REVIEWS OF MODERN PHYSICS VOLUME 32, NUMBER 4 OCTOBER, 1960

Shock Waves and Shock-Wave Structure in Magneto-Fluid Dynamics

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INTRODUCTION

THERE are two points of view in shock-wave theory. According to the first one, shock waves are discontinuities in a perfect-fluid flow. The shock-wave relations and the equation of state allow one to establish, using for instance the method of Weyl,[1] the fundamental properties of shocks. In the second, which may be referred to as "shock-layer theory,"[2] the fluid is not assumed to be perfect, but some dissipative mechanisms are allowed, each one being characterized by a dissipation coefficient which usually is assumed to be small. The shock layer is then the region in which the flow appeared as a result of a balance between nonlinear effects (arising from the inertia terms) and dissipative effects (which become important when gradients are themselves important).

This paper is devoted to the study of these two types of questions when the fluid is electrically conducting and subjected to a magnetic field. The following assumptions are made: the flow is plane and stationary; the magnetic field is also plane (in the same plane, say, xy); all the physical quantities depend on the coordinate x only and are uniformly bounded in x for all x $(-\infty<x<+\infty)$. The equation of state is subjected only to Weyl's conditions.[1] The dissipative effects are the viscosity (two coefficients), the thermal conductibility (one coefficient), and the inverse of the electrical conductivity (one coefficient). No assumption is required as far as the behavior of these coefficients with respect to the thermodynamical or physical quantities (for instance, temperature or magnetic field) is concerned. When writing the equations governing the problem, one gets a system of four differential equations with four unknowns (Sec. 1). However, this system may be written in a compact form (Sec. 2) by using the dissipation energy and a generalized thermodynamical potential which is independent of the dissipation effects.

Shock waves in a perfect-fluid flow theory depend only on a system of four ordinary algebraic equations when the constants of the flow, such as mass flux, momentum flux, and energy flux are fixed. It can be shown (Sec. 3) that there are at most four solutions S_1, S_2, S_3, S_4, each of these solutions being characterized by the value of the normal speed component relative to the three small-perturbation (or characteristic) speeds. The value of the index of a solution S is chosen so that the specific entropy is an increasing function of this index. Thus, according to this theory, a shock from a state S_i to a state S_j may exist if $i<j$.

Now the question arises: Is such a flow the limit of a flow with dissipation effects, when the dissipation coefficients become vanishingly small? To discuss this problem, the shock layer must be considered. The previous states S correspond to the singular points of the differential system; the eigenvalues of the linear system which give the local behavior of the solutions near such a singular point are easily found (Sec. 4). The *fast-shock* solution—transition from S_1 to S_2—is shown to be stable (Sec. 5). That means that there is one and only one integral curve from S_1 to S_2 (in a convenient four-dimensional space) and that, if the upper bound α of the dissipation coefficients tends towards zero, the flow in the physical plane tends (if conveniently normalized) to the corresponding perfect-fluid shock flow. For an *intermediate-shock*—transition from S_1 or S_2 to S_3 or S_4—such a result is not valid (Sec. 6). Moreover, it can be proved that an intermediate shock is *never* stable. Finally, no general conclusion can be stated for the *slow-shock* transition from S_3 to S_4—such a flow may be stable, but a counter-example may be constructed (Sec. 7).

Several papers are devoted to the shock-layer theory in magneto-fluid dynamics, for instance, those of Marshall,[3] Burgers,[4] and Ludford.[5] However, in these papers at least two of the four dissipation coefficients have been assumed to be zero and no discussion has been given of the stability of the flow. On the other hand, the problem of stability is considered by Akhiezer, Liubarskii, and Polovin,[6] but these authors discuss the question in the framework of the perfect-fluid flow theory. Their general conclusions generally agree with those of the present paper *except for the slow shock*, where they state that the slow shock is always stable. Such a difference is not surprising, for the definition of stability is not the same. Details of the proofs can be found elsewhere.[7]

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1. BASIC EQUATIONS

On using normalized units (mksq), Ohm's law and Maxwell's equations, when electrostatic forces and displacement currents are dropped, reduce to

$$\begin{aligned} \mathbf{J}&=\sigma(\mathbf{E}+\mathbf{V}\times\mathbf{B}) \quad &\operatorname{div}\mathbf{B}=0 \\ \partial\mathbf{B}/\partial t+\nabla\times\mathbf{E}&=0 \quad &\nabla\times\mathbf{B}=\mu\mathbf{J}. \end{aligned} \tag{1}$$

$\mathbf{V}$ is the velocity vector, $\mathbf{B}$ magnetic induction, $\mathbf{E}$ electrical field, $\mathbf{J}$ current density, μ magnetic inductive capacity of free space. When writing the momentum and energy equations, one must take account of the electromagnetic momentum source $\mathbf{J}\times\mathbf{B}$ and of the electromagnetic energy source $\mathbf{E}\cdot\mathbf{J}$. On the other hand, for convenience, viscous stresses are written (with obvious notations)

$$\tau_{ij}=(m_1-2m_2)V_{k,k}S_{ij}+m_2(V_{i,j}+V_{j,i}) \tag{2}$$

introducing the coefficients m_2, which is the usual viscosity coefficient and m_1, instead of the bulk viscosity. As usual, the heat flux is written

$$\mathbf{Q}=-k\nabla T. \tag{3}$$

Finally, the equation of state is taken as

$$p=G(\tau,s), \tag{4}$$

with p pressure, τ specific volume, s specific entropy. The function G is assumed to check the classical Weyl conditions

$$G_\tau<0, \quad G_{\tau\tau}>0, \quad G_s>0. \tag{5}$$

Let us recall that $c^2=-\tau^2G_\tau$, where c is the sound speed.

Now, according to the assumptions which have been stated in the Introduction, we may denote by u, v, 0 the components of $\mathbf{V}$, and by B_0, B, 0 those of $\mathbf{B}$. Thus $\mu\mathbf{J}$ is the vector $(0,0,dB/dx)$ and $\mathbf{E}$ has only one component—the third one, E—which is not zero. Equations (1) show that B_0 *and* E *are constant* and that

$$dB/dx=\sigma\mu(uB-vB_0+E).$$

The continuity equation shows that $u=M\tau$, where M, a positive constant, is the mass flux ($M=\rho u$). Conservation of momentum and energy give

$$\begin{aligned} p+\rho u^2+(2\mu)^{-1}B^2&=m_1(du/dx)+P \\ \rho uv-\mu^{-1}B_0B&=m_2(dv/dx)+P_2 \\ \rho u[h+\tfrac{1}{2}(u^2+v^2)]-(EB/\mu)& \\ =m_1u(du/dx)+m_2v(dv/dx)&+k(dT/dx)+MC. \end{aligned}$$

P, P_2, C are, as are M, B_0, E, integration constants; h is the specific enthalpy, T the temperature. Without loss of generality, P_2 may be taken to be zero. Thus, if we introduce the new constants E^* and $B_0{}^*$,

$$E=M\mu E^*, \quad B_0=M\mu B_0{}^*,$$

the basic system to be discussed has the following form:

$$\begin{aligned} (\sigma\mu^2)^{-1}dB/dx&=M[(B\tau/\mu)-B_0{}^*v+E^*] \\ m_2dv/dx&=M(v-B_0{}^*B) \\ m_1M^2d\tau/dx&=M[p+M^2\tau+(B^2/2\mu)-P] \\ kdT/dx&=M[e-\tfrac{1}{2}M^2\tau^2-\tfrac{1}{2}v^2-(B^2\tau/2\mu) \\ &\qquad -E^*B+B_0{}^*Bv+P\tau-C], \end{aligned} \tag{6}$$

e is the specific internal energy.

2. INTERPRETATION OF THE SYSTEM

The dissipation mechanisms give rise to some dissipated energy: $\sigma^{-1}J^2=(\sigma\mu^2)^{-1}(dB/dx)^2$ (dissipation by Joule effect), $m_1(du/dx)^2+m_2(dv/dx)^2$ (dissipation by viscous forces), $kT^{-1}(dT/dx)^2$ (dissipation by thermal conductivity). Let us define

$$\mathfrak{D}=\frac{1}{T}\left\{\frac{1}{\sigma\mu^2}\left(\frac{dB}{dx}\right)^2+m_1M^2\left(\frac{d\tau}{dx}\right)^2 +m_2\left(\frac{dv}{dx}\right)^2+\frac{k}{T}\left(\frac{dT}{dx}\right)^2\right\}, \tag{7}$$

a quantity proportional to this dissipated energy, which is shown later—see (11)—the entropy source in the fluid. Now, the following generalized potential is introduced:

$$\mathfrak{M}=\frac{2M}{T}\left\{\frac{B^2\tau}{2\mu}+\frac{M^2\tau^2}{2}+\frac{v^2}{2}-f(\tau,T) +E^*B-B_0{}^*Bv-P\tau+C\right\}, \tag{8}$$

where f is the Helmholtz free energy per unit mass, $\frac{1}{2}(u^2+v^2)=\frac{1}{2}(M^2\tau^2+v^2)$ the kinetic energy, and $(2\mu\rho)^{-1}B^2$ the magnetic energy. As

$$df=-sdT-pd\tau,$$

then

$$2Ms=\mathfrak{M}+T\partial\mathfrak{M}/\partial T. \tag{9}$$

To be more systematic, the variables, B, v, τ, T are, respectively, denoted by q_1, q_2, q_3, q_4, and dq_i/dx by $\dot{q}_i$. Then, the system (6) may be written in the more compact form

$$\partial\mathfrak{D}/\partial\dot{q}_i=\partial\mathfrak{M}/\partial q_i. \tag{10}$$

It is easily shown that

$$Mds/dx+(d/dx)[-(k/T)(dT/dx)]=\mathfrak{D} \tag{11}$$

and that

$$\mathfrak{M}(x_2)-\mathfrak{M}(x_1)=2\int_{x_1}^{x_2}\mathfrak{D}dx. \tag{12}$$

Thus $\mathfrak{M}$ is a nondecreasing function of x. The precise question may be stated as follows: In the space $(\mathcal{E})$ defined by the q_i, find an integral arc curve (L) of

(10)—say, S_i, S_j—along which x increases from $-\infty$ to $+\infty$. When the point q_i of (L) tends towards S_i or S_j, $\mathfrak{D}$ must tend towards zero, because $\mathfrak{M}$ remains finite. Then $\dot{q}_i$ and $\partial\mathfrak{D}/\partial\dot{q}_i$ tend towards zero, and consequently:

Theorem 1. The points S of $(\mathcal{E})$ which may be end points of arcs (L) are points which make $\mathfrak{M}$ stationary.

These points will be denoted by S_1, S_2, $\cdots$. As a result of (12) $\mathfrak{M}(+\infty)\geqslant\mathfrak{M}(-\infty)$ and according to (9), it may be stated:

Theorem 2. If S_i and S_e are end points of an arc (L) such that these points correspond, respectively, to $x=-\infty$ and $x=+\infty$, then $s(S_i)\leqslant s(S_e)$.

The first question to be answered is the following: Find the points S_i which make $\mathfrak{M}$ stationary and choose the index in such a way that the entropy s is a nondecreasing function of the index. This question is precisely the one which has to be considered in the theory of shocks as discontinuities arising in a perfect-fluid flow when writing the shock relations (conservation of mass, momentum, and energy) and the shock inequality arising from the second principle of thermodynamics.

3. SHOCK WAVES WITHOUT DISSIPATION

In classical gas dynamics general results may be obtained by *variation of constants of integration*. In order to generalize these results let us consider the manifold $(\mathcal{V})$ defined in a space B, v, τ, T, M, B_0^*, E^*, P, C, by

$$\partial\mathfrak{M}/\partial B=\partial\mathfrak{M}/\partial v=\partial\mathfrak{M}/\partial\tau=\partial\mathfrak{M}/\partial T=0. \quad (13)$$

On such a manifold

$$(2M)^{-1}Td\mathfrak{M}=Tds=BdE^*+dC-\tau dP -BvdB_0^*+(\tau^2/2)dM^2. \quad (14)$$

Two points of $(\mathcal{V})$ are images of states connected through a shock if their coordinates M, B_0^*, E^*, P, C are the same. Now, according to (13), if the new constant $\tau_*=\mu B_0^{*2}$ is introduced,

$$v=B_0^*B, \quad B(\tau-\tau_*)+\mu E^*=0 \quad (15)$$

and if $E^*\neq 0$,

$$p+M^2\tau+\frac{\mu E^*}{2(\tau-\tau_*)^2}=P,$$
$$h+\tfrac{1}{2}M^2\tau^2+\frac{\mu E_*^2(2\tau-\tau_*)}{2(\tau-\tau_*)^2}=C. \quad (16)$$

(a) B_0^*, M, E^*, P Fixed, C is Variable

The corresponding points of $(\mathcal{V})$ define a curve whose image $(\mathcal{T})$ in the (τ,p) plane is given by the first of Eqs. (16). Along such a curve, sketched in Fig. 1,

$$Tds=dC. \quad (17)$$

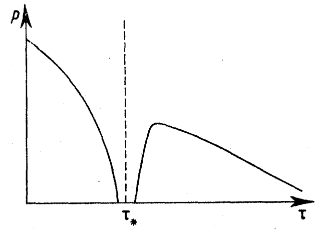


FIG. 1.

But taking account of (5), it may be shown that along a connected subarc of $(\mathcal{T})$ s can have only one stationary point—in fact one maximum. Thus (17) allows one to prove that there are at most four points S. More precisely, along $(\mathcal{T})$

$$G_s ds/d\tau=-M^2-G_\tau+[\mu E_*^2/(\tau-\tau_*)^3]=-\tau^{-2}R(u),$$

if we set

$$\alpha^2=B^2\tau/\mu, \quad \alpha_n^2=B_0^2\tau/\mu=M^2\tau_*\tau$$
$$R(u)=u^2-c^2-[\alpha^2u^2/(u^2-\alpha_n^2)]. \quad (18)$$

α and α_n are *Alfvên speeds computed with the tangential and normal components of the magnetic field*. At a point of $(\mathcal{T})$ which is a relative maximum of s, u equals either A or a, $(A>a)$, *A and a being the characteristic speeds or speed of propagation of small perturbances* $R(a)=R(A)=0$. From this result, one easily deduces:

Theorem 3. For given values of constants of integration, there exist at most four points S: S_1, S_2, S_3, S_4. Each of the corresponding states is characterized by the relative value of the normal component u with respect of the propagation speeds A, α_n, a:

$$S_1: \quad u\geqslant A \qquad S_3: \; a\leqslant u\leqslant\alpha_n$$
$$S_2: \; \alpha_n\leqslant u\leqslant A \qquad S_4: \quad u\leqslant a.$$

In the limiting case $E^*=0$, S_2, and S_3 (if they exist) have the same image in the (τ,p) plane—values of v and B are opposite—and correspond to $u=\alpha_n$. This special case corresponds in particular to the switch-off and switch-on shocks.

(b) Hugoniot Curve

Now E^* and B_0^* are kept fixed, M, P, C varying in such a way that in the (τ,p) plane, the curves $(\mathcal{T})$ and $(\mathcal{C})$ defined by (16) pass by a given point (τ_b,p_b). The image in (τ,p) of the curve defined on $(\mathcal{V})$ by these requirements is a generalization of the so-called *Hugoniot curve*. Then

$$dP=\tau_b dM^2, \quad dC=\tfrac{1}{2}\tau_b dM^2$$

and consequently (15):

$$2Tds=(\tau-\tau_b)^2dM^2.$$

By an obvious generalization of the classical argument of Weyl, it can easily be shown that $s(S_1)\leqslant s(S_2)$ and $s(S_3)\leqslant s(S_4)$. Moreover, for a very weak shock ($u\simeq A$

or $u \simeq a$), the entropy jump is found to be proportional to the third power of the specific-volume jump.

(c) B_0^*, M, P, C Fixed, E^* Variable

These conditions define on $(\mathcal{V})$ a curve whose image in the (τ, p) plane is given by

$$K(\tau,p)=h+\tfrac{1}{2}(M^2\tau)-C-(2\tau-\tau_*)(p+M^2\tau-P)=0,$$

along which, according to (14) and (15),

$$(\tau_*-\tau)ds=\mu E^* dE^*.$$

With such a result, it may be shown that $s(S_2)\leqslant s(S_3)$. In fact, this last result is obtained, when assuming not only (5) to be valid but also that $\partial h/\partial p-2\tau>0$. Thus it is possible to state:

Theorem 4. The specific entropy for the states S_1, S_2, S_3, S_4, is a nondecreasing function of the index i of these points S_i.

In conclusion, within the framework of a perfect-fluid flow theory, it may be expected that a shock transition is possible with a state S_i in front of the shock and a state S_j behind it, provided $i<j$.

4. BEHAVIOR OF THE SOLUTIONS IN THE NEIGHBORHOOD OF A POINT S

Let us denote with the subscript i every physical quantity whose value is considered at a point S_i. In order to build the linearized system associated with (6) which gives the behavior of the integral curves near the singular point S_i, we introduce nondimensional variables by

$$\begin{aligned} B&=B_i(1+\bar{B}), & \tau&=\tau_i(1+\bar{\tau}),\\ v&=v_i+u_i\bar{v}, & T&=T_i(1+\bar{T}). \end{aligned} \tag{19}$$

In this section and in the next one B_0^* and B_i are assumed to be positive. Obviously, the results still remain valid in the other cases.

The quadratic terms in the expansion of $\mathfrak{D}$ and $\mathfrak{M}$ in the neighborhood of S_i may be written

$$\mathfrak{D}_2=\frac{1}{T_i}\left\{\frac{B_i^2}{\sigma\mu^2}\left(\frac{d\bar{B}}{dx}\right)^2+m_1u_i^2\left(\frac{d\bar{\tau}}{dx}\right)^2 +m_2u_i^2\left(\frac{d\bar{v}}{dx}\right)^2+kT_i\left(\frac{d\bar{T}}{dx}\right)^2\right\}, \tag{20}$$

$$\mathfrak{M}_2=\frac{M}{T_i}\left\{\alpha_i^2(2\bar{B}\bar{\tau}+\bar{B}^2)+u_i^2(\bar{v}^2+\bar{\tau}^2) -2\alpha_i\alpha_{n_i}\bar{B}\bar{v}+\left(\frac{\partial p}{\partial\tau}\right)_i\tau_i^2\bar{\tau}^2 +2\left(\frac{\partial s}{\partial\tau}\right)_i\tau_iT_i\bar{\tau}\bar{T}+\left(\frac{\partial s}{\partial T}\right)_iT_i^2\bar{T}^2\right\}. \tag{21}$$

If r_1, r_2, r_3, r_4 are used to denote $\bar{B}$, $\bar{v}$, $\bar{\tau}$, $\bar{T}$, respectively, the linearized system to be considered is

$$\partial\mathfrak{D}_2/\partial\dot{r}_j=\partial\mathfrak{M}_2/\partial r_j. \tag{22}$$

It is important to remark that $\mathfrak{M}_2$ may be expressed as a sum of "squares":

$$\mathfrak{M}_2=\frac{M}{T_i}\left\{(\alpha_i\bar{B}+\bar{\tau}\alpha_i-\alpha_{n_i}\bar{v})^2 +(u_i^2-\alpha_{n_i}^2)\left(\bar{v}+\frac{\alpha_i\alpha_{n_i}\bar{\tau}}{u_i^2-\alpha_{n_i}^2}\right)^2 +\left[1\Big/\left(\frac{\partial s}{\partial T}\right)_i\right]\left[\left(\frac{\partial s}{\partial T}\right)_iT_i\bar{T}+\left(\frac{\partial s}{\partial\tau}\right)_i\tau_i\bar{\tau}\right]^2 +R(u_i)\bar{\tau}^2\right\}. \tag{23}$$

Now there exist four eigensolutions of (22) which may be written $r_j=r_j^*\exp(\lambda x)$, where the λ are the eigenvalues of the system. The following result is easily proved, thanks to (23) and theorem 3:

Theorem 5. At a point S_1, $0\leqslant\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\lambda_4$; at S_2 or at S_3, $\lambda_1\leqslant0\leqslant\lambda_2\leqslant\lambda_3\leqslant\lambda_4$; at S_4, $\lambda_1\leqslant\lambda_2\leqslant0\leqslant\lambda_3\leqslant\lambda_4$.

On using known results pertaining to linear systems of differential equations, used in particular in vibration theory, a more precise result may quite generally be derived for the eigensolution corresponding to the smallest eigenvalue λ_1. Without going into the details, it can be stated:

Theorem 6. The components of the eigenvector (B^*, v^*, τ^*, T^*), which corresponds to the smallest eigenvalue λ_1, satisfy either

$$B^*>0,\quad v^*>0,\quad \tau^*<0,\quad T^*>0$$

or the opposite inequalities.

5. EXISTENCE, UNIQUENESS, AND STABILITY OF THE FAST SHOCK

As was mentioned previously, the fast shock is a transition from S_1 to S_2. First it must be noted that there exists in $(\mathcal{E})$ an open connected domain (D) in which

$$\partial\mathfrak{M}/\partial B>0,\quad \partial\mathfrak{M}/\partial v>0,\quad \partial\mathfrak{M}/\partial\tau<0,\quad \partial\mathfrak{M}/\partial T>0 \tag{24}$$

such that the points S_1 and S_2 belong to its closure $(\bar{D})$ and no other singular point lies in $(\bar{D})$. Then, by theorem 6, it is clear that one of the eigenvectors at S_1 and at S_2, corresponding to the smallest eigenvalue λ_1, points inwards toward (D). The third preliminary remark states that every integral curve of (6), orientated in such a way that x is increasing, when crossing the boundary of (D) in a point different from S_1 or S_2 must go out of (D). This can easily be proved with the following identities:

$$\frac{d}{dx}\left(T\frac{\partial\mathfrak{M}}{\partial B}\right)=2M\left\{\frac{\tau}{\mu}\frac{dB}{dx}-B_0{}^*\frac{dv}{dx}+\frac{B}{\mu}\frac{d\tau}{dx}\right\}$$

$$\frac{d}{dx}\left(T\frac{\partial\mathfrak{M}}{\partial v}\right)=2M\left\{\frac{dv}{dx}-B_0{}^*\frac{dB}{dx}\right\}$$

$$\frac{d}{dx}\left(T\frac{\partial\mathfrak{M}}{\partial \tau}\right)=2M\left\{\frac{B}{\mu}\frac{dB}{dx}+\left(M^2+\frac{\partial p}{\partial\tau}\right)\frac{d\tau}{dx}+\frac{1}{T}\frac{\partial s}{\partial T}\frac{dT}{dx}\right\} \tag{25}$$

$$\frac{d}{dx}\left(T\frac{\partial\mathfrak{M}}{\partial T}\right)=2M\left\{\frac{\partial s}{\partial\tau}\frac{d\tau}{dx}+\frac{\partial s}{\partial T}\frac{dT}{dx}\right\}-\frac{d\mathfrak{M}}{dx}.$$

Now, the unique integral curve which arrives at S_2, inside (D), along the eigenvector which corresponds to λ_1, must necessarily start at the node S_1, and x increases from $-\infty$ to $+\infty$ when following this arc from S_1 to S_2. On the other hand, it is easy to show that it is the only integral curve having this property. Finally, it is clear that B, v, τ, T, are monotonic functions of x. On using (5) and some classical identities of thermodynamics, it can be shown that the same result is also valid for the pressure p. Thus we have the following:

Theorem 7. When every dissipation coefficient is not zero, there exists one, and only one, continuous flow (up to an obvious translation along the x axis), which allows one to pass from the state S_1 (for $x=-\infty$) to the state S_2 (for $x=+\infty$). In this flow, B, v, τ, T, p are monotonic functions of x.

Just in order to fix this translation, the origin of the x axis will be chosen at a point where $2\tau=\tau_1+\tau_2$.

Now, it remains to consider the behavior of such a flow as function of the dissipation coefficients, especially when these coefficients tend towards zero. The following relations are found useful. From (7) and (10) we obtain

$$\frac{d\mathfrak{M}}{dx}=\frac{T}{2}\left\{\sigma\mu^2\left(\frac{\partial\mathfrak{M}}{\partial B}\right)^2+\frac{1}{m_2}\left(\frac{\partial\mathfrak{M}}{\partial v}\right)^2+\frac{1}{m_1M^2}\left(\frac{\partial\mathfrak{M}}{\partial \tau}\right)^2+\frac{T}{k}\left(\frac{\partial\mathfrak{M}}{\partial T}\right)^2\right\}=\mathfrak{D}^*,$$

and consequently, if Q_1 and Q_2 are two points of the integral curve (L),

$$\mathfrak{M}(S_2)-\mathfrak{M}(S_1)>\mathfrak{M}(Q_2)-\mathfrak{M}(Q_1)=\int_{x(Q_1)}^{x(Q_2)}\mathfrak{D}^*dx. \tag{26}$$

Let ϵ, arbitrarily small, be given, and consider (D_ϵ) the subdomain of (D) which does not contain cubes with center S_1 and S_2 and length 2ϵ. Inside (D_ϵ),

$$\frac{T}{2}\left\{\left(\frac{\partial\mathfrak{M}}{\partial B}\right)^2+\left(\frac{\partial\mathfrak{M}}{\partial v}\right)^2+\frac{1}{M^2}\left(\frac{\partial\mathfrak{M}}{\partial \tau}\right)^2+T\left(\frac{\partial\mathfrak{M}}{\partial T}\right)^2\right\}>\epsilon',$$

ϵ' being strictly positive and well determined when ϵ is known. Choose Q_1 and Q_2 inside (D_ϵ), but with $x(Q_1)<0$, $x(Q_2)>0$. Let α be an upper bound of m_1, m_2, k, $(\sigma\mu^2)^{-1}$. Along Q_1Q_2, $\mathfrak{D}^*>\epsilon'/\alpha$, and according to (26),

$$x(Q_2)-x(Q_1)<(\alpha/\epsilon')[\mathfrak{M}(S_2)-\mathfrak{M}(S_1)].$$

From this, it is easy to deduce:

Theorem 8. If α tends towards zero, the corresponding flows tend towards the state defined by S_1 for $x<0$, the state defined by S_2 for $x>\infty$, the convergence being uniform outside every open interval including $x=0$.

Thus, it can be said that the last shock is stable.

6. INSTABILITY OF AN INTERMEDIATE SHOCK

By definition, the state in front of the shock is either S_1 or S_2, the state behind the shock either S_3 or S_4. In this section, E^* is assumed to be nonzero. We want to know if such a perfect-fluid shock flow is the limit of a set of flows with dissipation, whatever be the dissipation coefficients with α as upper bound, when α tends towards zero, all the physical quantities remaining bounded. It is proved that the answer is no. For, let B_M, $T_M\cdots$ be uniform upper bounds for $|B|$, $|T|\cdots$. As

$$\frac{\partial M}{\partial B}+B_0{}^*\frac{\partial\mathfrak{M}}{\partial v}=\frac{2M}{T}\left[\frac{B}{\mu}(\tau-\tau_*)+E^*\right],$$

if δ is defined by

$$2B_M\delta=\mu|E^*|,$$

it may be stated that in $\tau_*-\delta\leqslant\tau\leqslant\tau_*+\delta$,

$$(T/2)\{(\partial\mathfrak{M}/\partial B)^2+(\partial\mathfrak{M}/\partial v)^2\}>K/T_M,$$

where K depends on the integration constants only. Let x_1 and x_2 be two values of x such that

$$\tau(x_1)=\tau_*+\delta \quad \tau(x_2)=\tau_*-\delta$$

and

$$\tau_*-\delta\leqslant\tau(x)\leqslant\tau_*+\delta \quad \text{for} \quad x_1\leqslant x\leqslant x_2.$$

It can be shown from (26), by a previous argument, that

$$x_2-x_1<K_1\max[(\sigma\mu^2)^{-1},m_2].$$

But there exists a constant C_1 independent of the dissipation coefficients, such that for every x between x_1 and x_2,

$$|d\tau/dx|<C_1/\min(m_1).$$

Then

$$2\delta/C_1K_1<\max[(\sigma\mu^2)^{-1},m_2]/\min(m_1). \tag{27}$$

The first member is independent of the dissipation coefficients. Thus, whatever be α, values of these

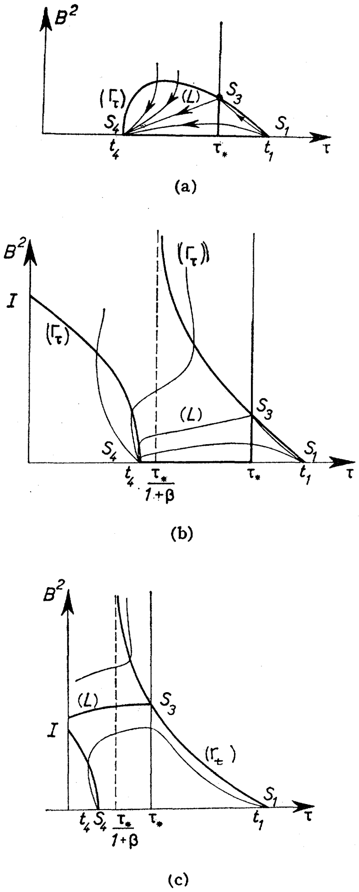


FIG. 2. (a) Case (i); (b) case (ii); (c) case (iii).

coefficients may be found which do not satisfy this inequality (27). That proves that *an intermediate shock is not stable*, in the precise sense that is used in this paper. This is illustrated by a particular example in the following section.

7. SPECIAL EXAMPLE

This example is, in fact, a slight generalization of the case originally considered by Ludford. The coefficients m_2 and k are supposed to be zero. It is easy to eliminate v and then to obtain the system

$$\frac{1}{\sigma\mu^2}\frac{dB}{dx}=M\left[\frac{B(\tau-\tau_*)}{\mu}+E^*\right]$$

$$m_1M^2\frac{d\tau}{dx}=M\left[p+M^2\tau+\frac{B^2}{2\mu}-P\right] \qquad (28)$$

$$e-\frac{M^2\tau^2}{2}+\frac{B_0{}^{*2}B^2}{2}-\frac{B^2\tau}{2\mu}-E^*B+P\tau-C=0.$$

By using the method of Sec. 4, one can show that S_1 is a node with positive eigenvalues, S_2 and S_3 are saddle points, S_4 a node with negative eigenvalues. To be more specific, let us assume the fluid to be a perfect gas, i.e. $e=\gamma p\tau$, $[\beta(\gamma-1)=1]$. Thus in a (τ,B) plane, the isocline $dB=0$ is the hyperbola (Γ_B):

$$B(\tau-\tau_*)+\mu E^*=0 \qquad (29)$$

and the isocline $d\tau=0$ is the cubic (Γ_τ):

$$M^2(\tfrac{1}{2}+\beta)\tau^2+(B^2/2\mu)[\tau(1+\beta)-\tau_*]-P(1+\beta)\tau+C+E^*B=0.$$

Assume that P and C are such that (Γ_τ) cuts the positive τ axis in two points $\tau=t_1$, $\tau=t_4$ $(t_1>t_4>0)$. With K a positive constant, the equation of (Γ_τ) may be written

$$B^2[\tau(1+\beta)-\tau_*]+K(\tau-t_1)(\tau-t_4)+2\mu E^*B=0. \quad (30)$$

To begin with, one assumes $E^*=0$ and one uses a (τ,B^2) plane. The singular points $B=0$, $\tau=t_1$ and $B=0$, $\tau=t_4$ are precisely S_1 and S_4. Then, let us assume $t_1<\tau_*<t_4$.

In order to discuss various possibilities for slow shocks, three cases are considered (Fig. 2):

(i) $\tau_*<(1+\beta)t_4$

(ii) $\tau_*>(1+\beta)t_4$, $(t_1-\tau_*)(\tau_*-t_4)-\beta t_1t_4<0$

(iii) $\tau_*>(1+\beta)t_4$, $(t_1-\tau_*)(\tau_*-t_4)-\beta t_1t_4>0$.

The difference between the first case and the others lies in the relative position of the asymptote of the hyperbola (Γ_τ), $\tau(1+\beta)=\tau_*$, with respect to S_1 and S_4. In case (ii), the ordinate of S_3 is less than that of I, I being the point of intersection of the B^2 axis with (Γ_τ); in case (iii), one has the opposite situation: the former is greater than the latter. In case (i), there exists one and only one integral curve (L) joining S_3

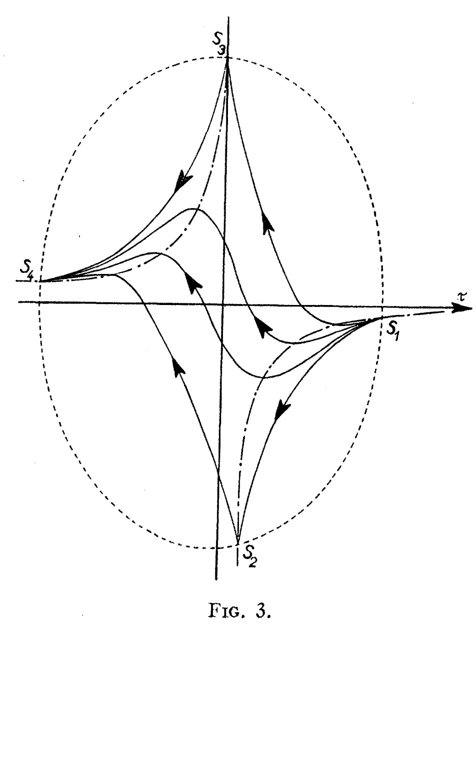


FIG. 3.

to S_4, whatever be σ and m_1. Moreover, it is seen that, when $(\sigma)^{-1}$ and m_1 tend toward zero, the corresponding flows tend toward the state S_3 in front of the shock and the state S_4 behind the shock. Then, in this case, *the slow shock is stable.* In case (ii), the arc (L) still exists and is unique whatever be σ and m_1. But, it must be noted that this arc (L) does not remain in a domain bounded by (Γ_B) and (Γ_τ). Consequently, it is clear that the argument used in Sec. 5 for the fast shock, which allows one to prove that the integral curve lies *a priori* in a certain domain, cannot be extended to the case of the slow shock. Finally, in case (iii), it is seen that the arc (L), if $m_1\sigma$ is small enough, cuts the B^2 axis before the curve (Γ_τ). Then, in such a case, which may happen for arbitrarily small α, no slow shock is possible. *This discussion shows that the slow shock may or may not be stable.*

It is interesting to consider what happens—for instance, in case (i)—if one assumes $E^*>0$ but small. Figure 3 shows a typical possible case where two intermediate shocks S_1S_3 and S_2S_4 (as well as one infinity of S_1S_4 shocks) may exist for suitable values of σ and m_1. But, for *the same values of the constants of integration*, if σ^{-1} is assumed to be very small with respect to m_1, it is easily understood, by looking at the shape of the integral curves sketched in Fig. 4, why only fast (S_1S_2) and slow (S_3S_4) shocks may be stable.

8. PARTICULAR CASES

Great simplifications arise in this general theory if $B_0{}^*=0$, i.e., if the normal component of the magnetic field is zero. It is easily seen that $v=0$. Consequently, the value of m_2 is irrelevant. Then $\tau_*=0$ and, as in

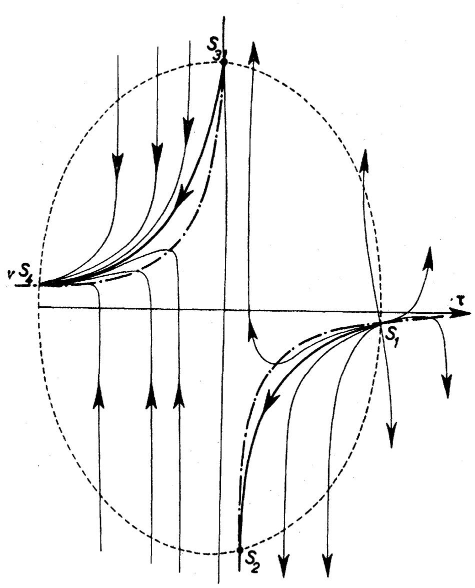


FIG. 4.

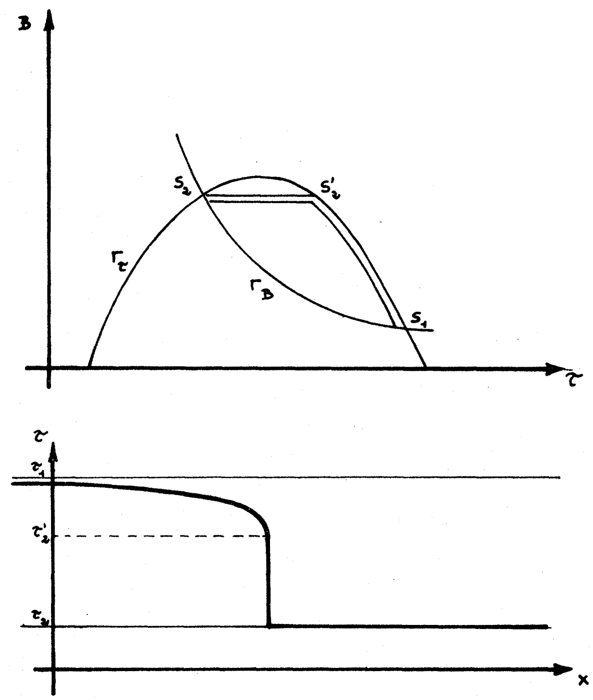


FIG. 5.

classical gas dynamics, only S_1 and S_2 may exist. Shocks are necessarily fast shocks and thus are always stable.

In the special cases where some of the dissipation coefficients are assumed to be zero, one can expect singularities of the type arising in singular-perturbation problems. In particular, the profile of any quantity as function of x may have some discontinuity. Let us assume for instance that the system (6), written in the form $(n=4)$,

$$F_i(q_j)=0,\quad 1\leqslant i\leqslant r;\quad dq_i/dx=F_i(q_j)\quad r<i\leqslant n,\quad (31)$$

admits a solution with a discontinuity for $x=0$. A singular perturbation of (31) may be written

$$\epsilon dq_i/dx=F_i(q_j)\quad 1\leqslant i\leqslant n;\ dq_i/dx=F_i(q_j)\quad r<i\leqslant n,\quad (32)$$

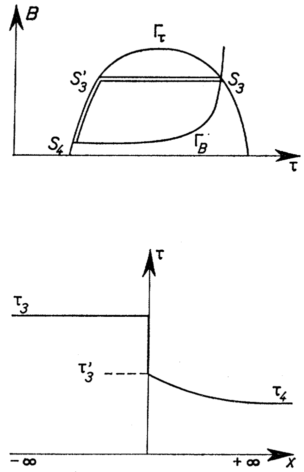
with ϵ a small positive parameter. The convergence of the convenient solution of (32) towards the corresponding solution of (31) is not uniform in the neighborhood of $x=0$. In order to study this discontinuity, let $x=\epsilon\xi$ and set $\epsilon=0$ in the result; one obtains

$$dq_i/d\xi=F_i(q_j)\quad 1\leqslant i\leqslant r;\quad q_i=\text{const},\quad r<i\leqslant n.\quad (33)$$

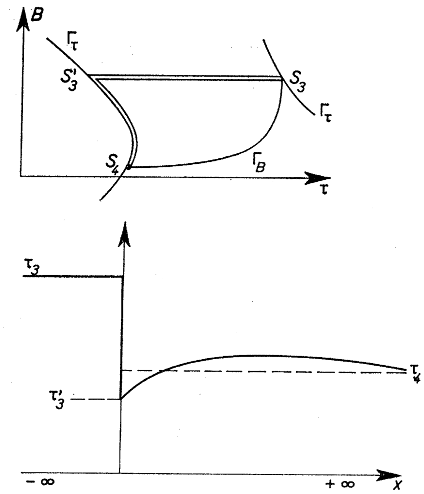
For the given values of q_i $(r<i\leqslant n)$, the solution of the differential system (33)—r equations—must satisfy

$$F_i(q_j{}^{(2)})=F_i(q_j{}^{(1)})\quad 1\leqslant i\leqslant r.\qquad (34)$$

$q_j{}^{(1)}$ and $q_j{}^{(2)}$ denote the values of the q_j on both sides



(a) FIG. 6. (a) Case (i); (b) case (ii). (b)



of the discontinuity $x=0$. The variations of the q_i $(1\leqslant i\leqslant r)$ inside the discontinuity are given by the solution of (33), for $-\infty\leqslant\xi\leqslant+\infty$.

Without discussing any further the general case, some special ones are mentioned. Assume σ to be the only nonzero dissipation coefficient. First, if $B_0{}^*=0$, the flow is continuous if the velocity in S_2 is supersonic $(c_2\leqslant u_2\leqslant A_2)$, but discontinuous, (Fig. 5), if $u_2\leqslant c_2$. Other possibilities arise as limiting cases of slow shocks considered in the previous section—cases (i) and (ii); in such cases (Fig. 6) the continuously varying part of the flow is located behind the shock.

When considering conditions (34), it can be verified that when $m_2=k=0$, $\sigma^{-1}\neq 0$, $m_2\neq 0$, no discontinuity may arise in the solution. This provides a justification of the special example which has been considered in Sec. 7.

DISCUSSION

Session Reporter: J. E. McCUNE

N. H. Kemp, *Avco-Everett Research Laboratory, Everett Massachusetts:* I have two things I would like to ask about. First, according to my interpretation of your "intermediate shocks," these are the ones that cotate the magnetic field only, is that correct? Is your classification the same as that given in the Russian papers to which you refer?

P. Germain: No, my "intermediate shock" is a shock for which the tangential magnetic field changes its sign. The shock that rotates the magnetic field only is a special intermediate shock for which $E^*=0$. My general argument is given here for $E^*\neq 0$.

N. H. Kemp: Then I take it this is not what they call an intermediate shock. The Russians have a corner in your diagram where the shock is stable.

P. Germain: Yes, only the corner—they have just this special case. And my statement that the intermediate shock is not stable is not necessarily valid in that special case. I do not believe I can simply extend my results to the special case $E^*=0$ without looking at that problem more carefully.

N. H. Kemp: The other point I wanted to ask about is this. You say that all the fast shocks are stable. But there is a limiting case of fast shocks called the "switch-on shock." Do you conclude that these are also stable?

P. Germain: Yes, I think so. As you say, that is a limiting case which must be investigated carefully, but my first impression is that it would be stable.

N. H. Kemp: Well, I believe there is a paper in the recent Russian literature which draws the conclusion that "switch-on" shocks are unstable—unstable to Alfvén waves.

H. Grad, *Institute of Mathematical Sciences, New York University, New York, New York:* The conclusion of an informal discussion which I had yesterday with Dr. Germain was that probably "switch-on" shocks are stable, but you can't study them in this way—they are isolated from the general approach. I would like to add a comment, however: the "switch-on" shocks *must* turn out to be stable or the whole structure of magnetohydrodynamic shock theory is worthless. One must have "switch-on" shocks to solve problems. For example, with the piston problem, one must have one to satisfy the boundary conditions.

N. H. Kemp: But cannot there be a different solution—one we have not looked at so far?

H. Grad: No, I do not believe one can solve the piston problem without a "switch-on" shock.

P. Germain: Let me emphasize again that I have not tried to study the "switch-on" shock in my general approach. The "switch-on" shock occurs for $E^*=0$, and the question of the stability of that special case I cannot answer for sure.