^2=1-(n_1/n_2)^2, \tag{96}$$

where the so-called quantum limit frequency ω_g is given by the relation

$$\omega_g=mv_0^2/2\hbar. \tag{97}$$

[7] Other quantum mechanical calculations by Oppenheimer (1929), Sugiura (1929), Gaunt (1930), and Maue (1932) are discussed in Sommerfeld's book. Their work was complemented by Weinstock (1942) and Kirkpatrick and Wiedmann (1945). Astrophysical applications were first made by Menzel and Pekeris (1935). On this paper, for instance, Henyey and Keenan (1940) based their interpretation of galactic radio radiation. A series of papers by Ginzburg (1946–1949) and the articles by Unsöld (1946), Townes (1947), Burkhardt, Elwert, and Unsöld (1948), Martyn (1948), Denisse (1950), Shklovskiy, and Pikel'ner (1950), Kulsrud (1954), and Rudkjøbing (1959), were mainly concerned with astrophysical applications.

Elwert then introduces his "assumption I"

$$\omega/\omega_g \ll 1 \tag{98}$$

which means that the energy loss due to the radiation during the encounter is small compared to the total kinetic energy. It does *not* mean, of course, that $\hbar \to 0$.

Elwert's "assumption II" reads

$$|n_1 - n_2| \ll 1 \tag{99}$$

which leads to the condition[8]

$$\tfrac{1}{2}(\omega/\omega_g)\ |n_1| = \omega(Ze^2/mv_0^3) \ll 1. \tag{100}$$

Recalling the meaning of our quantity Ω used in the classical treatment and the definition of ω_0 from Eq. (81), we can rewrite condition (100) as

$$\Omega = \omega/\omega_0 \ll 1, \tag{101}$$

which defines the low-frequency range discussed in Sec. 5. Condition (101) by no means implies that $\hbar \to 0$. Elwert's calculation then proceeds through some mathematical transformations to the "Gaunt-factor"

$$g = -(\sqrt{3}/\pi)\{\ln(\omega/4\omega_g) + \gamma^* + \text{R.P.}\psi(i\,|n_1|)\}. \tag{102}$$

In Eq. (102), γ^* is Euler's constant from Eq. (49), ψ the logarithmic derivative of the Gamma function as defined in Appendix D. Conventionally, the Gaunt factor g for free-free transitions is defined as the frequency dependent part of the total emission.[9] The Gaunt factor in our expression for Q_ω, including the integration over the collision parameters [cf. Eq. (68)], may be put in evidence by writing

$$\pi\langle Q_\omega\rangle_b d\omega = N_i(16\pi e^2/3\sqrt{3}c^3)v_0^3(Ze^2/mv_0^2)^2 g\, d\omega. \tag{103}$$

Equations (102) and (103) contain the desired information on quantum corrections, subject to the frequency restrictions of Eqs. (99) and (100). This completes the general quantum mechanical derivation.

We now specialize the general quantum mechanical expression for the low-frequency range, i.e., Eq. (102) for the classical limit $n_1 \to \infty$ [see Eq. (105) below] and for the opposite limiting case $n_1 \to 0$ where the classical computation breaks down [see Eq. (108)]. In the classical limit, n_1 and n_2 tend to infinity according to Eq. (92). In this case we expand the ψ function and obtain [Erdélyi *et al.*, (1953), Vol. I, p. 15]

$$\text{R.P.}\psi(i\,|n_1|) \approx \ln|n_1|. \tag{104}$$

From Eq. (102) we find [γ is defined in Appendix D]

$$\lim_{n_1\to\infty} g = -(\sqrt{3}/\pi)\{\ln(\omega\gamma/4\omega_g) + \ln|n_1|\}$$
$$= (\sqrt{3}/\pi)\ln[2mv_0^3/\gamma\omega Ze^2]. \tag{105}$$

[8] In Elwert's notation, Eq. (100) reads $\tfrac{1}{2}\omega\alpha Z/\omega_g\beta_1 \ll 1$.

[9] In the case of bound-free transitions, the Gaunt factor does not contain all frequency dependent factors. A general "definition" would be that the Gaunt factor comprises all terms by which the quantum mechanical expressions differ from the classical ones.

Eq. (105) combined with Eq. (103) is exactly the same as the classical result, Eq. (68).

On the other hand, for $|n_1|$ not exceeding unity, or rather

$$|n_1| \ll 1, \tag{106}$$

which is the case for large velocities (or, in terms of average velocities, for very high temperatures), we use the series expansion representing the ψ-function (Erdélyi *et al.*, Vol. I, p. 15), and obtain the following real part

$$\text{R.P.}\psi(i\,|n_1|) = -\gamma^* + \sum_{m=1}^{\infty} |n_1|^2/m(m^2 + |n_1|^2). \tag{107}$$

In the limit of Eq. (106), the only term to be retained is the constant γ^*, and hence, the Gaunt factor is

$$\lim_{n_1\to 0} = -(\sqrt{3}/\pi)\ln(\omega/4\omega_g) = (\sqrt{3}/\pi)\ln[2mv_0^2/\omega\hbar]. \tag{108}$$

This expression was first given by Gaunt (1930). It shows that in this approximation the quantum effects enter only by way of the argument of the logarithmic function.

In the intermediate range

$$|n_1| \approx 1, \tag{109}$$

one must go back to the general formula [Eq. (102)]. As Elwert (1948) has shown by a simple numerical calculation, the classical solution remains useful for values of $|n_1|$ as low as one. This remark completes our discussion of quantum effects. In Sec. 11, we review some mathematical procedures which, though irrelevant to the physical analysis, facilitate the subsequent evaluation of the quantum corrections for the radiation of an assembly of particles with a velocity spread.

11. A USEFUL FORM OF QUANTUM CORRECTIONS

A certain practical interest attaches to a scheme which substitutes for the rigorous quantum mechanical expression [Eq. (102)], one of its limiting values and reduces the error thereby committed through the artifice of applying a correction factor to the argument of the logarithms in Eqs. (105) and (108). For this purpose we form their difference which is

$$\text{const}\cdot\ln A, \text{ with } A = v_0\hbar/\gamma Ze^2. \tag{110}$$

Note that the argument A of this difference is independent of frequency.

Next, we consider the classical angle $2\theta_{cl}$ by which the electron is deflected as a result of the encounter with an ion. In the limit of *small* values for the deflection, i.e., when

$$\epsilon \gg 1, \tag{111}$$

we obtain from Eq. (4)

$$\theta_{cl} = \pi - \theta_0 \approx Ze^2/mbv_0^2. \tag{112}$$

Furthermore, according to Marshak (1940), an electron wave passing a circular diaphragm of radius b is deflected by an angle $2\theta_{qu}$, with

$$\theta_{qu}=\hbar/mv_0b. \tag{113}$$

The ratio θ_{qu}/θ_{cl} is precisely the quantity A defined by Eq. (110), which therefore may be interpreted as follows: When quantum effects are dominant, i.e., in the range defined by Eq. (106) for high electron velocities, classical and quantum mechanical calculations yield differing deflection angles.

In this manner, Spitzer (1956) applied quantum corrections to the dc conductivities. He proposed to use the uncorrected classical expression up to the *average* velocity $\langle v_0\rangle$ corresponding to a temperature of a few hundred thousand degrees. At higher temperatures, Spitzer recommends multiplying the argument of the logarithmic term in the average dc conductivity by an average value of A, which would be the average of our expression (110),

$$\langle A\rangle=(\hbar/\gamma Ze^2)\langle v_0\rangle, \tag{114}$$

but for the appearance of a certain numerical factor, on which we shall comment in Sec. 12. This procedure corresponds to neglecting the transition region where the complete Eq. (107) is required, or to extending the expansion of Eq. (106) up to $|n_1|$-values near unity.

A potential cutoff does not affect the scheme we have outlined (Oster 1957). This contention is confirmed by an early quantum mechanical computation by Wentzel (1927) of the equivalent of Rutherford's scattering formula for the potential

$$\Phi=(Ze^2/r)\exp(-r/b_m), \tag{115}$$

which is essentially Coulombian out to a cut-off distance b_m (identified in his work on the shielding effects of atomic electrons on the nucleus with roughly Bohr's atomic radius). Wentzel's expression for the differential cross section $q(\theta, v_0)$ is

$$q(\theta, v_0)=[Ze^2/2mv_0^2]^2\{\sin^2(\theta/2)+\delta^2/4\}^{-2} \tag{116}$$

with $(Z=1)$

$$\delta=\hbar/mv_0b_m\approx\sin\delta. \tag{117}$$

Whereas the integral over all scattering angles θ in the pure Coulomb case diverges [as does our Eq. (68) in the limit $\omega\rightarrow0$], the integral over Eq. (116) is bounded. As a matter of fact, for $\theta\rightarrow0$, the cross section

$$q(0, v_0)\rightarrow\sin^{-4}(\delta/2). \tag{118}$$

The argument of the sine function is exactly the limiting angle of θ_{qu} in Eq. (113) for $b\rightarrow b_m$.

12. CLASSICAL EMISSION COEFFICIENT OF AN ASSEMBLY OF PARTICLES WITH A VELOCITY SPREAD

The final step in evaluating the phenomenological coefficients required for practical purposes is to average over an assembly of electrons with prescribed velocity distribution interacting with an equal number of positive charges. The positions of these ions are supposed to be uncorrelated (cf. Secs. 7 and 8).

To avoid any misunderstanding, we repeat that the microscopic mechanisms analyzed so far, in no way predicate the velocity distribution of electrons and that our analysis does apply to any non-equilibrium state. In this section we prefer a Maxwellian distribution for the sole reason that it is the one most likely to occur under laboratory conditions as well as in many astrophysical problems. Nevertheless it is possible to make a rigorous prediction of the instantaneous emission coefficient for an arbitrary velocity distribution of the electrons, because this computation does not rely on any assumption as to the mechanisms by which such a non-equilibrium distribution is established or maintained.

Given N_e electrons and N_i ions per unit volume, the number of electrons with absolute values of velocity between v_0 and v_0+dv_0 is

$$N_ef(v_0)dv_0=4\pi N_e[m/2\pi KT]^{\frac{3}{2}}\exp[-(mv_0^2/2KT)]\cdot v_0^2dv_0. \tag{119}$$

T defines a "kinetic temperature" or a mean energy of the electrons. The ions need not have either the same velocity distribution or the same mean energy, provided the latter does not differ too much from that of the electrons, since we had equated the relative velocity electron-ion with the electron velocity proper.

The rate of emission between frequency ω and $\omega+d\omega$ per unit volume is

$$4\pi\epsilon_\omega d\omega=d\omega\pi N_e\int_0^\infty\langle Q_\omega\rangle_bf(v_0)dv_0, \tag{120}$$

where $\langle Q_\omega\rangle_b$ is the appropriate expression for the single encounter spectrum, for instance the classical expression, Eq. (65). The factor 4π on the left-hand side results from the normalization of ϵ_ω to unit solid angle.

It is convenient to use as integration variable

$$\Omega\equiv\omega(Ze^2/m)v_0^{-3}, \tag{121}$$

instead of v_0. With the abbreviation

$$M_0\equiv(m/2KT)[\omega Ze^2/m]^{\frac{2}{3}}, \tag{122}$$

we find, after a few transformations,

$$4\pi\epsilon_\omega d\omega=N_eN_i(16\pi^3e^2/9c^3)(Ze^2/m)^2(m/2\pi KT)^{\frac{3}{2}}\cdot\left(\frac{\omega Ze^2}{m}\right)^{\frac{1}{3}}\int_0^\infty i\Omega H_{i\Omega}^{(1)}(i\Omega)\frac{d}{du}H_{i\Omega}^{(1)}(u)|_{u=i\Omega}\cdot\exp(-M_0\Omega^{-\frac{2}{3}})\Omega^{-5/3}d\Omega d\omega. \tag{123}$$

Equation (123) is complete except for quantum effects which will be added later (Sec. 13).

It is assumed that all ions present have the same charge $+Ze$ [cf. Eqs. (59) and (60)]. If several species of ions are present, the integral in Eq. (123) has to be computed for each species and a weighted mean is then formed according to their relative abundances.

The integral of Eq. (123), to our knowledge, cannot be expressed in terms of tabulated functions. However, for most cases of practical interest, the approximate expressions to be given presently should suffice.

First, we recall from Sec. 4 that the straight-line approximation should hold throughout the range of astrophysical and most laboratory applications. Nevertheless, since we are now dealing with a spread in velocities, we must justify the use of this approximation for *all* values of electron velocities by proving that for the *majority* of electrons with a Maxwellian distribution, the straight-line approximation can safely be used. This proof is given in Appendix I.

With the straight-line approximation and the formula (Watson, p. 74)

$$(d/du)H_0^{(1)}(u)|_{u=i\Omega}=-H_1^{(1)}(i\Omega) \tag{124}$$

we write for the integral in Eq. (123),

$$-\int_0^\infty i\Omega H_0^{(1)}(i\Omega)H_1^{(1)}(i\Omega)\exp[-M_0\Omega^{-\frac{2}{3}}]\Omega^{-5/3}d\Omega. \tag{125}$$

Moreover we know (see Appendix I) that for all interesting values of temperature and frequency, the major contributions come from a range where

$$\Omega \ll 1, \tag{126}$$

so that, according to the formulas of Sec. 5 and Appendix D, the Hankel functions can be expanded. Then the integral becomes

$$-(4/\pi^2)\int \ln(\gamma\Omega/2)\exp[-M_0\Omega^{-\frac{2}{3}}]\Omega^{-5/3}d\Omega, \tag{127}$$

provided that the range of integration is not extended to ∞. We therefore terminate the integration at an upper limit which to some extent is arbitrary. We tentatively adopt the limit

$$\Omega^*=2/\gamma \tag{128}$$

and shall verify that the integral is insensitive to the precise location of the terminal point (Appendix J).

We introduce a new variable $s=\Omega^{-\frac{2}{3}}$ into Eq. (127) and find after some minor reductions

$$-9/\pi^2\int_0^{\Omega^*} \ln[(\gamma/2)^{-\frac{2}{3}}\Omega^{-\frac{2}{3}}]\exp[-M_0\Omega^{-\frac{2}{3}}]d(\Omega^{-\frac{2}{3}})$$

$$=9/\pi^2\int_{s^*}^\infty \ln(s/s^*)\exp(-M_0 s)ds. \tag{129}$$

The lower limit of integration is given by the relation

$$s^*=(\Omega^*)^{-\frac{2}{3}}=(2/\gamma)^{-\frac{2}{3}}. \tag{130}$$

Integrating the right-hand side of Eq. (129) by parts yields

$$-(9/\pi^2M_0)[\ln(s/s^*)\exp(-M_0s)]_{s^*}^{\infty}+(9/\pi^2M_0)$$

$$\cdot\int_{s^*}^\infty \exp(-M_0s)\frac{ds}{s}=-\frac{9}{\pi^2M_0}Ei(-M_0s^*), \tag{131}$$

where $Ei(-M_0s^*)$ is the exponential integral in the notation of Watson and Erdélyi. The argument

$$-M_0s^*\equiv-(m/2KT)[(\gamma Ze^2/2m)w]^{\frac{2}{3}}\ll 1, \tag{132}$$

whenever $\Omega\ll 1$, because, according to Eq. (121), M_0s^* is proportional to the $\frac{3}{2}$ power of the average of Ω with respect to v_0. The magnitude of this average has been discussed in detail in Appendix I. Finally, we expand the exponential integral (Erdélyi *et al.*, Vol. 2, p. 143) in the range $M_0s^*\ll 1$:

$$Ei(-M_0s^*)=\ln(\gamma M_0s^*)+o(M_0s^*), \tag{133}$$

and find at once the emission coefficient per unit volume

$$4\pi\epsilon_\omega d\omega=N_eN_i[32Z^2e^6/3(2\pi)^{\frac{1}{2}}m^2c^3](m/KT)^{\frac{1}{2}}$$

$$\cdot\ln\{(2KT/\gamma m)^{\frac{3}{2}}(2m/\gamma Ze^2\omega)\}d\omega. \tag{134}$$

This equation was given previously by Oster (1959) and Scheuer (1960) but neither derivation used a correct averaging procedure. The same deficiency is inherent in all current expressions for the dc conductivities.

Comparing Eq. (134) with the spectrum coefficient before integrating over the electron velocities [Eq. (68)] we find that the *average* emission of N_e particles is N_e times the emission from a single particle, provided that the average velocity

$$\langle v_0\rangle=(2KT/\gamma m)^{\frac{1}{2}}, \tag{135}$$

is substituted for v_0 in the logarithmic term. The correct expression for the dc conductivities, likewise, should contain the argument $\langle v_0\rangle$ [Eq. (135)] instead of the currently used value $(3KT/m)^{\frac{1}{2}}$.

13. QUANTUM CORRECTIONS TO THE EMISSION COEFFICIENT

In Sec. 10 we reviewed the quantum mechanical derivation of the emission coefficient for a single particle and have proved the classical calculations correct for

$$|n_1|=Ze^2/\hbar v_0\gtrsim 1. \tag{136}$$

Significant deviations from the classical result arise only for $|n_1|\ll 1$. We now turn to the corrections to be applied to the emission coefficient of Eq. (137), if a certain fraction of electrons have energies that violate condition (136). We expect that such corrections will be required only if the *average* velocity defined by Maxwell's distribution law falls into the range excluded by Eq. (136).

The most general expression which contains the classical and the quantum mechanical formulas as limiting cases and is the analog of the integral (127) reads, according to Eq. (102),

$$-\frac{4}{\pi^2}\int_0^{\Omega^*}\left\{\ln\left[\frac{\gamma}{2}\frac{\omega\hbar}{m}\left(\frac{Ze^2\omega}{m}\right)^{-\frac{2}{3}}\Omega^{\frac{2}{3}}\right]+\text{R.P.}\psi(i\,|\,n_1\,|)\right\}\cdot\exp[-M_0\Omega^{-\frac{2}{3}}]\Omega^{-5/3}d\Omega. \quad (137)$$

The ψ function depends on n_1 and therefore on Ω. The integral cannot be expressed by tabulated functions. Since the numerical corrections expected at higher temperatures are not very large, we follow with some modifications a method adopted by Spitzer (1956) for computation of dc conductivities.

Splitting the range of integration at $n_1=1/\gamma$, we assume the classical calculations to be correct for

$$v_0\leqq\gamma(Ze^2/\hbar),\quad \Omega\geqq(\omega\hbar^3/\gamma^3Z^2e^4m)\equiv\Omega_0. \quad (138)$$

For all values

$$\Omega<\Omega_0 \quad (139)$$

we take the quantum *limit*, i.e., Eq. (108), as the correct value for the logarithmic term. The subdivision of the range of integration to some extent is arbitrary. The choice actually made avoids introducing a discontinuity at the point where we switch from the classical limit [Eq. (105)] to the quantum formula [Eq. (108)]. In fact the classical formula and the quantum limit are equal at the very value of v_0 defined by Eq. (138).

We therefore replace the integral (137) by the expression

$$-\frac{4}{\pi^2}\int_0^{\Omega_0}\ln\left[\frac{2m}{\omega\hbar}\left(\frac{m}{\omega Ze^2}\right)^{-\frac{2}{3}}\Omega^{-\frac{2}{3}}\right]\exp[-M_0\Omega^{-\frac{2}{3}}]\Omega^{-5/3}d\Omega$$
$$-\frac{4}{\pi^2}\int_{-\Omega_0}^{\Omega^*}\ln(\tfrac{1}{2}\gamma\Omega)\exp[-M_0\Omega^{-\frac{2}{3}}]\Omega^{-5/3}d\Omega. \quad (140)$$

That the second integral in Eq. (140) is meaningful, i.e., that $\Omega_0<\Omega^*$, is obvious from the definition of Ω_0, Eq. (138), and the expression for Ω^*, Eq. (128).

Performing the same type of transformation that led from Eq. (127) to Eq. (129), we obtain

$$+\frac{6}{\pi^2}\int_\infty^{s_0}\ln(s/s^{**})\exp(-M_0s)\,ds$$
$$+\frac{9}{\pi^2}\int_{s_0}^{s^*}\ln(s/s^*)\exp(-M_0s)\,ds, \quad (141)$$

with s^* defined by Eq. (130),

$$s^{**}=\omega\hbar/2m[m/\omega Ze^2]^{\frac{2}{3}} \quad (142)$$

and

$$s_0\equiv\Omega_0^{-\frac{2}{3}}=[\gamma^3Z^2me^4/\omega\hbar^3]^{\frac{2}{3}}. \quad (143)$$

Integrating Eq. (141) by parts, we obtain

$$-(6/\pi^2M_0)\ln(s_0/s^{**})\exp(-M_0s_0)+(9/\pi^2M_0)\cdot\ln(s_0/s^*)\exp(-M_0s_0)-\frac{6}{\pi^2M_0}\int_{s_0}^{\infty}\exp(-M_0s)\frac{ds}{s}$$
$$+\frac{9}{\pi^2M_0}\int_{s_0}^{s^*}\exp(-M_0s)\frac{ds}{s}. \quad (144)$$

The fourth term in Eq. (144) can be expressed as a difference of two exponential integrals, so that combining the third and the fourth term, we obtain

$$+(6/\pi^2M_0)\,Ei(-M_0s_0)-(9/\pi^2M_0)\cdot[Ei(-M_0s_0)-Ei(-M_0s^*)]. \quad (145)$$

The argument

$$M_0s_0=m/2KT(\gamma Ze^2/\hbar)^2 \quad (146)$$

is not necessarily small with respect to unity. However, it is *consistent* with the scheme of approximation to postulate that M_0s_0 is *negligibly* small with respect to one, whenever

$$M_0s_0\leqq 1. \quad (147)$$

That M_0s^* is small compared with unity had already been stated in Eq. (132).

Expanding the integral exponential function [Eq. (133)] we obtain for Eq. (145)

$$-(3/\pi^2M_0)\,\ln(\gamma M_0s_0)+(9/\pi^2M_0)\,\ln(\gamma M_0s^*). \quad (148)$$

Reasoning as before, we set the exponential function in Eq. (144),

$$\exp(-M_0s_0)\approx 1, \quad (149)$$

and recall from Eqs. (130) and (142) that

$$s/s^*=[(\gamma Z^2e^4/\hbar^3)(2m/\omega)]^{\frac{2}{3}}=(s_0/s^{**})^{\frac{2}{3}}. \quad (150)$$

Hence, the first two terms in Eq. (144) cancel out. The remaining terms, namely Eq. (148), can be written as

$$-(3/\pi^2M_0)\,\ln[\gamma^2M_0^2(s^*)^3s_0^{-1}]=-(6/\pi^2M_0)\cdot\ln[(\gamma/2)(\omega\hbar/2KT)]. \quad (151)$$

The definitive form of the emission coefficient that corresponds to the classical Eq. (134) then reads

$$4\pi\epsilon_\omega d\omega=N_eN_i[32Z^2e^6/3(2\pi)^{\frac{1}{2}}m^2c^3](m/KT)^{\frac{1}{2}}\cdot\ln[4KT/\gamma\omega\hbar]d\omega. \quad (152)$$

We have reviewed the intermediate steps in sufficient detail in order to gain insight into the physical correlates of the mathematical approximations inherent in our rather crude treatment. In practice the respective ranges of validity of Eq. (134) and (152) are as follows: From Eqs. (146) and (147) we infer that the quantum formula, Eq. (152), is appropriate whenever the temperature

$$T\geqq m/2K(\gamma Ze^2/\hbar)^2\approx 550\,000°\text{K},\quad Z=1. \quad (153)$$

Below 550 000°, the classical formula from Eq. (134) ought to be used. We repeat that at 550 000°, where the ranges of validity of Eqs. (134) and (152) meet, these equations give identical results.

Comparing once more the outcome of the averaging procedure over a Maxwellian distribution of electron velocities with the original formula [Eq. (108)], we see that, as in the classical case, the velocity v_0 has been replaced by the average $(2KT/\gamma m)^{\frac{1}{2}}$.

Finally, as in Sec. 11, we divide the arguments of the two logarithmic functions of Eq. (134) and (152) and find

$$\hbar/\gamma Ze^2(2KT/\gamma m)^{\frac{1}{2}}=\langle A\rangle. \qquad (154)$$

The average velocity to be substituted in Eq. (114) emerges from Eq. (154) in a most natural way. Since the same scheme of approximation underlies the evaluation of dc conductivities, it would be appropriate to use Eq. (154) in this context also, rather than the smaller value adopted by Spitzer (1956). The value of $\langle A\rangle$ we recommend has the advantage of complying naturally with either of the limiting equations at the point where their ranges of validity overlap.

14. ABSORPTION COEFFICIENT AND CONDUCTIVITY

In this section we summarize the relations between the emission coefficient and the coefficients of absorption and conductivity. We also show how to compute the power absorbed by the plasma from incident radio waves and the rate of re-emission from an optically thick layer.

The power $4\pi P_\omega d\omega$ absorbed per volume element of the plasma from an incident field of specific radiation intensity I_ω per cm^2 per steradian in an angular frequency interval $d\omega$ centered on ω is given by the expression,

$$4\pi P_\omega d\omega=4\pi\kappa_\omega I_\omega d\omega. \qquad (155)$$

The phenomenological absorption coefficient κ_ω has the dimension of a reciprocal of a length. Eq. (155) does not specify the mechanism by which the radiation field I_ω is generated or maintained, nor does it imply any particular mechanism of absorption. No assumption is made as to the presence of thermal equilibrium. However, the form of Eq. (155) neglects deviations of the refractive index from unity which have been ignored throughout this review.

The same power absorption can be defined alternately through a coefficient of electrical conductivity. Obviously, P_ω can be written in the form

$$P_\omega V=\left|\int_V \mathbf{j}_\omega\cdot\mathbf{E}_\omega dV\right|=\sigma_\omega\int_V E_\omega^2 dV, \qquad (156)$$

where the integration is carried out over a certain volume V in which the quantities determining the electrical field $\mathbf{E}_\omega$ at angular frequency ω, and the specific current $\mathbf{j}_\omega$ as well as the (real part of the) conductivity σ_ω are constant.

The integral in Eq. (156) represents the energy *density* of the incident field, customarily called $u_\omega d\omega$. Energy density and intensity are connected by the relation

$$u_\omega d\omega=(4\pi/c)I_\omega d\omega. \qquad (157)$$

Incidentally, Eq. (157) shows how the refractive index enters. The vacuum velocity of light c here stands for the velocity at which *energy* moves through the volume element. This "energy velocity" equals the phase velocity c only if the refractive index is unity.

The absorption coefficient κ_ω and the conductivity σ_ω, in fact, denote the same quantity in the context, or language, of radiative transfer and electromagnetic theory, respectively. Accordingly,

$$\sigma_\omega=(c/4\pi)\kappa_\omega, \qquad (158)$$

as comparison between Eqs. (155) and (157) reveals.

Before we discuss the relations between the coefficients of absorption and emission, we comment on the proposal repeatedly made to compute the absorption coefficient or the conductivity *directly* from equations such as the Boltzmann equation or the equation of motion of an (average) electron together with Maxwell's equations.

Clearly the derivation by way of a hydrodynamic equation of motion is an inferior approach because essentially it specifies the absorption coefficient in terms of a damping constant or collision frequency whose numerical value must be secured by some other procedure. For example, Smerd and Westfold (1949), as an alternative to their previously discussed derivation of the emission coefficient, used a collision cross section derived originally for the dc limit. Although one obtains in this manner the correct order of magnitude of the absorption coefficient, the suppression of a distribution function in the hydrodynamic equations amounts to discarding valuable information.

Logically, the use of Boltzmann's equation or the Fokker-Planck equation for deriving an absorption coefficient is equivalent to our (classical) derivation of the emission coefficient. In practice however, no computation on this basis has been carried out that would be comparable to the calculations in the dc limit, as undertaken by Spitzer and his collaborators and by Landshoff (cf. Sec. 7). The formal introduction of a collision frequency with a postulated velocity dependence, instead of the complete collision integral, may be an improvement over the hydrodynamic treatment of the Lorentz type, but it is still more or less arbitrary. We therefore believe that at present it is preferable, and more profitable, to derive the emission coefficient directly and to deduce the absorption coefficient in the manner outlined below.

In strict thermodynamic equilibrium, emission and absorption are related by Kirchhoff's law. However, in any kind of steady state, not only in thermodynamic equilibrium, the rates of *total* absorption and emission per unit volume must balance everywhere:

$$4\pi\int_0^\infty [P_\omega-\epsilon_\omega]d\omega=0. \qquad (159)$$

It is only in thermodynamic equilibrium that absorption and emission balance separately for *each frequency*, i.e.

$$P_\omega=\epsilon_\omega=\kappa_\omega B_\omega, \qquad (160)$$

where B_ω is a universal function of the equilibrium temperature. Equation (160) is Kirchhoff's law. The explicit form of the universal temperature function B_ω is known as Planck's law, for which we may substitute in the rf region the (classical) Rayleigh-Jeans approximation, i.e.,

$$B_\omega(T)d\omega=(\hbar\omega^3/4\pi^3c^2)[\exp(\hbar\omega/KT)-1]^{-1}d\omega$$
$$\approx(\omega^2/4\pi^3c^2)KTd\omega. \qquad (161)$$

It will be remembered that in thermodynamic equilibrium the *same* temperature T appears in Planck's law and in Maxwell's velocity distribution.

Substitution of Eq. (161) in Eq. (160) gives the absorption coefficient in thermodynamic equilibrium. For temperatures less than 550 000°K [cf. Eq. (153)], at which the classical treatment of the emission coefficient is permissible, we obtain from Eq. (134)

$$\kappa_\omega=\epsilon_\omega/B_\omega=(N_eN_i/\omega^2)[32\pi^2Z^2e^6/3(2\pi)^{\frac{1}{2}}m^3c](m/KT)^{\frac{3}{2}}$$
$$\cdot\ln\{(2KT/\gamma m)^{\frac{3}{2}}(2m/\gamma Ze^2\omega)\}. \qquad (162)$$

If quantum effects are appreciable, i.e., at temperatures above 550 000°K, we find from Eq. (152)

$$\kappa_\omega=(N_eN_i/\omega^2)[32\pi^2Z^2e^6/3(2\pi)^{\frac{1}{2}}m^3c](m/KT)^{\frac{3}{2}}$$
$$\cdot\ln[4KT/\gamma\omega\hbar]. \qquad (163)$$

The corresponding conductivities follow from Eqs. (162) and (163) by multiplication with $c/4\pi$.

The formal computation of the total emission along a given line of sight along which the distance from the observer is called s, is carried out by the following equation;

$$I_\omega=\int_0^\infty\epsilon_\omega(s)\exp[-\tau_\omega(s)]ds. \qquad (164)$$

I_ω is the specific intensity perceived by the observer, ϵ_ω is the emission coefficient at distance s from the observer, and the exponential factor expresses the attenuation of the radiation on its way from the point of emission to the observer. The quantity τ_ω is called the optical distance from the observer and is defined by the relation

$$\tau_\omega(s)=\int_0^s\kappa_\omega(s')ds'. \qquad (165)$$

Depending on the geometry, for instance if radiation is received at a given point inside the plasma, the integral (164) requires appropriate modification.

Evaluation of the integral (164) requires knowledge of the local coefficients of emission and absorption. The coefficient of absorption, in general, differs from the one given by Eq. (162) or (163), which strictly apply only to thermodynamic equilibrium, that is to say, to material which is enclosed in a thermostat. Material in thermal communication with its surroundings is adequately described as being in a steady nonequilibrium state. In the absence of a comprehensive physical theory of nonequilibrium states, it is customary to substitute in the integral (164) the equilibrium expressions of Eqs. (162) or (163). This procedure is often referred to as the postulate of "local thermodynamic equilibrium," (LTE).

However, it is well to keep in mind that at present it is impossible to make quantitative estimates of the deviations of steady nonequilibrium states from local thermodynamic equilibrium, except in a few special cases. Space does not permit us to outline all possible factors to be considered if one wants to make an estimate of how good an approximation the concept of LTE is likely to be in a given physical situation.

These nonequilibrium considerations apart, we believe the equilibrium expressions [Eqs. (162) and (163)] to be superior to those previously given in the literature.

15. SUMMARY

1. Classical and quantum mechanical calculations are compared, yielding the emission spectrum of free electrons in the Coulomb field of positive ions. The resulting classical spectrum is given by Eq. (65) in the most general case and by Eq. (68) in a suitable approximation valid throughout the rf region. Quantum corrections can be applied with the aid of Eqs. (102) and (107).
2. The spectrum has been integrated over a Maxwellian distribution of velocities. The resulting emission coefficient per unit volume is given by Eq. (134) at temperatures less than 550 000°K and by Eq. (152) at temperatures exceeding 550 000°K.
3. Absorption and conductivity coefficients are likewise derived for a Maxwellian distribution of velocities [Eqs. (162) and (163)].

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APPENDIX A

Whereas the term that arises from the endpoints of integration in deriving Eq. (27) vanishes, the corresponding term to be added to Eq. (26) reads

$$\text{const}[\sinh x/(\epsilon\cosh x-1)]\cos[\Omega(\epsilon\sinh x-x)] \qquad (A1)$$

to be taken at the limits $+\infty$ and $-\infty$. Since this is an odd function, (A1) does not vanish identically. The combination of hyperbolic functions preceding the cosine function tends to unity in the limit $x\to\infty$, so that we are left with

$$\lim_{x\to\infty}\cos[\Omega(\epsilon\sinh x-x)]\approx\cos(\Omega e^{x})|_{x=\infty}, \quad \text{(A2)}$$

which is indeterminate. To prove that this term can be neglected, we verify that the integrand of Eq. (24) vanishes at the endpoints of integration, so that the contribution to the integral from the range beyond a certain value can be made smaller than any prescribed quantity. Hence, in the limit ∞, the contribution tends to zero.

APPENDIX B

In order to reformulate Eqs. (26) and (27) in terms of Hankel functions, we replace the trigonometric functions by exponential functions. Obviously it makes no difference if a cosine term is added under the integral of Eq. (26) and i times a sine in Eq. (27):

$$\ddot{x}(\omega)=(v_0\Omega/i\pi)\int\sinh\xi\,\exp[i\Omega(\xi-\epsilon\sinh\xi)]d\xi, \quad \text{(B1)}$$

and

$$\ddot{y}(\omega)=-\frac{v_0\Omega}{\pi}\frac{(\epsilon^2-1)^{\frac{1}{2}}}{\epsilon}\int\exp[i\Omega(\xi-\epsilon\sinh\xi)]d\xi. \quad \text{(B2)}$$

Watson (1958), p. 182, defines the following integral:

$$K_\nu(u)=\tfrac{1}{2}\exp(\tfrac{1}{2}\pi\nu i)\int_{-\infty}^{+\infty}\exp(-iu\sinh\xi+\nu\xi)d\xi. \quad \text{(B3)}$$

The K functions and the Hankel functions are related by the expression (Watson, p. 78)

$$K_\nu(u)=\tfrac{1}{2}\pi i\exp(\tfrac{1}{2}\pi\nu i)H_\nu^{(1)}(iu). \quad \text{(B4)}$$

Using Eqs. (B3) and (B4) we obtain the representation given in Eqs. (28) and (29).

APPENDIX C

According to Watson (p. 185 and p. 78),

$$\int\cos(\Omega\epsilon u)[du/(1+u^2)^{\frac{1}{2}}]=2K_0(\Omega\epsilon)=i\pi H_0^{(1)}(i\Omega\epsilon). \quad \text{(C1)}$$

Hence,

$$\ddot{y}(\omega)=-(Ze^2/mv_0^2)\omega iH_0^{(1)}(i\Omega\epsilon), \quad \text{(C2)}$$

with

$$\Omega\epsilon=\omega b/v_0. \quad \text{(C3)}$$

Furthermore

$$\int\cos(\Omega\epsilon u)[du/(1+u^2)^{\frac{3}{2}}]=(2\Omega\epsilon)K_1(\Omega\epsilon) \quad \text{(C4)}$$

and

$$K_1(\Omega\epsilon)=-K_0'(\Omega\epsilon)=+\pi/2[(d/ds)H_0^{(1)}(s)]_{s=i\Omega\epsilon}. \quad \text{(C5)}$$

Finally

$$\ddot{x}(\omega)=(Ze^2/mv_0^2)\omega(d/ds)H_0^{(1)}(s)|_{s=i\Omega\epsilon}. \quad \text{(C6)}$$

Equation (41) is now readily obtained from Eqs. (C2) and (C6).

APPENDIX D

Watson (p. 80) derives the expansion of the Hankel functions in terms of Bessel functions of the first kind and imaginary argument $I(\Omega\epsilon)$:

$$H_0^{(1)}(i\Omega\epsilon)=(2/\pi i)-\ln(\Omega\epsilon/2)I_0(\Omega\epsilon)+\sum_{m=0}^{\infty}(\tfrac{1}{2}\Omega\epsilon)^{2m}[\Gamma(m+1)]^{-2}\psi(m+1). \quad \text{(D1)}$$

In writing down Eq. (D1), use has been made of Eq. (B4). Γ denotes the Gamma function. The auxiliary function ψ (the logarithmic derivative of the Gamma function) is defined by the relation (Watson, p. 60);

$$\psi(m+1)=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}-\gamma^*;\quad \psi(1)=-\gamma^*. \quad \text{(D2)}$$

The expansion of $I_0(\Omega\epsilon)$ for small arguments is given by Watson (p. 77):

$$I_0(\Omega\epsilon)=\sum_{m=0}^{\infty}(\tfrac{1}{2}\Omega\epsilon)^{2m}[\Gamma(m+1)]^{-2}. \quad \text{(D3)}$$

Equation (48) follows immediately.

APPENDIX E

In passing from Eq. (63) to Eq. (64), we have used the relation

$$(uH_\nu H_\nu')'=u\{H_\nu'^2+[(\nu^2/u^2)-1]H_\nu^2\}, \quad \text{(E1)}$$

with primes denoting derivatives with respect to the argument u. H_ν stands for any solution of Bessel's equation

$$H_\nu''+H_\nu'+u[1-(\nu^2/u^2)]H_\nu=0, \quad \text{(E2)}$$

for instance, the Hankel functions $H_{i\Omega}^{(1)}(u)$.

Writing for the left-hand side of Eq. (E1)

$$H_\nu H_\nu'+u(H_\nu')^2+uH_\nu H_\nu'' \quad \text{(E3)}$$

reveals that Eq. (E1) is in fact a variant of Eq. (E2).

APPENDIX F

To prove Eq. (65), it has to be shown that the upper limit of the integral in Eq. (64) gives no contribution, i.e., that

$$\lim_{u\to i\infty}[uH_{i\Omega}(u)H_{i\Omega}'(u)]\to 0. \quad \text{(F1)}$$

This follows from the asymptotic expansion for $H_{i\Omega}$ reading for large pure imaginary values $u=iz$ (Watson, p. 198):

$$H_{i\Omega}\propto(2/\pi iz)^{\frac{1}{2}}\exp(-z). \quad \text{(F2)}$$

APPENDIX G

Erdélyi *et al.* [(1953), Vol. 2, pp. 87–88] give several expansions for the Bessel functions of the second kind and pure imaginary argument that are related to the Hankel functions by Eq. (B4). Denoting the order by ip,

$$K_{ip}(z)=2^{-\frac{1}{2}}(z^2-p^2)^{-\frac{1}{4}}\exp\{-(z^2-p^2)^{\frac{1}{2}}-p[\sin(p/z)^{-1}\}\cdot\left\{\sum_{m=0}^{M-1}(-1)^m 2^m b_m\Gamma(m+\tfrac{1}{2})(z^2-p^2)^{-m/2}+O(z^{-M})\right\}. \quad \text{(G1)}$$

The b_m's are polynomials with $(1-z^2/p^2)^{-1}$ as argument

$$b_m=\sum_{\mu=0}^{\infty}(-1)^\mu\,|\,a_\mu\,|(1-z^2/p^2)^{-\mu},\quad |\,a_\mu\,|<1. \quad \text{(G2)}$$

Because in the case under consideration, $z=\Omega\epsilon$, $p=\Omega$ with $z\gg p$ over most of the integration interval, p can be set zero in the combination z^2-p^2. This is the desired result which corresponds to the straight-line approximation. We emphasize that the reciprocal of the sine function entering the exponential in Eq. (G1) does not affect this reasoning, since

$$p[\sin(p/z)]^{-1}\approx z\approx[z^2-p^2]^{\frac{1}{2}}. \quad \text{(G3)}$$

When $z\approx p$, i.e., for the small values of ϵ, the order of the Hankel functions can still be put equal to zero while retaining a finite value for the argument. To justify this claim, we quote another expansion (Erdélyi *et al.*, Vol. 2, p. 88):

$$K_{ip}(z)\sim\tfrac{1}{3}\pi\exp(-\tfrac{1}{2}\pi p)\sum_{m=0}^{\infty}(-1)^m c_m(nz)\sin[(m+1)\tfrac{1}{3}\pi]\cdot\Gamma(\tfrac{1}{2}m+\tfrac{1}{3})(z/6)^{-(m+1)/3} \quad \text{(G4)}$$

which applies if

$$z\approx p,\quad n=1-p/z=o(z^{-\frac{2}{3}}). \quad \text{(G5)}$$

Without repeating the definition of the polynomials c_m, it suffices to mention that c_m may be represented by zero if m is odd, by a pure number <1 if m is even.

Because the function K_{ip} is a product of two factors, one depending on p and the other one depending on z, it is possible to let p tend to zero without disturbing the dependence of K_{ip} on z.

APPENDIX H

We give a few numerical examples illustrating the statement that under most physical conditions the shielding effects can be neglected in the rf region, so that the contribution from Eq. (83) greatly outweighs the contribution from Eq. (79). We consider only the straight-line approximation, i.e., we substitute for the Hankel function $H_{i\Omega}$ the function of order zero. Then, for typical values in the rf region, the argument

$$\omega/\omega_0=\omega Ze^2/mv_0^3\approx 2.5\times10^8\omega v_0^{-3}\ll 1. \quad \text{(H1)}$$

For instance, for $\omega\approx10^{10}$ and $v_0\approx10^8$, a value which corresponds to the average velocity of a Maxwellian distribution with a temperature of about 100 000°K, ω/ω_0 is about 2.5×10^{-6}. Hence the Hankel functions can be expanded according to the "low-frequency limit" in which the contribution from Eq. (83) increases logarithmically with decreasing argument.

Next, we consider the ratio of the arguments

$$(\omega/\omega_0)/(\omega/\omega_p)=\frac{2\pi^{\frac{1}{2}}e^3Z}{m^{\frac{3}{2}}v_0^3}N_e^{\frac{1}{2}}\approx10^{13}N_e^{\frac{1}{2}}v_0^{-3}\ll1. \quad \text{(H2)}$$

For $N_e=10^{10}$ and the same value of v_0 as above, the ratio is of the order 10^{-6}.

For the same values of v_0 and ω, the integrals (83) and (79) amount to approximately 13 and 10^{-2}, respectively.

Evidently, for higher electron densities and lower velocities the ratio of Eq. (H2) is closer to one, and the shielding effects might become more important. On the other hand, an increase of ω (in the example given above, ω is only about 2.5 times the plasma frequency), will produce a smaller shielding effect, since the Hankel functions decrease rapidly for arguments >1, while the main contribution to the coefficient, Eq. (83), still decreases logarithmically.

APPENDIX I

In order to justify Eq. (125), i.e., the use of the straight-line approximation for the hyperbolic orbit for *all* particle velocities, we must prove that for the *majority* of electrons having a Maxwellian distribution the assumption

$$\epsilon\approx mbv_0^2/Ze^2\gg1 \quad \text{(I1)}$$

is valid. Strictly speaking, this should be carried out for each collision parameter separately and the result integrated over all b values. We intend to make only a qualitative check by substituting an average collision parameter into Eq. (I1). Relying on the same arguments that led to Eq. (34), we choose as average collision parameter

$$\langle b\rangle=v_0/\omega. \quad \text{(I2)}$$

We then justify the condition

$$\Omega=\omega Ze^2/v_0^3m\ll1,\quad v_0^3\gg\omega Ze^2/m \quad \text{(I3)}$$

by proving the inequality

$$J_1\ll J_2 \quad \text{(I4)}$$

with

$$J_1=\int_0^{v^*}e^{-\mu v_0^2}v_0^2dv_0,\quad J_2=\int_{v^*}^{\infty}e^{-\mu v_0^2}v_0^2dv_0, \quad \text{(I5)}$$

where

$$v^*\equiv(\omega Ze^2/m)^{\frac{1}{3}} \quad \text{(I6)}$$

follows from Eq. (I3), and

$$\mu\equiv m/2KT. \quad \text{(I7)}$$

These integral may be expressed in terms of error functions:

$$J_1=(v^*/2\mu)\exp(-\mu v^{*2})+(1/2\mu^{\frac{3}{2}})\,\text{Erf}(\mu^{\frac{1}{2}}v^*) \quad \text{(I8)}$$

and

$$J_2 = v^*/2\mu \exp(-\mu v^{*2}) + 1/2\mu^{\frac{3}{2}} \operatorname{Erfc}(\mu^{\frac{1}{2}} v^*) \quad \text{(I9)}$$

[cf. Erdélyi *et al.* (1953), Vol. 2, p. 147].

With the aid of the relation

$$\operatorname{Erfc}(\mu^{\frac{1}{2}} v^*) = \pi^{\frac{1}{2}}/2 - \operatorname{Erf}(\mu^{\frac{1}{2}} v^*) \quad \text{(I10)}$$

and the expansion

$$\operatorname{Erf}(\mu^{\frac{1}{2}} v^*) = \exp(-\mu v^{*2}) \left[\mu^{\frac{1}{2}} v^* + \sum_{m=1}^{\infty} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+m)} (\mu^{\frac{1}{2}} v^*)^{2m+1} \right] \quad \text{(I11)}$$

we obtain from Eqs. (I4) and (I9) as condition for the validity of the proposed approximation

$$(\mu^{\frac{1}{2}} v^*)^3 = (m/2KT)^{\frac{3}{2}} (\omega Z e^2/m) \ll 1, \quad \text{(I12)}$$

or, numerically,

$$10^{-3} T^{-\frac{1}{2}} \omega^{\frac{1}{3}} \ll 1. \quad \text{(I13)}$$

This inequality is satisfied in all applications.

APPENDIX J

We must verify that the value of the integral (127) is insensitive to the precise value of the upper limit Ω^* adopted in Eq. (128). We change the limit of integration from s^* to

$$s = s^* - \delta \quad \text{(J1)}$$

Let δ be restricted in such a manner that $s^* - \delta$ still satisfies the condition for the expansion of the Hankel functions. Instead of Eq. (131), we then have

$$(9/\pi^2 M_0) \ln[(s^* - \delta)/s^*] - Ei[-M_0(s^* - \delta)], \quad \text{(J2)}$$

since the exponential function in the first term of Eq. (131) can be put equal to one. Expanding again the Ei function, we obtain

$$(9/\pi^2 M_0)\{+\ln[(s^* - \delta)/s^*] - \ln[\gamma M_0(s^* - \delta)]\} = -(9/\pi^2 M_0) \ln(\gamma M_0 s^*) \quad \text{(J3)}$$

from which δ has vanished. This completes the proof.

BIBLIOGRAPHY

H. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146,** 83 (1934).

G. Burkhardt, G. Elwert, and A. Unsöld, Z. Astrophys. **25,** 310 (1948).

S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, New York, 1955), 2nd Ed.

R. S. Cohen, L. Spitzer, and P. M. Routly, Phys. Rev. **80,** 230 (1950).

P. Debye and E. Hückel, Physik. Z. **24,** 185 (1923).

J. F. Denisse, J. Phys. radium **11,** 164 (1950).

A. S. Eddington, Monthly Notices Roy. Astron. Soc. **86,** 1 (1925).

G. Elwert, Ann. Physik **34,** 178 (1939).

G. Elwert, Z. Naturforsch. **3a,** 477 (1948).

A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Function* (McGraw-Hill Book Company, Inc., New York, 1953).

I. A. Gaunt, Proc. Roy. Soc. (London) **A126,** 654 (1930).

V. L. Ginzburg, Doklady Akad. Nauk S.S.S.R. **52,** 491 (1947).

V. L. Ginzburg, Uspekhi Fiz. Nauk **32,** 26 (1947).

V. L. Ginzburg, Astron. Zhur. **26,** 84 (1949).

J. Greene, Astroph. J. **129,** 693 (1959).

L. G. Henyey and P. C. Keenan, Astroph. J. **91,** 625 (1940).

W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954), 3rd Ed.

P. Kirkpatrick and L. Wiedmann, Phys. Rev. **67,** 321 (1945).

H. A. Kramers, Phil. Mag. **46,** 836 (1923).

R. M. Kulsrud, Astroph. J. **119,** 386 (1954).

L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951).

R. Landshoff, Phys. Rev. **76,** 904 (1949).

R. Landshoff, Phys. Rev. **82,** 442 (1951).

R. E. Marshak, Astroph. J. **92,** 321 (1940).

D. F. Martyn, Proc. Roy. Soc. (London) **A193,** 44 (1948).

A. W. Maue, Ann. Physik **13,** 161 (1932).

D. H. Menzel and C. L. Pekeris, Monthly Notices Royal Astron. Soc. **96,** 77 (1935).

R. Oppenheimer, Z. Physik **55,** 725 (1929).

L. Oster, Z. Astrophys. **42,** 228 (1957).

L. Oster, Z. Astrophys. **47,** 169 (1959).

L. Oster, Phys. Rev. **119,** 1444 (1960a).

L. Oster, Rev. Modern Phys. **32,** 141 (1960b).

E. Persico, Monthly Notices Royal Astron. Soc. **86,** 93 (1926).

M. Rudkjøbing, Ann. astrophys. **22,** 118 (1959).

F. Sauter, Ann. Physik **20,** 404 (1934).

P. A. G. Scheuer, Monthly Notices Royal Astron. Soc. **120,** 231 (1960).

G. A. Schott, *Electromagnetic Radiation* (Cambridge University Press, New York, 1912).

I. S. Shklovskiy and S. B. Pikel'ner, Izvest. Krymsk. Astrof. Observat. **6,** 29 (1950).

S. F. Smerd and K. C. Westfold, Phil. Mag. **40,** 831 (1949).

A. Sommerfeld, *Atombau und Spektrallinien,* (Fredrich Vieweg and Sohn, Braunschweig, Germany, 1939), Vol. 2.

L. Spitzer and R. Härm, Phys. Rev. **89,** 977 (1953).

L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

Y. Sugiura, Phys. Rev. **34,** 858 (1929).

C. H. Townes, Astroph. J. **105,** 235 (1947).

A. Unsöld, *Physik der Sternatmosphären* (J. Springer, Heidelberg, Germany, 1955), 2nd ed.

A. Unsöld, Naturwissenschaften **33,** 37 (1948).

G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1958), 2nd Ed.

R. Weinstock, Phys. Rev. **61,** 584 (1942).

G. Wentzel, Z. Physik **40,** 590 (1927).