On Flows of Conducting Fluids past Bodies

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FIRST, the general three-dimensional flow of a compressible, inviscid, perfectly conducting fluid in the presence of uniform magnetic field is considered when the undisturbed flow and magnetic field are parallel. It is shown that the flow is essentially similar to the ordinary flow of a hypothetical nonconducting gas with an appropriate pressure-density relation. Thus, approximate methods of treating such magnetohydrodynamic flows can be developed along a line similar to the von Kármán-Tsien approximation for plane gas flows. As an example, Resler's equation for small-perturbation compressible plane flows can be derived as a special case.

Second, general considerations are given to the small-perturbation theory of magnetohydrodynamic flows at large or small magnetic Reynolds numbers. In particular, using the Stokes-type approximation, the general expressions are obtained for the velocity and magnetic field as well as for the force and moment acting on a body placed in the flow at small magnetic and ordinary Reynolds numbers. As an example of application, the force and moment experienced by a sphere moving in an arbitrary manner (translating and rotating) in a uniform magnetic field are explicitly calculated.

I. STEADY THREE-DIMENSIONAL MOTION OF A COMPRESSIBLE, INVISCID, PERFECTLY CONDUCTING FLUID WITH PARALLEL VELOCITY AND MAGNETIC FIELD

The small-perturbation theory of magnetohydrodynamics has recently been developed by Resler[1,2] for the case of steady plane motion of an inviscid, perfectly conducting, compressible fluid. He has established the basic equation in a very elegant form, which bears a remarkable resemblance to Prandtl-Glauert's equation. Taniuti[3] has independently found the basic equation for plane flows, without making the assumption of small perturbation.

Here, Resler and Taniuti's equation is extended to the general case of three-dimensional flow.

Reduction of Magnetohydrodynamics to Conventional Gas Dynamics

The rationalized mks system of units is used. The steady magnetohydrodynamic flow of an inviscid, perfectly conducting, compressible fluid is governed by the following system of equations:

$$\mathrm{div}\rho\mathbf{v}=0 \tag{1}$$

$$\mathrm{grad}(\tfrac{1}{2}q^2+P+\Omega)=\mathbf{v}\times\boldsymbol{\omega}-\rho^{-1}\mathbf{B}\times\mathbf{j} \tag{2}$$

$$\mathrm{curl}(\mathbf{v}\times\mathbf{B})=0 \tag{3}$$

$$\mathrm{div}\mathbf{B}=0, \tag{4}$$

where

$$\boldsymbol{\omega}=\mathrm{curl}\mathbf{v} \tag{5}$$

$$\mathbf{j}=\mu^{-1}\,\mathrm{curl}\mathbf{B} \tag{6}$$

$$P=\int\frac{dp}{\rho}. \tag{7}$$

Equations (1)–(3) are, respectively, the equations of continuity, of motion, and of induction. $\mathbf{v}$ is the velocity vector, q is its magnitude, $\mathbf{B}$ is the magnetic induction, μ is the magnetic permeability, and Ω is the potential of the external force such as gravity. Here it is assumed that the density ρ is a certain definite function of the pressure p. $\boldsymbol{\omega}$ and $\mathbf{j}$ are the vorticity and the electric-current density, respectively.

As a particular solution of Eq. (3), let us consider the case of parallel fields: $\mathbf{v}\times\mathbf{B}=0$. Then we may write

$$\rho\mathbf{v}=\lambda\mathbf{B}, \tag{8}$$

where λ is a certain position function.

Substituting Eq. (8) in Eq. (1) and using Eq. (4), we have

$$B\partial\lambda/\partial s=0, \tag{9}$$

where $\partial/\partial s$ denotes differentiation along a streamline. Hence

$$\lambda=\mathrm{const}\quad\text{along each streamline.} \tag{10}$$

Let us assume that the conditions at infinity upstream are

$$\mathbf{B}\to B_\infty\mathbf{e},\quad \mathbf{v}\to U\mathbf{e},\quad \rho\to\rho_\infty, \tag{11}$$

the subscript ∞ indicating the condition at infinity. Then Eq. (8) gives

$$\lambda=\rho_\infty U/B_\infty. \tag{12}$$

Thus λ is constant throughout the flow field.

Now, Eq. (2) can be written as

$$\mathrm{grad}(\tfrac{1}{2}q^2+P+\Omega)=(\mathbf{B}/\rho)\times(\lambda\boldsymbol{\omega}-\mathbf{j}). \tag{13}$$

Multiplying this scalarly by $\mathbf{B}$, we have

$$B(\partial/\partial s)(\tfrac{1}{2}q^2+P+\Omega)=0. \tag{14}$$

Hence

$$\tfrac{1}{2}q^2+P+\Omega=\mathrm{const}\quad\text{along each streamline.} \tag{15}$$

[1] E. L. Resler and J. E. McCune, in *The Magnetodynamics of Conducting Fluids*, Daniel Bershader, Editor (Stanford University Press, Stanford, California, 1959), p. 120.
[2] W. R. Sears, Proc. 8th Japan Natl. Congr. Appl. Mech. 1958, 1 (1959).
[3] T. Taniuti, Progr. Theoret. Phys. (Kyoto) **19**, 749 (1958).

In view of the uniform flow conditions at infinity upstream, the constant must be an absolute constant. Therefore, Eq. (13) leads to

$$\lambda\boldsymbol{\omega}-\mathbf{j}=\kappa\mathbf{B}, \tag{16}$$

κ being a certain position function. From Eq. (16) we have

$$0=\mathrm{div}\kappa\mathbf{B}=\kappa\,\mathrm{div}\mathbf{B}+\mathbf{B}\cdot\mathrm{grad}\kappa=B\partial\kappa/\partial s.$$

Thus, the same reasoning as before gives

$$\kappa=\mathrm{const}=0,$$

so that

$$\mathbf{j}=\lambda\boldsymbol{\omega}. \tag{17}$$

Substituting Eqs. (5) and (6) into Eq. (17), we have

$$\mathrm{curl}\{[1-(\mu\lambda^2/\rho)]\mathbf{B}\}=0. \tag{18}$$

Let us define the local Alfvén velocity V_a and the local Alfvén number A as

$$V_a=B/(\mu\rho)^{\frac{1}{2}},\quad A^2=q^2/V_a^2=\lambda^2\mu/\rho=A_\infty^2\rho_\infty/\rho, \tag{19}$$

respectively, where Eq. (9) has been used. Then we can introduce a vector $\mathbf{b}$ by the relations

$$\begin{aligned}\mathbf{v}/U&=[(1-A_\infty^{-2})/(1-A^{-2})]\mathbf{b}\\ \mathbf{B}/B_\infty&=[(1-A_\infty^2)/(1-A^2)]\mathbf{b}.\end{aligned} \tag{20}$$

Thus Eqs. (4) and (18) can be expressed as

$$\mathrm{div}\sigma\mathbf{b}=0,\quad \mathrm{curl}\mathbf{b}=0, \tag{21}$$

where

$$\sigma=(1-A_\infty^2)/(1-A^2). \tag{22}$$

Let us consider, for simplicity, the case of no external force: $\Omega=0$. Then we have the Bernoulli theorem in the form

$$\tfrac{1}{2}q^2+P=\mathrm{const}. \tag{23}$$

Therefore,

$$\begin{aligned}q&=f_1(P)=f_2(\rho)=f_3(A^2)\\ b&=|\mathbf{b}|=qf_4(A^2)=f_5(A^2),\end{aligned}$$

so that

$$\sigma=f_6(A^2)=f_7(b), \tag{24}$$

where f_i denotes a certain definite function.

Equations (21) and (24) imply that $\mathbf{b}$ is essentially equivalent to the velocity vector of some irrotational flow of an ordinary nonconducting fluid whose density σ varies with the velocity b in accordance with the relation (24). Thus, the magnetohydrodynamics in this case is completely reduced to the conventional gas dynamics of a hypothetical gas. For example, the basic equation for the magnetohydrodynamic flows can be readily obtained by employing the well-known procedures of conventional gas dynamics.

Basic Equation

For ordinary gas dynamics, Bernoulli's equation is written in the differential form as

$$qdq+dp/\rho=0, \tag{25}$$

and the sound velocity c is given by

$$c^2=dp/d\rho=-\rho qdq/dp. \tag{26}$$

Therefore, for our hypothetical gas, the wave velocity a, which might be called the pseudosound velocity, is given by

$$a^2=-\sigma b(db/d\sigma)=-\tfrac{1}{2}(db^2/dA^2)(d\log\sigma/dA^2)^{-1}. \tag{27}$$

Now, by Eq. (20), we have

$$b^2=[(1-A^{-2})/(1-A_\infty^{-2})]^2(q/U)^2. \tag{28}$$

By use of Eqs. (22) and (28) and the relations

$$d\log\sigma/dA^2=-A^{-2},\quad dq^2/dA^2=2c^2/A^2,$$

we obtain

$$a^2=[(A^2+M^2-1)/A^2M^2]b^2, \tag{29}$$

where

$$M=q/c, \tag{30}$$

M being the local Mach number. By analogy, we may define the pseudo-Mach number m as

$$m^2=b^2/a^2=A^2M^2/(A^2+M^2-1). \tag{31}$$

The equation of motion can be expressed in terms of the pseudovelocity potential ϕ, pseudovelocity $\mathbf{b}$, and pseudosound velocity a in the form

$$(a^2-b_x^2)\frac{\partial^2\phi}{\partial x^2}+(a^2-b_y^2)\frac{\partial^2\phi}{\partial y^2}+(a^2-b_z^2)\frac{\partial^2\phi}{\partial z^2}-2b_yb_z\frac{\partial^2\phi}{\partial y\partial z}-2b_zb_x\frac{\partial^2\phi}{\partial z\partial x}-2b_xb_y\frac{\partial^2\phi}{\partial x\partial y}=0, \tag{32}$$

where

$$\mathbf{b}=(b_x,b_y,b_z)=\mathrm{grad}\phi. \tag{33}$$

For the case of plane flows, Eq. (32) reduces to what Taniuti has obtained by quite a different method.

For small perturbations,

$$b_x\doteqdot 1,\quad b_y\doteqdot 0,\quad b_z\doteqdot 0,$$

we have, from Eq. (29),

$$a^2\doteqdot(A_\infty^2+M_\infty^2-1)/A_\infty^2M_\infty^2=1/m_\infty^2.$$

Hence Eq. (32) becomes

$$(1-m_\infty^2)\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2=0. \tag{34}$$

Since

$$1-m_\infty^2=(1-M_\infty^2)(A_\infty^2-1)/(A_\infty^2+M_\infty^2-1), \tag{35}$$

Eq. (34) is nothing but Resler's equation (originally given in the two-dimensional form).

It is seen from Eq. (31) that the level curves of m^2

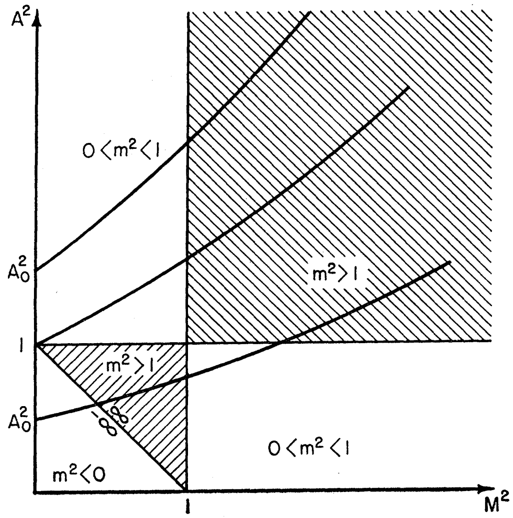

FIG. 1.

in the (M^2,A^2) plane are hyperbolas. There are five regions to be distinguished (Fig. 1):

I:	$A^2>1,\ M^2<1$	$0<m^2<1$
II:	$A^2<1,\ M^2>1$	
III:	$A^2>1,\ M^2>1$	$m^2>1$
IV:	$A^2+M^2>1,\ A^2<1,\ M^2<1$	
V:	$A^2+M^2<1$	$m^2<0.$

For the regions I and II the flow is of elliptic type, the regions III and IV are hyperbolic and the region V is again elliptic. However, it should be noted that the wave velocity for the region V is imaginary, since $m^2<0$ there.

In the small-perturbation theory, the values of A^2 and M^2 vary from point to point in the flow field, following a certain curve in the (M^2,A^2) plane, which is fixed by the free-stream conditions $(M_\infty{}^2,A_\infty{}^2)$. For example, for an ideal gas,

$$M^2=q^2/c^2=(q^2/c_0{}^2)\{1-[(\gamma-1)/2](q^2/c_0{}^2)\}^{-1}, \quad (36)$$

$$A^2=A_0{}^2(\rho_0/\rho)$$
$$=A_0{}^2\{1-[(\gamma-1)/2](q^2/c_0{}^2)\}^{-1(\gamma-1)}, \quad (37)$$

where the subscript 0 denotes the stagnation condition. On eliminating q^2, we obtain

$$A^2=A_0{}^2\{1+[(\gamma-1)/2]M^2\}^{1/(\gamma-1)}. \quad (38)$$

Thus A^2 is a monotonically increasing function of M^2. It is seen that if $A_0{}^2>1$, m^2 increases monotonically with M^2, but if $A_0{}^2<1$, m^2 varies in a complicated manner; that is, it first decreases to $-\infty$, then jumps up to $+\infty$, and then continues to decrease down to a minimum, which is less than 1, and then increases monotonically to $+\infty$.

Suggested Methods of Calculation

If the pressure-density relation $p=f(\rho)$ of the gas is given, we can find the relation $\sigma=f(b)$, so that the calculation of flow quantities proceeds in quite the same manner as for ordinary gas dynamics. In fact, the boundary condition is also the same:

$$\partial\phi/\partial n=0 \quad \text{on the body surface.}$$

In general, the relation $\sigma=f(b)$ can be found only by numerical means. But we can develop approximate methods similar in principle to von Kármán-Tsien's approximation.

(*i*) *Flow at free-stream Mach number M_∞ and Alfvên number A_∞.* The free-stream pseudo-Mach number m_∞ is given by Eq. (31) as

$$m_\infty{}^2=A_\infty{}^2M_\infty{}^2/(A_\infty{}^2+M_\infty{}^2-1). \quad (39)$$

Calculate the pseudovelocity $\mathbf{b}$ of the hypothetical gas, treating it as if it were a real gas at free-stream Mach number m_∞. Then the velocity $\mathbf{v}$ is found by means of Eqs. (20) and (37).

(*ii*) *Small-perturbation theory.* On putting

$$\mathbf{v}=(U+v_x',\ v_y',\ v_z'),\quad \mathbf{v}'=O(\epsilon)$$
$$\mathbf{b}=(1+b_x',\ b_y',\ b_z'),\quad \mathbf{b}'=O(\epsilon),$$

substituting in Eq. (28), and neglecting second-order small quantities, we readily have

$$v_x'/U=\{1-\tfrac{1}{2}[M_\infty{}^2A_\infty{}^{-2}/(1-A_\infty{}^{-2})]\}^{-1}b_x'. \quad (40)$$

Here b_x' can be obtained by the conventional small-perturbation theory such as that of Prandtl and Glauert.

(*iii*) *M^2-expansion method.* For thick bodies at low Mach number, we can employ the M^2-expansion method for the calculation of $\mathbf{b}$-field. Thus, we can easily find

$$\mathbf{v}/U=\mathbf{b}_0+[M_\infty{}^2/(1-A_\infty{}^{-2})]$$
$$\times[\mathbf{b}_1-\tfrac{1}{2}A_\infty{}^{-2}(b_0{}^2-1)\mathbf{b}_0], \quad (41)$$

where $\mathbf{b}_0$ and $\mathbf{b}_1$ are the incompressible-flow solution and the first-order correction in the M^2-expansion method, respectively. In fact, for $M_\infty\ll1$, we have

$$m_\infty{}^2=A_\infty{}^2M_\infty{}^2/(A_\infty{}^2-1)=M_\infty{}^2/(1-A_\infty{}^{-2}) \quad (42)$$

and

$$\mathbf{b}=\mathbf{b}_0+m_\infty{}^2\mathbf{b}_1. \quad (43)$$

Substituting Eqs. (42) and (43) into Eq. (20) gives Eq. (41).

It should be emphasized that the foregoing procedures are applicable to the three-dimensional problem, whereas the hodograph method well known in conventional gas dynamics is only applicable to two-dimensional problems.

II. CONSIDERATIONS ON SMALL-PERTURBATION METHODS

The study of magnetohydrodynamic flows of an electrically conducting fluid was initiated by Chester,[4] who treated the case of slow motion of a sphere in the direction parallel to a uniform magnetic field. His treatment is of the Stokes-type approximation. Later, Yosinobu and Kakutani[5] applied Chester's method of analysis to the two-dimensional flow past a circular cylinder, the imposed magnetic field being either parallel or perpendicular to the undisturbed-flow direction. Also, the Oseen-type approximation has been introduced for the study of flows at small magnetic Reynolds number; the flow past a circular cylinder was dealt with by Yosinobu[6] and the flow past a sphere by Gotoh,[7] both for the case of parallel fields. Quite recently Gotoh[8] has extended Chester's analysis to cover the case of general three-dimensional flow, and, in particular, treated the flow past a sphere perpendicular to the magnetic field. Here the electric field induced by the conducting fluid across the magnetic field plays an essential part, in contrast to the fact that in the two-dimensional or axisymmetric case such an induced electric field does not make its appearance.

Considerable developments in the field of linearized magnetoaerodynamics have recently been made by Sears and his group,[1,2,9,10] for inviscid fluids.

In the following, some general considerations are given to the small-perturbation treatment of an incompressible fluid with finite viscosity and electrical conductivity, and basic equations suitable for the treatment of some representative limiting cases are derived.

The equations of motion and of induction are, respectively, written in the form

$$\rho\frac{\partial \mathbf{V}}{\partial t}=-\rho(\mathbf{V}\cdot\text{grad})\mathbf{V}+\frac{1}{\mu}(\mathbf{B}\cdot\text{grad})\mathbf{B}-\text{grad}\left(p+\frac{B^2}{2\mu}\right)+\rho\nu\Delta\mathbf{V} \quad (44)$$

and

$$\frac{\partial \mathbf{B}}{\partial t}=\text{curl}(\mathbf{V}\times\mathbf{B})-\frac{1}{\sigma\mu}\,\text{curl curl}\mathbf{B}, \quad (45)$$

where ν is the kinematic viscosity and σ is the electrical conductivity; ρ, ν, and σ are assumed to be constant.

Let us write

$$\mathbf{B}=B(\mathbf{e}+\mathbf{b}^*),\quad \mathbf{V}=U(\mathbf{e}'+\mathbf{v}), \quad (46)$$

where $\mathbf{e}=(1,0,0)$, $\mathbf{e}'$ is an arbitrary unit vector, and $\mathbf{b}^*,\mathbf{v}=O(\epsilon)$. On neglecting $O(\epsilon^2)$, Eq. (44) becomes

$$\left(\Delta-\frac{1}{\nu}\frac{\partial}{\partial t}-\frac{U}{\nu}\frac{\partial}{\partial s}\right)\mathbf{v}=-\frac{B^2}{\rho\nu\mu U}\frac{\partial \mathbf{b}^*}{\partial x}+\text{grad}\bar{p}, \quad (47)$$

$$\bar{p}=[p+(B^2/\mu)b_x^*]/\rho\nu U, \quad (48)$$

while the induction equation (45) reduces to

$$\left(\Delta-\frac{1}{\nu_m}\frac{\partial}{\partial t}-\frac{U}{\nu_m}\frac{\partial}{\partial s}\right)\mathbf{b}^*=-\frac{U}{\nu_m}\frac{\partial \mathbf{v}}{\partial x}, \quad (49)$$

where $\nu_m=1/\sigma\mu$, and $\partial/\partial s\equiv\mathbf{e}'\cdot\text{grad}=$ differentiation along $\mathbf{e}'$. The form of Eq. (49) suggests putting

$$\mathbf{b}^*=R_m\mathbf{b},\quad R_m=UL/\nu_m, \quad (50)$$

L being the characteristic length (taken $L=1$, for simplicity). Thus Eqs. (47) and (49) become, in nondimensional forms,

$$\{\Delta-R(\partial/\partial t+\partial/\partial s)\}\mathbf{v}=-H^2\partial\mathbf{b}/\partial x+\text{grad}\bar{p} \quad (51)$$

$$\{\Delta-R_m(\partial/\partial t+\partial/\partial s)\}\mathbf{b}=-\partial\mathbf{v}/\partial x, \quad (52)$$

where

$$R=UL/\nu,\qquad R_m=UL/\nu_m=\sigma\mu UL$$
$$H^2=B^2\sigma L^2/\rho\nu=R_mR/A^2,\quad A=U/V_a=(U/B)(\rho\mu)^{\frac{1}{2}}. \quad (53)$$

R, R_m, H, and A are the Reynolds number, magnetic Reynolds number, Hartmann number, and Alfvén number, respectively.

Since

$$\text{div}\mathbf{v}=0,\quad \text{div}\mathbf{b}=0, \quad (54)$$

we have, from Eq. (51),

$$\Delta\bar{p}=0. \quad (55)$$

Now $\bar{p}$ can be eliminated from Eqs. (51) and (52) by putting

$$\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1,\quad \mathbf{b}=\mathbf{b}_0+\mathbf{b}_1, \quad (56)$$

where

$$\mathbf{v}_0=R_m(\partial/\partial t+\partial/\partial s)\,\text{grad}\phi,\quad \mathbf{b}_0=(\partial/\partial x)\,\text{grad}\phi \quad (57)$$

$$\bar{p}=\{H^2\partial^2/\partial x^2-RR_m(\partial/\partial t+\partial/\partial s)^2\}\phi \quad (58)$$

$$\Delta\phi=0. \quad (59)$$

Thus, we finally have a system of equations

$$\{\Delta-R(\partial/\partial t+\partial/\partial s)\}\mathbf{v}_1=-H^2\partial\mathbf{b}_1/\partial x \quad (60)$$

$$\{\Delta-R_m(\partial/\partial t+\partial/\partial s)\}\mathbf{b}_1=-\partial\mathbf{v}_1/\partial x \quad (61)$$

$$\text{div}\mathbf{v}_1=0,\quad \text{div}\mathbf{b}_1=0. \quad (62)$$

Special Cases

(*i*) *Small-perturbation theory of inviscid flow at small magnetic Reynolds numbers* $[R\rightarrow\infty, R_m=O(1)]$. In

[4] W. Chester, J. Fluid Mech. **3**, 304 (1957).
[5] H. Yosinobu and T. Kakutani, J. Phys. Soc. Japan **14**, 1433 (1959).
[6] H. Yosinobu, J. Phys. Soc. Japan **15**, 175 (1960).
[7] K. Gotoh, J. Phys. Soc. Japan **15**, 189 (1960).
[8] K. Gotoh, J. Phys. Soc. Japan **15**, 696 (1960).
[9] J. E. McCune, J. Fluid Mech. **7**, 449 (1960).
[10] W. R. Sears and E. L. Resler, J. Fluid Mech. **5**, 257 (1959).

this case Eqs. (60) and (61) become, respectively,

$$(\partial/\partial t+\partial/\partial s)\mathbf{v}_1=(1/A^2)\partial\mathbf{b}_1^*/\partial x, \tag{63}$$

$$\{\Delta-R_m(\partial/\partial t+\partial/\partial s)\}\mathbf{b}_1^*=-R_m\partial\mathbf{v}_1/\partial x, \tag{64}$$

while Eqs. (57)–(59) become

$$\mathbf{v}_0=(\partial/\partial t+\partial/\partial s)\ \mathrm{grad}\phi^*, \quad \mathbf{b}_0^*=(\partial/\partial x)\ \mathrm{grad}\phi^* \tag{65}$$

$$\bar{p}=R\{(1/A^2)\partial^2/\partial x^2-\partial/\partial t+\partial/\partial s^2\}\phi^* \tag{66}$$

$$\Delta\phi^*=0. \tag{67}$$

Finally, we have

$$\mathbf{V}=U(\mathbf{e}'+\mathbf{v}), \quad \mathbf{v}=\mathbf{v}_0+\mathbf{v}_1 \tag{68}$$

$$\mathbf{B}=B(\mathbf{e}+\mathbf{b}^*), \quad \mathbf{b}^*=\mathbf{b}_0^*+\mathbf{b}_1^* \tag{69}$$

$$p=\rho\nu U\bar{p}-(B^2/\mu)b_x^*. \tag{70}$$

(*ii*) *Small-perturbation theory of inviscid flow at large magnetic Reynolds numbers* ($R\to\infty, R_m\gg 1$). In this case, it is convenient to rewrite Eqs. (63) and (64) in the form

$$\partial\mathbf{v}_1/\partial x=(\partial/\partial t+\partial/\partial s)\mathbf{b}_1^*-(1/R_m)\Delta\mathbf{b}_1^*, \tag{71}$$

$$(\partial/\partial t+\partial/\partial s)\mathbf{v}_1=(1/A^2)\partial\mathbf{b}_1^*/\partial x. \tag{72}$$

(*iii*) $R=R_m$; *hence* $H=R/A=R_m/A$. For simplicity, let us first consider the steady flow. If we define $\mathbf{u}_1$ and $\mathbf{u}_2$ by

$$\mathbf{u}_1=\mathbf{v}_1+H\mathbf{b}_1, \quad \mathbf{u}_2=\mathbf{v}_1-H\mathbf{b}_1, \tag{73}$$

so that

$$\mathbf{v}_1=\tfrac{1}{2}(\mathbf{u}_1+\mathbf{u}_2), \quad \mathbf{b}_1=(1/2H)(\mathbf{u}_1-\mathbf{u}_2), \tag{74}$$

then Eqs. (60) and (61) lead to

$$(\Delta-R\partial/\partial s+H\partial/\partial x)\mathbf{u}_1=0 \tag{75}$$

and

$$(\Delta-R\partial/\partial s-H\partial/\partial x)\mathbf{u}_2=0. \tag{76}$$

By orthogonal transformations,

$$(x,y)\to(x',y') \quad \text{and} \quad (x,y)\to(x'',y''),$$

with

$$\left.\begin{aligned}\tan\theta'&=A\sin\alpha/(A\cos\alpha-1)\\ \tan\theta''&=A\sin\alpha/(A\cos\alpha+1),\end{aligned}\right\} \tag{77}$$

$\theta'=\angle xOx'$, $\theta''=\angle xOx''$, $(0\leqq\theta',\theta''<\pi)$, Eqs. (75) and (76) reduce, respectively, to

$$(\Delta-H'\partial/\partial x')\mathbf{u}_1=0, \quad (\Delta-H''\partial/\partial x'')\mathbf{u}_2=0, \tag{78}$$

where

$$\left.\begin{aligned}H'&=H(1+A^2-2A\cos\alpha)^{\frac{1}{2}}\\ H''&=H(1+A^2+2A\cos\alpha)^{\frac{1}{2}}.\end{aligned}\right\} \tag{79}$$

Here α is the angle between $\mathbf{e}$ and $\mathbf{e}'$, i.e., the angle between the undisturbed flow and magnetic field (Fig. 2). By analogy to the Oseen approximation, Eqs. (78) imply that two wake regions W_1, W_2 develop along the x' and x'' axes, whose widths are proportional to $(H')^{-\frac{1}{2}}$ and $(H'')^{-\frac{1}{2}}$, respectively. For the unsteady flow, we have only to replace Δ by $\Delta-R\partial/\partial t$ in Eqs. (78).

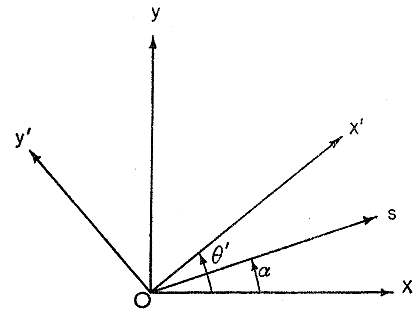

FIG. 2.

(*iv*) *Parallel fields at infinity* ($\mathbf{e}=\mathbf{e}'$). In this case we can put

$$ds=dx.$$

On putting

$$\mathbf{u}_i=\mathbf{v}_1+\lambda_i\mathbf{b}_1 \quad (i=1,2), \tag{80}$$

we have

$$(\Delta-R_i\partial/\partial x)\mathbf{u}_i=0, \tag{81}$$

where

$$R_{1,2}=\tfrac{1}{2}(R+R_m)\mp[\tfrac{1}{4}(R-R_m)^2+H^2]^{\frac{1}{2}} \tag{82}$$

$$\lambda_{1,2}=\tfrac{1}{2}(R-R_m)\pm[\tfrac{1}{4}(R-R_m)^2+H^2]^{\frac{1}{2}}. \tag{83}$$

It is obvious that $R_1\gtrless 0$ as $A^2\gtrless 1$, whereas $R_2>0$. This implies that the wake W_2 is situated always around the positive x axis, while the wake W_1 appears around the negative or positive x axis, as A^2 is smaller or greater than 1.

In particular, when $R\to 0$, $R_m\to 0$ (H being kept finite),

$$R_1\to -H, \quad R_2\to H$$

$$\lambda_1\to H, \quad \lambda_2\to -H.$$

This case corresponds to the Stokes approximation.

(*v*) *Stokes approximation* ($R\to 0, R_m\to 0$). This case can be obtained as a special case of either (iii) or (iv). Thus we have

$$\mathbf{v}=\tfrac{1}{2}(\mathbf{u}_1+\mathbf{u}_2) \tag{84}$$

$$\mathbf{b}=(\partial/\partial x)\ \mathrm{grad}\phi+(1/2H)(\mathbf{u}_1-\mathbf{u}_2) \tag{85}$$

$$\bar{p}=H^2\partial^2\phi/\partial x^2 \tag{86}$$

$$p/\rho\nu U=\bar{p}-H^2b_x=(H/2)(u_{2x}-u_{1x}), \tag{87}$$

where

$$(\Delta+H\partial/\partial x)\mathbf{u}_1=0, \quad (\Delta-H\partial/\partial x)\mathbf{u}_2=0 \tag{88}$$

$$\Delta\phi=0. \tag{89}$$

III. GENERAL SOLUTIONS OF THE STOKES APPROXIMATION

In the preceding section, the Stokes-type approximation for the magnetohydrodynamic flows at small magnetic and ordinary Reynolds numbers have been formulated in the form given by Eqs. (84)–(89). At first sight, it might appear that this formulation would be more complicated than Chester's, which deals only with the velocity field and considers the magnetic field to be unaffected by the slow motion of the fluid. However, as is seen in the following, the governing equations are essentially the same as for Chester's formulation. Rather, in Chester's formulation it is somewhat embarrassing that the number of boundary conditions is too small in comparison with the order of the governing differential equations, whereas in our formulation in terms of **v** and **b**, the boundary conditions for **v** and **b** just suffice to determine the solution uniquely. As a matter of fact, Gotoh[8] has extended Chester's analysis to the three-dimensional flows, and shown that it is necessary in the general case to take the induced electric field potential as a further unknown variable besides the flow velocity **v**. At any rate, it seems to the author that the formulation in terms of **v** and **b** is most convenient, since it leads to the general expressions for magnetohydrodynamic flows as well as for the force and moment acting on a submerged body in a very natural and straightforward manner.

General Expressions for the Velocity and Magnetic Field

From Eqs. (84) and (85), we have

$$\mathbf{u}_1=\mathbf{v}+H[\mathbf{b}-(\partial/\partial x)\operatorname{grad}\phi]. \tag{90}$$

Hence, by Eq. (86)

$$u_{1x}=v_x+Hb_x-H^{-1}\bar{p}. \tag{91}$$

This is regular and one-valued in the flow field. Since u_{1x} satisfies Eq. (88) and vanishes at infinity, it must be of the form

$$u_{1x}=\sum_{m,n=0}^{\infty}\left(A_{mn}{}^{(0)}+A_{mn}{}^{(1)}\frac{\partial}{\partial x}\right)\frac{\partial^{m+n}}{\partial y^m\partial z^n}\left(\frac{e^{-k(r+x)}}{r}\right). \tag{92}$$

Since $\bar{p}$ is a one-valued harmonic function, we can write

$$\bar{p}=H^2\frac{\partial^2\phi}{\partial x^2}=\sum_{m,n=0}^{\infty}\left(D_{mn}{}^{(0)}+D_{mn}{}^{(1)}\frac{\partial}{\partial x}\right)\frac{\partial^{m+n}}{\partial y^m\partial z^n}\left(\frac{1}{r}\right), \tag{93}$$

whence, on integration we obtain

$$H^2\frac{\partial\phi}{\partial x}=\sum\frac{\partial^{m+n}}{\partial y^m\partial z^n}\left\{D_{mn}{}^{(0)}\log(r+x)+D_{mn}{}^{(1)}\frac{1}{r}\right\}+f(y,z), \tag{94}$$

$f(y,z)$ being an arbitrary harmonic function

$$\partial^2 f/\partial y^2+\partial^2 f/\partial z^2=0. \tag{95}$$

Now, from Eq. (90), we have

$$u_{1y}=v_y+H[b_y-\partial^2\phi/\partial x\partial y].$$

Let us consider the expression

$$u_{1y}+\frac{1}{H}\sum\frac{\partial^{m+n}}{\partial y^m\partial z^n}D_{mn}{}^{(0)}\frac{\partial}{\partial y}E[k(r+x)]+\frac{1}{H}\frac{\partial f}{\partial y}$$
$$=\frac{1}{H}\sum\frac{\partial^{m+n}}{\partial y^m\partial z^n}\frac{\partial}{\partial y}(D_{mn}{}^{(0)}\{E[k(r+x)]-\log(r+x)\}-D_{mn}{}^{(1)}/r)+v_y+Hb_y, \tag{96}$$

where $E(\xi)$ is a function defined by

$$E(\xi)=\int_\infty^\xi\frac{e^{-\xi}}{\xi}d\xi=\gamma+\log\xi+\sum_{n=1}^{\infty}\frac{(-1)^n\xi^n}{n\cdot n!}, \tag{97}$$

$\gamma=0.5772\cdots$ being Euler's constant. $E[k(r+x)]$ is a particular solution of

$$(\Delta+2k\partial/\partial x)\mathbf{u}_1=0;\quad H=2k. \tag{98}$$

The lhs of Eq. (96) satisfies Eq. (98), and moreover it is one-valued and regular [as seen from the rhs of Eq. (96)]. Hence it must be expressible in the form similar to Eq. (92). Thus

$$u_{1y}=\sum\frac{\partial^{m+n}}{\partial y^m\partial z^n}\left\{\left(B_{mn}{}^{(0)}+B_{mn}{}^{(1)}\frac{\partial}{\partial x}\right)\frac{e^{-k(r+x)}}{r}-\frac{1}{2k}D_{mn}{}^{(0)}\frac{\partial}{\partial y}E[k(r+x)]\right\}-\frac{1}{2k}\frac{\partial f}{\partial y}. \tag{99}$$

u_{1z} and u_2 can be obtained in a similar manner. Thus, finally we have, by Eqs. (84)–(86),

$$\mathbf{v}=\tfrac{1}{2}\sum\frac{\partial^{m+n}}{\partial y^m\partial z^n}\left\{\left[\left(\mathbf{C}_{mn}{}^{(0)}+\mathbf{C}_{mn}{}^{(1)}\frac{\partial}{\partial x}\right)\frac{e^{-k(r+x)}}{r}+\left(\mathbf{C}'_{mn}{}^{(0)}+\mathbf{C}'_{mn}{}^{(1)}\frac{\partial}{\partial x}\right)\frac{e^{-k(r-x)}}{r}\right]-\frac{1}{k}D_{mn}{}^{(0)}\left(0,\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)g_e\right\} \tag{100}$$

$$\mathbf{b}=\frac{1}{4k}\sum\frac{\partial^{m+n}}{\partial y^m\partial z^n}\left\{\left[\left(\mathbf{C}_{mn}{}^{(0)}+\mathbf{C}_{mn}{}^{(1)}\frac{\partial}{\partial x}\right)\frac{e^{-k(r+x)}}{r}-\left(\mathbf{C}'_{mn}{}^{(0)}+\mathbf{C}'_{mn}{}^{(1)}\frac{\partial}{\partial x}\right)\frac{e^{-k(r-x)}}{r}\right]+\frac{1}{k}\left(0,\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times\left(D_{mn}{}^{(0)}g_o+D_{mn}{}^{(1)}\frac{1}{r}\right)+\frac{\partial^2\phi}{\partial x^2}\mathbf{e}\right\} \tag{101}$$

$$\bar{p}=4k^2\frac{\partial^2\phi}{\partial x^2}=\sum\frac{\partial^{m+n}}{\partial y^m\partial z^n}\left(D_{mn}{}^{(0)}+D_{mn}{}^{(1)}\frac{\partial}{\partial x}\right)\frac{1}{r}, \tag{102}$$

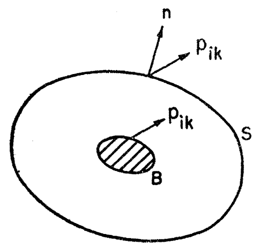

FIG. 3.

where

$$g_e=\tfrac{1}{2}\{E[k(r+x)]+E[k(r-x)]-\log(y^2+z^2)\} \tag{103}$$

$$g_O=\tfrac{1}{2}\{E[k(r-x)]-E[k(r+x)] -\log[(r-x)/(r+x)]\}. \tag{104}$$

On putting

$$f_e=\tfrac{1}{2}\{[e^{-k(r+x)}/r]+[e^{-k(r-x)}/r]\} \tag{105}$$

$$f_O=\tfrac{1}{2}\{[e^{-k(r+x)}/r]-[e^{-k(r-x)}/r]\}, \tag{106}$$

Eqs. (100) and (101) can also be written in the form

$$\begin{aligned}\mathbf{v}=\cdots\{[(\mathbf{C}_{mn}{}^{(0)}+\mathbf{C}'_{mn}{}^{(0)})f_e &+(\mathbf{C}_{mn}{}^{(1)}+\mathbf{C}'_{mn}{}^{(1)})\partial f_e/\partial x\\ &+(\mathbf{C}_{mn}{}^{(0)}-\mathbf{C}'_{mn}{}^{(0)})f_O\\ &+(\mathbf{C}_{mn}{}^{(1)}-\mathbf{C}'_{mn}{}^{(1)})\partial f_O/\partial x]-\cdots\}\end{aligned} \tag{107}$$

$$\begin{aligned}\mathbf{b}=\cdots\{[(\mathbf{C}_{mn}{}^{(0)}-\mathbf{C}'_{mn}{}^{(0)})f_e &+(\mathbf{C}_{mn}{}^{(1)}-\mathbf{C}'_{mn}{}^{(1)})\partial f_e/\partial x\\ &+(\mathbf{C}_{mn}{}^{(0)}+\mathbf{C}'_{mn}{}^{(0)})f_O\\ &+(\mathbf{C}_{mn}{}^{(1)}+\mathbf{C}'_{mn}{}^{(1)})\partial f_O/\partial x]+\cdots\};\end{aligned} \tag{108}$$

f_e and g_e are even and f_O and g_O are odd with respect to x. The expressions (107) and (108) are convenient for the treatment of symmetrical bodies such as a sphere.

Because of the relations $\text{div}\mathbf{v}=0$ and $\text{div}\mathbf{b}=0$, the coefficients $\mathbf{C}_{mn}{}^{(0)}(A_{mn}{}^{(0)},B_{mn}{}^{(0)},C_{mn}{}^{(0)})$, $\mathbf{C}'_{mn}{}^{(0)}(A'_{mn}{}^{(0)}, B'_{mn}{}^{(0)},C'_{mn}{}^{(0)})$, are not entirely independent, but are subjected to the conditions

$$\begin{aligned}&D_{mn}{}^{(0)}\\ &=A_{m-2,n}{}^{(1)}+A_{m,n-2}{}^{(1)}-B_{m-1,n}{}^{(0)}-C_{m,n-1}{}^{(0)}\\ &D_{mn}{}^{(0)}\\ &=-2k(A_{mn}{}^{(0)}-2kA_{mn}{}^{(1)}+B_{m-1,n}{}^{(1)}+C_{m,n-1}{}^{(1)})\\ &D_{mn}{}^{(0)}\\ &=A'_{m-2,n}{}^{(1)}+A'_{m,n-2}{}^{(1)}-B'_{m-1,n}{}^{(0)}-C'_{m,n-1}{}^{(0)}\\ &D_{mn}{}^{(0)}\\ &=2k(A'_{mn}{}^{(0)}+2kA'_{mn}{}^{(1)}+B'_{m-1,n}{}^{(1)}+C'_{m,n-1}{}^{(1)}).\end{aligned} \tag{109}$$

Here constants with a negative subscript should be taken to be zero. For example,

$$\begin{aligned}&D_{00}{}^{(0)}=0\\ &A_{00}{}^{(0)}=2kA_{00}{}^{(1)}\\ &A'_{00}{}^{(0)}=-2kA'_{00}{}^{(1)}\\ &B_{00}{}^{(0)}=B'_{00}{}^{(0)}=-D_{10}{}^{(0)}\\ &C_{00}{}^{(0)}=C'_{00}{}^{(0)}=-D_{01}{}^{(0)},\cdots.\end{aligned} \tag{110}$$

From expression (100), it is obvious that the $\mathbf{v}$ field consists of two wake regions and outer region, namely, the terms containing $\exp[-k(r+x)]$ or $E[k(r+x)]$ represent a paraboloidal wake extending in the negative x direction, while those containing $\exp[-k(r-x)]$ or $E[k(r-x)]$, the wake extending in the positive x direction. The outer region is that influenced by the $D_{mn}{}^{(0)}$ terms, and has a character of two-dimensional irrotational motion, since the flow there is given by

$$\left(0,\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\frac{1}{4k}\sum D_{mn}{}^{(0)}\frac{\partial^{m+n}}{\partial y^m\partial z^n}\log(y^2+z^2), \tag{111}$$

which represents a uniform distribution of various kinds of multiplets on the x axis.

General Formulas for the Force and Moment Acting on a Body Placed in an Electrically Conducting Fluid Flow in the Presence of Uniform Magnetic Field

By considering the conservation of momentum, the force $\mathbf{F}$ acting on a body B can be calculated by

$$F_i=\int\int_S(p_{ik}-\rho V_iV_k)n_kdS, \tag{112}$$

where S is an arbitrary closed surface enclosing the body B, and $\mathbf{n}$ its outward normal (Fig. 3). p_{ik} is the stress tensor given by

$$p_{ik}=-p\delta_{ik}+\rho\nu e_{ik}+T_{ik}, \tag{113}$$

where

$$e_{ik}=\partial V_i/\partial x_k+\partial V_k/\partial x_i \tag{114}$$

$$T_{ik}=\mu^{-1}(B_iB_k-\tfrac{1}{2}B^2\delta_{ik}), \tag{115}$$

T_{ik} being the Maxwell stress in the magnetohydrodynamic approximation. Since

$$\mathbf{V}=U(\mathbf{e}'+\mathbf{v}),\quad \mathbf{B}=B_\infty(\mathbf{e}+R_m\mathbf{b}), \tag{116}$$

we have, on neglecting second-order small quantities,

$$e_{ik}=(\partial v_i/\partial x_k+\partial v_k/\partial x_i) \tag{117}$$

$$T_{ik}=\frac{B_\infty{}^2}{\mu}R_m\begin{bmatrix}(2R_m)^{-1}+b_x & b_y & b_z\\ b_y & -(2R_m)^{-1}-b_x & 0\\ b_z & 0 & -(2R_m)^{-1}-b_x\end{bmatrix}. \tag{118}$$

In the Stokes approximation, $\rho V_i V_k$ can be neglected in Eq. (112), so that

$$F_i=\iint p_{ik}n_k dS, \tag{119}$$

where

$$\frac{p_{ik}}{\rho\nu U}=e_{ik}-H^2\frac{\partial^2\phi}{\partial x^2}\delta_{ik}+H^2\begin{bmatrix}2b_x & b_y & b_z\\ b_y & 0 & 0\\ b_z & 0 & 0\end{bmatrix}. \tag{120}$$

Similarly, the moment $\mathbf{M}$ acting on the body is given by

$$M_i=\iint (x_j p_{kl}-x_k p_{jl})n_l dS, \tag{121}$$

(i,j,k) being even permutation of the numbers 1, 2, 3.

On remembering the wake property of the $\mathbf{v}$ and $\mathbf{b}$ fields, it is convenient to adopt as the control surface S a very large circular cylinder with its axis coinciding with the x axis. Its cylindrical surface S_3 is to be taken so large as to be completely away from the magnetic wakes. Then the diameter $2h$ and the length $2l$ of the cylinder can be taken to be in such a relation

$$hk^2\sim l \tag{122}$$

(Fig. 4). On S_3, the terms containing $\exp[-k(r\pm x)]$ or $E[k(r\pm x)]$ are negligible, and the normal vector has the components

$$\mathbf{n}=(0, n_y, n_z),$$

while on the bases S_1 and S_2,

$$\mathbf{n}=(\pm 1, 0, 0).$$

Thus the evaluation of the integrals (119) and (121) are considerably simplified.

On employing the expressions for $\mathbf{v}$ and $\mathbf{b}$ as given by Eqs. (100) and (101), it is found that the viscous stress $\rho\nu e_{ik}$ contributes nothing to $\mathbf{F}$ and $\mathbf{M}$. The final result is

$$\begin{aligned}F_x&=-4\pi\rho\nu U(A_{00}{}^{(0)}+A'_{00}{}^{(0)}+D_{00}{}^{(1)})\\ F_y&=4\pi\rho\nu U D_{10}{}^{(0)}\\ F_z&=4\pi\rho\nu U D_{01}{}^{(0)};\end{aligned} \tag{123}$$

$$\begin{aligned}M_x&=2\pi\rho\nu U(C_{10}{}^{(0)}+C'_{10}{}^{(0)}-B_{01}{}^{(0)}-B_{01}{}'^{(0)})\\ M_y&=2\pi\rho\nu U\{D_{01}{}^{(1)}+2(A_{01}{}^{(0)}+A'_{01}{}^{(0)})\}\\ M_z&=-2\pi\rho\nu U\{D_{10}{}^{(1)}+2(A_{10}{}^{(0)}+A'_{10}{}^{(0)})\}.\end{aligned} \tag{124}$$

Thus $\mathbf{F}$ and $\mathbf{M}$ depend only on the first few terms of the expressions for $\mathbf{v}$. It may be interesting to compare this situation with the Kutta-Joukowski theorem in conventional aerodynamics.

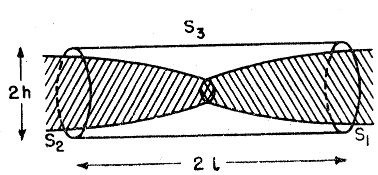

FIG. 4.

Arbitrary Motion of a Sphere

As an example of application of our formulation, the magnetohydrodynamic flow due to an arbitrary slow motion of a sphere in a conducting fluid in the presence of a uniform magnetic field has been investigated. Because of the linearity of the Stokes approximation, we have only to consider the four special cases, (i) translation parallel to the x axis, (ii) translation parallel to the y axis, (iii) rotation about the x axis, and (iv) rotation about the y axis. Obviously the translation parallel and rotation about the z axis can be immediately found from (ii) and (iv), respectively, in view of the symmetry of the geometry.

Boundary conditions are as follows: (1) the velocity $\mathbf{v}$ is given on the body surface, and (2) the magnetic field $\mathbf{b}$ must be continuously joined to that in the body, which is given by $\mathbf{b}^{(i)}=\text{grad}\chi$, where $\Delta\chi=0$, for an insulator.

For small values of the Hartmann number H or k, we can use the method of expansion in powers of k; the calculation is straightforward. Here, for brevity, only the results concerning the force and moment are given:

(i) $F_x=6\pi(1+\frac{3}{4}k)\rho\nu Ua$

(ii) $F_y=6\pi[1+(9/8)k]\rho\nu Ua$

(iii) $M_x=-8\pi[1+(4/15)k^2]\rho\nu\Omega a^3$

(iv) $M_y=-8\pi[1+(4/45)k^2]\rho\nu\Omega a^3$.

All the other force and moment components vanish. Here a is the radius of the sphere, U and Ω are the magnitude of linear and angular velocity, respectively. The results (i) and (ii) are in agreement with Chester's and Gotoh's results, respectively. The results (iii) and (iv) are believed to be new.

The resisting force and moment have different magnitude depending on the direction of translation and rotation. Therefore, if the sphere is left free in the fluid with some initial translational and angular velocities, it moves on a curved path with its axis of rotation changing continuously, in such a manner that it tends to move in the direction parallel to the magnetic field, rotating about an axis perpendicular to the field, until at last it comes to a standstill.