Electron Polarization Operators*

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CONTENTS

I. INTRODUCTION

1. Preliminary Remarks

THAT the polarization of a free electron should be defined as the expectation value of the spin operator $\boldsymbol{\sigma}$ in the Lorentz frame in which the electron is at rest was originally suggested by Darwin.[1] This definition was used in the Mott scattering theory[2] and has been further applied and reviewed by Tolhoek and de Groot.[3] The polarization direction is treated in a way implicitly; for a given linear combination of the two independent electron states of a specific momentum, the polarization direction is determined by the expansion coefficients [see Eq. (6.16)].

However, it has been realized recently that the polarization can also be discussed very well explicitly in terms of certain operators. These operators commute with the Hamiltonian and so can be used to remove the twofold degeneracy that remains after momentum and charge are specified. There are three possible starting points: (1) a four-vector operator shown by Bargmann and Wigner[4] to generate the little group, a subgroup of the group of Lorentz transformations; (2) a three-vector operator defined by Stech[5] which for an electron is $\boldsymbol{\sigma}$ in the direction of the momentum and $\beta\boldsymbol{\sigma}$ otherwise; (3) A scalar operator of the form $i\gamma_5\gamma_\mu n_\mu$, introduced by Michel and Wightman.[6] The relations between these points of view have been discussed to some extent by Bouchiat and Michel,[7] by Werle,[8] and by Good and Rose.[9]

The understanding of the basic properties of these operators seems now to be complete and therefore a résumé of their properties might be of some value. This paper gives a consistent account of the theory of electron polarization, showing the relations between the various approaches, but without going into the applications. The three-vector polarization operator is appropriate for calculations involving plane-wave states, whereas the four-vector polarization operator is convenient for taking account of external electromagnetic fields. This paper is correspondingly divided into two parts.

The theory of electron polarization effects in Mott and Compton scattering has been reviewed by Tolhoek.[3] Calculations of polarization have been made for internal conversion electrons by Becker and Rose,[10] and for beta-decay electrons and positrons by Jackson, Treiman, and Wyld,[11] by Ebel and Feldman,[12] and by Good and Rose.[9] The problem of precession of polarization in external electromagnetic fields has been treated in the small-field limit by Tolhoek[3] and in the classical (non-quantum) approximation by Bargmann, Michel, and Telegdi,[13] starting from the classical equations of motion in the rest frame. A treatment of the classical precession problem from first principles is given in Secs. 16 and 17.

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[3] H. A. Tolhoek and S. R. deGroot, Physica **17**, 1, 17 (1951); see also H. A. Tolhoek, Revs. Modern Phys. **28**, 277 (1956).

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[10] R. L. Becker and M. E. Rose, Nuovo cimento **13**, 1182 (1959).

[11] J. D. Jackson, S. B. Treiman, and H. W. Wyld, Jr., Phys. Rev. **106**, 517 (1957); Nuclear Phys. **4**, 206 (1957).

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[13] V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Letters **2**, 435 (1959).

Another development of the basic theory of the polarization operator and several applications are given by Rose.[14] A review of polarization phenomena and experimental techniques has been given by Page.[15]

2. Notation

Units are used for which $m=c=1$. Latin indexes range from 1 to 3 and Greek indexes from 1 to 4; $x_4=it$. The symbols A^*, $A^\dagger$, and $\tilde{A}$ denote the complex conjugate, Hermitian conjugate, and transpose of any matrix A, respectively. The symbols $\mathbf{e}$ and $\mathbf{s}$ are reserved for unit vectors.

Abstractly, the Dirac matrices are defined by

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}, \quad \text{where} \quad \gamma_\mu{}^\dagger=\gamma_\mu.$$

Auxiliary matrices are defined by

$$\beta=\gamma_4, \quad \boldsymbol{\alpha}=i\beta\boldsymbol{\gamma}, \quad \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4,$$
$$\boldsymbol{\sigma}=i\gamma_4\gamma_5\boldsymbol{\gamma}=-\tfrac{1}{2}i(\boldsymbol{\gamma}\times\boldsymbol{\gamma}).$$

A specific representation that is referred to is

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}, \quad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \gamma_5=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},$$

where the 2×2 $\boldsymbol{\sigma}$ are the usual Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The electromagnetic field is described by

$$A_\mu=(\mathbf{A}, A_4=i\phi), \quad F_{\mu\nu}=\partial A_\nu/\partial x_\mu-\partial A_\mu/\partial x_\nu,$$
$$B_i=\tfrac{1}{2}\epsilon_{ijk}F_{jk}, \quad E_k=iF_{k4}.$$

The Dirac equation is written as

$$H\Psi=i\hbar(\partial/\partial t)\Psi,$$

where the Hamiltonian is given by

$$H=\boldsymbol{\alpha}\cdot(\mathbf{p}-e\mathbf{A})+\beta+e\phi. \tag{2.1}$$

Here, $\mathbf{p}$ is $-i\hbar\boldsymbol{\nabla}$ and e is the actual charge, negative for the electron. Equivalently, one may write

$$(\gamma_\mu\pi_\mu-i)\Psi=0,$$

where $\pi_\mu=p_\mu-eA_\mu$ and p_μ is $-i\hbar\partial/\partial x_\mu$.

The charge conjugation matrix satisfies

$$C^*\gamma_\mu{}^*C=-\gamma_4\gamma_\mu\gamma_4, \quad C^{-1}=C^*=C^\dagger.$$

The charge conjugates of a wave function and an operator are

$$\Psi^C=C^*\Psi^*, \quad \Omega^C=C^*\Omega^*C.$$

[14] M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, New York, 1961).

[15] L. A. Page, Revs. Modern Phys. **31**, 759 (1959).

3. Physical Interpretations

For the free particle, as an alternative to Dirac's hole theory, one may interpret the four solutions of the Dirac equation as actually describing electrons and positrons (instead of electrons alone). One must then assign the operators $|H|$, $(H/|H|)\mathbf{p}$, and $(H/|H|)\times(\mathbf{x}\times\mathbf{p}+\frac{1}{2}\hbar\boldsymbol{\sigma})$, to be the energy, momentum, and angular momentum, respectively. The operator $|H|$ is defined in momentum space by $|H|=(p^2+1)^{\frac{1}{2}}$, the positive root to be taken. This point of view is carried through consistently in what follows. When equations apply nonuniformly to electrons and positrons, the upper signs apply coherently for electrons and the lower for positrons. The plane-wave solutions for the free particle are therefore written as

$$\Psi_{\pm,\lambda}=\psi_{\pm,\lambda}\exp[i\hbar^{-1}(\mathbf{p}\cdot\mathbf{x}-Wt)]$$
$$=\psi_{\pm,\lambda}\exp[\pm i\hbar^{-1}(\mathbf{q}\cdot\mathbf{x}-Et)],$$

where $\mathbf{p}$ and W are the eigenvalues of the operators $\mathbf{p}$ and H, and where $\mathbf{q}$ and E are the eigenvalues of the momentum $(H/|H|)\mathbf{p}$ and the energy $|H|$. Here $\mathbf{p}$ and W satisfy the equation

$$p^2-W^2=-1.$$

The $\pm$ subscript on ψ denotes the sign of W, and λ characterizes the twofold polarization degeneracy. It is clear that the operator $(H/|H|)$ is $+1$ for an electron state and -1 for a positron state. Also, the energy eigenvalues $E=\pm W$ are positive.

The wave equation for the free particle is covariant with regard to charge conjugation. Also, one finds that

$$H^C=-H,$$

so the operators for energy, momentum, and angular momentum are self-charge conjugate:

$$|H|^C=|H|, \quad [(H/|H|)\mathbf{p}]^C=(H/|H|)\mathbf{p},$$

and similarly for the angular momentum. The theory is therefore covariant with regard to charge conjugation both for the wave equation and for the physical assignments. Since $H^C=-H$, one sees that $(H/|H|)^C$ is $-H/|H|$, so the charge conjugate of an electron state with momentum $\mathbf{q}$ is a positron state with momentum $\mathbf{q}$. Finally, it is seen that

$$(\pm\boldsymbol{\alpha}\cdot\mathbf{q}+\beta)\psi_{\pm,\lambda}=\pm E\psi_{\pm,\lambda}$$

is the equation satisfied by the plane wave amplitudes in terms of the physical momentum $\mathbf{q}$ and energy E.

II. THREE-VECTOR POLARIZATION OPERATOR

4. Definition

For the free particle, the three-vector polarization operator is defined as

$$\mathbf{O}=p^{-2}[(\boldsymbol{\sigma}\cdot\mathbf{p})(H/|H|)\mathbf{p}+\mathbf{p}\times(\beta\boldsymbol{\sigma}\times\mathbf{p})]$$
$$=\beta\boldsymbol{\sigma}+p^{-2}(\boldsymbol{\sigma}\cdot\mathbf{p})[(H/|H|)-\beta]\mathbf{p}, \tag{4.1}$$

Thus, for electrons/positrons, the three-vector polarization operator is $\pm\boldsymbol{\sigma}$ in the direction of motion and $\beta\boldsymbol{\sigma}$ perpendicular to the motion. Explicitly writing out the Hamiltonian and expanding, one may alternatively express the defining equation as

$$\mathbf{O}=\beta\boldsymbol{\sigma}-|H|^{-1}\gamma_5\mathbf{p}-[|H|(|H|+1)]^{-1}(\beta\boldsymbol{\sigma}\cdot\mathbf{p})\mathbf{p}. \quad (4.2)$$

One finds that

$$\mathbf{O}^C=C^*\mathbf{O}^*C=\mathbf{O}, \quad (4.3)$$

so the interpretation of $\mathbf{O}$ as the polarization operator also is covariant with respect to charge conjugation.

5. Algebraic Properties

If one introduces a right-handed orthogonal coordinate system $\mathbf{e}_i$ such that $\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij}$, $\mathbf{e}_i\times\mathbf{e}_j=\epsilon_{ijk}\mathbf{e}_k$, then the components of $\mathbf{O}$ in this system, $\mathrm{O}_i=\mathbf{O}\cdot\mathbf{e}_i$, have an algebra similar to that of the Pauli matrices:

$$\mathrm{O}_i\mathrm{O}_j=\delta_{ij}+i\epsilon_{ijk}\mathrm{O}_k(H/|H|), \quad (5.1)$$

where H is the free-particle Hamiltonian.

For any unit vector $\mathbf{s}$, Eq. (5.1) implies

$$(\mathbf{O}\cdot\mathbf{s})^2=1, \quad (5.2)$$

so $\mathbf{O}\cdot\mathbf{s}$ has eigenvalues ±1. Also, it is easily verified that $\mathbf{O}$ is Hermitian, and that any component of $\mathbf{O}$ commutes with the (free-particle) Hamiltonian

$$[\mathbf{O},H]_-=0. \quad (5.3)$$

Therefore $\mathbf{O}$ corresponds to an integral of the motion, and a complete set of eigenfunctions may be found which are simultaneously eigenfunctions of the Hamiltonian and $\mathbf{O}\cdot\mathbf{s}$.

6. Eigenfunctions

Since $\mathbf{O}$ commutes with the Hamiltonian, a complete set of plane-wave eigenfunctions,

$$\Psi_{\epsilon,\lambda}(\mathbf{s})=\psi_{\epsilon,\lambda}(\mathbf{s})\exp[i\hbar^{-1}(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)],$$

may be found such that

$$H\psi_{\epsilon,\lambda}(\mathbf{s})=\epsilon E\psi_{\epsilon,\lambda}(\mathbf{s}), \quad \mathbf{O}\cdot\mathbf{s}\psi_{\epsilon,\lambda}(\mathbf{s})=\lambda\psi_{\epsilon,\lambda}(\mathbf{s}), \quad (6.1)$$

where ϵ and λ are independently ±1. It is clear that if $\psi(\mathbf{s})$ is an eigenfunction of $\mathbf{O}\cdot\mathbf{s}$ with eigenvalue $+1$, then $\psi(-\mathbf{s})$ is an eigenfunction with eigenvalue -1. One may therefore replace $\psi_{\epsilon,\lambda}(\mathbf{s})$ by $\psi_\epsilon(\lambda\mathbf{s})$. A system in an eigenstate of $\mathbf{O}\cdot\mathbf{s}$ with eigenvalue $+1$ is said to be polarized in the $\mathbf{s}$ direction.

One may relate the plane-wave eigenfunctions of arbitrary momentum to eigenfunctions in the rest system. Let these eigenfunctions be simultaneous eigenfunctions of the Hamiltonian and $\mathbf{O}\cdot\mathbf{s}$ in the rest system, so

$$\psi_\epsilon^{0\dagger}(\mathbf{s})\psi_\epsilon^0(\mathbf{s})=1, \quad (6.2)$$

$$\beta\psi_\epsilon^0(\mathbf{s})=\epsilon\psi_\epsilon^0(\mathbf{s}), \quad (6.3)$$

$$\beta\boldsymbol{\sigma}\cdot\mathbf{s}\psi_\epsilon^0(\mathbf{s})=\psi_\epsilon^0(\mathbf{s}). \quad (6.4)$$

On defining the projection operators P_ϵ by

$$P_\epsilon=\tfrac{1}{2}[1+\epsilon(H/E)] = (2E)^{-1}(E+\epsilon\boldsymbol{\alpha}\cdot\mathbf{p}+\epsilon\beta), \quad (6.5)$$

where H, E, and $\mathbf{p}$ are the eigenvalues in the laboratory system, one finds, by using Eqs. (6.3), (6.4), and (4.1), that

$$H(P_\epsilon\psi_\epsilon^0(\mathbf{s}))=\epsilon E(P_\epsilon\psi_\epsilon^0(\mathbf{s})), \quad (6.6)$$

$$\mathbf{O}\cdot\mathbf{s}(P_\epsilon\psi_\epsilon^0(\mathbf{s}))=\mathbf{O}\cdot\mathbf{s}P_\epsilon(\beta\boldsymbol{\sigma}\cdot\mathbf{s})\psi_\epsilon^0(\mathbf{s})=(P_\epsilon\psi_\epsilon^0(\mathbf{s})). \quad (6.7)$$

Also it is known that

$$P_\epsilon^\dagger=P_\epsilon, \quad P_\epsilon P_\epsilon=P_\epsilon,$$

so

$$(P_\epsilon\psi_\epsilon^0)^\dagger(P_\epsilon\psi_\epsilon^0)=\psi_\epsilon^{0\dagger}P_\epsilon\psi_\epsilon^0=(2E)^{-1}(E+1),$$

where Eqs. (6.2) and (6.3) and the fact that

$$\psi_\epsilon^{0\dagger}\boldsymbol{\alpha}\cdot\mathbf{p}\psi_\epsilon^0=0,$$

which is easily proved from Eq. (6.3), are used. Therefore, the wave function given by

$$\psi_\epsilon(\mathbf{s})=[2E/(E+1)]^{\frac{1}{2}}P_\epsilon\psi_\epsilon^0(\mathbf{s}) \quad (6.8)$$

satisfies

$$\psi_\epsilon^\dagger(\mathbf{s})\psi_\epsilon(\mathbf{s})=1, \quad (6.9)$$

$$H\psi_\epsilon(\mathbf{s})=\epsilon E\psi_\epsilon(\mathbf{s}), \quad (6.10)$$

$$\mathbf{O}\cdot\mathbf{s}\psi_\epsilon(\mathbf{s})=\psi_\epsilon(\mathbf{s}). \quad (6.11)$$

The functions $\psi_\epsilon(\mathbf{s})$ actually are proportional to the rest-system functions $\psi_\epsilon^0(\mathbf{s})$ Lorentz-transformed to the laboratory frame. The wave-function amplitudes, ψ^0 in the rest system and ψ in the laboratory system, are related by

$$\psi=\Lambda\psi^0,$$

where

$$\Lambda^{-1}\gamma_\mu\Lambda=a_{\mu\rho}\gamma_\rho,$$

and the transformation coefficients are

$$a_{11}=a_{44}=E, \quad a_{14}=-a_{41}=-iq,$$

the x axis having been chosen in the $\mathbf{q}$ direction. In this case the transformation matrix is found to be

$$\Lambda=[2(E+1)]^{-\frac{1}{2}}(E+\alpha_x q+1).$$

When this is applied to the function $\psi_\epsilon^0(\mathbf{s})$, one can replace q by ϵp and 1 by $\epsilon\beta$ so that

$$\begin{aligned}\psi&=\Lambda\psi_\epsilon^0(\mathbf{s})\\&=[2(E+1)]^{-\frac{1}{2}}(E+\epsilon\alpha_x p+\epsilon\beta)\psi_\epsilon^0(\mathbf{s})\\&=E^{\frac{1}{2}}[2E/(E+1)]^{\frac{1}{2}}P_\epsilon\psi_\epsilon^0(\mathbf{s}),\end{aligned}$$

and this proves the assertion. One sees that if a particle has polarization $\mathbf{s}$ in the laboratory system, then it has the same polarization $\mathbf{s}$ in the rest system. In other words, the polarization of an electron beam is the same no matter from which Lorentz frame the beam is viewed.

The explicit plane-wave eigenfunctions satisfying Eqs. (6.2)–(6.4), in the specific representation of Sec. 2, are

$$\Psi_+{}^0(\mathbf{s})=\begin{bmatrix}\cos\frac{1}{2}\theta e^{-\frac{1}{2}i\phi}\\ \sin\frac{1}{2}\theta e^{\frac{1}{2}i\phi}\\ 0\\ 0\end{bmatrix}e^{-it/\hbar},\quad \Psi_-{}^0(\mathbf{s})=\begin{bmatrix}0\\ 0\\ -\sin\frac{1}{2}\theta e^{-\frac{1}{2}i\phi}\\ \cos\frac{1}{2}\theta e^{\frac{1}{2}i\phi}\end{bmatrix}e^{it/\hbar}, \tag{6.12}$$

where θ and ϕ are the polar and azimuthal angles of $\mathbf{s}$. From Eq. (6.8), the corresponding arbitrary Lorentz frame eigenfunctions are

$$\begin{aligned}\Psi_+(\mathbf{p},\mathbf{s})=&[\cos\tfrac{1}{2}\theta e^{-\frac{1}{2}i\phi}U_{+\frac{1}{2}}(\mathbf{p})\\ &+\sin\tfrac{1}{2}\theta e^{\frac{1}{2}i\phi}U_{-\frac{1}{2}}(\mathbf{p})]e^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar},\\ \Psi_-(\mathbf{p},\mathbf{s})=&[\cos\tfrac{1}{2}\theta e^{\frac{1}{2}i\phi}V_{-\frac{1}{2}}(\mathbf{p})\\ &-\sin\tfrac{1}{2}\theta e^{-\frac{1}{2}i\phi}V_{+\frac{1}{2}}(\mathbf{p})]e^{i(\mathbf{p}\cdot\mathbf{x}+Et)/\hbar},\end{aligned} \tag{6.13}$$

where $U_{\pm\frac{1}{2}}$ and $V_{\pm\frac{1}{2}}$ are the functions

$$U_{\pm\frac{1}{2}}(\mathbf{p})=[2E(E+1)]^{-\frac{1}{2}}\begin{pmatrix}(E+1)\chi_{\pm\frac{1}{2}}\\ \boldsymbol{\sigma}\cdot\mathbf{p}\chi_{\pm\frac{1}{2}}\end{pmatrix},\quad V_{\pm\frac{1}{2}}(\mathbf{p})=[2E(E+1)]^{-\frac{1}{2}}\begin{pmatrix}-\boldsymbol{\sigma}\cdot\mathbf{p}\chi_{\pm\frac{1}{2}}\\ (E+1)\chi_{\pm\frac{1}{2}}\end{pmatrix}, \tag{6.14}$$

and $\chi_{\pm\frac{1}{2}}$ are the familiar "spin-up," "spin-down" functions of nonrelativistic theory,

$$\chi_{+\frac{1}{2}}=\begin{pmatrix}1\\0\end{pmatrix},\quad \chi_{-\frac{1}{2}}=\begin{pmatrix}0\\1\end{pmatrix}. \tag{6.15}$$

The solutions given in Heitler[16] are $\Psi_\epsilon(p_z\mathbf{e}_3,\lambda\mathbf{e}_3)$ in the present notation, where Heitler's $E\gtrless 0$ corresponds to $\epsilon=\pm1$ here, and where Heitler's ↑ and ↓ correspond to $\epsilon\lambda=1$ and $\epsilon\lambda=-1$ here. The functions $\Psi_\pm$ of Eq. (6.13) are a similarity transformation from those of deGroot and Tolhoek.[17] They write A and B in place of $\cos\frac{1}{2}\theta e^{-\frac{1}{2}i\phi}$ and $\sin\frac{1}{2}\theta e^{\frac{1}{2}i\phi}$, and express the functions in terms of the physical momentum. Since they define the direction of polarization θ, ϕ by

$$B/A=\tan\tfrac{1}{2}\theta e^{i\phi}, \tag{6.16}$$

it is clear that their direction of polarization coincides with the vector $\mathbf{s}$ used here.

In general, if the system is in a state described by the wave function $\psi_\epsilon(\mathbf{s})$, then the expectation value of $\mathbf{O}$ is $\mathbf{s}$,

$$\Psi_\epsilon{}^\dagger(\mathbf{s})\mathbf{O}\Psi_\epsilon(\mathbf{s})=\mathbf{s}. \tag{6.17}$$

This may be easily proved since, from Eq. (5.1),

$$(\mathbf{O}\cdot\mathbf{s})\mathbf{O}+\mathbf{O}(\mathbf{O}\cdot\mathbf{s})=2\mathbf{s}$$

and operation with $\Psi_\epsilon{}^\dagger(\mathbf{s})$ and $\Psi_\epsilon(\mathbf{s})$ on the left- and right-hand sides yields the result by virtue of Eq. (6.11).

Also, any plane-wave electron or positron state is a state completely polarized in some direction $\mathbf{s}$. To show this consider the expansion of an electron state X in terms of $\Psi_+(\mathbf{e}_3)$ and $\Psi_+(-\mathbf{e}_3)$, which for convenience are studied in the representation of Eq. (6.13). Then

$$\mathrm{X}(\mathbf{p})=[ae^{i\alpha}U_{-\frac{1}{2}}(\mathbf{p})+be^{i\beta}U_{+\frac{1}{2}}(\mathbf{p})]e^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar}, \tag{6.18}$$

where a, b, α, and β are positive real numbers and $a^2+b^2=1$. One may write

$$ae^{i\alpha}=e^{\frac{1}{2}i(\alpha+\beta)}\sin\tfrac{1}{2}\theta e^{\frac{1}{2}i\phi},\quad be^{i\beta}=e^{\frac{1}{2}i(\alpha+\beta)}\cos\tfrac{1}{2}\theta e^{-\frac{1}{2}i\phi},$$

since this has a solution θ, ϕ, where $\sin\frac{1}{2}\theta=a$, $\cos\frac{1}{2}\theta=b$, and $\phi=\alpha-\beta$. Consequently, Eq. (6.18) reads

$$\mathrm{X}(\mathbf{p})=e^{\frac{1}{2}i(\alpha+\beta)}\Psi_+(\mathbf{p},\mathbf{s}), \tag{6.19}$$

and, since wave functions are defined only to within a phase factor, the assertion is proved.

7. Foldy-Wouthuysen Representation

The three-vector polarization operator assumes an especially simple form in the Foldy-Wouthuysen[18] representation and many of its properties become evident in that representation. In the specific matrix representation of Sec. 2, the free-particle Hamiltonian contains even and odd operators—odd operators being matrix operators that mix the upper and lower two-component spaces of the wave function (e.g., γ_5, $\boldsymbol{\alpha}$), even operators being those which do not effect this mixing (e.g., β, $\boldsymbol{\sigma}$). The purpose of the FW transformation is to obtain a representation in which the Hamiltonian is an even operator, so that electron and positron solutions are separated into the two-component spaces.

Any operator A in the FW representation is

$$A_{\mathrm{FW}}=e^{iS}Ae^{-iS}, \tag{7.1}$$

where the desired unitary transformation is explicitly

$$e^{\pm iS}=[2|H|(|H|+1)]^{-\frac{1}{2}}\beta[\beta(|H|+1)\pm\boldsymbol{\alpha}\cdot\mathbf{p}]. \tag{7.2}$$

On performing the indicated transformation, one obtains

$$\mathbf{O}_{\mathrm{FW}}=\beta\boldsymbol{\sigma},\quad H_{\mathrm{FW}}=\beta|H|. \tag{7.3}$$

The upper/lower two-component spaces in the FW representation are associated with the Pauli nonrelativistic two-component theory of the electron/positron. One sees, therefore, that the Pauli theory limit of the three-vector polarization operator is $\pm\boldsymbol{\sigma}$ ($\beta=\pm1$ for electrons/positrons). It is also seen that the algebraic properties of the three-vector polarization operator follow easily from Eq. (7.3).

[16] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), p. 107.

[17] S. R. deGroot and H. A. Tolhoek, Physica **16**, 461 (1950), Eqs. (12) and (14); **17**, 1 (1951), Eq. (11).

[18] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

8. Density Matrix

In scattering problems, where incoming and outgoing particles are treated asymptotically as free-particle wave states, it is useful to have an expression for the statistical density matrix[19] as a function of the average polarization $\mathbf{P}$ of the considered ensemble of particles.

In general, the electron/positron density matrix for an ensemble of single-particle systems with definite energy E and momentum $\mathbf{q}$ is given by

$$\rho_{\pm}=\sum_{\lambda=1,-1}p_{\pm\lambda}\psi_{\pm}(\lambda\mathbf{s})\psi_{\pm}^{\dagger}(\lambda\mathbf{s}), \tag{8.1}$$

where $p_{\pm\lambda}$ is the probability that the particle is in polarization state $\lambda\mathbf{s}$. On using Eq. (6.8), one may write

$$\rho_{\pm}=[2E/(E+1)]P_{\pm}(\sum_{\lambda=1,-1}p_{\pm\lambda}\psi_{\pm}^{0}(\lambda\mathbf{s})\psi_{\pm}^{0\dagger}(\lambda\mathbf{s}))P_{\pm},$$

which, in consequence of Eq. (6.3), is equivalent to

$$\rho_{\pm}=E[2(E+1)]^{-1}P_{\pm}(1\pm\beta)(\sum p_{\pm\lambda}\psi_{\pm}^{0}(\lambda\mathbf{s}) \times\psi_{\pm}^{0\dagger}(\lambda\mathbf{s}))(1\pm\beta)P_{\pm}. \tag{8.2}$$

In the specific representation of the Dirac matrices given in Sec. 2, $(1\pm\beta)\sum p_{\pm\lambda}\psi_{\pm}^{0}(\lambda\mathbf{s})\psi_{\pm}^{0\dagger}(\lambda\mathbf{s})(1\pm\beta)$ is of the form

$$\begin{pmatrix} X & 0 \\ 0 & 0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & X\end{pmatrix}$$

for upper and lower signs, respectively. Here X is a 2×2 matrix and therefore may be written as

$$4(A_{\pm}+\mathbf{B}_{\pm}\cdot\boldsymbol{\sigma}),$$

where $A_{\pm}$ and $\mathbf{B}_{\pm}$ are still to be determined. This gives

$$\rho_{\pm}=E[2(E+1)]^{-1}P_{\pm}(1\pm\beta)(A_{\pm}+\mathbf{B}_{\pm}\cdot\boldsymbol{\sigma})(1\pm\beta)P_{\pm}. \tag{8.3}$$

Finally, $A_{\pm}$ and $\mathbf{B}_{\pm}$ are evaluated from the relations

$$\mathrm{Tr}\,(\rho_{\pm})=1,\quad \mathrm{Tr}\,(\rho_{\pm}\mathbf{O}_{\pm})=\mathbf{P},\quad |\mathbf{P}|\leq 1,$$

which yield the result that $A_{\pm}=\frac{1}{2}$, $\mathbf{B}_{\pm}=\pm\frac{1}{2}\mathbf{P}$. Consequently, the plane-wave density matrix is

$$\begin{aligned}\rho_{\pm}&=E[4(E+1)]^{-1}P_{\pm}(1\pm\beta)(1\pm\mathbf{P}\cdot\boldsymbol{\sigma})(1\pm\beta)P_{\pm}\\ &=(4E)^{-1}[E\pm\beta+\boldsymbol{\alpha}\cdot\mathbf{q}\pm\mathbf{P}\cdot\boldsymbol{\sigma}\mp\gamma_5\mathbf{P}\cdot\mathbf{q}\\ &\quad+E(\mathbf{P}\cdot\beta\boldsymbol{\sigma})+i\beta\boldsymbol{\alpha}\cdot(\mathbf{P}\times\mathbf{q})\pm(E+1)^{-1}(\mathbf{P}\cdot\mathbf{q})(\boldsymbol{\sigma}\cdot\mathbf{q})\\ &\quad-(E+1)^{-1}(\mathbf{P}\cdot\mathbf{q})(\beta\boldsymbol{\sigma}\cdot\mathbf{q})].\end{aligned} \tag{8.4}$$

The expression for the density matrix containing the projection operators was given by Mühlschlegel and Koppe,[20] the expanded form by Tolhoek and deGroot.[3]

9. Covariant Description

Michel and Wightman[6] introduced the operator

$$\mathrm{O}_{\mathrm{MW}}=i\gamma_5\gamma_\mu n_\mu \tag{9.1}$$

to describe the polarization of a plane-wave state of a free particle. Here n_μ is defined to be a four-vector with components $(\mathbf{s},0)$ in the rest system of the particle. [It is clear that $n_\mu n_\mu$ is unity and that $q_\mu n_\mu$ is zero since q_μ is $(0,i)$ in the rest system.] This operator is equivalent to $\mathbf{O}\cdot\mathbf{s}$ as shown in the following.

The components of n_μ in the laboratory system are

$$\mathbf{n}=\mathbf{s}+(E+1)^{-1}(\mathbf{q}\cdot\mathbf{s})\mathbf{q},\quad n_4=i\mathbf{q}\cdot\mathbf{s}. \tag{9.2}$$

As long as n_μ directly multiplies a plane-wave solution of the Dirac equation, one can replace it by $n_{\mu\,\mathrm{op}}$, defined by

$$\mathbf{n}_{\mathrm{op}}=\mathbf{s}+(|H|+1)^{-1}(\mathbf{p}\cdot\mathbf{s})\mathbf{p},\qquad n_{4\,\mathrm{op}}=i\mathbf{p}\cdot\mathbf{s}(H/|H|). \tag{9.3}$$

Here $\mathbf{p}$ is $-i\hbar\boldsymbol{\nabla}$, and these operators have the properties

$$n_{\mu\,\mathrm{op}}n_{\mu\,\mathrm{op}}=1,\quad \mathbf{n}_{\mathrm{op}}\cdot\mathbf{p}+in_{4\,\mathrm{op}}H=0. \tag{9.4}$$

A direct consequence of Eq. (4.2) is that

$$i\gamma_5\gamma_\mu n_{\mu\,\mathrm{op}}=\mathbf{O}\cdot\mathbf{s}. \tag{9.5}$$

One sees then that O_{MW}, defined for a plane-wave state, is equivalent to $\mathbf{O}\cdot\mathbf{s}$ when operating on the state function.

III. FOUR-VECTOR POLARIZATION OPERATOR

10. Definition

For the free particle, the four-vector polarization operator T_μ is defined to be

$$\begin{aligned}\mathbf{T}&=\gamma_5(i\boldsymbol{\gamma}-\mathbf{p})\\ &=\beta\boldsymbol{\sigma}-\gamma_5\mathbf{p},\\ T_4&=\gamma_5(i\gamma_4-iH)\\ &=i\boldsymbol{\sigma}\cdot\mathbf{p}.\end{aligned} \tag{10.1}$$

This is closely related to the operator

$$T_{\mu\,\mathrm{BW}}=-\tfrac{1}{2}\epsilon_{\mu\rho\pi\nu}\gamma_\rho\gamma_\pi p_\nu \tag{10.2}$$

which was first discussed by Bargmann and Wigner.[4] In fact, as a consequence of the relation

$$\gamma_\mu\gamma_\nu p_\nu=p_\mu-\tfrac{1}{2}\epsilon_{\mu\rho\pi\nu}\gamma_5\gamma_\rho\gamma_\pi p_\nu, \tag{10.3}$$

one finds that

$$T_{\mu\,\mathrm{BW}}=\gamma_5(i\gamma_\mu-p_\mu)+\gamma_5\gamma_\mu(\gamma_\nu p_\nu-i). \tag{10.4}$$

Therefore, when applied to solutions of the Dirac equation, the operators are equivalent.

11. Generators of the Little Group

The components T_μ are the generators of the little group; the subgroup of homogeneous Lorentz transformations that leave the four-vector p_μ of a plane-wave state unchanged. This was pointed out as a specialization of the general case of arbitrary spin and mass by Bargmann and Wigner.[4]

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[19] See, for example, U. Fano, Revs. Modern Phys. **29**, 74 (1957).
[20] B. Mühlschlegel and H. Koppe, Z. Physik **150**, 474 (1958).

To see this in detail, consider the infinitesimal Lorentz transformation

$$x_\mu' = a_{\mu\nu}x_\nu = (\delta_{\mu\nu}+\xi_{\mu\nu})x_\nu, \quad (11.1)$$

where $\xi_{\mu\nu}=-\xi_{\nu\mu}$ are infinitesimals. The corresponding wave-function transformation is

$$\Psi'(x')=\Lambda\Psi(x), \quad (11.2)$$

where, infinitesimally,

$$\Lambda=1+\tfrac{1}{4}\xi_{\sigma\rho}\gamma_\sigma\gamma_\rho. \quad (11.3)$$

On substituting Eqs. (11.1) and (11.3) into (11.2) and expanding $\Psi(x_\mu-\xi_{\mu\nu}x_\nu)$ about $\Psi(x_\mu)$ in a Taylor's series, one obtains (to first order in infinitesimals)

$$\Psi'(x)=[1+\xi_{\sigma\rho}(\tfrac{1}{4}\gamma_\sigma\gamma_\rho-x_\rho\partial/\partial x_\sigma)]\Psi(x). \quad (11.4)$$

For a plane-wave state of a free particle of specified four-vector p_μ, the wave function has the form

$$\Psi(x)=\psi(p_\nu)e^{ip_\mu x_\mu/\hbar}. \quad (11.5)$$

If only those homogeneous proper Lorentz transformations that leave p_μ unchanged are considered, then

$$\xi_{\mu\nu}p_\nu=0, \quad (11.6)$$

so that

$$\Psi'(x)=(1+\tfrac{1}{4}\xi_{\sigma\rho}\gamma_\sigma\gamma_\rho)\Psi(x).$$

On using Eqs. (10.2) and (11.6), it may easily be verified that

$$\epsilon_{\nu\mu\rho\lambda}\xi_{\mu\rho}T_\lambda\Psi(x)=\xi_{\sigma\rho}\gamma_\sigma\gamma_\rho p_\nu\Psi(x).$$

Thus, for the eigenvalue $p_\nu\neq 0$, one obtains

$$\Psi'(x)=[1+(4p_\nu)^{-1}\epsilon_{\nu\mu\rho\lambda}\xi_{\mu\rho}T_\lambda]\Psi(x); \quad (11.7)$$

no sum on ν. Equation (11.6) implies that only three infinitesimal parameters are independent, which, for a given ν, may be taken as $\epsilon_{\nu\mu\rho\lambda}\xi_{\mu\rho}$. Therefore, the operators T_λ are the generators of the little group.

12. Algebraic Properties

The operators T_μ satisfy the equations

$$T_\mu T_\mu=3, \quad (12.1)$$

$$\mathbf{T}\cdot\mathbf{p}+iT_4H=0, \quad (12.2)$$

$$[T_\mu,H]_-=0. \quad (12.3)$$

The operators T_i are of primary interest because Eq. (12.2) can be used to express T_4 in terms of them. Their algebra is involved with that of the operators S_i, defined by

$$S_i=\sigma_i+\epsilon_{ijk}\gamma_j p_k. \quad (12.4)$$

In detail, one finds

$$[T_i,T_j]_-=2i\epsilon_{ijk}S_k,$$
$$[T_i,S_j]_-=2i(\epsilon_{ijk}T_k+\epsilon_{jlm}T_lp_mp_i),$$
$$[S_i,S_j]_-=2i\epsilon_{ijk}(S_k+S_lp_lp_k),$$
$$[T_i,T_j]_+=2(\delta_{ij}+p_ip_j),$$
$$[S_i,T_j]_+=2\delta_{ij}H,$$
$$[S_i,S_j]_+=2[\delta_{ij}(1+p^2)-p_ip_j],$$
$$[T_i,H]_+=2(S_i+S_jp_jp_i),$$
$$[S_i,H]_+=2(1+p^2)T_i-2T_jp_jp_i.$$

There is the relation

$$\mathbf{T}\cdot\mathbf{J}=\mathbf{J}\cdot\mathbf{T}=\hbar(K+\tfrac{1}{2}H),$$

between the polarization operators, the angular momentum operator

$$\mathbf{J}=\mathbf{x}\times\mathbf{p}+\tfrac{1}{2}\hbar\boldsymbol{\sigma},$$

and Dirac's operator[21]

$$\hbar K=\beta[\boldsymbol{\sigma}\cdot(\mathbf{x}\times\mathbf{p})+\hbar].$$

The charge-conjugated four-vector polarization operator is

$$\mathbf{T}^C=C^*\mathbf{T}^*C=\mathbf{T},\quad T_4^C=C^*T_4^*C=-T_4.$$

13. Connection with Three-Vector Operator

The relation between the operators is

$$\begin{aligned}\mathbf{T}&=\mathbf{O}+(|H|+1)^{-1}(\mathbf{O}\cdot\mathbf{p})\mathbf{p},\\ T_4&=i(H/|H|)\mathbf{O}\cdot\mathbf{p},\end{aligned} \quad (13.1)$$

as is easily verified. The connection between T_μ and $\mathbf{O}$ is the same as the one between $n_{\mu\,\mathrm{op}}$ and $\mathbf{s}$, Eq. (9.3). On combining Eqs. (10.1), (9.4), and (9.5), one finds that

$$T_\mu n_{\mu\,\mathrm{op}}=\mathbf{O}\cdot\mathbf{s}. \quad (13.2)$$

Therefore the wave function $\Psi_\pm(\mathbf{s})$ describing a plane-wave state polarized in the $\mathbf{s}$ direction is also an eigenstate of $T_\mu n_{\mu\,\mathrm{op}}$,

$$T_\mu n_{\mu\,\mathrm{op}}\Psi_\pm(\mathbf{s})=\Psi_\pm(\mathbf{s}). \quad (13.3)$$

To find the expected value of T_μ, one observes from Eq. (5.1) that

$$[\mathbf{O},\mathbf{O}\cdot\mathbf{s}]_+=2\mathbf{s},$$

so that Eqs. (13.1) yield

$$\begin{aligned}[T_\mu,T_\nu n_{\nu\,\mathrm{op}}]_+&=[T_\mu,\mathbf{O}\cdot\mathbf{s}]_+\\ &=2n_{\mu\,\mathrm{op}}.\end{aligned} \quad (13.4)$$

The result of taking the expected value of this last equation is

$$\begin{aligned}\Psi_\pm^\dagger(\mathbf{s})T_\mu\Psi_\pm(\mathbf{s})&=\psi_\pm^\dagger(\mathbf{s})n_{\mu\,\mathrm{op}}\psi_\pm(\mathbf{s})\\ &=n_\mu,\end{aligned} \quad (13.5)$$

where n_μ and $\mathbf{s}$ are related by Eq. (9.2).

It is clear from Eqs. (6.17), (13.5), and (9.2) that

[21] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, New York, 1958), 4th ed., p. 268.

the following interpretation of these operators can be made: For a plane-wave state, the three-vector polarization operator $\mathbf{O}$ is the laboratory-system operator corresponding to the direction of polarization $\mathbf{s}$ in the rest system of the particle; the four-vector polarization operator is the laboratory-system operator corresponding to the four-vector which is the Lorentz transform of $(\mathbf{s},0)$ from the rest system.

14. Lorentz Transformation Properties

Since T_μ commutes with the Hamiltonian, the expectation values

$$\langle T_\mu\rangle=\int\Psi^\dagger T_\mu\Psi, \tag{14.1}$$

where Ψ is any solution of the Dirac equation and the integral extends over all space, are constant in time. The $\langle T_i\rangle$ are real and $\langle T_4\rangle$ is pure imaginary. It is interesting to inquire into the tensor transformation properties of these quantities.

Let the expectation values be defined in a different coordinate system by

$$\langle T_\mu\rangle'=\int\Psi'^\dagger T_\mu\Psi'. \tag{14.2}$$

It is immediately clear that for space rotations $\langle T_i\rangle$ is a vector and $\langle T_4\rangle$ a scalar. Pure Lorentz transformations can be easily discussed infinitesimally. The transformation is

$$\mathbf{x}'=\mathbf{x}-\mathbf{v}t,\quad t'=t-\mathbf{v}\cdot\mathbf{x},$$
$$\Psi'=\Psi+\mathbf{v}\cdot(t\nabla\Psi+\mathbf{x}\partial\Psi/\partial t-\tfrac{1}{2}\boldsymbol{\alpha}\Psi). \tag{14.3}$$

On substituting Eq. (14.3) into Eq. (14.2), replacing $\partial\Psi/\partial t$ by $-i\hbar^{-1}H\Psi$, and simplifying, one finds

$$\langle T_i\rangle'=\langle T_i\rangle+iv_i\langle T_4\rangle,\quad \langle T_4\rangle'=\langle T_4\rangle-iv_j\langle T_j\rangle,$$

which are the correct rules for a Lorentz four-vector. It is clear that the same proof applies for off-diagonal matrix elements.

For the space reflection

$$\mathbf{x}'=-\mathbf{x},\quad t'=t,$$

one may consider either the usual wave-function transformation

$$\psi'(x')=i\gamma_4\psi(x)$$

or the Wigner-Landau combined inversion

$$\psi'(x')=i\gamma_4C^*\psi^*(x).$$

In either case the result is

$$\langle T_i\rangle'=\langle T_i\rangle,\quad \langle T_4\rangle'=-\langle T_4\rangle.$$

Finally, for the time reflection

$$\mathbf{x}'=\mathbf{x},\quad t'=-t,$$

the wave-function transformation rule is

$$\psi'(x')=\gamma_4\gamma_5C^*\psi^*(x),$$

and it is found that

$$\langle T_i\rangle'=-\langle T_i\rangle,\quad \langle T_4\rangle'=\langle T_4\rangle.$$

In summary, for the general Lorentz transformation

$$x_\mu'=a_{\mu\nu}x_\nu,$$

the expectation values transform according to the rule

$$\langle T_\mu\rangle'=(\det a)a_{\mu\nu}\langle T_\nu\rangle.$$

The reflection properties of $\langle T_i\rangle$ are the same as those of angular momentum.

15. Effect of External Fields

The four-vector polarization operator can be generalized to the case of a Dirac particle in an external electromagnetic field. The operator is then defined by

$$\begin{aligned}T_i&=\gamma_5(i\gamma_i-\pi_i)\\&=\beta\sigma_i-\gamma_5\pi_i,\\T_4&=\gamma_5(i\gamma_4-iH+ie\phi)\\&=i\boldsymbol{\sigma}\cdot\boldsymbol{\pi},\end{aligned} \tag{15.1}$$

where H is given by Eq. (2.1). This operator has the properties

$$T_\mu T_\mu=3+e\hbar\boldsymbol{\sigma}\cdot\mathbf{B}, \tag{15.2}$$

$$\boldsymbol{\pi}\cdot\mathbf{T}+i(H-e\phi)T_4=e\hbar\boldsymbol{\alpha}\cdot\mathbf{B}, \tag{15.3}$$

$$[\mathbf{T},H]_-=ie\hbar\boldsymbol{\sigma}\times\mathbf{B}-ie\hbar\gamma_5(\mathbf{E}+\partial\mathbf{A}/\partial t), \tag{15.4}$$

$$[T_4,H]_-=-e\hbar\boldsymbol{\sigma}\cdot(\mathbf{E}+\partial\mathbf{A}/\partial t). \tag{15.5}$$

Consequently, the Heisenberg equations of motion are found to be

$$d\mathbf{T}/dt=e(\boldsymbol{\sigma}\times\mathbf{B}-\gamma_5\mathbf{E}),\quad dT_4/dt=ie\boldsymbol{\sigma}\cdot\mathbf{E}. \tag{15.6}$$

These equations can be accumulated into the form

$$dT_\mu/dt=ie\gamma_4\gamma_5\gamma_\nu F_{\mu\nu}. \tag{15.7}$$

In these three special cases there are polarization integrals of the motion:

(1) If $\mathbf{E}$ is zero, T_4 is an integral.

(2) If $\mathbf{E}$ is zero and the magnetic field has a fixed direction $\mathbf{B}/B$, then T_4 and $\mathbf{T}\cdot\mathbf{B}/B$ are integrals.

(3) If $\mathbf{B}$ is zero and the component of $\mathbf{E}$ in some fixed direction $\mathbf{e}$ is zero, then $\mathbf{T}\cdot\mathbf{e}$ is an integral.

16. Classical Equations of Motion

Equation (15.7) gives the equations of motion of the four-vector polarization. One is often interested in the analogous equations of motion for the expectation value of the polarization of a particle which is localized so that the wave function is a classical packet. In this limit the rate of change of the polarization can be expressed in terms of the external fields and the polarization itself.

The following conditions are sufficient to make the classical limit applicable:

(1) The wave function $\Psi(\mathbf{x},t)$ is negligible except in a small region of space, defining the position of the classical particle.

(2) The wave function has a rather narrow spread in momentum and H value; therefore, it applies to a specific charge, and one may write

$$\pi_\mu\Psi(\mathbf{x},t)=\langle\pi_\mu\rangle\Psi(\mathbf{x},t), \tag{16.1}$$

where $\langle\pi_\mu\rangle$ is the classical value, varying along the orbit but factorable out of integrals on the wave function. The equations following are written for an electron packet so that $\langle H-e\phi\rangle$ is positive for an unbound particle. For a positron packet one makes a charge conjugation at the outset, and then the same argument applies identically except that e is replaced by $-e$. The requirement of a narrow spread of momentum imposes a condition on the size of the packet. If a is a characteristic dimension, the spread in momentum is $\hbar/a$, and this must be small compared to the average value so that

$$\hbar^2/a^2<\langle\pi_i\rangle\langle\pi_i\rangle. \tag{16.2}$$

(3) The fields and potentials vary negligibly across the packet so that only their values at the position of the particle are pertinent.

(4) The quantities $e\hbar B$ and $e\hbar E$ are negligible compared to unity. The point is that, as long as a solution of the Dirac equation is considered so that $i\hbar\partial/\partial t$ may be replaced by H, the equation

$$\pi_\mu\pi_\mu\Psi=(e\hbar\boldsymbol{\sigma}\cdot\mathbf{B}-ie\hbar\boldsymbol{\alpha}\cdot\mathbf{E}-1)\Psi \tag{16.3}$$

applies; thus, if $e\hbar B$ and $e\hbar E$ are negligible, the classical result

$$\langle\pi_\mu\rangle\langle\pi_\mu\rangle=-1. \tag{16.4}$$

is valid. This is not a stringent requirement; at 10^4 gauss, $e\hbar B$ is 10^{-10} and at 10^6 v/cm, $e\hbar E$ is 10^{-10}.

These conditions can be met when $\hbar$ is small compared to the classical actions in the problem. To see this in detail, one considers a packet with characteristic dimension a of the order of $\hbar$ as required by conditions (1) and (2). Equation (16.1) can be satisfied because the commutators between the operators are proportional to $\hbar$. The neglect of the variation in a field component F across the packet introduces an error of the order of $aF^{-1}\partial F/\partial x$. This error and those arising from the terms disregarded in condition (4) are of the order of $\hbar$ divided by some classical action.

In view of Eq. (16.4), one introduces the abbreviations

$$\mathbf{b}=i\langle\boldsymbol{\pi}\rangle/\langle\pi_4\rangle, \tag{16.5}$$

$$\bar{\gamma}=-i\langle\pi_4\rangle=\langle H-e\phi\rangle=(1-b^2)^{-\frac{1}{2}}, \tag{16.6}$$

so that $\mathbf{b}$ is the classical particle velocity and $\bar{\gamma}$ is positive.

In determining classical equations of motion, it is convenient to observe that, for any Hermitian operator Q,

$$\int\Psi^\dagger Q(H-e\phi)\Psi d\mathbf{x}+\int[Q(H-e\phi)\Psi]^\dagger\Psi d\mathbf{x}$$
$$=2\langle H-e\phi\rangle\int\Psi^\dagger Q\Psi d\mathbf{x},$$

which may be written in the form

$$\langle[Q,H-e\phi]_+\rangle=2\bar{\gamma}\langle Q\rangle. \tag{16.7}$$

Immediate consequences are

$$\langle\boldsymbol{\alpha}\rangle=\mathbf{b}, \tag{16.8}$$

$$\langle\beta\rangle=\bar{\gamma}^{-1}, \tag{16.9}$$

$$\langle i\gamma_4\gamma_5\gamma_\mu\rangle=\bar{\gamma}^{-1}\langle T_\mu\rangle. \tag{16.10}$$

The equations determining the orbit follow from the equations of motion in the Heisenberg picture for $\boldsymbol{\pi}$ and $\mathbf{x}$,

$$d\boldsymbol{\pi}/dt=e\mathbf{E}+e\boldsymbol{\alpha}\times\mathbf{B},\quad d\mathbf{x}/dt=\boldsymbol{\alpha}, \tag{16.11}$$

by taking expectation values

$$d(\bar{\gamma}\mathbf{b})/dt=e\mathbf{E}+e\mathbf{b}\times\mathbf{B},\quad d\langle\mathbf{x}\rangle/dt=\mathbf{b}. \tag{16.12}$$

On using Eq. (16.6) it is readily established that

$$d\bar{\gamma}/dt=\mathbf{b}\cdot d(\bar{\gamma}\mathbf{b})/dt=e\mathbf{E}\cdot\mathbf{b}.$$

These are the relativistic equations of motion for a charged particle subject to the Lorentz force. The orbit is independent of the polarization. In terms of the proper time τ, defined by

$$d/d\tau=\bar{\gamma}d/dt,$$

one may write

$$d^2x_\mu/d\tau^2=eF_{\mu\nu}dx_\nu/d\tau, \tag{16.13}$$

where $x_\mu=(\langle\mathbf{x}\rangle,it)$ is the location of the particle. By combining Eqs. (15.7) and (16.10), one finds equations of motion for the average polarization,

$$d\langle T_\mu\rangle/d\tau=eF_{\mu\nu}\langle T_\nu\rangle. \tag{16.14}$$

As written, $\langle T_\mu\rangle$ is the expectation value of the operator T_μ in a single-particle state of existence. However, since every term in Eq. (16.14) is proportional to $\langle T_\mu\rangle$, one can carry out an average incoherently over a complete set of states and so interpret $\langle T_\mu\rangle$ also as the average polarization of a beam of particles.

Equations (16.13) and (16.14) give directly two constants of the motion:

$$(d/d\tau)[\langle T_\mu\rangle\langle T_\mu\rangle]=0, \tag{16.15}$$

$$(d/d\tau)[(dx_\mu/d\tau)\langle T_\mu\rangle]=0. \tag{16.16}$$

Equations (16.13) and (16.14) apply in any Lorentz reference frame, so it is clear that x_μ and $\langle T_\mu\rangle$ are classical Lorentz four-vectors (a detailed proof can be made using the method of Sec. 14). Consequently, the conserved quantities have the same values in all Lorentz coordinate systems. In the system moving instantaneously with the particle, for which $\langle\boldsymbol{\pi}\rangle=0$, it follows from Eqs. (15.1) that $\langle\mathbf{T}\rangle$ is $\langle\beta\boldsymbol{\sigma}\rangle$ and $\langle T_4\rangle$ is zero. One may therefore interpret $\langle T_\mu\rangle$ as the components in the laboratory system of the four-vector which is $(\langle\beta\boldsymbol{\sigma}\rangle,0)$ in the instantaneous rest system. The discussion leading to Eq. (6.19) applies here to the extent that the packet is dominated by a single momentum eigenfunction and any single particle state is an eigenstate of $\beta\boldsymbol{\sigma}\cdot\mathbf{s}$ for some direction $\mathbf{s}$. Choose the z axis in this direction so that $\langle\beta\sigma_z\rangle$ is unity and the other components are zero. It is clear then that

$$\langle T_\mu\rangle\langle T_\mu\rangle=1 \tag{16.17}$$

when a single particle is under discussion. For an incompletely polarized beam, one chooses the z axis in the direction of $\langle\beta\boldsymbol{\sigma}\rangle$ in the rest system, and it is then seen that

$$\langle T_\mu\rangle\langle T_\mu\rangle=(p_+-p_-)^2, \tag{16.18}$$

where $p_\pm$ is the probability that the particle be observed to be polarized up/down in its rest system. To evaluate the other integral implied by Eq. (16.16), one observes that $d\mathbf{x}/d\tau$ and T_4 are zero in the instantaneous rest system; hence, in all systems,

$$(dx_\mu/d\tau)\langle T_\mu\rangle=0. \tag{16.19}$$

This means that $\langle T_4\rangle$ may be eliminated from the problem:

$$\langle T_4\rangle=i\mathbf{b}\cdot\langle\mathbf{T}\rangle. \tag{16.20}$$

Equation (16.14) then yields these equations of motion for $\langle\mathbf{T}\rangle$:

$$\bar{\gamma}d\langle\mathbf{T}\rangle/dt=e\langle\mathbf{T}\rangle\times\mathbf{B}+e(\mathbf{b}\cdot\langle\mathbf{T}\rangle)\mathbf{E}. \tag{16.21}$$

In principle, this determines the polarization, given the external fields and starting conditions. One solves Eqs. (16.12) to find the orbit, evaluates the fields at the particle to obtain $\mathbf{E}$ and $\mathbf{B}$ as functions of the time alone, and then solves Eq. (16.21) for the polarization.

There is a simplification when the polarization in the rest system is used as the dependent variable. Let $\langle\mathbf{O}\rangle$ denote $\langle\beta\boldsymbol{\sigma}\rangle$ evaluated in the particle's rest system. It is related to $\langle T_\mu\rangle$ by

$$\langle\mathbf{T}\rangle=\langle\mathbf{O}\rangle+\bar{\gamma}^2(\bar{\gamma}+1)^{-1}(\mathbf{b}\cdot\langle\mathbf{O}\rangle)\mathbf{b}, \tag{16.22}$$

$$\langle T_4\rangle=i\bar{\gamma}\mathbf{b}\cdot\langle\mathbf{O}\rangle, \tag{16.23}$$

$$\langle\mathbf{O}\rangle=\langle\mathbf{T}\rangle-\bar{\gamma}(\bar{\gamma}+1)^{-1}(\mathbf{b}\cdot\langle\mathbf{T}\rangle)\mathbf{b}. \tag{16.24}$$

The equations of motion are found, from Eqs. (16.12) and (16.21), to be

$$\bar{\gamma}d\langle\mathbf{O}\rangle/dt=e\langle\mathbf{O}\rangle\times[\mathbf{B}+\bar{\gamma}(\bar{\gamma}+1)^{-1}\mathbf{E}\times\mathbf{b}]. \tag{16.25}$$

The vector $\langle\mathbf{O}\rangle$ has constant length $\langle T_\mu\rangle\langle T_\mu\rangle$ given by Eq. (16.17) or (16.18), and its motion consists of a precession about $\mathbf{B}+\bar{\gamma}(\bar{\gamma}+1)^{-1}\mathbf{E}\times\mathbf{b}$. It is interesting that, in the extreme relativistic region, an electric field is as effective as a magnetic field in disturbing the polarization. If only the magnetic field need be considered, the polarization $\langle\mathbf{O}\rangle$ precesses in the right-hand sense about $-e\mathbf{B}$ with frequency $|e|B/\bar{\gamma}$. If only the electric field need be considered, the precession is about $-e\mathbf{E}\times\mathbf{b}$ with frequency $|e\mathbf{E}\times\mathbf{b}|/(\bar{\gamma}+1)$. In the extreme relativistic region, if $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{b}$ are all perpendicular and arranged so as to produce no deflection of the particle, then according to Eq. (16.25) the polarization is undisturbed also.

The original discussions of the precession of polarization were made by Tolhoek and deGroot,[3] and by Bargmann, Michel, and Telegdi.[13]

17. Anomalous Magnetic Moment Considerations

When the Pauli[22] anomalous moment term is included, the wave equation is

$$(\gamma_\mu\pi_\mu+\tfrac{1}{8}e\hbar\mu F_{\mu\nu}\gamma_\mu\gamma_\nu-i)\Psi=0, \tag{17.1}$$

where μ is a dimensionless number measuring the strength of the anomalous contribution. The Hamiltonian is then

$$H=\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\beta+e\phi-\tfrac{1}{4}e\hbar\mu\beta\boldsymbol{\sigma}\cdot\mathbf{B}+\tfrac{1}{4}ie\hbar\mu\beta\boldsymbol{\alpha}\cdot\mathbf{E}. \tag{17.2}$$

The polarization operator may be defined by

$$\mathbf{T}=\beta\boldsymbol{\sigma}-\gamma_5\boldsymbol{\pi}, \tag{17.3}$$

$$T_4=i\boldsymbol{\sigma}\cdot\boldsymbol{\pi}. \tag{17.4}$$

It satisfies the following equations of motion:

$$d\mathbf{T}/dt=\tfrac{1}{2}ge(\boldsymbol{\sigma}\times\mathbf{B}-\gamma_5\mathbf{E})-\tfrac{1}{4}ie\mu\beta[\boldsymbol{\pi},\,\boldsymbol{\alpha}\cdot\mathbf{B}]_+ -\tfrac{1}{4}e\mu\beta[\boldsymbol{\pi},\,\boldsymbol{\sigma}\cdot\mathbf{E}]_+, \tag{17.5}$$

$$dT_4/dt=ie\boldsymbol{\sigma}\cdot\mathbf{E}-\tfrac{1}{4}ie\mu\epsilon_{ijk}[\pi_i,B_j]_+\beta\sigma_k -\tfrac{1}{4}e\mu\epsilon_{ijk}[\pi_i,E_j]_+\beta\alpha_k, \tag{17.6}$$

where the g factor is given by

$$g=2+\mu, \tag{17.7}$$

and where the fact that $\nabla\cdot\mathbf{E}$ and $\nabla\cdot\mathbf{B}$ are zero was used to simplify the T_4 equation.

The classical limit may be found the same way as in Sec. 16. One takes expectation values for a packet satisfying Eqs. (16.1) and (16.2), and discards all terms with factors of $\hbar$ in them. Equations (16.4) and (16.13) for the orbit again apply here. The following intermediate steps are used in calculating the polarization equations:

$$\langle\boldsymbol{\sigma}\rangle=\bar{\gamma}^{-1}\langle\mathbf{T}\rangle, \tag{17.8}$$

$$\langle\gamma_5\rangle=i\bar{\gamma}^{-1}\langle T_4\rangle, \tag{17.9}$$

$$\langle\beta\boldsymbol{\sigma}\rangle=\langle\mathbf{T}\rangle+i\bar{\gamma}^{-1}\langle T_4\rangle\langle\boldsymbol{\pi}\rangle, \tag{17.10}$$

$$\langle\beta\boldsymbol{\alpha}\rangle=i\bar{\gamma}^{-1}\langle\boldsymbol{\pi}\rangle\times\langle\mathbf{T}\rangle. \tag{17.11}$$

[22] W. Pauli, *Handbuch der Physik* (Springer-Verlag, Berlin, 1958), Vol. V/1, p. 157.

The equations of motion for $\langle T_\mu\rangle$ are found to be

$$\bar{\gamma}d\langle\mathbf{T}\rangle/dt=\tfrac{1}{2}ge[\langle\mathbf{T}\rangle\times\mathbf{B}-i\langle T_4\rangle\mathbf{E}] -\tfrac{1}{2}\mu e\langle\boldsymbol{\pi}\rangle[(\langle\boldsymbol{\pi}\rangle\times\mathbf{B})\cdot\langle\mathbf{T}\rangle +\bar{\gamma}\mathbf{E}\cdot\langle\mathbf{T}\rangle+i(\langle\boldsymbol{\pi}\rangle\cdot\mathbf{E})\langle T_4\rangle], \quad (17.12)$$

$$\bar{\gamma}d\langle T_4\rangle/dt=ie\langle\mathbf{T}\rangle\cdot\mathbf{E}-\tfrac{1}{2}ie\mu\bar{\gamma}(\langle\boldsymbol{\pi}\rangle\times\mathbf{B})\cdot\langle\mathbf{T}\rangle -\tfrac{1}{2}ie\mu(\langle\boldsymbol{\pi}\rangle\times\mathbf{E})\cdot(\langle\boldsymbol{\pi}\rangle\times\langle\mathbf{T}\rangle). \quad (17.13)$$

From Eqs. (17.4) and (17.8), one sees that

$$\bar{\gamma}\langle T_4\rangle=i\langle\mathbf{T}\rangle\cdot\langle\boldsymbol{\pi}\rangle; \quad (17.14)$$

by using this result and Eq. (16.4), one may rewrite Eq. (17.13) as

$$\bar{\gamma}d\langle T_4\rangle/dt=\tfrac{1}{2}ige\langle\mathbf{T}\rangle\cdot\mathbf{E}-\tfrac{1}{2}i\mu e\bar{\gamma}[(\langle\boldsymbol{\pi}\rangle\times\mathbf{B})\cdot\langle\mathbf{T}\rangle +\bar{\gamma}\mathbf{E}\cdot\langle\mathbf{T}\rangle+i(\langle\boldsymbol{\pi}\rangle\cdot\mathbf{E})\langle T_4\rangle]. \quad (17.15)$$

Equations (17.12) and (17.15) combine into the form

$$d\langle T_\mu\rangle/d\tau=\tfrac{1}{2}egF_{\mu\nu}\langle T_\mu\rangle+\tfrac{1}{2}e\mu\langle\pi_\mu\rangle\langle\pi_\nu\rangle F_{\nu\lambda}\langle T_\lambda\rangle \quad (17.16)$$

and were first given by Bargmann, Michel, and Telegdi[13] from purely classical considerations. Equation (17.14) reads that $\langle T_\mu\rangle\langle\pi_\mu\rangle$ is zero and in consequence Eq. (17.16) implies that $\langle T_\mu\rangle\langle T_\mu\rangle$ is an integral of the motion. The discussion of Sec. 16 applies equally well here so that the size of $\langle T_\mu\rangle$ may be identified with the net amount of polarization in the instantaneous rest system according to Eq. (16.18). In Eq. (17.12), $\langle T_4\rangle$ may be eliminated and $\langle\boldsymbol{\pi}\rangle$ replaced by $\bar{\gamma}\mathbf{b}$ to obtain

$$\bar{\gamma}d\langle\mathbf{T}\rangle/dt=\tfrac{1}{2}ge[\langle\mathbf{T}\rangle\times\mathbf{B}+(\mathbf{b}\cdot\langle\mathbf{T}\rangle)\mathbf{E}] -\tfrac{1}{2}\mu e\bar{\gamma}^2\mathbf{b}[(\mathbf{b}\times\mathbf{B})\cdot\langle\mathbf{T}\rangle+\mathbf{E}\cdot\langle\mathbf{T}\rangle -(\mathbf{b}\cdot\mathbf{E})(\mathbf{b}\cdot\langle\mathbf{T}\rangle)] \quad (17.17)$$

as the equation for the time development of $\langle\mathbf{T}\rangle$.

As before, there is a simplification when the polarization in the rest system, defined by Eq. (16.24), is used as the dependent variable. The equation of motion is found to be

$$\bar{\gamma}d\langle\mathbf{O}\rangle/dt=\tfrac{1}{2}ge\langle\mathbf{O}\rangle\times[\mathbf{B}+\bar{\gamma}(\bar{\gamma}+1)^{-1}\mathbf{E}\times\mathbf{b}] +\tfrac{1}{2}\mu e\bar{\gamma}^2(\bar{\gamma}+1)^{-1}\langle\mathbf{O}\rangle\times[(\mathbf{E}+\mathbf{b}\times\mathbf{B})\times\mathbf{b}], \quad (17.18)$$

and this makes it geometrically clear how the polarization varies in time. In the relativistic region the second term is of order $(\mu\bar{\gamma}/g)$ compared to the first, so at sufficiently high energies it becomes dominant and the precession is about the normal to the plane of the orbit. If $\mathbf{E}$ is zero and $\mathbf{B}$ is parallel to $\mathbf{b}$, the polarization precesses about the $-e\mathbf{B}$ direction with frequency $\frac{1}{2}g|e|B/\bar{\gamma}$. If $\mathbf{E}$ is zero and $\mathbf{B}$ is perpendicular to $\mathbf{b}$, the polarization precesses about the $-e\mathbf{B}$ direction with frequency $(1+\frac{1}{2}\mu\bar{\gamma})|e|B/\bar{\gamma}$. These last two results were found by Carrassi[23] and by Mendlowitz and Case[24] to have validity even before the classical average is taken.

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