Classical Electrodynamic Equations of Motion with Radiative Reaction*

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I. INTRODUCTION

A FUNDAMENTAL property of all charged particles is that electromagnetic energy is radiated whenever they are accelerated. Abundant experimental verification has been provided from many different branches of physics. When quantum phenomena are unimportant and the classical theory is valid, it might seem that the exact equations of motion for a radiating charged particle should be understood at the present stage in the development of physics. Yet the view has often been expressed that the customary equations which represent this phenomenon provide only an approximate classical description.

The previous objections to the theory of radiative reaction are discussed here and are shown to be largely invalid. When this evidence is considered together with the new solutions which are presented here, there no longer appears to be any reason for not accepting these equations as providing an exact classical description of a radiating body.

Lorentz[1] chose a small charged sphere as a model for the electron. He calculated the force of radiative reaction by considering the retarded action of one part of the particle on another. The result can be expressed in a power series expansion in which the radius of the electron is the parameter. The first term in the series is independent of the radius and thus represents the force of radiative reaction for a point particle.

Certain difficulties occur when the Lorentz model is used for a particle of finite size. The higher terms in the series expansion depend on arbitrary assumptions about the shape and charge distribution of the particle. These higher-order terms become more and more important as the time required for a spatial oscillation of the charged particle becomes small compared to the time taken by a light signal to travel a distance equal to the particle radius. Furthermore, the proposed model is not stable and would require nonelectric forces to hold it together. Rohrlich[2] has recently shown that it is possible to formulate a consistent relativistically invariant theory for a particle of finite size if the energy and momentum of the field are correctly defined. He also shows that all divergences for a point particle may be eliminated in a unique manner by a renormalization procedure based on the relativistic invariance of the theory.

Dirac[3] has given a well-defined and relativistically invariant prescription for the calculation of the force of radiative reaction. He assumes that Maxwell's equations

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[1] H. A. Lorentz (1892), republished in his *Collected Papers*, Vol. II, pp. 281, 343; *The Theory of Electrons* (Leipzig, 1909), pp. 49, 253.

[2] F. Rohrlich, Am. J. Phys. **28**, 639 (1960).

[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).

are valid right up to the point singularity. Then he shows that there is only one Lorentz invariant procedure by which the infinities at the position of the particle may be subtracted out. The force in question is found to be proportional to the difference between the retarded and advanced fields of the particle. When this quantity is evaluated in terms of the velocity of the particle and its derivatives, an expression is obtained for the force of radiative reaction. In the nonrelativistic approximation this result agrees with the leading term in the Lorentz expression. A derivation of the relativistic force of radiative reaction which does not use tensor notation was given by Schott.[4]

Wheeler and Feynman[5] have given a physical interpretation of the mathematical results of Dirac. The retarded minus the advanced field at the position of the particle appears in Dirac's derivation. Wheeler and Feynman show that this combination of fields arises from the interaction of the charged particle in question with all of the other charged particles in the universe. If it is assumed that our universe is completely absorbing, then Wheeler and Feynman show that the forces which occur in the equation of motion of a charged particle are the sum of the force from the usual retarded electric and magnetic fields and of the force of radiative reaction. The latter force agrees exactly with the expression obtained by Dirac.

The derivation and interpretation of the force of radiative reaction for a point particle which have been given by Dirac[3] and by Wheeler and Feynman[5] are based only on assumptions of great generality. Thus it would seem that their expression for this force should be adopted as an exact mathematical representation for the force of radiative reaction within the framework of classical theory; however, this conclusion does not seem to have gained general acceptance. Most textbooks[6] have stressed that this expression for the force has only a limited range of applicability. In particular, it is often stated that this expression cannot be used over arbitrarily large time intervals nor when the radiative reaction forces are large compared to the other forces which act upon the particle.

The belief in the limited applicability of this expression has probably arisen through a combination of two factors. First, the solution of the equations of motion with radiative reaction for a particular force always contains terms which require that the acceleration of the particle must eventually increase exponentially with time. These solutions have been called "self-accelerated," "run-away," and "nonphysical." The particle does not obtain its added energy from any physical force which acts upon it. Clearly, these are absurd solutions when applied to our real physical world. Dirac[3] considered the simplest possible case: the motion of a free particle. He pointed out that the physical solution with the acceleration equal to zero could be obtained by choosing the particular value zero for one of the integration constants. Although a satisfactory physical solution for this particular example is obtained by this procedure, no method was suggested for the determination of this integration constant for other force fields.

Second, Eliezer wrote an extensive series of papers, most of which are referred to in his review article,[7] in which he claimed to have shown that no physical solutions exist for three particular force fields: (1) the field of a thin infinite charged plate; (2) an attractive Coulomb field; (3) a repulsive Coulomb field. Since it appeared that no exact physical solution could be obtained for these force fields, there was no reason to obtain a general, physically acceptable solution for the equations of motion.

Unfortunately Eliezer overlooked the physical solution among the infinite number of nonphysical solutions in most of the cases which he considered. It is shown in the following sections that there are physical solutions for the motion in the field of a thin infinite charged plate, for three-dimensional motion in both attractive and repulsive Coulomb fields, and for one-dimensional motion in a repulsive Coulomb field. When the motion is constrained to one dimension along a line which contains the source of an attractive Coulomb potential, a physical solution in terms of ordinary functions cannot be obtained; however, a satisfactory solution can be given in terms of the generalized functions known as distributions.

This article considers the problem of determining the motion of a charged body when the concepts of classical physics are valid. Effects due to quantum mechanics or to the possible finite sizes of fundamental particles are not considered. A classical charged body has nothing to do with a quantum particle from the real physical world; nevertheless, it may be a useful model which within certain limits describes the trajectory and radiation loss of a quantum particle.

In Sec. II, the general physical solution for the motion of a classical charged body is given when the force is an explicit function of time. For one-dimensional motion the general physical solution can be obtained for the relativistic equations of motion; however, for three-dimensional motion the exact physical solution can be given only in the nonrelativistic limit. For three-dimensional relativistic motion an integral equation is derived. A proof is given that a nondivergent solution of this equation exists over a range of initial volocities for any force which is bounded.

Since a higher derivative is introduced into the equations of motion by the force of radiative reaction, extra constants appear in the solution. As in all physical

[4] G. A. Schott, Phil. Mag. 29, 49 (1915).

[5] J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. 17, 157 (1945).

[6] E.g., L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), p. 221.

[7] C. J. Eliezer, Revs. Modern Phys. 19, 147 (1947).

problems, these constants are evaluated by the application of appropriate boundary conditions. When the requirement is formulated that the body must not acquire more energy over long time intervals than it obtains from the physical forces which act upon it, it is found that the exponentially divergent terms can easily be eliminated from the general solution. The resulting nondivergent physical solution always exists when the force along the trajectory is finite. Furthermore, a physically acceptable solution also exists for certain singular forces.

The exact physically acceptable solutions of the equations of motion are given in Sec. III for many different force fields. A number of examples are given where exact analytic solutions can be obtained for forces which are explicit functions of either time or position. Some approximate solutions are given for cases where exact analytic solutions cannot be obtained. Finally, some numerical solutions are given for the important case of a Coulomb force and the results are interpreted.

II. GENERAL SOLUTIONS OF THE CLASSICAL EQUATIONS OF MOTION WITH RADIATIVE REACTION

1. Equations of Motion

The classical equations of motion of a charged particle including radiative reaction as derived by Dirac[3] and by Wheeler and Feynman[5] are

$$\dot{u}_i=\frac{e}{mc}F_{ik}u^k+\frac{1}{b}\left(\ddot{u}_i-\frac{1}{c^2}u_i\dot{u}^k\dot{u}_k\right), \qquad (1)$$

where F_{ik} is the electromagnetic field tensor,

$$F_{ik}=\begin{bmatrix} 0 & H_z & -H_y & E_x \\ -H_z & 0 & H_x & E_y \\ H_y & -H_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{bmatrix}, \qquad (2)$$

b is the reciprocal of the time that it takes a light signal to travel a distance equal to two-thirds of the classical electron radius,

$$1/b=\tfrac{2}{3}(e^2/mc^3), \qquad (3)$$

m is the rest mass of the particle with charge e, c is the velocity of light, and $\mathbf{E}$ and $\mathbf{H}$ are the retarded electric and magnetic fields.

The following notation is used in this article: the four-velocity, denoted by u_i, is the derivative of position with respect to proper time τ,

$$u_i=dx_i/d\tau, \qquad (4)$$

and the velocity denoted by v_i is the derivative of position with respect to time,

$$v_i=dx_i/dt. \qquad (5)$$

Proper time and ordinary time t are connected by the relation

$$d\tau/dt=[1-(v^2/c^2)]^{\frac{1}{2}}=[1+(u^2/c^2)]^{-\frac{1}{2}}, \qquad (6)$$

where u^2 and v^2 represent the sum of the squares of the three spatial components of the corresponding velocities. The coordinates are chosen so that $x^1=x_1=x$; $x^2=x_2=y$; $x^3=x_3=z$; $x^4=-x_4=ct$. Thus

$$\begin{aligned} u^1= & \; u_1=v_x[1-(v^2/c^2)]^{-\frac{1}{2}}=v_x[1+(u^2/c^2)]^{\frac{1}{2}}, \\ \vdots \; & \; \vdots \qquad \vdots \qquad \vdots \\ u^4= & -u_4= c[1-(v^2/c^2)]^{-\frac{1}{2}}= c[1+(u^2/c^2)]^{\frac{1}{2}}. \end{aligned} \qquad (7)$$

The following useful relations are needed later:

$$\begin{aligned} u^iu_i&=-c^2, \\ u^i\dot{u}_i&=0, \\ \dot{u}_i\dot{u}^i&=-u_i\ddot{u}^i. \end{aligned} \qquad (8)$$

In relativistic equations dots over letters always indicate derivatives with respect to proper time; in nonrelativistic equations they represent derivatives with respect to ordinary time. The summation convention is used for any repeated index. In this notation all quantities have the same dimensions as their nonrelativistic counterparts, i.e., u_i and τ have the dimensions of velocity and time, respectively. The fourth component of Eq. (1) is not an independent equation as it can be derived from the first three components and Eq. (8).

When $v/c\ll1$, the nonrelativistic equations of motion as obtained from Eqs. (1) and (2) are

$$\frac{d\mathbf{v}}{dt}=\frac{e}{m}\mathbf{E}+\frac{e}{mc}\mathbf{v}\times\mathbf{H}+\frac{1}{b}\frac{d^2\mathbf{v}}{dt^2}. \qquad (9)$$

The last term is the familiar nonrelativistic expression for the force of radiative reaction divided by the mass of the particle.

Equations (1) and (9) are the exact equations of motion of a radiating charged point particle within the framework of classical physics. These equations can be derived from very general and basic assumptions[3,5] and thus must be accepted as providing a complete description of the phenomenon within their range of validity. The remainder of this article is devoted to the physical solutions of these equations, both for general and particular force fields.

2. Exact Solution for One-Dimensional Motion without Radiative Reaction

(a) *Nonrelativistic Equation*

First, consider the nonrelativistic equation of motion when the force of radiative reaction is neglected, the particle is constrained to move in one dimension, and the force $f(t)$ is an explicit function of time and is independent of the particle velocity. Then Eq. (9)

reduces to Newton's familiar equation of motion

$$ma(t)=f(t), \tag{10}$$

where a is the acceleration of the particle.

If the position and velocity of the particle are x_0 and v_0 at $t=0$, then the solution of this well-known equation may be written in the form

$$v=v_0+m^{-1}\int_0^t f(t')dt' \tag{11}$$

and

$$x=x_0+v_0t+m^{-1}\int_0^t (t-t')f(t')dt'. \tag{12}$$

Whenever the force acting on the particle is given as an explicit function of time, the complete solution for the motion of the particle can be obtained from these equations. The solution is written in this form so that it can be compared later with the solution for the equations of motion including radiative reaction.

(b) *Relativistic Equation*

The relativistic equation of motion which corresponds to Eq. (10) can be obtained from Eq. (1) and is found to be

$$m\dot{u}=[1+(u^2/c^2)]^{\frac{1}{2}}f(\tau). \tag{13}$$

The square-root factor represents the variation of mass with velocity.

Introduce a new variable w defined by the equation

$$u=c\sinh[w(\tau)/c]. \tag{14}$$

When this is substituted into Eq. (13), it is found that

$$m\dot{w}=f(\tau). \tag{15}$$

Thus, for a given force, w satisfies an equation of the same form as the nonrelativistic velocity v. An exact solution for the relativistic velocity of the particle is obtained if one takes a solution for the corresponding nonrelativistic equation of motion, substitutes τ for t, uses Eq. (14), and appropriately adjusts the initial conditions. From Eqs. (11) and (14) it is found that

$$u=c\sinh\left[\sinh^{-1}\left(\frac{u_0}{c}\right)+(mc)^{-1}\int_0^\tau f(\tau')d\tau'\right]. \tag{16}$$

The acceleration and the position of the particle as a function of proper time are obtained by the appropriate differentiation or integration of this result.

3. Exact Solutions for One-Dimensional Motion with Radiative Reaction

(a) *Nonrelativistic Equation*

The equation of motion of a charged body which is moving slowly compared to the velocity of light, which is constrained to move in one dimension, and which is acted upon by a force $f(t)$ independent of the particle velocity is found from Eq. (9) to be

$$ma-mb^{-1}\dot{a}=f(t). \tag{17}$$

The general solution of this equation is

$$a(t)=e^{bt}\left[a(0)-\left(\frac{b}{m}\right)\int_0^t e^{-bt'}f(t')dt'\right]. \tag{18}$$

In general, for an arbitrary initial acceleration, the particle acceleration eventually increases as e^{bt}. This is an entirely unacceptable physical solution. These "run-away" solutions have been discussed extensively in the literature.[3,7]

The second derivative of the position of a body occurs in Newton's equation of motion. In order to obtain a particular solution of this equation, it is necessary to specify the initial position and velocity of the particle. The equation of motion with the force of radiative reaction, Eq. (17), contains a third derivative of the position of the body. Thus an additional adjustable constant, the initial value of the acceleration, occurs in the mathematical solution of this equation. In physical problems the value of such constants is always determined from appropriate initial conditions. In this particular problem, the initial value of the acceleration is determined from the following physical boundary condition: as the time approaches infinity, the acceleration cannot increase idefinitely unless a corresponding physical force exists which supplies the particle with the required energy.

From this boundary condition it follows that the square bracket in Eq. (18) must approach zero as the time approaches infinity. Otherwise the acceleration of the body would eventually increase at least as rapidly as e^{bt} which would result in a nonphysical solution. Thus it is found that the value of the initial acceleration which satisfies this boundary condition is

$$ma(0)=b\int_0^\infty e^{-bt'}f(t')dt'. \tag{19}$$

The initial value of the acceleration when radiative reaction is not taken into account is directly proportional to the initial force, so that $ma(0)=f(0)$. When radiative reaction is included in the equations of motion, the initial acceleration is given by Eq. (19); however, for any physically realizable force the difference in the two values is very small, since the time constant in the exponential is so small ($b^{-1}=6.27\times10^{-24}$ sec). Thus, in most cases the initial value of the acceleration is determined by the force in a time interval of the order of a few times b^{-1}. Longer time intervals can be of importance only when the force varies extremely rapidly with time or actually has a singularity. However, there is an important difference in the interpretation of these two results. When radiative reaction is taken into account, the initial acceleration is determined not only by the force at the initial time, but also by the force which is encountered by the particle at each future instant of time multiplied by e^{-bt}. The unusual feature of this solution is that the present motion of the particle is determined by the forces which act upon it over a short time interval in the future. This phenomenon of preacceleration is discussed further following Eq. (21).

The exact solution of the one-dimensional equation of motion as obtained from Eqs. (18) and (19) is

$$ma(t)=b\int_t^\infty e^{-b(t'-t)}f(t')dt' \qquad (20)$$

or

$$ma(t)=b\int_0^\infty e^{-bt'}f(t+t')dt'. \qquad (21)$$

It follows from Eq. (21) that there is a physical nondivergent solution of the equations of motion whenever the Laplace transform of the force, $f(t+t')$, exists. Thus, from the theory of Laplace transforms it follows that a physical solution exists when (1) $f(t+t')$ is continuous or piecewise continuous; (2) $|t'-t_0|^n f(t_0+t')$ is bounded near a singularity at $t'=t_0$ for some number n, where $n<1$; (3) $|f(t+t')|<Ae^{\gamma t'}$ for large values of t', where A is a constant and $\gamma<b$.

In particular, if the force which acts on the particle is everywhere finite, then a physical nondivergent solution always exists. This result has been proved when the force is an explicit function of time. The corresponding general solution cannot be given when the force is an explicit function of position; however, it seems reasonable to assume that a solution also exists whenever a force which is a function of position is everywhere finite. Moreover, it can be shown that a physically reasonable answer can be obtained when the nonsingular force always acts in the same direction. For example, let us assume that the force always acts in the direction of increasing x, so that $f[x(t)]>0$. Then it follows from Eq. (20) that $a(t)>0$ for all t. Thus, the acceleration of the particle is always in the direction of increasing x. The particle never turns around or executes some unreasonable maneuver.

When radiative reaction is not considered, the acceration at a particular time is proportional to the force which acts on the particle at that time. When radiative reaction is included in the equations of motion, it is seen from Eq. (21) that the acceleration at the time t is determined by the force which acts on the particle at all future times multiplied by e^{-bt}. Unless the force varies appreciably in a time interval of the order of b^{-1}, the acceleration calculated from the equations of motion with and without radiative reaction is very nearly the same; however, the motion of the particle is influenced by the forces which act on it over a time interval of the order b^{-1} which extends into the future. For example, if a force is suddenly applied at a certain time, the particle begins accelerating over a time interval of the order b^{-1} *before* the force is applied. This phenomenon of preacceleration is clearly illustrated in some of the examples which are given in Sec. III and always occurs when radiative reaction is considered.

The subject of preacceleration has been discussed in detail by Wheeler and Feynman.[5] They find that it is not possible to separate the advanced and retarded interactions between particles in the universe over time intervals of the order of b^{-1}; however, over longer time intervals the usual relations of physics are valid which contain only retarded interactions. Moreover, they show that it is not possible to use the phenomenon of preacceleration to propagate a disturbance at a speed greater than the velocity of light over a distance which is large compared to the classical electron radius. Their conclusion is that the phenomenon of preacceleration does not violate any of the fundamental physical concepts nor is it contradicted by any available experimental evidence.

The position and velocity of the particle can be determined from the integration of Eqs. (20) and (21). The results may be written in many alternate ways. Perhaps the following form is the most instructive, since the equations can be compared directly with Eqs. (11) and (12) for the velocity and position without radiative reaction:

$$mv(t)=mv_0+\int_0^t f(t')dt' + \int_0^\infty e^{-bt'}[f(t+t')-f(t')]dt' \qquad (22)$$

and

$$mx(t)=x_0+v_0t+\int_0^t (t-t')f(t')dt'+b^{-1}\int_0^t f(t')dt' + b^{-1}\int_t^\infty e^{-b(t'-t)}f(t')dt' - b^{-1}(1+bt)\int_0^\infty e^{-bt'}f(t')dt'. \qquad (23)$$

The exact solution to the equation of motion of a charged particle in one dimension is given by Eqs. (20)–(23). The exponential divergence of the form e^{bt} does not occur in these equations. The position, velocity, and acceleration of the body are always finite unless the force is sufficiently strong so that one or more of these quantities would also become infinite when radiative reaction is neglected.

(b) *Relativistic Equation*

The relativistic equation of motion of a charged particle which is moving in one dimension and is acted upon by a force $f(\tau)$ independent of the particle velocity is found from Eqs. (1)-(8) to be

$$\dot{u}-\frac{1}{b}\ddot{u}+\frac{1}{b}\frac{u\dot{u}^2}{c^2+u^2}=\frac{1}{m}\left(1+\frac{u^2}{c^2}\right)^{\frac{1}{2}}f(\tau). \tag{24}$$

This nonlinear differential equation can be solved by the introduction of a new dependent variable defined by the equation

$$u(\tau)=c\sinh[w(\tau)/c]. \tag{25}$$

When this is substituted into Eq. (24), it is found that

$$m\dot{w}-mb^{-1}\ddot{w}=f(\tau). \tag{26}$$

The form of the differential equation for w is the same as that for the nonrelativistic velocity v as given by Eq. (17). Thus, an exact solution for the relativistic equation of motion can be obtained from the expression for the nonrelativistic velocity, Eq. (22). In this expression replace t by τ, substitute the result into Eq. (25), and introduce the proper velocity u_0 at the proper time $\tau=0$. The following result is obtained for the exact solution to the relativistic equation of motion:

$$u=c\sinh\left\{\sinh^{-1}\left(\frac{u_0}{c}\right)+(mc)^{-1}\int_0^\tau f(\tau')d\tau' +(mc)^{-1}\int_0^\infty e^{-b\tau'}[f(\tau+\tau')-f(\tau')]d\tau'\right\}. \tag{27}$$

The acceleration of the particle at the proper time τ as obtained by differentiation of Eq. (27) is

$$m\dot{u}(\tau)=b\int_\tau^\infty e^{-b(\tau'-\tau)}f(\tau')d\tau'\cosh T(\tau), \tag{28}$$

where $T(\tau)$ is the argument of the hyperbolic sine in Eq. (27). Whenever $T^2\ll 1$, then $\cosh T\approx 1$ and the solution given by Eq. (28) is formally the same as the nonrelativistic solution, Eq. (20), Similarly, whenever $\sinh T$ in Eq. (27) can be replaced by T, the solution for the velocity is the same as the nonrelativistic solution, Eq. (22).

The exact relativistic nondivergent solution for the velocity and acceleration of a radiating charged body which is constrained to move in one dimension is given by Eqs. (27) and (28). It is interesting to compare these equations with the similar results when radiative reaction is neglected, Eq. (16) together with its derivative.

4. Perturbation Solution for One-Dimensional Motion

(a) *Nonrelativistic Equation*

The acceleration of a radiating charged particle at a given time depends on the forces which act on it at future times, as is shown by Eq. (21). When the force which acts on the particle can be represented by a Taylor series expansion at the present time, the acceleration of the particle can be expressed in terms of the force and its derivatives evaluated at the *present* time only. Thus, from Eq. (21) and the Taylor series expansion of $f(t)$, it is found that

$$ma(t)=b\sum_{n=0}^{\infty}\frac{f^{(n)}(t)}{n!}\int_0^\infty e^{-bt'}t'^n dt', \tag{29}$$

where $f^{(n)}(t)$ is the nth derivative of f with respect to t.

After evaluation of this integral, the acceleration may be written as

$$ma(t)=\sum_{n=0}^{\infty}\frac{1}{b^n}f^{(n)}(t). \tag{30}$$

This equation is equivalent to Eq. (21) and may be derived from it by expansion of the differential operator. The acceleration for a particular force field can sometimes be obtained more readily from one or the other of these expressions. For example, if all of the derivatives of the force are known and can be expressed in a reasonably simple form, it is usually more convenient to obtain the solution from Eq. (30). Furthermore, since b^{-1} is an extremely small number, a sufficiently accurate solution for most problems can be obtained from the first two terms in the summation alone. The first term is the Newtonian solution while the second term approximately represents the effects of radiative reaction. Thus, a perturbation solution for the acceleration has been obtained as a power series in b^{-1}. It is instructive to substitute Eq. (30) into Eq. (17) and thus to verify directly that it is a solution of the latter equation.

The velocity of the particle as obtained from the integration of Eq. (30) is

$$mv(t)=mv_0+\int_0^t f(t')dt' +b^{-1}\sum_{n=0}^{\infty}b^{-n}[f^{(n)}(t)-f^{(n)}(0)]. \tag{31}$$

Solutions for the equations of motion are sometimes found for a mathematically interesting but physically unrealizable force which always increases with time. If the time variation of the force is not too rapid, a sufficiently accurate solution may be obtained by retaining only terms through b^{-1} in Eq. (31). Now, since $f^{(1)}(t)\geq 0$ for all t, it follows that $a(t)\geq a_N(t)$ and $v(t)\geq v_N(t)$, where a_N and v_N are the solutions of the Newtonian equation without radiative reaction. Thus, if the force always increases with time from $t=0$, in this approximation the particle always gains energy from the radiation field. On the other hand, if the force decreases with time from $t=0$, in this approximation the particle always loses energy to the radiation field. A physically realizable force must become zero after a sufficiently long time interval has passed. In this case the particle always loses energy by radiation as is shown in Sec. II.5.

(b) *Relativistic Equation*

A similar power series solution can be given for the relativistic equation of motion. If the Taylor series expansion for the force is substituted into Eq. (27), it is found that

$$u=c\sinh\left\{\sinh^{-1}\left(\frac{u_0}{c}\right)+(mc)^{-1}\int^{\tau}f(\tau')d\tau' + (mc)^{-1}\sum_{n=0}^{\infty}b^{-n}\left[f^{(n)}(\tau)-f^{(n)}(0)\right]\right\}. \quad (32)$$

The solution for the relativistic velocity which is correct through terms of the order b^{-1} may be written

$$u=c\sinh S(\tau)+(bm)^{-1}[f(\tau)-f(0)]\cosh S(\tau), \quad (33)$$

where

$$S(\tau)=\sinh^{-1}\left(\frac{u_0}{c}\right)+(mc)^{-1}\int_0^{\tau}f(\tau')d\tau'. \quad (34)$$

The first term of Eq. (33) is identical with Eq. (16).

5. Energy Equations for One-Dimensional Motion

(a) *Time Rate of Change of the Energy*

The time rate of change of the energy can readily be obtained from the relativistic equation of motion. If Eq. (24) is multiplied by $u[1+(u/c)^2]^{-\frac{1}{2}}$, it may be rewritten in the form

$$\frac{d}{d\tau}\left\{mc^2\left(1+\frac{u^2}{c^2}\right)^{\frac{1}{2}}+V(x)-\frac{2e^2}{3c^3}u\dot{u}\left(1+\frac{u^2}{c^2}\right)^{-\frac{1}{2}}\right\} = -\frac{2e^2}{3c^3}\dot{u}^2\left(1+\frac{u^2}{c^2}\right)^{-1}. \quad (35)$$

The first term in the brackets on the left-hand side of the equation is the relativistic energy of the particle, while the second term is the potential energy. The third term was called the acceleration energy by Schott.[4] It represents a reversible loss or gain of energy by radiation during the acceleration of the particle. If either the velocity or acceleration is zero at both the beginning and end of the time interval under consideration, then this term is identically zero. The term on the right-hand side of the equation represents an irreversible loss of energy by radiation. This term is always negative.

This equation can be written in a more convenient form for some purposes by the introduction of the new variable $w(\tau)$ defined by Eq. (25). After integration from the proper time 0 to τ, it is found that

$$\mathcal{E}-\mathcal{E}_0=V[x(0)]-V[x(\tau)]+\frac{2e^2}{3c^3}\left[\dot{w}(\tau)\sinh\frac{w(\tau)}{c}-\dot{w}(0)\sinh\frac{w(0)}{c}-\frac{1}{c}\int_0^{\tau}\dot{w}^2\cosh\frac{w}{c}d\tau\right], \quad (36)$$

where $\mathcal{E}$ and $\mathcal{E}_0$ are the relativistic kinetic energy of the particle at the proper times τ and 0, respectively.

The corresponding nonrelativistic expression for the change in the kinetic energy T is

$$T-T_0=V[x(0)]-V[x(t)]+\frac{2e^2}{3c^3}\left[\dot{v}(t)v(t)-\dot{v}(0)v(0)-\int_0^{t}\dot{v}^2dt\right]. \quad (37)$$

The irreversible loss of energy by radiation is represented by the last term in Eqs. (36) and (37). The two terms before the last in these equations represent a reversible gain or loss of energy by the charged particle. These terms are zero if the velocity or acceleration is zero at the ends of the interval.

(b) *Differential Equation for the Energy*

When the force is given as a function of position, it is useful to have available a differential equation with the position as the independent variable. Let

$$y=[1+(u/c)^2]^{\frac{1}{2}}. \quad (38)$$

Thus y is the relativistic energy of the particle in units of mc^2. After the independent variable in Eq. (24) is changed from t to x and the new variable y is introduced, after some algebra it is found that

$$\begin{aligned} mc^2(dy/dx)&=f(x)-\tfrac{2}{3}e^2(y^2-1)^{\frac{1}{2}}(d^2y/dx^2), \quad u<0,\\ mc^2(dy/dx)&=f(x)+\tfrac{2}{3}e^2(y^2-1)^{\frac{1}{2}}(d^2y/dx^2), \quad u>0. \end{aligned} \quad (39)$$

The sign of the last term, which represents the effect of the radiative reaction, depends on whether the velocity of the particle is positive or negative.

6. Three-Dimensional Equations of Motion

(a) *Nonrelativistic Equation*

The three-dimensional motion of a charged body whose velocity is small compared to the velocity of light and which is acted upon by a force $f(t)$ independent of the particle velocity is described by the differential equation

$$m\mathbf{a}-(m/b)(d\mathbf{a}/d\tau)=\mathbf{f}(t). \quad (40)$$

The solution of this equation can be obtained by the same procedure which was used in Sec. II.3(a). The exact physical solution of this equation is

$$ma(t)=b\int_0^\infty e^{-bt'}\mathbf{f}(t+t')dt'. \tag{41}$$

A physical nondivergent solution of this equation exists whenever the three components of $f(t)$ satisfy the conditions given in Sec. II.3(a). In particular, a physical solution of Eq. (40) always exists for any force which is everywhere finite and continuous.

(b) *Relativistic Equation*

The most general form of the equation of motion with radiative reaction is given by Eq. (1). The general solution of the equations of motion has been given in the preceding sections for nonrelativistic motion and even for the case of relativistic motion in one dimension. It is not possible to give a general solution for relativistic motion in three dimensions since the last term on the right-hand side of Eq. (1) is nonlinear; nevertheless, the existence of nondivergent physical solutions can be proved for motion in any bounded electromagnetic field which is a function of the proper time and which acts over a finite time interval.

In order to prove the existence and uniqueness of these physical solutions, the equations of motion are written in intrinsic form. An equivalent set of integral equations is derived. From these it is shown that unique physical solutions exist for any bounded field which acts over a finite time interval. Finally the usual four-vector notation is used to derive a useful integral equation which has only bounded solutions when the fields are bounded.

1. Equations of motion in intrinsic form. In this section the equations of motion are obtained in terms of the intrinsic coordinates. The use of these coordinates usually simplifies any problem where the trajectory is a skew curve in three dimensions. For this reason alone it is interesting to put the equations in this form. Moreover, it turns out that the existence and uniqueness of physical solutions for a broad class of fields can be demonstrated directly from these equations.

Let $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ be the unit vectors in the direction of the tangent, principal normal, and binormal, respectively. Let $\mathbf{u}$ be the three-dimensional vector representing the proper velocity and U be its magnitude. Thus

$$\begin{aligned}\mathbf{u}&=U\mathbf{t},\\ d\mathbf{u}/d\tau&=\dot{U}\mathbf{t}+k_1U^2\mathbf{n},\\ d^2\mathbf{u}/d\tau^2&=(\ddot{U}-k_1^2U^3)\mathbf{t}+[3k_1U\dot{U}+(dk_1/ds)U^3]\mathbf{n}\\ &\quad -k_1k_2U^3\mathbf{b},\end{aligned} \tag{42}$$

where the Frenet formulas,

$$\begin{aligned}d\mathbf{t}/ds&=k_1\mathbf{n},\\ d\mathbf{b}/ds&=k_2\mathbf{n},\\ d\mathbf{n}/ds&=-k_1\mathbf{t}-k_2\mathbf{b},\end{aligned} \tag{43}$$

have been used; k_1 is the curvature, k_2 is the torsion, and s is the arc length along the trajectory.

First, let us derive the intrinsic equations when radiative reaction is neglected. Take the components of Eq. (1) in the direction of $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ and use Eqs. (2) and (42) to obtain

$$\dot{U}=(e/mc)(c^2+U^2)^{\frac{1}{2}}E_t, \tag{44}$$

$$k_1U^2=(e/mc)[(c^2+U^2)^{\frac{1}{2}}E_n-UH_b], \tag{45}$$

$$(c^2+U^2)^{\frac{1}{2}}E_b+UH_n=0, \tag{46}$$

where the subscripts on the fields denote their components in the directions of the intrinsic coordinates.

If $\mathbf{E}$ and $\mathbf{H}$ are given as functions of τ together with an initial value for $\mathbf{u}$, a unique trajectory for the particle can be obtained in the following manner. The initial direction of the tangent vector is known. The plane perpendicular to this vector contains $\mathbf{b}$ and $\mathbf{n}$. Their initial direction can be determined from Eq. (46) which always has a solution. From Eq. (44) the variation of U with proper time can be computed, while Eq. (45) gives the value of k_1. These equations can be integrated step by step to obtain a unique trajectory provided only that $\mathbf{E}$ and $\mathbf{H}$ are bounded. It is instructive to obtain the trajectories for a few simple electromagnetic fields from the preceding equations.

Next, consider the intrinsic equations with radiative reaction when the electric and magnetic fields are zero. It is evident from Eq. (1) that the solution $u_i=0$ satisfies the equations of motion. In order to prove that this is the *only* nondivergent physical solution, write the equations in intrinsic form. From Eqs. (1), (8), and (42) it follows that

$$\dot{U}=\frac{1}{b}\left[\ddot{U}-k_1^2U^3-\frac{U}{c^2}\left(\frac{c^2U^2}{c^2+U^2}+k_1^2U^4\right)\right], \tag{47}$$

$$k_1U^2=\frac{1}{b}\left[3k_1U\dot{U}+\frac{dk_1}{ds}U^3\right], \tag{48}$$

$$k_1k_2U^3=0. \tag{49}$$

From Eq. (49) it follows that either $k_1=0$ or $k_2=0$ or both. If $k_1=0$, the trajectory is a straight line and Eq. (47) reduces to

$$b\dot{U}=\ddot{U}-U\dot{U}^2(c^2+U^2)^{-1}. \tag{50}$$

The solution of this equation is

$$U=c\sinh(D+Ee^{b\tau}), \tag{51}$$

where D and E are constants. The constant E must be chosen as zero in order to have a nondivergent solution. The resulting solution represents a straight line trajectory with the particle moving at constant velocity.

Now, let us assume that $k_2=0$, but that k_1 may be different from zero. The motion takes place in a plane from the definition of the binormal. Equation (48) may be integrated to give

$$k_1 = QU^{-3}e^{b\tau}, \tag{52}$$

where Q is a constant. This may be verified by substitution into the original differential equation.

If this value of k_1 is substituted in Eq. (49), the terms may be rearranged to give

$$(d/d\tau)[c^2\dot{U}^2e^{-2b\tau}/(c^2+U^2)] = 2Q^2(\dot{U}/U^3). \tag{53}$$

This equation may then be integrated with the result that

$$\dot{U}^2 = (1+c^{-2}U^2)(M^2-Q^2U^{-2})e^{2b\tau}, \tag{54}$$

where M is a constant. The velocity can be obtained from a second integration as

$$U^2 = \frac{[N^2\exp(2b^{-1}c^{-1}Me^{b\tau})+Q^2M^{-2}-c^2]^2+4c^2Q^2M^{-2}}{4N^2\exp(2b^{-1}c^{-1}Me^{b\tau})}, \tag{55}$$

where N is a constant.

Let us consider the behavior of this solution for large values of τ when M is positive, negative, and zero. When $M>0$, then for large values of τ

$$U \sim \tfrac{1}{2}N\exp(b^{-1}c^{-1}Me^{b\tau}). \tag{56}$$

This always represents a divergent solution.

When $M<0$, it is found that for large values of τ

$$U \sim \tfrac{1}{2}N^{-1}(c^2+Q^2M^{-2})\exp(-b^{-1}c^{-1}Me^{b\tau}). \tag{57}$$

Again this always results in a divergent solution.

Finally if $M=0$, it follows from Eq. (54) that $Q=0$ and $\dot{U}=0$. Furthermore, $k_1=0$ from Eq. (52). Thus, when there are no forces acting on the particle indefinitely far into the future, the only nondivergent solution is a straight line trajectory with constant velocity.

When electric and magnetic fields act upon the particle, the intrinsic equations of motion as obtained from Eqs. (1), (2), (8), and (42) are

$$\dot{U} = (e/mc)(c^2+U^2)^{\frac{1}{2}}E_t + b^{-1}\{\ddot{U} - k_1^2U^3 - c^{-2}U[c^2\dot{U}^2(c^2+U^2)^{-1}+k_1^2U^4]\}, \tag{58}$$

$$k_1U^2 = (e/mc)[(c^2+U^2)^{\frac{1}{2}}E_n - UH_b] + b^{-1}[3k_1U\dot{U} + (dk_1/ds)U^3], \tag{59}$$

$$(e/mc)[(c^2+U^2)^{\frac{1}{2}}E_b + UH_n] - b^{-1}k_1k_2U^3 = 0. \tag{60}$$

Some exact solutions of these equations for motion on a skew curve at constant speed are given in Sec. III.3(d).

The remainder of this section is devoted to a proof of the uniqueness and existence of a physical solution for any bounded electromagnetic field which acts over a finite time interval. The reader is cautioned that thorough study is required to understand the many details in this proof. Because of its length only the major points can be given here, but the proof is believed to be complete.

The first step in the proof is the derivation of integral expressions for U, k_1, and k_2, [Eqs. (64–67)], which are nondivergent for large values of the proper time. If the fields are zero after $\tau=\tau_0$, it is shown next that a unique trajectory can be obtained by integrating backwards in proper time from this point. The particle which follows this trajectory has some particular velocity at $\tau=0$. This may be taken as the initial velocity for a particle which starts at $\tau=0$ and follows this physically acceptable trajectory forward in time. This same procedure may be repeated for a continuous range of velocities at $\tau=\tau_0$. The initial velocities at $\tau=0$ from these trajectories also cover a continuous range of values. In this manner it is established that a physically acceptable solution exists for a continuous range of initial velocities at $\tau=0$. It is in this sense that a proof is given of the existence of a solution. In particular it should be noted that it is not shown that a solution exists for *any* initial value of the velocity at $\tau=0$, but only for some continuous range of initial values; however, there seems to be no reason to doubt that a physical solution also exists for all initial values of the velocity.

The curvature of the trajectory can be obtained by formally integrating Eq. (59) with the result that

$$k_1 = \frac{e^{b\tau}}{U^3}\left\{Q - \frac{eb}{mc}\int_0^\tau e^{-b\tau'}U[(c^2+U^2)^{\frac{1}{2}}E_n - UH_b]d\tau'\right\}, \tag{61}$$

where Q is a constant which can be expressed in terms of the initial values of the velocity and curvature, $Q=k_1(0)U^3(0)$.

Similarly the torsion is obtained from Eq. (60) by replacing k_1 by its value as given by Eq. (61) with the result that

$$k_2 = \frac{eb}{mc}\,\frac{[(c^2+U^2)^{\frac{1}{2}}E_b + UH_n]e^{-b\tau}}{Q - \left(\dfrac{eb}{mc}\right)\displaystyle\int_0^\tau e^{-b\tau'}U[(c^2+U^2)^{\frac{1}{2}}E_n - UH_b]d\tau'}. \tag{62}$$

If the terms in Eq. (58) which involve derivatives are written in the form of the left-hand side of Eq. (53) and k_1 is replaced by its value from Eq. (61), the resulting equation can be integrated to obtain

$$\dot{U}^2(1+c^{-2}U^2)^{-1} = e^{2b\tau}\Bigg[M^2 + \int_0^\tau 2\dot{U}U^{-3}\Bigg(\Bigg\{Q - \left(\frac{eb}{mc}\right)\times\int_0^{\tau'} Ue^{-b\tau''}[(c^2+U^2)^{\frac{1}{2}}E_n - UH_b]d\tau''\Bigg\}^2 - em^{-1}cb(c^2+U^2)^{-\frac{1}{2}}U^3E_te^{-2b\tau'}\Bigg)d\tau'\Bigg], \tag{63}$$

where M^2 is a constant which can be evaluated in terms of the initial velocity and acceleration,

$$M^2=c^2\dot{U}^2(0)/[c^2+U^2(0)].$$

If six initial conditions are given and if **E** and **H** are bounded, Eqs. (61)–(63) can always be integrated step by step to obtain a trajectory. In general this is a divergent solution. The six initial conditions may be taken as the initial values of **u**, U, k_1, and **n**. From these the values of the constants Q and M can be determined.

Let us assume that **E** and **H** are given as functions of τ and are bounded. An actual physical force cannot act for an infinite time and therefore let us assume that the fields are zero after some proper time τ_0. A physical solution to the problem is obtained when the trajectory is a straight line and the particle has constant velocity for all proper times greater than τ_0.

There may be a step discontinuity in the fields at τ_0. From Eq. (58) it follows that a step discontinuity in U generates a Dirac delta function in $\dot{U}$ and the square of a Dirac delta function in $\dot{U}^2$. The coefficient of these terms must vanish since the order of infinity of the Dirac delta function and of its square are different and are different from ordinary functions. Thus U and $\dot{U}$ are continuous. In the same manner it follows from Eq. (59) that k_1 is continuous since a discontinuity in k_1 would generate only a Dirac discontinuity in dk_1/ds. Thus U, $\dot{U}$, and k_1 must be continuous and k_2 and dk_1/ds may have a step discontinuity at τ_0.

Thus $k_1=0$ at $\tau=\tau_0$ which determines the value of the constant Q. It follows that

$$k_1U^3=\left(\frac{eb}{mc}\right)e^{b\tau}\times\int_{\tau}^{\tau_0} e^{-b\tau'}U[(c^2+U^2)^{\frac{1}{2}}E_n-UH_b]d\tau', \quad (64)$$

when $\tau\leq\tau_0$ and $k_1=0$ when $\tau\geq\tau_0$.

The torsion is now given by the expression

$$k_2=\frac{[(c^2+U^2)^{\frac{1}{2}}E_b+UH_n]e^{-b\tau}}{\int_{\tau}^{\tau_0} e^{-b\tau'}U[(c^2+U^2)^{\frac{1}{2}}E_n-UH_b]d\tau'}, \quad (65)$$

when $\tau\leq\tau_0$. In order for the torsion to be finite at $\tau=\tau_0$, the condition

$$[(c^2+U^2)^{\frac{1}{2}}E_b+UH_n]_{\tau=\tau_0}=0 \quad (66)$$

must be satisfied.

Further $\dot{U}=0$ at $\tau=\tau_0$ since $\dot{U}$ is continuous. This determines the value of the constant M so that

$$\dot{U}^2(1+c^{-2}U^2)^{-1}=2\int_{\tau}^{\tau_0}\dot{U}e^{-2b(\tau'-\tau)}\times\left[\left(\frac{ecb}{m}\right)(c^2+U^2)^{-\frac{1}{2}}E_t-k_1U^3\right]d\tau'. \quad (67)$$

Equations (64)–(67) form a complete set of integral equations describing the physical motion of a charged particle acted upon by bounded electromagnetic fields which are zero after the proper time τ_0. All solutions of these equations are nondivergent and appropriately join with the solution for zero field when $\tau>\tau_0$. For a given set of initial conditions the solutions are unique since there is only one nondivergent solution in the field free region for each value of **u** at τ_0. All of the available constants in our equation have been determined in joining the two solutions properly at τ_0. The only remaining parameters are the three components of the velocity at $\tau=0$.

A group of physical solutions which cover a continuous range of initial values of the vector velocity at $\tau=0$ can be generated by the following procedure. Start at $\tau=\tau_0$ with an initial vector velocity and integrate backwards in proper time until $\tau=0$ is reached. The directions of the principal normal and binormal at $\tau=\tau_0$ are determined in terms of the given fields by Eq. (66). The values of k_1, k_2, and U are determined from the remaining equations which can be integrated step by step until the value $\tau=0$ is reached. The result of the calculation is a set of values for **u**, $\dot{U}$, k_1, and **n** at $\tau=0$. If this path is now described in reverse so that the proper time increases along the trajectory, then this is the actual trajectory for a charged particle which starts at $\tau=0$ with these initial conditions. Since the solution joins up correctly with the nondivergent solution in the field-free region when $\tau>\tau_0$, it follows that this is a nondivergent physical solution for the motion.

If the initial conditions at $\tau=\tau_0$ are varied over a continuous range and the corresponding trajectories are computed back to $\tau=0$, then a continuous range of values is obtained for **u**, $\dot{U}$, k_1, and **n** at $\tau=0$. All of these values can be used as the initial values for the physical trajectory of an actual charged particle. Thus it has been shown that nondivergent solutions exist over a continuous range of initial velocities which a particle may have at $\tau=0$. It has further been shown that the solution is unique since there is one and only one solution for $\tau<\tau_0$ which is physically acceptable and can be properly joined up with a given trajectory when there is no field.

2. Integral equation. In four-dimensional notation it is possible to obtain an integral equation which is equivalent to the original differential equation. Moreover, the limits of the integral can be chosen so that only physical solutions result. If Eq. (1) is multiplied by $\dot{u}^i$, it is found that

$$(d/d\tau)(\dot{u}_i\dot{u}^i)-2b\dot{u}_i\dot{u}^i=-(2eb/mc)F_{ik}u^k\dot{u}^i. \quad (68)$$

This equation may be formally integrated to obtain

$$\dot{u}_i\dot{u}^i=\frac{2eb}{mc}\int_\tau^\infty F_{ik}u^k\dot{u}^ie^{-2b(\tau'-\tau)}d\tau'. \quad (69)$$

The limits on the integral have been chosen to obtain a bounded solution when F_{ik} is bounded. This fact is proven after Eq. (71).

Next substitute Eq. (69) into Eq. (1) and solve the resulting equation for $\dot{u}_i$ to obtain

$$\dot{u}_i=\frac{eb}{mc}\int_\tau^\infty\left[F_{ik}u^k-\frac{2u_i}{c}\int_{\tau'}^\infty F_{ik}u^k\dot{u}^ie^{-2b(\tau''-\tau')}d\tau''\right]\times e^{-b(\tau'-\tau)}d\tau'. \quad (70)$$

This equation is entirely equivalent to the original differential equation. In order to prove that Eq. (70) is bounded, substitute $\dot{u}_k\dot{u}^k$ from Eq. (69) for the integral over $d\tau''$ in Eq. (70) so that

$$\dot{u}_i=\frac{eb}{mc}\int_\tau^\infty\left(F_{ik}u^k-\frac{mu_i}{ecb}\dot{u}_k\dot{u}^k\right)e^{-b(\tau'-\tau)}d\tau'. \quad (71)$$

It has already been shown that the only bounded solution when $F_{ik}=0$ is $\dot{u}_i=0$. When a force acts on a particle, let us assume that all of the components of F_{ik} are bounded, but that at least one of the $\dot{u}_i$ is not bounded and then show that there is no solution of Eq. (71) with this property. From our assumption it follows that

$$|F_{ik}u^k|\ll(m/ecb)|u_i\dot{u}_k\dot{u}^k|$$

for a sufficiently large value of τ. For larger values of τ the solution is then asymptotic with the case when $F_{ik}=0$. The only solution then is $\dot{u}_i=0$ which violates the original assumption. Thus $\dot{u}_i$ is always bounded when it satisfies Eq. (70) provided that F_{ik} is bounded. These integral equations are necessarily compatible with those in intrinsic form.

Thus it has been shown that a unique solution always exists for the general relativistic equations of motion provided that the fields are bounded and act only over some finite time interval. Solutions are known to exist for some fields which exhibit singularities, but we have not attempted to make the existence proof more general.

(c) *Integral for Central Fields*

When a central force acts on a charged body moving at relativistic velocities, the motion takes place in a plane even when radiative reaction is taken into consideration. From Eq. (1) it follows that for central forces

$$\frac{d^2\mathbf{r}}{d\tau^2}=\frac{f(r)}{mr}\mathbf{r}+\frac{1}{b}\frac{d^3\mathbf{r}}{d\tau^3}+\frac{1}{bc^3}\frac{d\mathbf{r}}{d\tau}\left[\left(\frac{d^2\mathbf{r}}{d\tau^2}\right)^2-c^2\left(\frac{d^2t}{d\tau^2}\right)^2\right]. \quad (72)$$

Take the vector product of this equation with $\mathbf{r}$ to obtain

$$\frac{d^2\mathbf{r}}{d\tau^2}\times\mathbf{r}=\frac{1}{b}\frac{d^3\mathbf{r}}{d\tau^3}\times\mathbf{r}+\frac{1}{bc^3}\left(\frac{d\mathbf{r}}{d\tau}\times\mathbf{r}\right)\left[\left(\frac{d^2\mathbf{r}}{d\tau^2}\right)^2-c^2\left(\frac{d^2t}{d\tau^2}\right)^2\right]. \quad (73)$$

If the vector product of this equation with $\mathbf{r}$ is taken and the order of the terms is rearranged, the result is obtained that

$$b\mathbf{r}\times\frac{d\mathbf{r}}{d\tau}\times\frac{d^2\mathbf{r}}{d\tau^2}=\mathbf{r}\times\frac{d\mathbf{r}}{d\tau}\times\frac{d^3\mathbf{r}}{d\tau^3}. \quad (74)$$

The physical nondivergent solution of this equation is

$$\mathbf{r}\times d\mathbf{r}/d\tau\times d^2\mathbf{r}/d\tau^2=0, \quad (75)$$

as may be verified by substitution. The general solution of Eq. (74) has $Ce^{b\tau}$ replacing zero in the preceding equation. The physical solution is obtained when the constant C is chosen equal to zero.

Since Eq. (75) is valid for all values of the time, it follows that the motion always takes place in a plane defined by the initial values of $\mathbf{r}$ and its derivative. The initial acceleration always lies in this same plane. This conclusion may also be verified for the nonrelativistic case from Eq. (41) when the requirement of a central force is imposed.

(d) *Energy Equation*

The time rate of change of the energy can be obtained by the same method as was used in Sec. II.5(a). When the forces are velocity independent, the spatial components of Eq. (1) may be written in the form

$$\dot{u}_i-\frac{1}{b}\ddot{u}_i+\frac{u_iu^2}{bc^2}-\frac{u_i(u_1\dot{u}_1+u_2\dot{u}_2+u_3\dot{u}_3)^2}{bc^2(c^2+u^2)}=\frac{1}{m}\left(1+\frac{u^2}{c^2}\right)^{\frac{1}{2}}f_i, \quad (76)$$

where $u^2=u_1^2+u_2^2+u_3^2$.

Now if Eq. (76) is multiplied by $u^i[1+(u/c)^2]^{-\frac{1}{2}}$ and the index i is summed over the three spatial coordinates only, the resulting equation may be written in the form

$$\frac{d}{d\tau}\left\{mc^2\left(1+\frac{u^2}{c^2}\right)^{\frac{1}{2}}+V(x_1,x_2,x_3)-\frac{2e^2}{3c^3}u^i\frac{du_i}{d\tau}\left(1+\frac{u^2}{c^2}\right)^{-\frac{1}{2}}\right\}=-\frac{2e^2}{3c^3}\left(1+\frac{u^2}{c^2}\right)^{-\frac{1}{2}}\left[\frac{du_i}{d\tau}\frac{du^i}{d\tau}+\frac{1}{c^2}\left(\mathbf{u}\times\frac{d\mathbf{u}}{d\tau}\right)^2\right], \quad (77)$$

where, in this equation only, all summations are over the three spatial components only. The potential energy V is derived from the external forces which act upon the particle. This equation could also be derived from the fourth component of Eq. (1).

The first and second terms in Eq. (77) are the relativistic kinetic energy and the potential energy. The third term is the acceleration energy and represents a reversible exchange of energy with the radiation field. The term on the right-hand side of Eq. (77) is always negative and represents the irreversible loss of energy by an accelerated charged body.

III. SOLUTIONS FOR PARTICULAR PROBLEMS

The solutions of the equations of motion for a radiating charged body acted upon by various particular force fields are presented in this section. Most of the solutions which are given here are exact and all are nondivergent physical solutions. These examples further illustrate the general results which are given in Sec. II. The phenomenon of preacceleration can be clearly seen in many of the examples. The energy loss by radiation is calculated for some of these examples.

1. One-Dimensional Motion for Time-Dependent Forces

(*a*) *Pulse of Radiation*

The motion of a charged particle which is disturbed by a momentary pulse of radiation has been considered by Dirac.[3] Let the force which acts on the particle be represented by a delta function, so that

$$f(t)=k\delta(t-t_0), \tag{78}$$

where k is a constant and the pulse of radiation arrives at the time t_0. From Eq. (21), it is found that

$$ma(t)=\begin{cases} bke^{-b(t_0-t)}, & t<t_0, \\ 0, & t>t_0, \end{cases} \tag{79}$$

when the velocity of the particle is small compared to the velocity of light. The particle velocity is obtained by direct integration as

$$mv(t)=\begin{cases} mv_0+ke^{-b(t_0-t)}, & t<t_0, \\ mv_0+k, & t>t_0, \end{cases} \tag{80}$$

where v_0 is the initial velocity of the particle at some time much earlier than t_0.

The loss of energy by radiation is found from Eq. (37) to be

$$T-T_0=-k^2/2m. \tag{81}$$

The time variation of the acceleration for the particular case when $bk/m=1$ is shown in Fig. 1. The pulse acts on the charged particle at $t_0=0$ in this example. The acceleration of the electron starts to increase before the pulse actually arrives at the position of the charged particle. This preacceleration has an appreciable value only over a time interval of the order of several times b^{-1}.

When the particle has relativistic velocities and the force is given by Eq. (78) with t replaced by τ, the

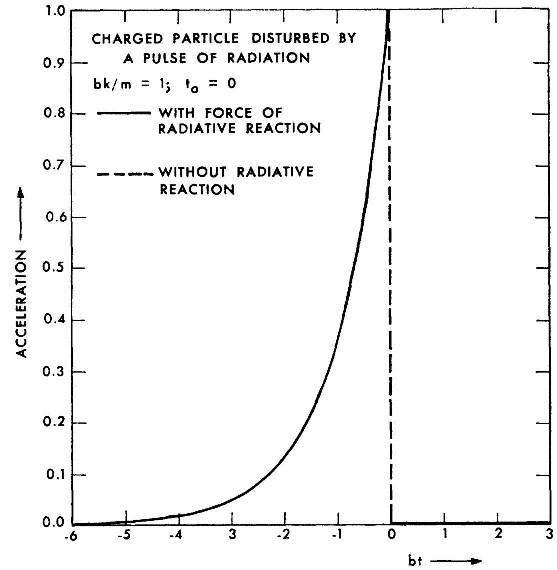

FIG. 1. Acceleration of a radiating charged body which is acted upon by a sharp pulse of radiation. The pulse acts at the time $t_0=0$. It is assumed that $bk/m=1$, where k is the force constant and m is the mass of the particle. The acceleration when radiative reaction is included in the equations of motion is shown by the solid curve. The dashed curve indicates that the acceleration is different from zero only at $t=0$ when radiative reaction is omitted from the equations of motion.

velocity and acceleration can be found from Eqs. (27) and (28). The acceleration in this case is

$$m\dot{u}(\tau)=\begin{cases} bke^{-b(\tau_0-\tau)}\cosh\{\sinh^{-1}(u_0/c)+(k/mc)e^{-b(\tau_0-\tau)}\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \tau<\tau_0, \\ 0, \qquad\qquad\qquad\qquad\qquad\quad \tau>\tau_0. \end{cases} \tag{82}$$

(*b*) *Constant Force*

A constant force which acts upon the charged particle may be written in the form

$$f(t)=mk. \tag{83}$$

When k is positive, the force tends to increase the value of the coordinate x. From Eq. (21), it is found that the nonrelativistic solution is

$$\begin{aligned} a(t)&=k, \\ v(t)&=v_0+kt, \end{aligned} \tag{84}$$

where v_0 is the initial velocity at $t=0$.

In this particular case, the motion is the same whether or not radiative reaction is included. This is because the time derivative of the acceleration is always zero and therefore the force of radiative reaction is zero; however, there is an irreversible loss of energy from the particle, as follows from the fact that the right-hand side of Eq. (35) is different from zero. The source of this radiated energy is the acceleration energy.

Because of the phenomenon of preacceleration, the particle knows that the acceleration remains constant for an indefinite time into the future. When this occurs, the acceleration energy can decrease without limit so that the kinetic energy of the particle is the same as if radiative reaction were omitted from the equations of motion. This rather strange behavior occurs only because the force acts forever. If the force is cut off after a certain elapsed time (as must happen with any physical force), the kinetic energy of the particle is reduced by an amount just equal to the radiated energy. The solution for this case is given in the next section. The problem of the radiation from a uniformly accelerated particle has been discussed further by Fulton and Rohrlich.[8]

The solution for relativistic velocities as obtained from Eqs. (27) and (28) is

$$\begin{aligned} \dot{u}(\tau) &= k\cosh[\sinh^{-1}(u_0/c)+(k\tau/c)], \\ u(\tau) &= c\sinh[\sinh^{-1}(u_0/c)+(k\tau/c)]. \end{aligned} \tag{85}$$

(c) *Constant Force Which Acts for a Specific Time*

Let a constant force act on the charged particle during the time interval from t_0 to t_1 so that

$$f(t)=\begin{cases} 0, & 0<t<t_0, \\ mk, & t_0<t<t_1, \\ 0, & t>t_1. \end{cases} \tag{86}$$

From Eq. (20) it is found that the nonrelativistic solution is

$$a(t)=\begin{cases} k(e^{-bt_0}-e^{-bt_1})e^{bt}, & 0<t<t_0, \\ k[1-e^{-b(t_1-t)}], & t_0<t<t_1, \\ 0, & t>t_1, \end{cases} \tag{87}$$

and

$$v(t)=\begin{cases} v_0+(k/b)(e^{-bt_0}-e^{-bt_1})(e^{bt}-1), & 0<t<t_0, \\ v_0+k(t-t_0)-(k/b)e^{-bt_1}(e^{bt}-1) \\ \qquad +(k/b)(1-e^{-bt_0}), & t_0<t<t_1, \\ v_0+k(t_1-t_0) \\ \qquad -(k/b)(e^{-bt_0}-e^{-bt_1}), & t>t_1, \end{cases} \tag{88}$$

where v_0 is the initial velocity at $t=0$.

If $k>0$, so that the force always tends to increase the coordinate x, then $a(t)\geq 0$ for any value of t_0, t_1, and k. Thus, the particle is always accelerated in the direction of the force and executes a physically acceptable motion. The final velocity is finite and is always less than the corresponding velocity when radiative reaction is omitted from the equations of motion; however, it should be noted that the work done by the external force is not the same in these two cases. This is because the particle has a different velocity at each instant of time when radiative reaction is included. Thus different amounts of work are done by the external force over the fixed time interval. Similar considerations hold when $k<0$. In this case the final velocity may be greater when radiative reaction is included than when it is not because the external force does a different amount of work in the two cases.

[8] T. Fulton and F. Rohrlich, Ann. Phys. **9**, 499 (1960).

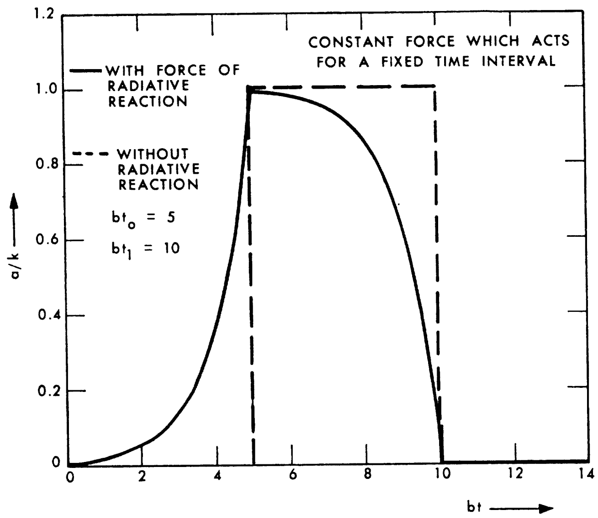

FIG. 2. Acceleration a of a radiating charged body when a constant force acts for a fixed time interval. The following values have been chosen for the parameters: $bt_0=5$; $bt_1=10$. The force constant is k. The acceleration is shown when the force of radiative reaction is and is not included in the equations of motion.

The acceleration for a particular choice of values of t_0 and t_1 is shown in Fig. 2. Because of the phenomenon of preacceleration, the particle begins to move before the force acts on it. The acceleration also begins to decrease before the force is turned off; however, these effects are noticeable only over a time interval of a few times b^{-1}. The solution of this same problem for a particle which is moving at relativistic velocities can easily be obtained from the above solution by the methods described in Sec. II.3(b). It is not given here because of its length.

(d) *Force Proportional to t^n*

Let us consider the motion under the influence of a force which varies as an integral power of time, so that

$$f(t)=mkt^n, \quad t>0. \tag{89}$$

In this case, the solution of the equations of motion can be obtained immediately from Eq. (30) as

$$\begin{aligned} a(t) &= k\sum_{l=0}^{n}\frac{1}{b^l}\frac{d^l t^n}{dt^l}, \\ v(t) &= v_0+\frac{k}{n+1}\sum_{l=0}^{n}\frac{1}{b^l}\frac{d^l t^{n+1}}{dt^l}, \end{aligned} \tag{90}$$

where v_0 is the initial velocity at the time $t=0$. The results given in Sec. III.1(b) are a special case of the foregoing with $n=0$.

When $k>0$, the acceleration and velocity are always greater than when radiative reaction is omitted from the equations of motion. This occurs because the force is assumed to increase indefinitely far into the future. The situation is quite different if the force approaches zero after a certain time. In an actual physical problem, the force could act on the particle only over some finite time interval. When this is the case, the particle always has a net loss of energy by radiation.

The relativistic solution for the force given by Eq. (89) with t replaced by τ is

$$\dot{u}(\tau)=k\sum_{l=0}^{n}\frac{1}{b^l}\frac{d^l\tau^n}{d\tau^l}\cosh\left\{\sinh^{-1}\left(\frac{u_0}{c}\right)+g(\tau)\right\},$$
$$u(\tau)=c\sinh\left\{\sinh^{-1}\left(\frac{u_0}{c}\right)+g(\tau)\right\}, \quad (91)$$

where

$$g(\tau)=\frac{k}{c(n+1)}\sum_{l=0}^{n}\frac{1}{b^l}\frac{d^l\tau^{n+1}}{d\tau^l}.$$

(e) Periodic Force

Let the charged particle be subjected to a periodic force so that

$$f(t)=mk\sin\omega t. \quad (92)$$

The acceleration and velocity of the particle can be obtained from Eq. (21) and are found to be

$$a(t)=k[1+(\omega/b)^2]^{-\frac{1}{2}}\sin[\omega t+\tan^{-1}(\omega/b)],$$
$$v(t)=v_0-(k/\omega)[1+(\omega/b)^2]^{-\frac{1}{2}}$$
$$\times\{\cos[\omega t+\tan^{-1}(\omega/b)]-[1+(\omega/b)^2]^{-\frac{1}{2}}\}, \quad (93)$$

where v_0 is the initial velocity at $t=0$.

If $\omega/b\ll 1$, the solution is very nearly the same whether or not radiative reaction is considered. The amplitude of the oscillation and the phase are changed only slightly when the radiative reaction is included. On the other hand, when $\omega/b\gg 1$, the magnitude of the acceleration decreases inversely as ω and the phase shift approaches 90°. The solution is illustrated in Fig. 3 for the particular case when $\omega=b$.

The energy radiated per unit time, W, can be calculated from Eq. (37). It is found that

$$W=-\{mk^2/4\pi b[1+(\omega/b)^2]\}. \quad (94)$$

Thus, the energy radiated per unit time is nearly

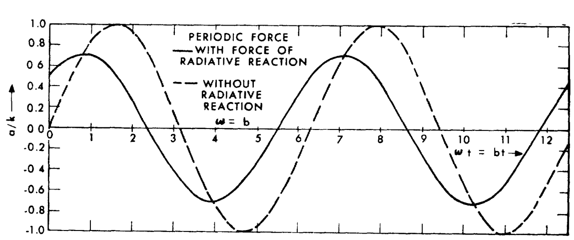

FIG. 3. Acceleration a of a radiating charged body when a sinusoidal force is applied for the particular case when $\omega=b$.

independent of the frequency of the applied force until ω is of the order of b.

For a particle which moves with a relativistic velocity and is subjected to the force given by Eq. (92) with t replaced by τ, the acceleration and velocity as obtained from Eqs. (27) and (28) are

$$\dot{u}(\tau)=\frac{k}{[1+(\omega/b)^2]^{\frac{1}{2}}}\sin\left(\omega\tau+\tan^{-1}\frac{\omega}{b}\right)\cosh g(\tau),$$
$$u(\tau)=c\sinh g(\tau), \quad (95)$$

where

$$g(\tau)=\sinh^{-1}(u_0/c)-(k/c\omega)[1+(\omega/b)^2]^{-\frac{1}{2}}$$
$$\times\{\cos[\omega\tau+\tan^{-1}(\omega/b)]-[1+(\omega/b)^2]^{-\frac{1}{2}}\}.$$

(f) Force Proportional to $\exp(-t^2)$

Next, let us consider the motion of a charged particle which is acted upon by the force

$$f(t)=\frac{km}{(2\pi\sigma^2)^{\frac{1}{2}}}\exp\left[-\frac{(t-t_0)^2}{2\sigma^2}\right]. \quad (96)$$

This represents a pulse of radiation which acts on the particle at the time t_0 and has a duration of the order of σ. In the limit as $\sigma\rightarrow 0$, this force becomes the delta function which is considered in Sec. III.1(a).

The exact nonrelativistic physical solution is found from Eq. (21) to be

$$a(t)=\tfrac{1}{2}bk\exp[\tfrac{1}{2}b^2\sigma^2+b(t-t_0)]$$
$$\times(1-\phi\{2^{-\frac{1}{2}}[b\sigma+\sigma^{-1}(t-t_0)]\}), \quad (97)$$

where

$$\phi(x)=2\pi^{-\frac{1}{2}}\int_0^x\exp(-y^2)dy.$$

When x is negative, the value of the integral is to be taken as negative.

The acceleration is shown in Fig. 4 for the particular case when $\sigma b=1$ and $t_0=0$. The pulse of radiation acts

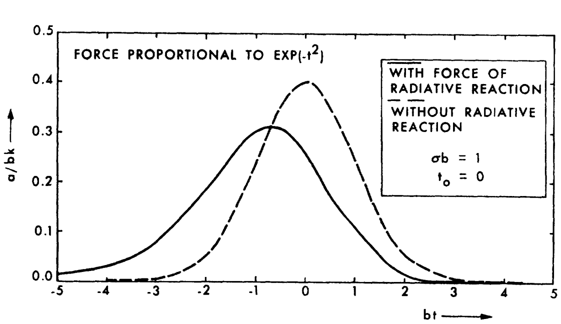

FIG. 4. Acceleration a of a radiating charged body when the applied force is proportional to $\exp(-t^2)$ for the particular case when $\sigma b=1$ and $t_0=0$. These quantities are defined by Eq. (96). The acceleration is no longer symmetrical about the maximum when radiative reaction is included because of the phenomenon of preacceleration.

over a time interval of the order of several times b^{-1}. Because of preacceleration, the curve for the acceleration of the particle is shifted ahead in time compared to the force. The maximum acceleration occurs approximately at the time, $bt \approx -0.75$ before the maximum force is applied. The curve for the acceleration is asymmetrical about its maximum value.

It is instructive to consider the expression for the acceleration in the limit as $\sigma \rightarrow 0$. As this limit is approached, it is found that the leading term in the expression for the acceleration increases exponentially with time until $t=t_0$. On the other hand, when $t>t_0$, the acceleration decreases very rapidly with a term proportional to the force itself. In the limit, the acceleration is zero when $t>t_0$ and only the exponential term due to the pre-acceleration exists when $t<t_0$.

(g) *Force Proportional to* e^{+ct}

An instructive example is obtained when it is assumed that the force varies exponentially with time, so that

$$f(t)=kme^{ct}. \quad (98)$$

When $c>0$, the force increases exponentially indefinitely far into the future and thus is physically unrealizable. The acceleration and velocity of a nonrelativistic charged particle which is acted upon by this force as obtained from Eqs. (21) and (22) is

$$\begin{aligned} a(t)&=[1-(c/b)]^{-1}ke^{ct}, \\ v(t)&=v_0+(k/c)[1-(c/b)]^{-1}(e^{ct}-1), \end{aligned} \quad (99)$$

where v_0 is the velocity at the time $t=0$.

The energy loss from radiation, W, from the time $t=0$ to the time t, is found from Eq. (37) to be

$$W=-\frac{mk}{b[1-(c/b)]}\left\{v_0(e^{ct}-1)+\frac{k(e^{ct}-1)^2}{2c[1-(c/b)]}\right\}. \quad (100)$$

W is chosen to be positive when there is a radiation loss.

In order to understand the physical meaning of these equations, assume that k is positive. Thus the force acts in the direction in which the particle coordinate increases. The same type of analysis can be given when k is negative. Consider the following four cases:

(1) $c<0$. In this case, the force decreases exponentially with time. The acceleration and velocity at a given time as calculated from Eq. (99) are less than the corresponding values when radiation is neglected. There is always a radiation loss; W is positive whenever c is negative, as is shown by Eq. (100).

(2) $c=0$. In this case, the force is constant and the results reduce to those given in Sec. III.1(b).

(3) $0<c<b$. In this case, the force increases exponentially indefinitely far into the future. An acceptable solution to the equations of motion still exists in this case. The third condition for the existence of a solution as given after Eq. (21) is that $|f(t+t')|<Ae^{\gamma t'}$ for large values of t', where $\gamma<b$. This condition is satisfied for the present case; however, this is the most rapid increase in the force at large values of the time which can occur and still have a physical solution to the equations of motion. The acceleration and velocity at a given time are always larger in this case than those calculated without radiative reaction. The radiation loss W is negative so that the particle gains energy from its surroundings. This occurs only because the force increases over an infinite time interval into the future.

(4) $c\geq b$. In this case, a physically acceptable solution should not be expected, since the time rate of variation of the force is too large. Direct substitution of Eq. (99) into the equation of motion shows that this expression is still a formal solution when $c>b$; however, the acceleration of the particle is in the opposite direction from the force. This solution is physically unacceptable.

Cases (3) and (4) represent physically unrealizable forces which increase continually for an indefinite time into the future. Thus, it is not surprising that the particle gains energy from the radiation field or has the acceleration in the opposite direction from the force for certain positive values of c. If the force is zero after some time t_0, as it must be for all actual forces, then there is always a physically acceptable solution for *any* value of c. For example, if

$$f(t)=\begin{cases} kme^{ct}, & t<t_0, \\ 0, & t>t_0, \end{cases} \quad (101)$$

then it is found from Eq. (21) that the acceleration is

$$a(t)=\begin{cases} k[1-(c/b)]^{-1}e^{ct_0}[e^{-c(t_0-t)}-e^{-b(t_0-t)}], \\ \qquad\qquad t<t_0, \\ 0, \qquad\qquad t>t_0. \end{cases} \quad (102)$$

In this case, the solution is physically acceptable for any value of c. If we assume that $k>0$, then the acceleration is always positive, even when $c>b$. If the limit is taken of Eq. (102) as c approaches b, the particular solution is obtained with $c=b$.

The relativistic equations comparable to Eqs. (99) and (102) can readily be obtained from Eqs. (27) and (28). Since they are somewhat lengthy and add nothing new to the discussion already given, they are not presented here.

(h) *Force Proportional to* $|t-t_0|^{-n}$

In Sec. III.1(g) a force is considered which has the strongest allowable singularity as the time approaches infinity. In this section the force is considered which has the strongest allowable singularity at a finite time t_0. Let

$$f(t)=\begin{cases} km/(t_0-t)^n, & t<t_0, \\ -[km/(t-t_0)^n], & t>t_0, \end{cases} \quad (103)$$

where $0 \leq n \leq 1$. This force represents the attraction of a charged body by a point divergent source which is located at the position of the body at $t=t_0$.

For nonrelativistic velocities it is found from Eq. (20) that

$$a(t)=kb^n e^{-b(t_0-t)}\left\{2\int_0^{b(t_0-t)} x^{-n}\sinh x dx - \Gamma[1-n,\, b(t_0-t)]\right\},\quad t<t_0, \tag{104}$$

$$a(t)=-kb^n e^{b(t-t_0)}\Gamma[1-n,\, b(t-t_0)],\qquad t>t_0,$$

where

$$\Gamma(l,x)=\int_x^\infty e^{-y}y^{l-1}dy.$$

As $t\rightarrow\infty$, the limiting value of the acceleration is $-k(t-t_0)^{-n}$. There is no term in this solution which diverges exponentially. Thus, this result represents a physically acceptable solution for the acceleration for any n such that $0\leq n\leq 1$. When $n=1$, the Cauchy principal value should be used for the divergent integrals. The strongest allowable infinity for an attractive force which can occur at a finite time and still have a physically acceptable nondivergent solution of the equations of motion occurs when $n=1$ in Eq. (103).

2. One-Dimensional Motion for Space-Dependent Forces

When the force is an explicit function of time, a general physical solution has been given for the equations of motion. The problem of finding a solution is more difficult when the force is an explicit function of position. Even when the force of radiative reaction is omitted from the equations of motion, an exact expression for the position of the particle can be obtained for only a small number of space-dependent forces. Thus, for space-dependent forces an exact solution to the equations of motion including radiative reaction can be expected in only a few cases.

(a) *Harmonic Oscillator*

An exact solution can be obtained for the problem of the radiating harmonic oscillator in the nonrelativistic limit. When the force is given by the expression

$$f(x)=-\alpha m x, \tag{105}$$

an exact solution of the equation of motion,

$$d^2x/dt^2-(1/b)d^3x/dt^3=-\alpha x, \tag{106}$$

exists in the form

$$x=A\sin\beta bt e^{-\gamma bt}. \tag{107}$$

There is no loss in generality in assuming that the particle is at the origin of coordinates at the time $t=0$. The relations between the parameters α, β, and γ are

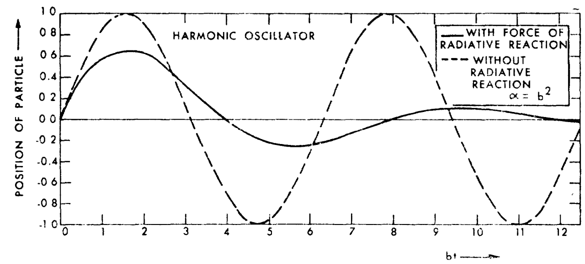

FIG. 5. Motion of a radiating harmonic oscillator when the force constant α as defined by Eq. (105) is equal to b^2. The motion is damped and the period is increased when radiative reaction is included in the equations of motion.

found by the substitution of Eq. (107) into Eq. (106). Thus,

$$\gamma^3+\gamma^2+\tfrac{1}{4}\gamma-\tfrac{1}{8}\alpha b^{-2}=0,\qquad \beta^2=\gamma(2+3\gamma). \tag{108}$$

The exact solution of this equation is

$$\gamma=-\frac{1}{3}+\frac{1}{6}\left[1+\frac{27}{2}\frac{\alpha}{b^2}+\left(\frac{27\alpha}{b^2}\right)^{\frac{1}{2}}\left(1+\frac{27\alpha}{4b^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}+\frac{1}{6}\left[1+\frac{27}{2}\frac{\alpha}{b^2}-\left(\frac{27\alpha}{b^2}\right)^{\frac{1}{2}}\left(1+\frac{27\alpha}{4b^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}. \tag{109}$$

A study of this expression shows that γ is real and positive for any positive value of α. Thus, the motion of the oscillator is always damped by the radiation of energy. Once γ has been calculated for a particular case, then the corresponding value of β is found from Eq. (108).

It is interesting to obtain the limiting solutions for small and large values of the force constant. If $\alpha\ll b^2$, it is found from either Eq. (108) or (109) that

$$x=A\sin\alpha^{\frac{1}{2}}t\exp[-\tfrac{1}{2}\alpha b^{-1}t]. \tag{110}$$

This expression agrees with the usual result for the radiation from an harmonic oscillator.[9] The frequency is not changed in first approximation and the motion is slowly damped compared to the frequency of the oscillation.

On the other hand, when $\alpha\gg b^2$, it is found that

$$x=A\sin[\tfrac{1}{2}\sqrt{3}(\alpha b)^{\frac{1}{3}}t]\exp[-\tfrac{1}{2}(\alpha b)^{\frac{1}{3}}t]. \tag{111}$$

In the limit of an exceptionally strong restoring force, not only is the frequency altered appreciably, but the particle radiates all except a minute fraction of its energy in a single oscillation.

The motion of a radiating harmonic oscillator for the intermediate case when $\alpha=b^2$ is shown in Fig. 5. In this particular case the values of the constants are $\beta=0.793$ and $\gamma=0.233$. The rapid damping of the motion and the decrease in the frequency when radiative reaction is included in the equations of motion is shown in Fig. 5.

[9] E.g., W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, England, 1936).

(b) *Linear Potential Wall*

An interesting problem which can be solved exactly is the reflection of a charged particle by a constant repulsive force which acts only when the coordinate x is greater than a certain value. In particular, let us assume that

$$f(x)=\begin{cases}0 & x<0,\\ -mk, & x>0,\end{cases} \tag{112}$$

where $k>0$. Thus, there is no force which acts on the particle when it is to the left of the origin, while a linearly increasing potential acts to the right of the origin. Let the velocity of the particle be v_0 as $x\rightarrow -\infty$, so that the particle is initially moving in the direction of increasing x. Choose the origin of time so that $t=0$ when $x=0$. The particle is then reflected by the linear potential wall and returns to the origin at some time $t=t_0$.

The nonrelativistic equations of motion are

$$\begin{aligned} a_1-b^{-1}\dot{a}_1&=0, && t<0,\\ a_2-b^{-1}\dot{a}_2&=-k, && 0<t<t_0,\\ a_3-b^{-1}\dot{a}_3&=0, && t>t_0,\end{aligned} \tag{113}$$

where a_1, a_2, and a_3 are the accelerations of the particle for the three time intervals indicated. The exact physical solution of these equations is

$$\begin{aligned} a_1&=-k(1-e^{-bt_0})e^{bt}, && t<0,\\ a_2&=-k[1-e^{-b(t_0-t)}], && 0<t<t_0,\\ a_3&=0, && t>t_0.\end{aligned} \tag{114}$$

In order to show that a physical solution always exists, it is necessary to solve for the position as a function of time. The result of the integration of Eq. (114) is that

$$\begin{aligned} x_1&=v_0t-kb^{-2}(1-e^{-bt_0})(e^{bt}-1), && t<0,\\ x_2&=v_0t-kb^{-1}t-\tfrac{1}{2}kt^2+kb^{-2}e^{-bt_0}(e^{bt}-1), && 0<t<t_0,\\ x_3&=(v_0-kt_0)(t-t_0), && t>t_0.\end{aligned} \tag{115}$$

Before the particle reaches the potential wall, it travels with a constant velocity, v_0, until this is modified by the effects of preacceleration. This occurs over a time interval of the order of several times b^{-1} before the particle reaches the origin. After crossing the origin, the particle slows down under the action of the potential wall, turns around, and reaches the origin again at the time t_0. The value of t_0 is determined from the solution of the transcendental equation

$$(1-bk^{-1}v_0)bt_0+\tfrac{1}{2}b^2t_0^2=1-e^{-bt_0}. \tag{116}$$

Whenever $v_0>0$, the initially free particle can reach the origin and there always exists a positive value of t_0 which satisfies Eq. (116). This follows from the fact that the right-hand side of Eq. (116) considered as a function of t_0 decreases in value monotonically from 1 to 0 as t_0 increases from the value 0, while the left-hand side increases monotonically from 0 to values greater than 1 as t_0 covers the same range of values. Thus a solution of this equation always exists for any positive values of k and v_0. Similarly it can be shown that the velocity and acceleration of the particle always have the correct sign, so that the preceding equations always represent a satisfactory physical solution for the motion.

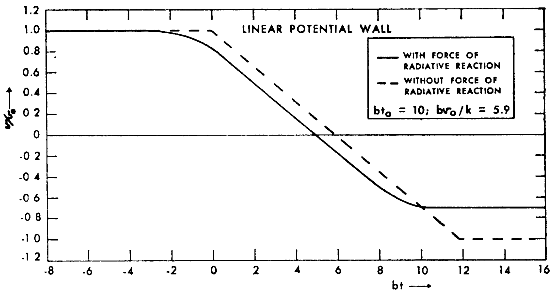

FIG. 6. Velocity of a radiating charged body which is reflected from a linear potential wall with the force given by Eq. (112) for the particular case when $bt_0=10$ and $bv_0/k=5.9$. The final velocity is $-0.695v_0$, where v_0 is the initial velocity.

The kinetic energy lost by the particle can be calculated from Eq. (37) and is found to be

$$T-T_0=-mkt_0(v_0-\tfrac{1}{2}kt_0). \tag{117}$$

The dimensionless parameter which determines the strength of the interaction is bv_0/k. It is instructive to study the energy loss as this parameter increases from a very small to a very large value. This corresponds to gradually reducing the strength of the repulsive force. When $bv_0/k\ll 1$, it is found from Eqs. (116) and (117) that almost all of the kinetic energy is radiated away. This corresponds to an exceedingly strong repulsive force. On the other hand, when $bv_0/k\gg 1$, it is found from the same equations that

$$(T-T_0)/T_0=-(4k/bv_0). \tag{118}$$

In this limit the fractional energy loss decreases as the strength of the repulsive force k decreases and as the initial velocity v_0 increases.

The velocity of a radiating charged particle acted upon by the force given in Eq. (112) is shown in Fig. 6 for the particular case when $bv_0/k=5.9$ or $bt_0=10$. The phenomenon of preacceleration is again clearly evident in this solution. The final velocity in this particular case is equal to $-0.695v_0$, where v_0 is the initial velocity. This value checks with the energy lost by radiation as calculated from Eq. (117).

An exact relativistic solution can also be given for the force given by Eq. (112). For example, the velocity is given by

$$\begin{aligned} u_1&=c\sinh\{\sinh^{-1}(u_0/c)\\ &\qquad -(k/b)(1-e^{-b\tau_0})e^{b\tau}\}, && \tau<0,\\ u_2&=c\sinh\{\sinh^{-1}(u_0/c)\\ &\qquad -k\tau-(k/b)[1-e^{-b(\tau_0-\tau)}]\}, && 0<\tau<\tau_0,\\ u_3&=c\sinh\{\sinh^{-1}(u_0/c)-k\tau_0\}, && \tau>\tau_0.\end{aligned} \tag{119}$$

The proper time τ_0 at which the particle returns to the origin is the solution of the equation

$$\int_0^{\tau_0} \sinh\left\{\sinh^{-1}\left(\frac{u_0}{c}\right)-k\tau -\frac{k}{b}[1-e^{-b(\tau_0-\tau)}]\right\}d\tau=0. \quad (120)$$

Since $u_2>0$ at $\tau=0$, the integrand is positive at $\tau=0$. On the other hand, as $\tau\rightarrow\tau_0$ the integrand becomes negative provided that τ is sufficiently large. Thus, it is always possible to find a value of τ_0 so that the integral has the required value of zero.

(c) *Potential Well*

Another problem which can be solved exactly is when the force is constant inside a given spatial region and zero elsewhere. Let

$$f(x)=\begin{cases}0, & x<0,\\ mk, & 0<x<x_0,\\ 0, & x>x_0.\end{cases} \quad (121)$$

As the initial condition, let the velocity of the particle be v_0 for large negative values of the position. The particle initially moves in the direction of increasing x and the origin of time is chosen so that $t=0$ when $x=0$. Let t_0 be the time when the particle reaches the point x_0. The solution of the nonrelativistic equations of motion is

$$\begin{aligned} a_1&=k(1-e^{-bt_0})e^{bt}, && t<0,\\ a_2&=k[1-e^{-b(t_0-t)}], && 0<t<t_0,\\ a_3&=0, && t>t_0 \end{aligned} \quad (122)$$

and

$$\begin{aligned} x_1&=v_0t+\left(\frac{k}{b^2}\right)(1-e^{-bt_0})(e^{bt}-1), && t<0,\\ x_2&=v_0t+\left(\frac{k}{b}\right)t+\frac{1}{2}kt^2-\left(\frac{k}{b^2}\right)e^{-bt_0}(e^{bt}-1), && 0<t<t_0,\\ x_3&=x_0+(v_0+kt_0)(t-t_0), && t>t_0. \end{aligned} \quad (123)$$

The solution of the transcendental equation

$$\tfrac{1}{2}kb^2t_0^2+(k+bv_0)bt_0-b^2x_0-k(1-e^{-bt_0})=0 \quad (124)$$

determines the time t_0. For an attractive force in the well ($k>0$), a solution for t_0 from Eq. (124) always exists such that $t_0>0$. Similarly, the velocity and acceleration of the particle always have the correct sign for a sensible physical solution of the problem.

For a repulsive force in the well ($k<0$), a solution of Eq. (124) always exists such that $t_0>0$ when the particle initially has sufficient energy to reach the other side of the barrier at x_0. The energy loss by radiation must be included in this calculation. When $bt_0\ll 1$, the particle can reach x_0 when $4|k|x_0/v_0^2<1$. When $bt_0\gg 1$, the particle can reach x_0 when

$$(2|k|x_0/v_0^2)<1-(2|k|/bv_0)+\cdots.$$

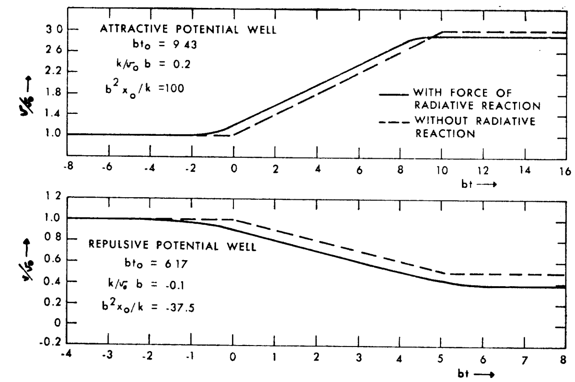

FIG. 7. Velocity of a radiating charged body which passes through either an attractive or a repulsive potential well. The force is defined by Eq. (121). The following particular values of the constants were chosen: attractive potential well, $bt_0=9.43$, $k/v_0b=0.2$, $b^2x_0/k=100$; repulsive potential well, $bt_0=6.17$, $k/v_0b=-0.1$, $b^2x_0/k=-37.5$.

A nonradiating particle has just sufficient energy to reach x_0 when $2|k|x_0v_0^{-2}=1$. The higher-order terms on the right-hand side of the preceding inequality represent the effect of radiation. When this inequality is not satisfied, the particle turns around before it reaches x_0 and moves in the direction of negative x. This case has already been treated in detail in Sec. III.2(b).

Particular solutions for attractive and repulsive potential wells are illustrated in Fig. 7. For the attractive potential well the following values of the parameters were chosen: $bt_0=9.43$, $k/v_0b=0.2$, $b^2x_0/k=100$; for the repulsive potential well: $bt_0=6.17$, $k/v_0b=-0.1$, $b^2x_0/k=-37.5$. For the attractive well, the velocity at a given time is greater during most of the interval when the particle is in the well when radiative reaction is included in the calculation than when it is not. This occurs because of the phenomenon of preacceleration; however, the final velocity is less with radiative reaction than without since the particle must lose energy by radiation. For the repulsive well, the velocity at a given time is always less with radiative reaction than without it. Thus, the final velocity is also less and the particle loses energy by radiation.

The solution for relativistic velocities is the same as given by Eq. (119) if the sign of k is changed. There are no new features of the problem which are introduced in this solution.

(d) *Field of a Thin Infinite Charged Plate*

From the examples already considered it seems reasonable to conclude that a physical solution can be found for any space-dependent force which has a different constant value in each of some finite number of regions of space. One more example of such a force must be discussed in detail since Eliezer[7] has claimed that only nonphysical solutions exist for this problem. The force due to an infinite, thin charged plate located in the yz plane and at the origin of the x coordinate is

$$f(x)=\begin{cases} km, & x<0, \\ -km, & x>0, \end{cases} \tag{125}$$

where k is a constant.

A particle in this force field oscillates with steadily decreasing amplitude until all of its energy has been lost by radiation. Let the particle start at the origin at the time t_0 with a positive velocity. Let the acceleration be a_1 in the time interval $t_0 \le t \le t_1$, where t_1 is the time at which the particle returns to the origin for the first time. Similarly a_n is the acceleration in the time interval $t_{n-1} \le t \le t_n$, where t_n is the time at which the particle returns to the origin for the nth time. The differential equation describing the nonrelativistic motion of the particle is

$$a_n - b^{-1}\dot{a}_n = (-1)^n k, \tag{126}$$

where the sign of the right-hand term depends on the time interval.

The physical nondivergent solution of Eq. (126) is

$$a_n(t) = (-1)^n k + 2k \sum_{i=n}^{\infty} (-1)^{i+1} e^{-b(t_i - t)}, \tag{127}$$

$$v_n(t) = v_n(t_{n-1}) + (-1)^n k(t-t_{n-1}) + 2kb^{-1} \sum_{i=n}^{\infty} (-1)^{i+1}[e^{-b(t_i-t)} - e^{-b(t_i-t_{n-1})}], \tag{128}$$

$$x_n(t) = (t-t_{n-1})v_n(t_{n-1}) + \tfrac{1}{2}(-1)^n k(t-t_{n-1})^2 + 2kb^{-2} \sum_{i=n}^{\infty} (-1)^{i+1}[e^{-b(t_i-t)} - e^{-b(t_i-t_{n-1})}] - 2kb^{-1}(t-t_{n-1}) \sum_{i=n}^{\infty} (-1)^{i+1} e^{-b(t_i-t_{n-1})}. \tag{129}$$

Since in every time interval the inequality $t \le t_n$ holds, it follows that the exponents of all of the exponential terms in the preceding equations are negative. Thus there are no exponentially increasing or divergent motions. The particle loses energy as it oscillates.

The numerical values for the times at which the particle returns to the origin are obtained by setting the the right-hand side of Eq. (129) equal to zero. In any practical problem, $bt_1 \gg 1$, in which case the value of the last two terms in Eq. (129) is very small compared to the first two terms. In this case the method of successive approximations provides a sequence of values which converges rapidly to the correct value for t_n. On the other hand, when $bt_1 \ll 1$, it can be shown that the equations always have a physical solution by expanding the exponentials in a power series.

The velocity of a particle as a function of time which is acted upon by the force given in Eq. (125) is shown in Fig. 8 for the particular case when $v_0 b/k = 4.8$. In this

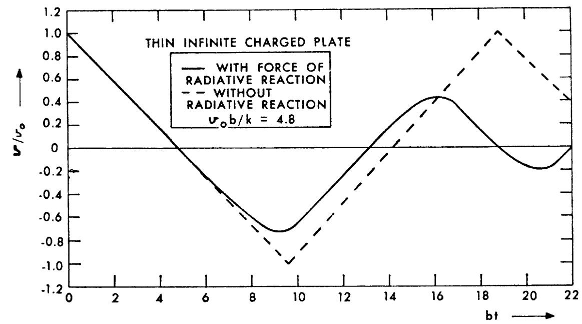

FIG. 8. Velocity of a radiating charged body as a function of time in the field of a thin infinite charged plate for the particular case when $v_0 b/k = 4.8$.

example the rapid damping of the motion due to radiation is clearly evident. There is no evidence of instability or of nonphysical behavior when the solution is computed numerically.

The solution of the same problem at relativistic velocities is so similar that the details need not be given. For example, the velocity is

$$u_n(\tau) = c \sinh\{\sinh^{-1}[c^{-1}u_n(\tau_{n-1})] + (-1)^n k(\tau-\tau_{n-1}) + 2kb^{-1} \sum_{i=n}^{\infty} (-1)^{i+1}[e^{-b(\tau_i-\tau)} - e^{-b(\tau_i-\tau_{n-1})}]\}. \tag{130}$$

The position can be written only as the integral of this expression. As a consequence of this, the equations which determine τ_n can be expressed only in integral form; however, since the integrand changes sign in the interval, a solution for τ_n always exists as was discussed in Sec. III.2(b).

3. Three-Dimensional Motion

(a) *Nonrelativistic Equation of Motion*

When the force is an explicit function of the time, the general solution for the acceleration of a particle which is moving at nonrelativistic velocities in three dimensions is given by Eq. (41). This equation has the same form as Eq. (21) for one-dimensional motion, except for the introduction of vector quantities. If the three components of the force have the same form as those in any of the examples given in Sec. III.1, then the complete three-dimensional solution can be written down at once. For example, for a constant force,

$$\mathbf{f}(t) = m\mathbf{k}, \tag{131}$$

where $\mathbf{k}$ is a constant vector, the solution to Eq. (41) is

$$\mathbf{a} = \mathbf{k}, \quad \mathbf{v} = \mathbf{v}_0 + \mathbf{k}t. \tag{132}$$

These equations have the same form as the one-dimensional solution given by Eq. (84). Similarly, any of the examples in Sec. III.1 may be generalized to the case of three-dimensional motion by making the constant k a vector quantity.

(b) *Uniform Magnetic Field*

The motion of a radiating particle in a uniform magnetic field involves a velocity dependent force and thus is essentially different from the previous examples. An exact, physical solution can be obtained when the velocities are nonrelativistic. If we assume that the magnetic field **H** is in the z direction, then the equations of motion are

$$\begin{aligned}(dv_x/dt)-b^{-1}(d^2v_x/dt^2)&=\omega v_y,\\ (dv_y/dt)-b^{-1}(d^2v_y/dt^2)&=-\omega v_x,\\ (dv_z/dt)-b^{-1}(d^2v_z/dt^2)&=0,\end{aligned} \tag{133}$$

where

$$\omega=eH/mc.$$

The exact nondivergent solution of these equations can be written in the form

$$\begin{aligned}v_x&=v_{x0}e^{-\alpha t}\cos\beta t,\\ v_y&=-v_{x0}e^{-\alpha t}\sin\beta t,\\ v_z&=v_{z0},\end{aligned} \tag{134}$$

where the phase factor is chosen so that $v_x(t=0)=v_{x0}$ and $v_y(t=0)=0$. The value of the constants α and β as determined by substitution in the original differential equation is

$$\begin{aligned}\alpha&=\tfrac{1}{2}b\{[\tfrac{1}{2}+\tfrac{1}{2}(1+16b^{-2}\omega^2)^{\frac{1}{2}}]^{\frac{1}{2}}-1\},\\ \beta&=\tfrac{1}{2}b[-\tfrac{1}{2}+\tfrac{1}{2}(1+16b^{-2}\omega^2)^{\frac{1}{2}}]^{\frac{1}{2}}.\end{aligned} \tag{135}$$

In most practical problems $b^{-1}\omega\ll 1$, in which case

$$\begin{aligned}\alpha&=b^{-1}\omega^2(1-5b^{-2}\omega^2+\cdots),\\ \beta&=\omega(1-2b^{-2}\omega^2+\cdots).\end{aligned} \tag{136}$$

Thus the coefficient α, which determines the exponential decay of the motion, increases in first approximation as the square of the Larmor frequency ω. The frequency of the circular motion of the particle is determined by β. It is equal to the Larmor frequency until that frequency is equal to a significant fraction of the constant b. Then the actual frequency of the circular motion begins to decrease.

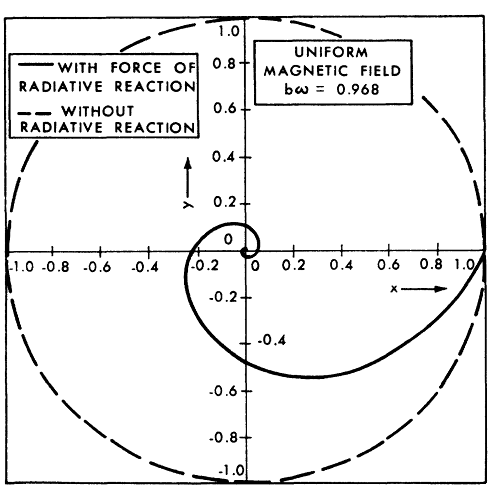

FIG. 9. Motion of a radiating charged body in a uniform magnetic field for the particular case when $b\omega=0.968$.

For a complete understanding of the problem it is also of interest to consider the limiting value when $b^{-1}\omega\gg 1$. In this limit

$$\alpha=\beta=(\tfrac{1}{2}b\omega)^{\frac{1}{2}}. \tag{137}$$

The decay constant and the actual frequency of the motion are equal. The particle radiates most of its energy in a single revolution.

The motion for the particular case when $\omega b=0.968$ is shown in Fig. 9. For this intermediate value of ωb the particle already loses a major fraction of its energy in a single revolution.

The fraction of the original energy lost in one revolution, ϵ, can readily be calculated from the foregoing results. It is found that

$$\epsilon=[1-\exp(-4\pi\alpha\beta^{-1})]\sin^2\theta, \tag{138}$$

where θ is the angle between the direction of the magnetic field and the velocity vector.

The limiting values for the fractional energy loss are

$$\begin{aligned}\epsilon&=4\pi b^{-1}\omega\sin^2\theta, & \omega&\ll b,\\ \epsilon&=(1-e^{-4\pi})\sin^2\theta, & \omega&\gg b.\end{aligned} \tag{139}$$

Thus, when $b^{-1}\omega\ll 1$, the fractional energy loss is essentially determined by the product of the Larmor frequency and the time it takes a light signal to cross the classical electron radius. On the other hand, when $b^{-1}\omega\gg 1$, it is seen that all but an insignificant fraction of the energy is radiated in one revolution. This completes the exact solution of the problem for nonrelativistic energies.

It is not possible to solve the relativistic equations of motion exactly since they involve cross products between the different velocity components. Although most of this article is concerned with the exact solutions of the equations of motion, this particular problem is of sufficient interest to warrant further consideration here. The relativistic equations for motion in the xy plane with a magnetic field in the z direction are found from Eqs. (1) and (2) to be

$$\begin{aligned}\dot{u}_x&=\omega u_y+b^{-1}\ddot{u}_x-b^{-1}c^{-2}u_x\dot{u}_i\dot{u}^i,\\ \dot{u}_y&=-\omega u_x+b^{-1}\ddot{u}_y-b^{-1}c^{-2}u_y\dot{u}_i\dot{u}^i,\end{aligned} \tag{140}$$

where, as in all relativistic equations in this article, dots indicate derivatives with respect to proper time and a repeated index is summed over all four components. A solution to these equations can be obtained by taking one derivative of the first equation in Eq. (140) and substituting for $\dot{u}_y$ from the second equation. The result can be written as

$$\frac{d^2u_x}{d\tau^2}+\omega^2u_x$$

$$=\frac{1}{b}\left\{\frac{d^3u_x}{d\tau^3}+\omega\frac{d^2u_y}{d\tau^2}-\frac{1}{c^2}\left(\frac{du_x}{d\tau}+\omega u_y\right)\frac{du^i}{d\tau}\frac{du_i}{d\tau}-\frac{u_x}{c^2}\frac{d}{d\tau}\left(\frac{du^i}{d\tau}\frac{du_i}{d\tau}\right)\right\}$$

$$\equiv\frac{1}{b}g\left(u_x,\frac{du_x}{d\tau},\frac{d^2u_x}{d\tau^2},\frac{d^3u_x}{d\tau^3},u_y,\frac{du_y}{d\tau},\frac{d^2u_y}{d\tau^2},\frac{d^3u_y}{d\tau^3}\right). \quad (141)$$

An approximate solution of this equation can be obtained by Picard's method, when the right-hand side of the equation is small compared to the individual terms on the left-hand side. The zero-order solution u_{x0} is obtained for the equation when the left-hand side is equated to zero. The same procedure is used for the corresponding equation in u_y. Thus it is found that

$$u_{x0}=A_0\cos\omega\tau,\quad u_{y0}=-A_0\sin\omega\tau. \quad (142)$$

There is no loss of generality in setting the phase factor equal to zero.

In the next approximation

$$u_{x1}=[A_0+b^{-1}A_1(\tau)]\sin\omega\tau, \quad (143)$$

where

$$\frac{dA_1}{d\tau}=-\frac{1}{2\pi}\int_0^{2\pi/\omega}g\left(u_{x0},\frac{du_{x0}}{d\tau},\frac{d^2u_{x0}}{d\tau^2},\frac{d^3u_{x0}}{d\tau^3},u_{y0},\frac{du_{y0}}{d\tau},\frac{d^2u_{y0}}{d\tau^2},\frac{d^3u_{y0}}{d\tau^3}\right)d\tau \quad (144)$$

and g is the right-hand side of Eq. (141) with the zero-order solution substituted for the components of the velocity and their derivatives. Thus it is found that

$$u_{x1}=A_0[1-\omega^2b^{-1}(1+c^{-2}A_0^2)\tau]\sin\omega\tau,\quad u_{y1}=-A_0[1-\omega^2b^{-1}(1+c^{-2}A_0^2)\tau]\cos\omega\tau. \quad (145)$$

This solution is valid as long as the second term is small compared to the first. It is an example of a series solution in powers of b^{-1}.

This result can be used to compute the energy loss per revolution, ΔE, as long as the particle does not lose a major fraction of its energy in one revolution. The proper time required for one revolution is $2\pi/\omega$ and the radius of curvature R is equal to u/ω, where u is the total proper velocity of the particle. Thus, when $\omega\ll b$, it is found that

$$\frac{\Delta E}{mc^2}=\frac{2\pi\omega}{b}\frac{u^2}{c^2}\left(1+\frac{u^2}{c^2}\right)^{\frac{1}{2}}=\frac{8\pi e^2}{3mc^2}\frac{u^3}{Rc^3}\left(1+\frac{u^2}{c^2}\right)^{\frac{1}{2}}. \quad (146)$$

In terms of the relativistic energy E of the particle, the energy loss per revolution is

$$\Delta E=\frac{4\pi}{3}\frac{e^2}{R}\frac{E}{mc^2}\left[\left(\frac{E}{mc^2}\right)^2-1\right]^{\frac{3}{2}}. \quad (147)$$

From the derivation, it follows that this result is valid at any particle energy provided that $\omega=eH/mc\ll b$.

At relativistic energies, Eq. (147) reduces to

$$\Delta E=\frac{4\pi}{3}\frac{e^2}{R}\left(\frac{E}{mc^2}\right)^4, \quad (148)$$

which agrees with the result previously obtained by Schwinger.[10]

The limiting expression for the energy loss at nonrelativistic energies as obtained from Eq. (147) agrees with Eq. (139) when $\omega\ll b$.

(c) *Oscillating Electromagnetic Field*

In this section another example is considered where only approximate solutions to the equations of motion can be obtained. The problem is the scattering of an electromagnetic wave by a charged particle. Let the electric field be in the x direction and the magnetic field be in the y direction so that

$$E_x=E_0\sin\omega t,\quad H_y=E_0\sin\omega t. \quad (149)$$

It is assumed that the amplitude of the oscillation of the particle is small compared to the wavelength of the radiation so that the spatial variation of the electromagnetic wave need not be considered. This is equivalent to requiring that the magnitude of the field E_0 be small compared to the electric field of a point electron at a distance equal to the classical electron radius.

The equations of motion of the charged particle at nonrelativistic velocities are

$$(dv_x/dt)-b^{-1}(d^2v_x/dt^2)=\mathcal{E}\sin\omega t-(v_z\mathcal{E}/c)\sin\omega t,$$
$$(dv_y/dt)-b^{-1}(d^2v_y/dt^2)=0, \quad (150)$$
$$(dv_z/dt)-b^{-1}(d^2v_z/dt^2)=(v_x\mathcal{E}/c)\sin\omega t,$$

where $\mathcal{E}=eE_0/m$. The y component of the velocity can be obtained immediately as

$$v_y=v_{y0}, \quad (151)$$

[10] J. Schwinger, Phys. Rev. **75**, 1912 (1949).

where v_{y0} is the initial component of the velocity in that direction.

Since it is assumed that $v_x \ll c$ and $v_z \ll c$, the equations may be solved by first obtaining a zero-order solution v_{x0} for the x component of the velocity from the equation

$$(dv_{x0}/dt) - b^{-1}(d^2v_{x0}/dt^2) = \mathcal{E}\sin\omega t. \tag{152}$$

The physically acceptable solution of this equation is

$$v_{x0} = \frac{\mathcal{E}}{b(1+b^{-2}\omega^2)}\left(\sin\omega t - \frac{b}{\omega}\cos\omega t\right). \tag{153}$$

This solution is then substituted into the right-hand side of the last of Eqs. (150). The zero-order approximation for the z component of the velocity v_{z0} is then obtained. It is found that

$$v_{z0} = \frac{\mathcal{E}^2 t}{2bc(1+b^{-2}\omega^2)} + \frac{(1-2b^{-2}\omega^2)\mathcal{E}^2(\cos 2\omega t - 1)}{4c\omega^2(1+b^{-2}\omega^2)(1+4b^{-2}\omega^2)} - \frac{3\mathcal{E}^2\sin 2\omega t}{4bc\omega(1+b^{-2}\omega^2)(1+4b^{-2}\omega^2)}. \tag{154}$$

This iteration procedure may be repeated any number of times. The next step would be to substitute this value of v_{z0} back into the differential equation for v_x in order to obtain the next approximation for v_x; however, the approximate solution just given is all that is needed in order to determine the leading term in the expression for the scattering cross section of electromagnetic radiation by a charged particle. The cross section is equal to the average energy loss of the charged particle per unit time divided by the incident electromagnetic energy per unit area and time. In the preceding approximation only v_{z0} has a term whose average over a complete cycle is different from zero. Thus, in the present notation,

$$\sigma = 8\pi m E_0^{-2}\langle \dot{v}_{z0}\rangle_{\mathrm{av}}. \tag{155}$$

From Eq. (154), it is found that

$$\langle \dot{v}_{z0}\rangle_{\mathrm{av}} = \frac{\mathcal{E}}{[2bc(1+b^{-2}\omega^2)]}. \tag{156}$$

Thus, the scattering cross section is

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{1}{1+b^{-2}\omega^2}. \tag{157}$$

The last term represents the radiative correction to the usual classical expression for the scattering cross section.

A large number of approximate solutions for the motion of a radiating charged particle have been given in the literature for a variety of different forces. A representative sample of such problems together with their approximate solutions has been given by Landau and Lifshitz.[6]

(d) *Motion along a Skew Curve with Constant Speed*

An exact solution for the relativistic motion along a skew curve can be obtained for certain particular oscillating electric and magnetic fields when the speed of the particle is constant. This provides an interesting application of the intrinsic equations of motion discussed in Sec. II.6(b).

If the speed of the particle, U, is assumed to be constant, it is found from Eqs. (58)–(60) that

$$E_t = \frac{2ek_1^2U^3}{3c^3}\left(1+\frac{U^2}{c^2}\right)^{\frac{1}{2}}, \tag{158}$$

$$k_1 = \frac{e}{mU^2}\left[\frac{U}{c}H_b - \left(1+\frac{U^2}{c^2}\right)^{\frac{1}{2}}E_n\right], \tag{159}$$

$$k_2 = \frac{b}{U}\,\frac{UH_n + c[1+(U/c)^2]^{\frac{1}{2}}E_b}{UH_b - c[1+(U/c)^2]^{\frac{1}{2}}E_n}, \tag{160}$$

where the subscripts t, n, and b refer to the components of the electric and magnetic fields along the tangent, normal, and binormal, respectively. The required electric and magnetic fields can be obtained from these equations in order that the particle may move at a constant speed along a skew curve which has a constant curvature and torsion (k_1 and k_2, respectively).

For example, if all the components of the electric and magnetic fields are zero except E_t and H_b, the particle moves in a circular path at constant speed. The energy which the particle loses by radiation is exactly compensated by the energy gained from the tangential electric field. A constant tangential field could be supplied by a rotating electric field which has a frequency equal to the Larmor frequency of the particle in the magnetic field H_b.

Whenever either H_n or E_b is different from zero, the particle follows a skew curve with a constant speed since the torsion k_2 is then different from zero. For any desired values of U, k_1, and k_2, possible values for the components of the electric and magnetic fields can be found from Eqs. (158)–(160). These components must remain constant at the position of the particle. This can be accomplished if the fields oscillate with the Larmor frequency.

This is probably one of the few cases where an exact relativistic solution to the equations of motion can be obtained for a skew trajectory. Thus, at least in this one case, there are exact nondivergent solutions for a particle moving at relativistic velocities along a skew curve.

4. Singular Forces

(a) *General Discussion*

It would seem reasonable that the actual physical force on a particle in our universe must always remain finite and cannot assume an infinite value. According to this viewpoint, singular forces arise only from mathematical abstractions which do not take into account all of the relevant physical factors. Nevertheless, singular forces, e.g., the Coulomb force, often provide an accurate representation of various types of physical interactions over a wide range of the appropriate physical parameters. Therefore the solutions of the equations of motion for a radiating charged particle acted upon by a singular force should be studied; however, we need not be surprised if there is no physical solution to the problem in certain particular cases.

As has been shown in the preceding sections, a physical solution to the equations of motion always exists when the force considered as a function of time is bounded. In classical mechanics a particle is acted upon only by the force along the actual trajectory of the particle. Thus, even with a singular force, if the particle does not pass through the spatial point which contains the singularity, the force along the trajectory is everywhere finite. In this case a physical solution of the equations of motion may be presumed to exist. This presumption is verified in the following sections by actual calculation of the trajectories for the cases of three-dimensional attractive and repulsive Coulomb fields. Thus it is reasonable to assume that a physically acceptable solution always exists for a repulsive singular force, since a particle moving along the classical trajectory can never reach the singularity.

Some examples have already been given of cases where the particle passes directly through a singularity in the force field. In Sec. III.1(h) a force proportional to $|t_0-t|^{-n}$ was considered. It was found that a physically acceptable solution could be found as long as $n\leq 1$. At a time t_0, the strongest allowable singularity for an attractive force for which there is still a physically acceptable solution to the equations of motion occurs when $n=1$ in this example. This force can be expressed as a function of position by solving the equations in the neighborhood of the singularity for the position of the particle. In this way it is found that a physical solution exists when an attractive force varies as strongly as x^{-1} near the singularity.

Another type of singular force discussed in Sec. III.1(g) is proportional to e^{+ct}. An acceptable solution exists when $c<b$. This represents the strongest allowable infinity as the time itself approaches infinity. In the limit as the time approaches infinity it is found that the force is proportional to $+x$.

(*b*) *Attractive Coulomb Force*

The scattering of nuclear particles under many conditions can approximately be represented by the trajectory of a body which is acted upon by a Coulomb force; however, this force cannot be used down to arbitrarily small distances, since it is certain that specific nuclear forces come into play and the phenomena can no longer be described by a classical theory. Furthermore, no force center can be regarded as exactly fixed in space with respect to a Galilean frame of reference, since this would require a particle with an infinite mass.

The three-dimensional electric field associated with an attractive Coulomb force is

$$\mathbf{E}=-er^{-3}\mathbf{r}. \tag{161}$$

Now it has been shown in Sec. II.6(c) that the motion is confined to the plane defined by: (1) the initial position vector $\mathbf{r}$ from the center of force to the particle; (2) the initial velocity vector.

The relativistic equations of motion obtained from Eqs. (1), (2), and (161) are

$$\frac{du_i}{d\tau}=-\frac{e^2}{mc}\frac{x_iu^4}{r^3}+\frac{1}{b}\left(\ddot{u}_i-\frac{u_i\dot{u}^k\dot{u}_k}{c^2}\right). \tag{162}$$

For simplicity assume that the motion takes place in the xy plane. The nonrelativistic equations of motion are then

$$\begin{aligned}\frac{d^2x}{dt^2}-\frac{1}{b}\frac{d^3x}{dt^3}&=-\frac{e^2x}{m(x^2+y^2)^{\frac{3}{2}}},\\ \frac{d^2y}{dt^2}-\frac{1}{b}\frac{d^3y}{dt^3}&=-\frac{e^2y}{m(x^2+y^2)^{\frac{3}{2}}}.\end{aligned} \tag{163}$$

It is not possible to obtain analytic solutions of these equations. The existence of physical solutions for the attractive Coulomb force can be demonstrated only by numerical techniques and by inference from Sec. II.6(b). There it is shown that a nondivergent solution exists when the force is bounded, is zero after some given time, and is given as an explicit function of time.

Numerical solutions were obtained for three different sets of initial conditions with the aid of an IBM 650 digital computer. First an accurate solution was obtained for a case where the initial energy was nonrelativistic. Because of the nature of the equations, it was necessary to integrate backward in time. When this was done, there were no instabilities whatsoever in the solutions. The initial value of the energy was determined at a point far from the scattering center.

In order to be certain that the relativistic Eq. (162) did not introduce any divergences, the calculations were repeated for the relativistic case, but with a coarser integration interval. No difference in the behavior of the solution was observed.

One of these solutions is shown in Fig. 10 where the initial energy (kinetic plus potential energy) was $0.010228mc^2$ and the final energy was $0.009316mc^2$. When distance is measured in units of two-thirds of the classical electron radius (equal to cb^{-1}), the particle comes within 33 units of the force center; it radiates 8.9% of its energy. The other cases are so similar that they are not given here.

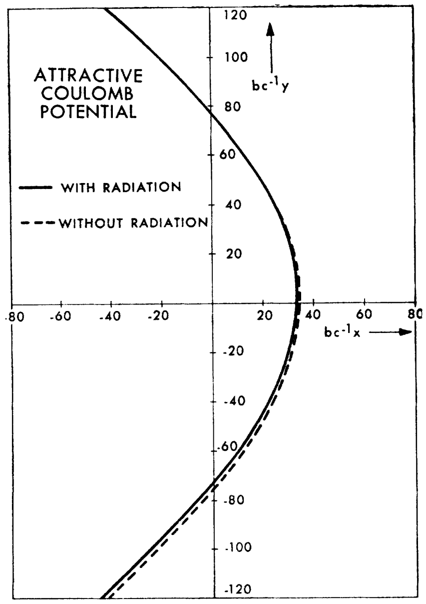

FIG. 10. Motion of a radiating charged particle which is acted upon by an attractive Coulomb force with its center located at the origin of the coordinates.

From these examples it is reasonable to assume that a physical solution exists for any two-dimensional motion under the influence of an attractive Coulomb force. The force is finite everywhere along the trajectory of the particle and therefore physical solutions to the equations of motion exist according to the discussion in Sec. II.

On the other hand, Eliezer[11] has given a mathematical proof that no physical solution exists for the one-dimensional problem where the particle moves directly toward the center of force. He assumes that the solution can be expressed by ordinary functions. The difficulty in this case occurs because the singularity in the force lies along the trajectory of the particle and this singularity is too strong to satisfy the criteria discussed in Sec. II for the existence of a physical solution. This is an artificial problem in the sense that: (1) a particle could not be projected exactly toward the center of the Coulomb force; (2) the Coulomb force does not extend down to arbitrarily small distances. Furthermore, it should be noted that there is always a physical solution to the equations of motion if the Coulomb force is cut off at some distance, no matter how small.

The solution to the mathematical problem of the one-dimensional motion under the influence of an attractive Coulomb force has recently been obtained by Clavier.[12] He has obtained a satisfactory solution in terms of the generalized functions known as distributions.[13] If the solutions are restricted to ordinary functions, there are none which actually reach the origin; however, many differential equations may be solved satisfactorily only when distributions are included among the admissible solutions. When this is done for the one-dimensional Coulomb problem, a satisfactory solution is obtained in which the particle reaches the origin. The velocity continually increases as the particle approaches the origin, but has a finite value at the origin. The discontinuity in the velocity at the origin arises from the strong infinity of the Coulomb force.

There has been some confusion in the literature in regard to the number of different cases for which Eliezer has provided a mathematical proof that only run-away solutions exist. A study of his papers shows that the *only* case where he has shown that physically acceptable solutions do not exist (when the solutions are restricted to ordinary functions) is for one-dimensional motion in an attractive Coulomb potential. In all other cases (three-dimensional motion in an attractive Coulomb potential and one- and three-dimensional motion in a repulsive Coulomb potential) he has merely shown that divergent solutions exist. These run-away solutions exist for every potential. In addition a physically acceptable nondivergent solution exists in each of these cases as is demonstrated by the numerical calculations reported in this and the following section and the general proof given in Sec. II.6(b).

An important physical problem is the motion of a charged particle in a nearly circular orbit about an attractive Coulomb potential which slowly spirals toward the center as it loses energy by radiation. It is not possible to follow the motion until the particle reaches the center by numerical integration since this would involve following an infinite number of revolutions; however, from the general proof of the existence of relativistic solutions of the equations of motion it follows that a physical solution can be obtained down to any arbitrarily small distance from the center, since the force which acts on the particle is always bounded in this case. In our actual universe the Coulomb force no longer describes the forces which act upon a particle when the distance becomes smaller than some amount of the order of the classical electron radius.

A particular solution can be given for the mathematical problem of spiral motion under the influence of an attractive Coulomb force. Clavier[12] has shown that a logarithmic spiral satisfies the differential equations at very small distances from the origin. The equation of a logarithmic spiral is

$$\begin{aligned} \rho &= \rho_0 e^{-(\cot\phi)\theta}, \\ \rho &= s\cos\phi, \qquad (164) \\ k_1 &= \rho^{-1}\sin\phi = s^{-1}\tan\phi, \end{aligned}$$

where ρ is the radius vector, ϕ is the constant angle between the radius vector and the tangent to the trajectory, θ is the polar angle, s is the arc length along the trajectory measured backwards from zero at the origin, and k_1 is the curvature.

[11] C. J. Eliezer, Proc. Cambridge Phil. Soc. 39, 173 (1943).
[12] P. A. Clavier (private communication, to be published).
[13] L. Schwartz, *Theorie des Distributions* (Hermann, Paris, 1950).

It is found[12] by substitution that the intrinsic equations of motion [Eqs. (58)–(60)] are satisfied in the limit as ρ approaches zero by a constant speed $U=U_0$. The angle which the spiral makes with the radius vector ϕ is found to satisfy the equation

$$4\cos^3\phi=3(1-\cos 2\phi), \qquad (165)$$

while

$$(U_0/c^3)=\tfrac{3}{2}(\sin\phi)^{-1}. \qquad (166)$$

Numerically, $\phi=29°\ 37'$ and $U_0=1.448c$.

A calculation of higher-order terms in the expansion of the solution shows that there are no free constants in the expansion. Thus, this is a particular solution where the particle starts at a given distance from the origin with a particular velocity. In the limit of small values of ρ, the particle has a limiting constant velocity and has a trajectory which is a logarithmic spiral. Thus one physically satisfactory trajectory has been obtained for this problem. The general solution which has not been found as yet either may involve other functions or it may be expressible only in terms of distributions as in the case of one-dimensional motion.

(c) *Repulsive Coulomb Force*

The case of a repulsive Coulomb force field acting upon a charged particle is considered next. As the particle moves along its classical trajectory it can never actually reach the point at which the force center is located. Thus, the force which acts upon the particle is always finite. For this reason, it is presumed that a physical solution always exists for this force field. This has indeed been found to be the case for every numerical example which has been investigated. It should be noted that Eliezer[11] has merely demonstrated that divergent solutions exist near the force center in this case. Since there are always an infinite number of divergent solutions for the equations of motion with radiative reaction, this fact alone does not disprove the existence of an appropriate physical solution in each case. Indeed one nondivergent physical solution does exist for each set of initial conditions.

The equations of motion for a repulsive Coulomb force are given by Eqs. (161)–(163) if the sign of the force is changed. In order to demonstrate the existence of physical solutions to this equation, the trajectories for three different sets of initial conditions were calculated on the IBM 650 digital computer. The same procedure was used that is described in the previous section.

One of these trajectories is shown in Fig. 11. The initial energy (kinetic plus potential energy) is $0.060850mc^2$ and the final energy is $0.060613mc^2$. The particle loses 0.39% of its energy by radiation. When distance is measured in units of two-thirds of the classical electron radius (equal to cb^{-1}), the particle comes within 52 units of the force center. The orbit of the radiating charged particle is very nearly the same as the hyperbolic orbit which is obtained when radiation is neglected.

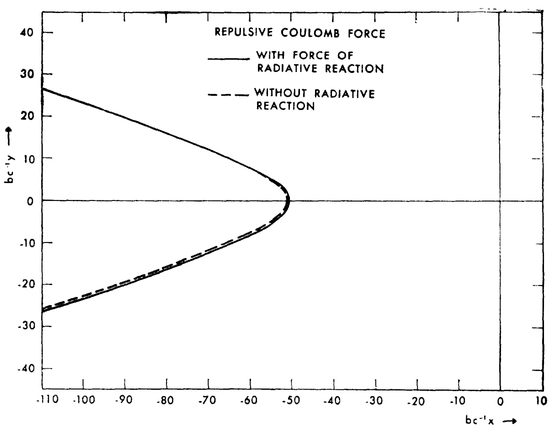

FIG. 11. Motion of a radiating charged particle which is acted upon by a repulsive Coulomb force with its center located at the origin of the coordinates.

A physical solution for a repulsive Coulomb potential exists even when the motion is confined to one dimension. In this case the particle has its velocity vector pointed directly at the center of the force field. Since the particle must turn around before it reaches the origin, the force which acts on the particle is always finite. In order to check this conclusion, the trajectory was obtained by numerical integration for the case when the initial value of u/c is 0.10521 at a distance of 100 units from the force center (with distance measured in units of two-thirds of the classical electron radius). At the same distance from the origin the value of u/c is 0.10500 after the particle has been scattered. The variation of the velocity with distance is shown in Fig. 12. The difference in the velocities when the particle is moving in each direction cannot be shown on the scale of this figure. For this reason, the velocities from 80 to 90 units from the origin are shown on an enlarged scale on the right-hand side of the figure. The particle has a smaller velocity when it is moving away from the origin than when it is moving toward it.

There was no evidence of divergent or unstable solutions obtained during the numerical integration of any of these examples. Furthermore, the force is always finite along the classical trajectory. The general proof of the existence of a physical solution given in Sec. II.6(b) should be valid, since the force may be regarded formally as an explicit function of time. Thus, a physical solution exists for all types of motion under the action of both attractive and repulsive Coulomb forces. For one-dimensional motion with an attractive Coulomb force the solution can be expressed only in terms of distributions rather than ordinary functions.

IV. CONCLUSION

The general physical solution for the equations of motion including radiative reaction has been found

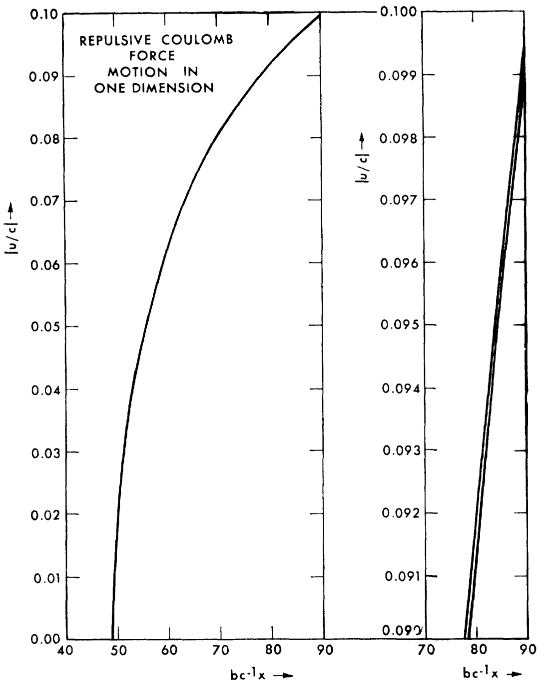

FIG. 12. Motion of a radiating charged particle constrained to move in one dimension and acted upon by a repulsive Coulomb force. The distance from the force center is measured in units of two-thirds of the classical electron radius ($bc^{-1}x$). The left-hand portion of the graph shows the magnitude of the velocity from the point where the particle turns around out to 90 units of distance. On this scale the difference in the velocities when the particle is moving in each direction cannot be shown. The right-hand portion of the graph shows the velocity from 70 to 90 units of distance when the particle is moving toward the origin (upper curve) and away from the origin (lower curve).

when the force is an explicit function of time. The exact solution has been given for the relativistic case and the nonrelativistic limit when the motion is confined to one dimension and for the nonrelativistic limit when the particle is free to move in three dimensions. For the general case of relativistic motion in three dimensions, a set of integral equations has been obtained in both a four-dimensional coordinate system and in intrinsic coordinates. It has been shown that a unique solution of these equations exists over a range of initial velocities. This solution is nondivergent when the force is a bounded, explicit function of time which is zero after a certain time interval has elapsed.

The extra constants of integration in the equations of motion which include the radiative reaction term are determined from a condition that the particle cannot increase its energy over a long period of time by more than the work which is done upon it by the external forces. These constants can always be determined so that a physical solution exists when the force is finite at all points along the particle trajectory. An acceptable solution can also be found for certain classes of forces which are not bounded. An acceptable physical solution has been found for all types of forces which have been investigated and which satisfy the three conditions given after Eq. (21).

Thus, there no longer appears to be any reason for not accepting the equation of motion including the force of radiative reaction, Eq. (1), as an exact equation for a charged point particle within the framework of classical theory.

ACKNOWLEDGMENTS

It is very difficult to give adequate credit to the many scientists who have contributed ideas toward the solution of this problem over the past 15 years. The author is especially indebted to Professor John A. Wheeler for valuable advice and many discussions during the early phases of this work. The general proof of the existence of a solution for three-dimensional relativistic motion as given in Sec. II.6(b) has been provided by Dr. Philippe A. Clavier. He has made several other important contributions to the mathematical theory, particularly in Secs. III.3(d) and 4(b). The general solution of the nonrelativistic equations of motion, Eq. (20), evolved during the course of a series of discussions with Dr. Sylvan Katz who made other useful suggestions. Professor F. Rohrlich first suggested to us the integral given in Eq. (71). The method used for obtaining the perturbation solution of Sec. III.3(b) was originally suggested by Dr. H. M. Watts.