# Attached Stationary Shock Waves in Ionized Gases

H. CABANNES*

*Faculté des Sciences, Université d'Aix Marseille, Marseille, France*

## 1. INTRODUCTION

THE equations governing the dynamics of compressible fluids that are electrically conductive combine the laws of electromagnetism with the gas-dynamical requirements for conservation of mass and energy. The laws of physics apply to finite nonzero quantities; it is, therefore, natural that the combined relations should appear initially in the form of integral equations. Since the unknown functions can be differentiated, one may immediately deduce from the integral forms the partial differential equations governing continuous flow. If the electrical conductivity of the fluid is infinite, the resultant partial differential equations take the same form as the equations of classical fluid mechanics (zero electrical conductivity). Specifically, if viscosity and thermal conductivity are neglected, there result from both cases first-order normal hyperbolic equations; and in both cases shock waves are able to propagate through the fluid. The equations for the transition across a shock front were first derived by de Hoffman and Teller[1] for steady flows and by Lüst[2] for the general case.

The object of the present work is the study of a particular problem relative to flows with shocks in ionized gases. We consider steady flows about a wedge in the case where there is an attached shock wave. In classical fluid mechanics the solution of the problem is well known; the difference between this solution and the one obtained here takes into account the influence of the magnetic field; the two solutions are identical when there is no magnetic field. The presence of a magnetic field involves many difficulties; the most simple case is that in which the magnetic field is parallel to the velocity of the fluid; the electric field is then zero. This work is a study of this case.

## 2. EQUATIONS OF SHOCK PHENOMENA

Let us consider a compressible fluid that is electrically conductive; the electric field and the magnetic field are indicated by $\mathbf{E}$ and $\mathbf{H}$, respectively, the velocity of the fluid is indicated by $\mathbf{V}$; the magnetic permeability $\mu$ is assumed constant. The functions $p$, $T$, and $\rho$ represent the pressure, the temperature, and the density; the adiabatic index $\gamma$ is assumed constant and the fluid is supposed to obey to the law of perfect gases, $p=R\rho T$. As the electrical conductivity is assumed infinite, the electric field $\mathbf{E}$ can be eliminated by the relation $\mathbf{E}=-\mathbf{V}\times\mu\mathbf{H}$, and the shocks equations (Faraday's law, fundamental dynamic law, principles of mass conservation, and energy conservation) indicate that the two following vectors and the two following scalar quantities remain constant across the shock wave[3]:

$$\begin{aligned}&(V_n-W)\mathbf{H}-H_n\mathbf{V}\\&(V_n-W)\rho\mathbf{V}+\boldsymbol{\pi}\\&(V_n-W)\rho\\&(V_n-W)\left(\rho\frac{V^2}{2}+\frac{p}{\gamma-1}+\frac{\mu H^2}{8\pi}\right)+\boldsymbol{\pi}\cdot\mathbf{V}\end{aligned}\tag{1}$$

with

$$\boldsymbol{\pi}=\mathbf{n}p+\mathbf{n}(\mu H^2/8\pi)-(\mu H_n\mathbf{H}/4\pi).\tag{2}$$

We denote by $\mathbf{n}$ the unit vector normal to the shock, by $H_n=\mathbf{n}\cdot\mathbf{H}$ and $V_n=\mathbf{n}\cdot\mathbf{V}$ the components of the vectors $\mathbf{H}$ and $\mathbf{V}$ along the normal to the shock. $W$ is the velocity of the displacement of the shock wave. The results just established permit us to evaluate the unknowns $\mathbf{H}$, $\mathbf{V}$, $p$, and $\rho$ when the values in front of the shock and the velocity of the displacement $W$ are known. When the magnetic field is zero we again obtain the classical fluid mechanics equations. From the shock equations we obtain Eq. (3); in the case of a stationary shock we find that the electrical field behind the shock is zero when the electrical field in front of the shock is zero:

$$\mathbf{E}_2-\mathbf{E}_1=\frac{\mathbf{h}_2-\mathbf{h}_1}{\mathbf{h}_1}\{\mathbf{n}E_{n1}-\mathbf{n}W\times\mu\mathbf{H}_1\},\tag{3}$$

with

$$\frac{\mathbf{h}_2-\mathbf{h}_1}{\mathbf{h}_1}=\frac{\rho_1(V_{n1}-W)^2-(\mu H_n^2/4\pi)}{\rho_2(V_{n2}-W)^2-(\mu H_n^2/4\pi)}-1;\tag{4}$$

$\mathbf{h}=\mathbf{n}\times(\mathbf{H}\times\mathbf{n})$ is the component of the magnetic field in a plane tangent to the shock wave. We indicate by subscript 1 the values before the shock, and by subscript 2 (or without subscript in the next section) the values after the shock.

## 3. PROBLEM OF THE ATTACHED SHOCK WAVE

Let us consider a uniform steady flow; the magnetic field, the velocity of the fluid, the pressure, and the density are constant. Within the fluid there is a plane stationary shock wave; because of the shock conditions

* Present address: Faculté des Sciences, Université de Paris, Paris, France.

[1] F. de Hoffmann and E. Teller, Phys. Rev. 80, 926 (1950).

[2] R. Lüst, Z. Naturforsch. 82, 277 (1953).

[3] H. Cabannes, Recherche aéronaut. (Paris) No. 71, 3 (1959).

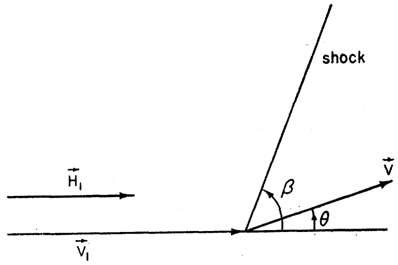


FIG. 1. *Note.* Letters in figure with overhead arrows correspond to boldface letters in the text.

and the equations of motion, the flow behind the shock is still uniform and must be determined.

If the electrical field is zero, a plane parallel to the velocity of the fluid and normal to the shock wave is a plane of symmetry for the figure. Let us consider such a plane (Fig. 1). Let us indicate by $\beta$ the angle of the shock wave with the velocity vector in front of the shock. By introducing the angle $\theta$ as the deviation of the velocity vector across the shock wave, we can express all the unknown functions **H**, **V**, $p$, and $\rho$ after the shock in terms of $\beta$ and $\theta$. We obtain further a relation between the two angles:

$$
\begin{aligned}
H&=H_1\frac{\sin\beta}{\sin(\beta-\theta)}\\
V&=V_1\frac{\cos\beta}{\cos(\beta-\theta)}+\frac{\mu H_1^2}{4\pi\rho_1V_1^2}\frac{\sin\theta}{\sin(\beta-\theta)\cos(\beta-\theta)}\\
p+\frac{\mu H^2}{8\pi}&=p_1+\rho_1V_1^2\frac{\sin\theta\sin\beta}{\cos(\beta-\theta)}+\frac{\mu H_1^2}{8\pi}\frac{\cos(\beta+\theta)}{\cos(\beta-\theta)}\\
\frac{\rho_1}{\rho}&=\frac{\tan(\beta-\theta)}{\tan\beta}+\frac{\mu H_1^2}{4\pi\rho_1V_1^2}\left\{1-\frac{\tan(\beta-\theta)}{\tan\beta}\right\}.
\end{aligned}
\tag{5}
$$

Let us introduce the parameters $x$ and $\epsilon$, nondimensional quantities proportional to the velocity of the fluid and to the magnetic field before the shock, respectively. The shock angle is determined as a function of $\theta$, the parameters $x$ and $\epsilon$, and the adiabatic index $\gamma$ by Eq. (6), obtained from the last of Eqs. (1):

$$x^2=\rho_1V_1^2/\gamma p_1 \quad \epsilon^2=\mu H_1^2/4\pi\gamma p_1$$

$$a\tan^5\beta+b\tan^4\beta+c\tan^3\beta+d\tan^2\beta+e\tan\beta+f=0 \tag{6}$$

$$a=(x^2-\epsilon^2)\left\{-\frac{\gamma-1}{2}x^2+\frac{\gamma+1}{2}\epsilon^2-1\right\}\tan\theta-\frac{\epsilon^4}{2}\tan^3\theta$$

$$b=(x^2-\epsilon^2)(x^2-1)+\left\{(\gamma-1)x^4-\left(\frac{3\gamma+2}{2}\epsilon^2-2\right)x^2+\epsilon^2\left(\frac{\gamma\epsilon^2}{2}-2\right)\right\}\tan^2\theta$$

$$
\begin{aligned}
c=&\left\{-\frac{\gamma+5}{2}x^4+\left(\frac{3\gamma+2}{2}\epsilon^2-1\right)x^2-\epsilon^2(\gamma\epsilon^2+\epsilon^2+1)\right\}\tan\theta\\
&+\left\{\frac{\gamma-1}{2}x^4+\left(\frac{\gamma+2}{2}\epsilon^2-1\right)x^2-\epsilon^2(\epsilon^2-1)\right\}\tan^3\theta
\end{aligned}
\tag{7}
$$

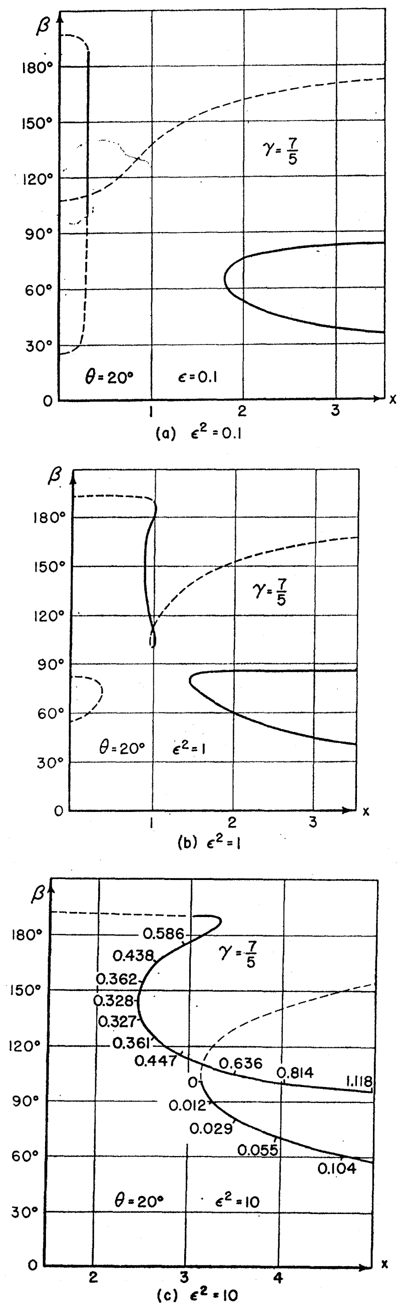


FIG. 2. Shock angle ($\theta=20°$, $\gamma=7/5$). (a) $\epsilon^2=0.1$; (b) $\epsilon^2=1$; (c) $\epsilon^2=10$.

$$d=-(\epsilon^2+1)x^2+\epsilon^2+\{(\gamma+2)x^4-(2\gamma\epsilon^2+\epsilon^2-1)x^2+\epsilon^2(\gamma\epsilon^2-1)\}\tan^2\theta$$

$$e=\left\{\left(\frac{\gamma+2}{2}\epsilon^2+2\right)x^2-\epsilon^2\left(\frac{\gamma+1}{2}\epsilon^2+2\right)\right\}\tan\theta+(x^2-\epsilon^2)\left\{-\frac{\gamma+1}{2}x^2+\frac{\epsilon^2}{2}-1\right\}\tan^3\theta$$

$$f=-(x^2-\epsilon^2)\left(\frac{\gamma\epsilon^2}{2}+1\right)\tan^2\theta.$$

Equation (6) is a fifth-degree equation in $\tan\beta$ and of the third degree in $\tan\theta$. When the magnetic field is zero, the degree is decreased by two in each case. This equation is investigated by giving particular values to certain variables.

Let us suppose that the adiabatic index has a constant value $\gamma=1.4$, and let us consider for a given wedge [$\theta$ is 20° in Figs. 2(a)–2(c)] the variation of shock angle $\beta$ as a function of the parameter $x$, the parameter $\epsilon$ being constant. Three cases are shown: in Fig. 2(a), $\epsilon^2=0.1$ (weak magnetic field): in Fig. 2(b), $\epsilon^2=1$; in Fig. 2(c), $\epsilon^2=10$ (strong magnetic field). For the parts of the curves drawn with solid lines, the entropy variation across the shock is positive; for the curves with broken lines, the entropy variation is negative, so that the second principle of thermodynamics is not fulfilled; the corresponding shocks therefore cannot have any physical reality and must be eliminated. While many shocks are possible, we are led to consider that the shock which occurs is that which corresponds to the smallest entropy increase; it happens that this shock corresponds to the smallest value of the angle $\beta$. [In Fig. 2(c) we have indicated the ratios of the specific entropy variation to the specific heat at a constant volume.]

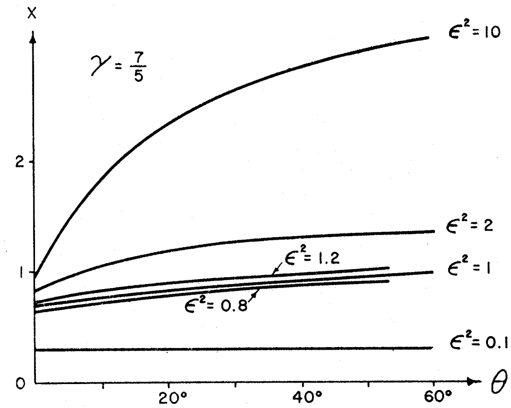


FIG. 4.

For a given wedge and a given magnetic field, we find there is no attached shock wave when Eq. (6) has no real root, corresponding to a positive variation of the entropy. To obtain in the $\theta$-$x$ plane the boundary curve of the regions in which the shock waves are attached (unshaded regions in Fig. 3) we have to eliminate $\beta$ from (6) and the equation which results by differentiating (6) with respect to $\beta$. For a weak

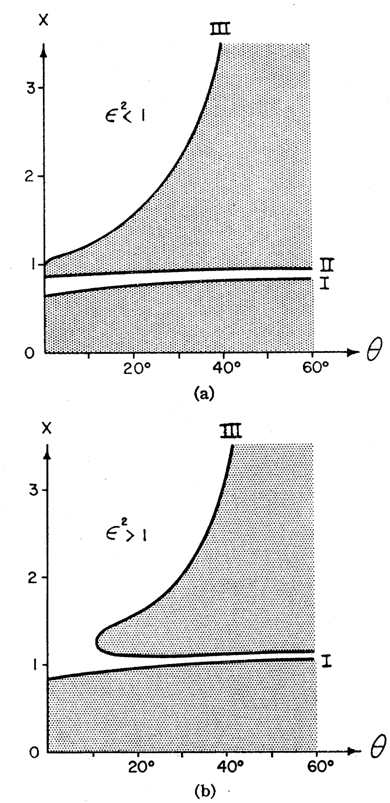


FIG. 3. (a) Weak magnetic field; (b) strong magnetic field.

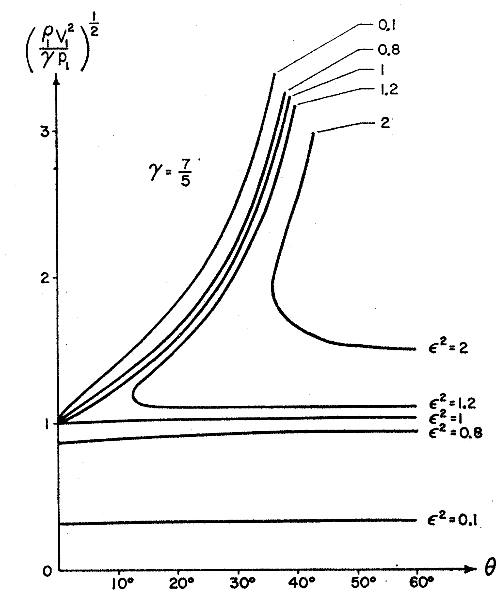


FIG. 5.

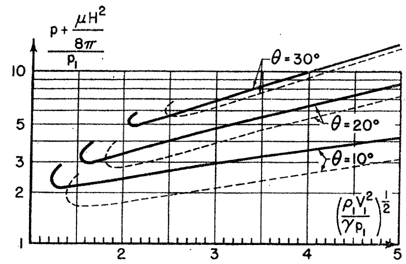


FIG. 6. Total pressure on the wedge for $\gamma=7/5$:

$$\frac{\mu H_1^2}{4\pi\gamma p_1}=0{,}1: - - -; \quad \frac{\mu H_1^2}{4\pi\gamma p_1}=1: ——.$$

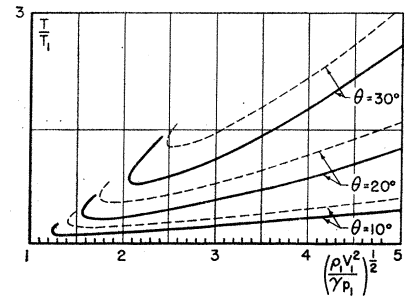


FIG. 7. Temperature on the wedge for $\gamma=7/5$:

$$\frac{\mu H_1^2}{4\pi\gamma p_1}=0{,}1: - - -; \quad \frac{\mu H_1^2}{4\pi\gamma p_1}=1: ——.$$

magnetic field ($\epsilon^2<1$) the curve $x(\theta)$ has three branches, I, II, III, whose ordinates at the origin, are, respectively, $\epsilon(1+\epsilon^2)^{-\frac{1}{2}}$, $\epsilon$, and 1: Fig. 3(a); when the magnetic field is zero, the branches I and II go to the axis $x=0$ and we find again the results of the classical aerodynamics. For strong magnetic field ($\epsilon^2>1$) the curve $x(\theta)$ has only two branches, I and III [Fig. 3(b)]. The family of curves corresponding to the branches I is drawn in Fig. 4. The family of curves corresponding to the branches II and III is drawn in Fig. 5; all the curves have as an asymptote the straight line defined by the relation $\gamma \sin\theta=1$.

## 4. CALCULATION OF PRESSURE AND TEMPERATURE

The aerodynamic pressure $p$ is defined by Eqs. (5). The presence of the magnetic field has the effect of decreasing the pressure: this result is paradoxical in appearance only. The pressure represents in effect only a part of the action of the fluid on the wedge; to obtain the total action, we have to add the electromagnetic forces. The force $\mathbf{F}$ exerted on the wedge by a unit surface is the vector defined by Eq. (8) in which $\boldsymbol{\nu}$ is the unit vector normal to the wedge (8):

$$\mathbf{F}=\boldsymbol{\nu}p+\boldsymbol{\nu}(\mu H^2/8\pi)-(\mu H_\nu \mathbf{H}/4\pi). \qquad (8)$$

In the actual problem, the magnetic field is parallel to the wedge, hence $H_\nu=\mathbf{H}\cdot\boldsymbol{\nu}=0$; the vector $\mathbf{F}$ is normal to the wedge. The variations of the quantity $p+(\mu H^2/8\pi)$, called "total pressure," as a function of the velocity are illustrated in Fig. 6. The presence of the magnetic field has the effect of increasing the "total pressure"; we can also say that the magnetic field has the effect of increasing the drag.

The temperature is defined by the law of perfect gases, $p=R\rho T$; its variations as a function of the velocity are shown in Fig. 7. The presence of the magnetic field has the effect of decreasing the temperature on the wedge.