Anomalous Thresholds*

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I. INTRODUCTION

THE word "anomalous" has a special meaning in physics. We always expect that the study of unanticipated effects will lead to a better understanding of the underlying principles. We should be surprised if the anomalous thresholds were not similar.

The ordinary thresholds refer to the threshold energies at which new physical processes become possible: the thresholds for the production of new particles. It is well known that ordinary thresholds are associated with singularities in the scattering amplitudes.[1] The anomalous thresholds are additional singularities, which do not correspond to inelastic thresholds. The possibility of additional singularities was first pointed out by Karplus, Sommerfield, and Wichman,[2] and Nambu.[3] Anomalous thresholds are not always present; there are none in the dispersion relations for pion-nucleon and nucleon-nucleon scattering. However, anomalous thresholds are a characteristic feature of production amplitudes, in which they always are present, and they even occur in elastic scattering when the masses of the particles have certain ratios.

In Sec. II the scattering of unstable particles is discussed. It is shown that anomalous thresholds arise very naturally in this problem and can in no sense be considered an unusual feature of the scattering matrix. Section III is a survey of recent work on the role of anomalous thresholds in dispersion relations. In Sec. IV it is shown how a Bethe-Salpeter equation may be derived from dispersion theory. This derivation shows explicitly, in terms of older physical concepts, how dispersion relations determine the dynamical behavior of interacting particles.

II. UNSTABLE PARTICLES

Consider first the inelastic scattering

$$K^+ + p \rightarrow \pi^+ + \pi^0 + p.$$

The amplitude associated with the graph in Fig. 1 is infinite when the intermediate state has the same energy

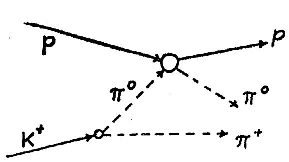

FIG. 1. A graph which leads to a pole at physical values of the momenta.

* Supported in part by the U. S. Atomic Energy Commission.

[1] R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952).

[2] R. Karplus, C. M. Sommerfield, and E. H. Wichman, Phys. Rev. **111**, 1187 (1958); **114**, 376 (1959).

[3] Y. Nambu, Nuovo cimento **9**, 1187 (1958).

as the initial state, which is possible because the K^+ is unstable. The scattering amplitude, considered as a function of the momenta of the final particles, has a pole. This pole is a consequence of the fact that in an experiment, the decay of the K^+ could take place as a real process at a distance far upstream from the target, depending on the precise experimental arrangement; the intensity of π mesons, relative to the intensity of "undecayed" K mesons, can be made arbitrarily large. (If we average over the momentum uncertainty provided by the localization of the particles by counters, there is no longer a pole for physical values of the average momenta.) If we imagine decreasing the K^+ mass until it becomes kinematically stable, the energy denominator cannot vanish when the final momenta are real, but when some of the components of the momenta are continued into the complex plane, the vanishing is still possible. Chew and Low[4] pointed out that these poles are a characteristic and important feature of production amplitudes.

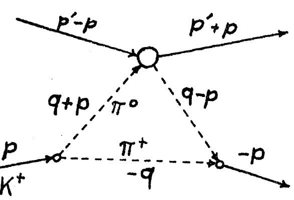

FIG. 2. The prototype of anomalous threshold graphs.

If the two π mesons recombine into the K^+ meson, there is a possible contribution to elastic scattering from the diagram in Fig. 2 in which energy is conserved in both intermediate states, so all three intermediate particles are able to propagate as free particles. Let us examine this possibility in the Breit frame in which the velocity of the unstable particle is reflected back by 180° by the scattering; the momenta are as indicated in the figure. If both intermediate states are to have the same energy, $\mathbf{q}\cdot\mathbf{p}=0$. The magnitudes of $\mathbf{q}$ and $\mathbf{p}$ are related through the equation

$$(M_0^2+p^2)^{\frac{1}{2}}=(M_1^2+q^2)^{\frac{1}{2}}+(M_2^2+q^2+p^2)^{\frac{1}{2}}, \qquad (1)$$

or

$$q^2=\frac{(M_0^2-M_1^2-M_2^2)^2-4M_1^2(M_2^2+p^2)}{4(M_0^2+p^2)}. \qquad (2)$$

Note that $-4p^2=t$, the conventional momentum transfer variable. The values of q^2 obtained from Eq. (2) are positive only if t is larger than the critical value

$$t_0=4M_2^2-M_1^{-2}(M_0^2-M_1^2-M_2^2)^2, \qquad (3)$$

[4] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

which is independent of the collision energy. If the scattering angle is sufficiently small, the process shown in Fig. 2 can contribute to the scattering with all three intermediate particles being on the mass shell, but if the scattering angle is larger than the critical angle, this is not possible. In the first case, the two intermediate states can both persist for indefinitely long times so the two π mesons can separate to an arbitrarily great distance during the scattering.

The critical momentum transfer t_0 is evidently a "threshold" of some type, although not an ordinary one; it is, in fact, an anomalous threshold. The nature of the singularity at this threshold can be studied in detail by the methods described later; it turns out to be a logarithmic singularity. That is, the scattering amplitude contains a term proportional to $\log(1-t/t_0)$. It is also easy to show this directly by integrating over $\mathbf{q}$ (for this calculation the amplitude for scattering of M_2 by M' can be considered to be constant). Since an unstable particle can be thought of as a narrow resonance in the scattering of other particles, we should expect similar logarithmic singularities in any multiparticle reaction. Whenever the initial and final states have more than two particles, the scattering matrix gets a contribution from diagrams such as that in Fig. 3, which is similar to that of Fig. 2. A calculation identical to that just sketched again shows there are logarithmic singularities for physical values of the energies and momenta. It is clear, therefore, that anomalous thresholds are just as general a feature of the scattering matrix as are ordinary thresholds.

Karplus, Sommerfield, and Wichman pointed out that even when we consider scattering of a particle which is stable, but loosely bound, there is an anomalous threshold at the point $t=t_0$ given by Eq. (3). In this case t_0 does not correspond to a real scattering angle, but the form of the scattering cross section for real angles might still be dominated by the existence of the anomalous threshold.

Bohr has given a simple physical picture which helps us understand why loosely bound particles behave somewhat like unstable particles.[5] We imagine a deuteron, bound to a heavy nucleus, which has virtually escaped beyond the nuclear surface (which we treat as a plane). The situation is depicted in Fig. 4. The deuteron's momentum is directed along the Z axis and has the magnitude $i\kappa_D$; in other words, the wave function is proportional to $\exp(-z\kappa_D)$. Now suppose that the deuteron "decays" while it is in this region. Conservation of the z component of momentum implies that the neutron and proton have imaginary momenta such that $\kappa_p+\kappa_n=\kappa_D$, so after decay the product of the neutron and proton densities has the same z dependence as the initial deuteron density. Let

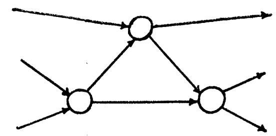

FIG. 3. A graph with six external lines, for which there are anomalous thresholds in the "physical region" of the momenta.

the neutron and proton momentum along the X axis (parallel to the nuclear surface) be $\pm p$. Conservation of energy requires that

$$[(p^2-\kappa_p^2)+(p^2-\kappa_n^2)]/2N=-B-(\kappa_D^2/4N), \qquad (4)$$

where B is the binding energy of the deuteron and N is the nucleon mass. From Eq. (4) we obtain

$$p^2=-NB+\tfrac{1}{4}(\kappa_D-2\kappa_n)^2. \qquad (5)$$

This shows that if the deuteron is bound sufficiently tightly to the nucleus so that $\kappa_D^2>4NB$, p^2 may be positive and the neutron and proton able to escape from each other. In other words, the deuteron is unstable in this situation.

We may also consider two nucleons in the nuclear fringe region and invert the foregoing. If these two nucleons scatter, they may form a deuteron as an excited resonant state which is exactly analogous to the resonances in ordinary scattering theory. If we have a deuteron which is virtually present beyond the nuclear surface, it may scatter against another nucleon in the same region; it is clear that there are anomalous thresholds in the scattering amplitude. This shows that when we consider the analytic continuation of a scattering amplitude to momenta which are not real, we must, in general, expect to find anomalous thresholds.

If we repeat the calculation leading to Eq. (5) using the relativistic expression for the energy, we find that a particle of mass M_0 can be made to be unstable against decay into particles of mass M_1 and M_2 by increasing its binding only when $M_0^2>M_1^2+M_2^2$. In this case we may describe the particle as "loosely bound"; when $M_0^2<M_1^2+M_2^2$, we may refer to it as "tightly bound."

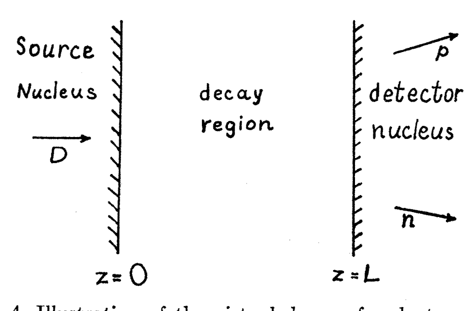

FIG. 4. Illustration of the virtual decay of a deuteron in the region beyond the surface (at $z=0$) of a very large nucleus. The proton and neutron could be detected in a second nucleus whose surface was placed at $z=L$. The emergent protons and neutrons could be considered to form three groups: two corresponding to decay in the neighborhood of $z=0$ and $z=L$, and the third to decay in the intervening region. The third group is what we are interested in; it has an intensity proportional to $L\exp(-2\kappa_D L)$ when L is large. The separation into these groups is, of course, not unique.

[5] Aage Bohr, preprint of "Lectures on dispersion relations," Theoretical Physics Institute, University of Colorado, Boulder, Colorado, Summer, 1960; and private communication. The argument presented here is a slight modification of Bohr's.

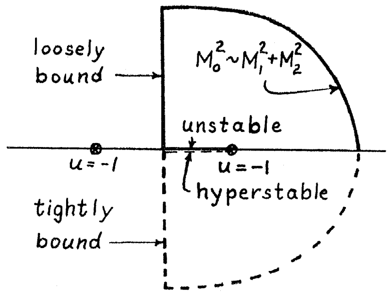

FIG. 5. Locus of the decay velocity when M_0 is varied.

The nature of this critical value of M_0 becomes somewhat clearer if we look at the invariant decay velocity which we define to be the velocity with which an observer moving along with the decay particle 1 sees particle 2 to be receding:

$$u = A(M_0^2 - M_1^2 - M_2^2)^{-1}, \qquad (6)$$

where

$$A^2 = M_0^4 + M_1^4 + M_2^4 - 2M_0^2M_1^2 - 2M_0^2M_2^2 - 2M_1^2M_2^2$$

(in units where $c=1$). When $M_0 > M_1 + M_2$, u is real and lies between 0 and 1. When $M_0 < M_1 + M_2$, u is imaginary. An imaginary decay velocity which corresponds to an exponentially decreasing wave function does not in itself preclude instability of the particle under all circumstances, as the example discussed before shows. It should also be remembered that, from a formal mathematical point of view, Lorentz transformations with complex velocities are perfectly well behaved; it is only $u^2 = 1$ that is singular.

Now note that if M_0 decreases further, $u \to i\infty$ when $M_0^2 \to M_1^2 + M_2^2$. In the nonrelativistic theory, the wave function corresponding to $u = i\infty$ is one which decreases infinitely rapidly so that particle 1 would see particle 2 as constrained to remain coincident with it. In relativity theory, the momentum is $k_2 = M_2 u(1-u^2)^{-\frac{1}{2}}$, so we obtain instead $k_2 \to iM_2$, which gives the maximum localization of particle 2 which is possible. If M_0 is decreased still further, the velocity u goes to the negative imaginary axis along the path shown in Fig. 5 (we give M_0 a small positive imaginary part). As M_0 decreases past the critical value, the decay velocity traverses the physically forbidden region $u > 1$, which we interpret as a branch cut in the complex velocity plane. When $M_0^2 < M_1^2 + M_2^2$, it is no longer possible to consider the particle to have "decayed." The energy of particle 2 (as seen by 1) is $E_2 = M_2(1-u^2)^{-\frac{1}{2}}$; since the branch point at $u=1$ has been encircled, E_2 is negative, which corresponds to absorption rather than emission.

The behavior of the anomalous threshold t_0 mirrors that of the decay velocity. We have seen that when a particle is unstable, there is an anomalous threshold at a physically accessible momentum transfer; when it is loosely bound, the anomalous threshold lies between the physical region and the ordinary threshold $t = 4M_2^2$. When the mass M_0 is decreased further, the anomalous singularity passes around the branch point at $t = 4M_2^2$ onto another sheet of the Riemann surface [see Eq. (3)] and disappears from the usual dispersion relations.

If we suppose the mass M_0 to be decreased still further (assuming $M_1 > M_2$) until $M_0 < M_1 - M_2$, the "decay" velocity becomes real again. Since the energy E_2 is negative, this evidently corresponds to the absorption of M_2 by M_0 to form the unstable particle M_1. When $M_0^2 < (M_1 - M_2)^2$, we say the particle is "hyperstable." This extreme case is of interest because dispersion relations often take a simpler form when one or more of the particles is hyperstable.

III. PROPERTIES OF THRESHOLDS AS DEDUCED FROM PERTURBATION THEORY

The first extension of dispersion relation techniques to a situation with anomalous thresholds was made by Mandelstam.[6] He started with the dispersion relations for scattering of tightly bound particles which do not have anomalous thresholds and made an analytic continuation in the masses, increasing them until the anomalous threshold popped out onto the principal sheet of the Riemann surface. It was necessary to approximate one of the matrix elements in the expression, using perturbation theory, to get a simple expression in which the analytic continuation could be easily studied. Mandelstam showed that in the anomalous case, a dispersion relation very much like the ordinary one was valid; the only difference was that there was an extra "anomalous" contribution to the dispersion integral.

The same technique of analytic continuation in the masses of the particles was applied by Blankenbecler and Nambu[7] to the study of the form factors of loosely bound particles. The "size" of a loosely bound particle, as represented by its form factor, is governed by the exponentially damped waves associated with its virtual decay, which, as we have already seen, are intimately related to the anomalous thresholds. There have been a number of other papers along similar lines[8–10]; that of Blankenbecler *et al.*,[11] in particular, has been invaluable in clarifying many of the mathematical features of anomalous thresholds and their role in dispersion relations. It is not yet known, however, whether the approach which starts from "ordinary" dispersion relations can be extended to the most general case; at any event, the assumptions about analyticity made by these authors are not yet known

[6] S. Mandelstam, Phys. Rev. Letters **4**, 84 (1960).
[7] R. Blankenbecler and Y. Nambu, Nuovo cimento (to be published).
[8] R. Oehme, Nuovo cimento **13**, 778 (1959).
[9] R. Blankenbecler and L. F. Cook, Jr., Phys. Rev. **119**, 1745 (1960).
[10] R. Oehme, preprint, University of Chicago, Chicago, Illinois, 1960.
[11] R. Blankenbecler, M. Goldberger, S. MacDowell, and S. Treiman (to be published); R. Blankenbecler, Proc. Ann. Rochester Conf. High Energy Phys. **10**, 247 (1960); also private communication.

to be valid except in perturbation theory, so all of our present knowledge is ultimately based on examination of the perturbation theory expansion.

The detailed study of these particular examples has been supplemented by the development of a general theory of singularities, based on the analysis of analytic properties of perturbation theory amplitudes. The general theory deals with an amplitude corresponding to a graph with n external lines. This amplitude is considered as a function of the $2+3(n-4)$ (for $n \geq 4$) complex variables corresponding to the invariants formed from the external momenta and, for some purposes, also of the n mass variables. A general method of classification of the singularities of the amplitude has been described by Landau,[12] Taylor,[13] Bjorken,[14] and others.[15–17] They derived equations from which the location of the singularities can be determined. Knowledge of the precise location of the singularities given by the LTB equations is essential in applications of dispersion relations, but here we are concerned mainly with some general qualitative concepts which arise directly from this work. This point of view adopted here is that the thresholds, both the ordinary and the anomalous kinds, are the most important and characteristic feature of quantum field theory and that the theory should be constructed in such a way that these thresholds are given first and equal emphasis.

First, the general theory shows that the singularities are isolated singularities and are either simple poles or branch points. This means that we can discuss each singularity of the Feynman amplitude by itself without any reference to its neighbors. It implies that for any amplitude F, in any variable z, we can obtain a dispersion relation from Cauchy's theorem:

$$F(z)=\frac{1}{2\pi i}\oint \frac{F(z')dz'}{z'-z}$$

$$=\sum_g \frac{R_g}{z-z_g}+\sum_G \int_{z_G}^{\infty} \frac{[F(z')]_G dz'}{2\pi i(z'-z)}, \qquad (7)$$

where R_g is the residue of the pole at $z=z_g$ and $[F(z)]_G$ is the jump across a branch cut which starts at the branch point z_G. [We have assumed that $F(z)$ vanishes at ∞ in all directions.] This shows that in any conceivable dispersion relation we may isolate (if we wish) a particular contribution from each singularity and treat it separately. We may also examine the dependence of $[F(z')]_G$ on the other variables. The jump $[F]_G$ can be thought of as the difference of two analytic functions, so its singularities are of the same type as those of F itself. By using Cauchy's theorem in the other variables, a multiple dispersion relation can be obtained in as many variables as one wishes. Every multiple dispersion relation has a form similar to Eq. (7) in that the contribution of the individual singularities can be displayed explicitly.

The second important result is that every singularity can be associated with a "reduced graph." A reduced graph is a kind of skeleton of a Feynman graph, which is obtained by drawing explicitly only a certain number of the lines and representing all the remaining lines and the vertexes by points which correspond to subgraphs of the original graph. Some examples are shown in Fig. 6. These reduced graphs summarize the content of the LTB equations which give the positions of the singularities: each of the lines of the reduced graph is to correspond to a four momentum which is on the mass shell and these momenta are also to satisfy certain additional geometrical conditions. None of the other lines of the original Feynman graph, those which are hidden in the vertexes of the reduced graph, enter into the determination of the location of the singularity. A general Feynman graph leads to a number of reduced graphs and to a corresponding number of singularities; on the other hand, a given reduced graph describes a particular singularity of infinitely many Feynman graphs.[18]

The third consequence of the general theory is that every discontinuity function $[F]_G$ can be written down immediately in terms of an integral associated with the reduced graph[17,19]:

$$[F]_G=\int \prod(dk)\prod(2\pi i\Lambda(q)\delta(q^2-M^2))\prod(F). \qquad (8)$$

The rules for writing down the integral (8) are very much like the familiar Feynman rules. They are: (a) there is an integration, $\int dk \equiv \int (2\pi)^{-4}d^4k$, for each independent closed loop; (b) for each line the integrand contains a factor $2\pi i\Lambda(q)\delta(q^2-M^2)$ [this is analogous to the Feynman propagator—$\Lambda(q)$ depends on the spin of the particle and is 1 if $S=0$, $\gamma q+M$ if $S=\frac{1}{2}$, etc.];

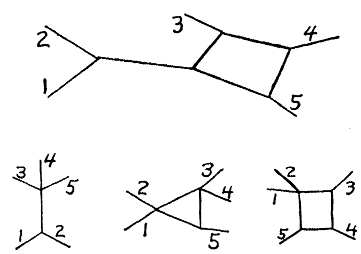

FIG. 6. A Feynman graph (top) and three reduced graphs which correspond to singularities of the amplitude.

[12] L. D. Landau, Nuclear Phys. **13**, 181 (1959).
[13] J. C. Taylor, Phys. Rev. **117**, 261 (1960).
[14] J. D. Bjorken, preprint, Stanford University, Stanford, California, 1959.
[15] J. C. Polkinghorne and G. R. Screaton, Nuovo cimento **15**, 289 (1960).
[16] J. Tarski, J. Math. Phys. **1**, 154 (1960).
[17] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).
[18] We are speaking here only about the "dynamical" singularities. There may also be various kinds of kinematical singularities, whose locations do not depend on the internal structure of the Feynman graphs. The non-Landauian singularities mentioned in reference 17 are of this type. Kinematical singularities appear, in all cases, to be avoidable nuisances in the dispersion relations.
[19] R. E. Cutkosky, Phys. Rev. Letters **4**, 532 (1960).

(c) for each vertex the integrand contains the *exact* transition amplitude F corresponding to the lines leading into that vertex [as a consequence of rule (b), this is always a physical transition amplitude since the momenta are on the mass shell but it is usually analytically continued to a region where the momentum components are not real]. These rules also give the residue R_g of a pole at $z=z_g$ if we interpret the "discontinuity" of a pole as $2\pi i\delta(z-z_g)R_g$. It can be shown that when the reduced graph corresponds to an ordinary threshold, these rules contain the usual unitarity condition on the S matrix. They may therefore be thought of as a generalization of unitarity although they refer to a property which cannot be expressed in terms of matrix algebra. It should be noted that the amplitudes corresponding to the vertices are analytic functions with many branches and some care must usually be exercised in choosing the branch. Moreover, only one root of $q_i^2=M_i^2$ must be allowed to contribute to the integral; the easiest way to insure this is to use the q_i^2 themselves as integration variables.

The preceding discussion, especially Eqs. (7) and (8), shows that the most general possible dispersion relation has the form of a nonlinear integral equation relating various transition amplitudes. This set of integral equations is, moreover, complete. The rules (a)-(c) are all that is needed to generate the perturbation series expansion to all orders, provided they are supplemented by the locations of all the LTB singularities (including rules for determining the branches of the analytic functions) and a statement about the asymptotic behavior of the amplitudes for large values of the invariants; they may, therefore, be considered to contain, in a certain sense, a complete specification of the dynamical behavior of interacting particles. It is particularly worth noting that it is possible to achieve completeness without introducing any amplitudes except those for particles on the mass shell, in other words, amplitudes which are directly measurable, or their analytic continuations. This means that even though there are anomalous thresholds, we are able to achieve through the dispersion relations a true S-matrix theory of the type originally suggested by Heisenberg.[20,21]

In the applications of dispersion relations to practical problems one always makes the assumption that it is only the nearest singularities that one needs to worry about explicitly; if everything goes well, the farther singularities can be taken care of by one or more phenomenological parameters. These distant singularities correspond to new effects which show up directly only at very high energies or, equivalently, at very small distances. It is reasonable to assume (but this has not yet been proved in the general case) that reduced graphs which are very complicated always correspond to distant singularities. This means that we can start to make a phenomenological description of experimental data by looking only at very simple reduced graphs. The simplest graphs are those with two vertexes connected by a single line (Fig. 1) which lead to the Chew-Low poles. Theoretical predictions based on these poles are in agreement with the experimental data in a number of cases. The next simplest are those with three vertexes connected by three lines, the original anomalous threshold graphs (Fig. 2). These singularities dominate the elastic scattering of loosely bound particles at small angles as we noted earlier. The importance of these anomalous thresholds is extremely well verified in scattering from deuterons and other nuclei and even in atomic scattering. It has not been customary to think of these scatterings in terms of anomalous thresholds but it can be shown that the well-known impulse approximation is equivalent to the anomalous threshold contribution.[22]

IV. DYNAMICAL EQUATIONS

The contribution of the simplest reduced graphs can be written down explicitly in terms of coupling constants and other more simple scattering amplitudes. In going beyond this approximation one obtains from the dispersion relations not an answer in closed form, but an integral equation which must first be solved. It has been repeatedly emphasized, especially by Mandelstam and Chew,[23] that from dispersion relations one does indeed obtain solutions which describe in a unique way (apart from a few parameters) the dynamical behavior of interacting particles. The essential point, as they have shown, is that the effective "potential" which acts between two particles is determined by the amplitude for the so-called "crossed" reactions. This is made particularly clear if we solve the dispersion integral equations by the device of introducing, as an intermediate step, a modified Bethe-Salpeter equation[24–26] whose solution gives the scattering amplitude. The method we follow is similar to that of Charap and Fubini.[27]

The modified Bethe-Salpeter equation is derived entirely from dispersion relations. In attempting to construct such an equation, however, we are departing from the pure S-matrix approach in which the only quantities introduced refer to particles which are on the mass shell. It is not yet certain that this equation has practical advantages. It is presented here in the

[20] W. Heisenberg, Z. Physik **120**, 513, 673 (1943).

[21] C. Moller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **23**, No. 1 (1945); **22**, No. 9 (1946).

[22] R. E. Cutkosky, Proc. Ann. Rochester Conf. High Energy Phys. **10**, 236 (1960).

[23] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); G. F. Chew and S. Mandelstam, *ibid.* **119**, 467 (1960); G. F. Chew, Lectures at Les Houches and Edinburgh (UCRL-9289) (1960); S. Mandelstam, paper presented at this conference (unpublished); G. F. Chew, Revs. Modern Phys. 33, 467 (1961), this issue.

[24] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

[25] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

[26] G. C. Wick, Phys. Rev. **96**, 1124 (1954).

[27] J. Charap and S. Fubini, Nuovo cimento **14**, 540 (1959).

hope that it will help to provide a bridge between pure dispersion theory and older methods based on Hamiltonians and wave functions. The form factors as calculated by pure dispersion theory techniques show that there is a very close relation between the anomalous threshold contribution and the ordinary Schrödinger wave function.[22] This suggests that the modified Bethe-Salpeter amplitude defined here will be useful in the calculation of form factors and similar quantities.

The ordinary Bethe-Salpeter amplitude

$$\psi(1,2)=\langle 0|\,T[\psi(1)\psi(2)]\,|\alpha\rangle$$

satisfies an equation of the form

$$\psi(1,2)=\delta_{12}+S_F'(1)S_F'(2)\int dkI(1,2;1',2')\psi(1',2'). \quad (9)$$

The term δ_{12} describes the incident plane wave of a scattering state; it is absent from the bound-state equation. The propagators S_F' are the complete propagation functions, not the "bare" propagators S_F. The kernel I is a function of six scalar variables: the energy and momentum transfer variables s and t, and the four virtual masses. We introduce a modified amplitude which satisfies a similar equation, but with the free particle propagators S_F, and in which the kernel is a function of only two variables. We assume the particles are distinguishable so we do not need to symmetrize the equation. We also assume that all discrete states with the same quantum numbers can be considered as composite particles in the sense that they arise from a nonsingular interaction between the two original particles.

The potential is the sum of two terms,

$$I(s,t,u)=I_o(s,t)+I_e(s,u),$$

where s is the energy variable, $t=(p_1-p_1')^2$, and $u=(p_1-p_2')^2$. On the mass shell, $s+t+u=2M_1^2+2M_2^2$. We represent I_o and I_e in the form

$$I_0=\int_{t_1}^{\infty}\frac{\rho_0(s,t')dt'}{t'-t},$$

$$I_e=\int_{u_1}^{\infty}\frac{\rho_e(s,u')dt'}{u'-u}.$$

In other words, I_o takes the form of a superposition of Yukawa potentials corresponding to the exchange of single quanta of mass $t'^{\frac12}$; I_e takes a similar form in which the "quantum" of mass $u'^{\frac12}$ also causes the two particles to be interchanged.

This potential is used in the integral equation

$$\phi_s(t,p_1^2,p_2^2)=I(s,t,u)+\int dkI(s,t'',u'')S_F(q_1)\times S_F(q_2)\phi_s(t',q_1^2,q_2^2), \quad (11)$$

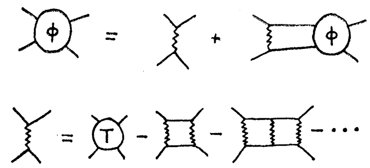

FIG. 7. Graphical representation of Eq. (11).

which is represented graphically in Fig. 7. The solution has the property that

$$\phi_s(t,M_1^2,M_2^2)\equiv T(s,t), \quad (12)$$

where T is the scattering amplitude. That is, the amplitude $\psi=\delta_{12}+S_F(1)S_F(2)\phi$ satisfies an equation of the Bethe-Salpeter type and is identical to the usual amplitude when the particles are on the mass shell. However, this identity does not hold for other values of p_1^2 and p_2^2.

If we solve Eq. (11) by iteration, (12) takes the form

$$T(s,t)=I(s,t,u)+\Sigma_2^{\infty}I^{(n)}(s,t). \quad (13)$$

We therefore require that when the initial and final particles are on the mass shell, the spectral representation of I should be consistent with the equation

$$I_o(s,t)+I_e(s,u)=T(s,t)-\Sigma_2^{\infty}I^{(n)}(s,t). \quad (14)$$

This gives an iterative method of constructing I. Let t_1 and u_1 denote the position of the singularities of $T(s,t)$ which lie closest to each side of $t=0$ (they may be either poles or branch points). We start the iteration by requiring that the spectral representation of $I_e(s,u)$ coincide with that of $T(s,t)$ for $u^{\frac12}<u_1^{\frac12}+t_1^{\frac12}$, and that the spectral representation of $I_o(s,u)$ coincide with that of $T(s,t)$ for $t^{\frac12}<\min(2t_1^{\frac12},2u_1^{\frac12})$. When the initial and final particles are on the mass shell, $I^{(n)}(s,t)$ satisfies a dispersion relation in the variable t with spectral thresholds at the points

$$t_{n,m}^{\frac12}=(n-m)t_1^{\frac12}+mu_1^{\frac12} \quad (m \text{ even}), \quad (15)$$

and

$$u_{n,m}^{\frac12}=(n-m)t_1^{\frac12}+mu_1^{\frac12}, \quad (m \text{ odd}). \quad (16)$$

Therefore, when we calculate $\rho_o(s,t)$ or $\rho_e(s,u)$ at given values of t or u, only a finite number of the iterated terms in Eq. (14) need to be subtracted. This iterative method of constructing I is reminiscent of ordinary perturbation theory, but it is somewhat more physical in that it corresponds to calculating I at successively smaller and smaller distances.

It might be supposed that a subtracted representation should be used in place of Eq. (10). However, the kernel $I(s,t,u)$ must be sufficiently regular at $t\to\infty$ and $u\to\infty$ so that Eq. (11) has solutions. In particular, a function $g(s)$ cannot be added to I without making the equation meaningless for S states. Therefore, even if $T(s,t)$ has a pole at $s=s_a$, $I(s,t,u)$ should not have a pole and the pole must be compensated in Eq. (14) by a corresponding divergence of the subtracted series.

If we multiply both sides of Eq. (11) by $s-s_a$, the inhomogeneous term disappears so $\lim(s-s_a)\phi_s(t,p_1^2,p_2^2)$ satisfies the homogeneous Bethe-Salpeter equation. In other words, since it is only in the case in which the unsubtracted dispersion relation for I can be used that the Bethe-Salpeter equation is meaningful, it is necessary for us to assume that all the poles of $T(s,t)$, which we presume represent discrete states, arise through the action of a sufficiently regular binding potential.

Let us now illustrate in somewhat more detail the technique of calculating $I(s,t,u)$ by examining the contribution of the one and two meson exchange graphs to the nuclear potential. The scattering amplitude has a pole at $t=m^2$ and a branch cut at $t=4m^2$, where m is the meson's mass (there are also exchange singularities). It follows that $I_o(s,t)$ also has a pole at $t=m^2$ so the one meson term in the potential is just that of the ordinary second-order perturbation theory without any radiative corrections. The iterated one-meson potential also has a branch point at $t=4m^2$; so after we calculate the discontinuity of $T(s,t)$, we must subtract the discontinuity of this iterated term. If we now calculate the discontinuity with respect to the variable s of these discontinuities of T and $I^{(2)}$, we obtain the well-known Mandelstam spectral functions and the corresponding reduced graphs. We see that the effect of the iteration is to eliminate one of the reduced graphs, the fourth-order ladder graph. Therefore, the two meson part of I possesses a Mandelstam representation which differs from that of T only in the removal of one term. We may note that while the reduced graphs in which the meson lines are not crossed contribute to either I_o or I_e, the crossed graphs contribute to both.

The general method we have outlined may also be applied to scattering problems in which there are anomalous thresholds in the t variable, as in the scattering of pions or nucleons from deuterons. In this case we obtain a description of multiple scattering effects in terms of a simple equivalent potential.

It is well known that the analytic continuations of $T(s,t)$ give the scattering matrix also for the channels in which t or u are the energy variables. Our method of calculating an effective potential, being based on a dispersion relation in t for fixed s, uses very directly and explicitly the fact that the potential for the s channel is related to the dynamical behavior of the particles which correspond to the t and u channels.

V. SUMMARY

Anomalous thresholds are a very general feature of the scattering matrix and their principal characteristics can be understood from simple examples which do not require formal arguments based on analytic continuation. The role of anomalous thresholds in S-matrix theory is shown completely, however, only when we consider the analytic continuations to complex momenta and energies. The study of anomalous thresholds has led to a clarification of the role of elementary wave mechanics in a pure S-matrix theory. In a certain sense, the Schrödinger equation can be considered as just a reflection of the anomalous thresholds. This is already quite clear from the works of Karplus, Sommerfield, and Wichman,[2] Nambu,[3] Oehme,[8,10] and especially Blankenbecler and Cook.[7]

On the other hand, the study of singularities in perturbation theory amplitudes has pointed out the existence of previously unnoticed relations between S-matrix elements, [Eq. (8)]. These relations are necessary to a calculation of the S matrix from the assumption of analyticity. The construction of the S matrix from these rules may be expressed in terms of a graphical calculus as suggested by Landau.[12] This graphical calculus is equivalent to the ordinary formulations of field theory.

DISCUSSION

S. F. Tuan, *Brown University, Providence, Rhode Island:* Would you like to make some comments about partial wave equations when you have anomalous thresholds?

R. E. Cutkosky: I do not think there is any difficulty, as one can calculate the partial wave amplitude with just an integral over the usual amplitude. Of course, one has to be very careful about the analytic continuations, etc.

S. Weinberg, *University of California, Berkeley, California:* My question concerns the cut that you calculated according to your rules—is this the discontinuity as you cross the real axis in one variable holding the other scalars fixed and either real or complex, or must the other scalars be specifically real? The reason I ask is because in some papers, like Tarski's, there are singularities all over the place if you use just the Landau-Bjorken rules and they are nevertheless irrelevant physically.

R. E. Cutkosky: Well, the other variables are held fixed but they may have arbitrary real or complex values and they may be even thought of as being on an arbitrary sheet of the Riemann surface that we want to consider.

S. Weinberg: I gather from your answer that in using these rules you get a lot of singularities which are not on the physical sheet and you then need the multiple dispersion relations to tell which singularities are on the physical sheet; or does that follow from your rules?

R. E. Cutkosky: No, it does not follow from the rules. You have to get them by looking at something else; for instance, you have to look at the Landau-Bjorken equations in more detail, or something like that. But it is, of course, very important and very necessary to know exactly where the singularities are and, in particular, on which sheet they occur. This also comes up in these discontinuities $[F]_G$. They themselves have singularities and you have to be careful to pick the right branch for them.

K. Symanzik, *Stanford University, Stanford, California:* One of the virtues of the ordinary Bethe-Salpeter equation is that the solution automatically satisfies unitarity, provided only that the kernel satisfies the standard irreducibility conditions. Now are there analogous properties here in your case?

R. E. Cutkosky: Yes. In fact, one can say that introducing the Bethe-Salpeter equation is one way of putting unitarity into the calculation. Of course unitarity is in here in a somewhat complicated way because if you go above the threshold for producing mesons, etc., the potential which you get out of this automatically becomes complex and has some of the inelastic channels taken into account indirectly.

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Recent Progress in the Dispersion Theory of Pion-Nucleon Interactions

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DURING the last two years a considerable amount of work has been dedicated to the application of dispersion theory to different problems of pion physics, such as pion-pion, pion-nucleon, and nucleon-nucleon scattering, and photopion production, and very interesting results have been obtained.

Here the attempt is made to explain the basic physical ideas which are at the origin of this development and to discuss the main points where our theoretical understanding of the experimental situation has improved. Only the case of pion-nucleon scattering is considered in detail, because the success of any attempt of treating the other problems, such as nuclear forces and photopion production, depends on our understanding of this fundamental problem.

The first successful approach to pion-nucleon scattering was the one based on the existence of a strong resonant interaction in the $J=\frac{3}{2}$, $T=\frac{3}{2}$ state. This approach was first based on the static model of Chew and Low, then on the relativistic dispersion treatment of Chew, Goldberger, Low, and Nambu (CGLN). In this model the pion-nucleon interaction is of short range, taking place essentially in the P-wave state. Since the details of such a short-range interaction are not known, the position of the (33) resonance cannot be determined by the theory but can be fitted to the experimental data.

One may ask why it is necessary to make use of dispersion theory and not simply to try the experimental data by means of a Breit-Wigner formula. One of the reasons is that dispersion theory allows a clear theoretical comparison between the phenomenological constants appearing in different phenomena involving the same basic interaction. For example, it has been possible to verify that the same renormalized coupling constant is obtained by comparison with experimental data on P-wave π-N scattering, S- and P-wave photoproduction, and high l nucleon-nucleon scattering.

One of the important problems treated recently is the study of possible corrections of the resonant model and, in particular, investigation as to whether, in addition to the short-range interaction of the pion with the core, there is a long-range pion-nucleon potential due to the interaction of the incoming pion with the pions of the nucleon cloud. This potential is the analog of the nucleon-nucleon potential due to two-pion exchange and has the same origin in the meson cloud effect which is responsible for the electromagnetic structure of the nucleon. The success of the resonant model in explaining the main features of low-energy pion-nucleon scattering and photoproduction might indicate that such a long-range term is negligible. However, there are many reasons that suggest the existence of the long-range pion-cloud interaction.

(1) The high-energy pion-nucleon cross sections are rather large and can be interpreted by means of an optical model with a nucleon radius of the order of the pion Compton wavelength. A similar radius appears in the optical model for high-energy nucleon-nucleon scattering and in the Hofstadter form factors of the nucleon.

(2) The existence of the $d\frac{3}{2}$ and $f\frac{5}{2}$ resonances shows the importance of the scattering with high l at energies of the order 600–900 Mev. This fact is difficult to understand on the basis of a short-range pion-nucleon interaction.

(3) At low energy the prediction of the CGLN theory for waves different from the (33) resonant one are in disagreement with experiment.

(4) The CGLN theory applied to the electromagnetic form factors of the nucleon gives results which are very difficult to reconcile with the experimental findings. Frazer and Fulco have shown that a satisfactory explanation of the data can be obtained by assuming the existence of a strong pion-pion interaction in the $T=J=1$ state.

Let us now discuss how one can evaluate the effect of the pion-pion interaction on pion-nucleon scattering. Here also the use of the dispersion method has definite advantages because it allows one to use the parameters which specify the strength and range of the pion-nucleon potential in connection with other problems, such as nucleon electromagnetic structure and nuclear forces.

Let us see what new singularities in the S