Idealized Problems of Plasma Dynamics Relating to Geomagnetic Storms

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1. INTRODUCTION

As with hydrodynamics, progress in plasma dynamics and hydromagnetics seems likely to come along two main lines—from general theorems, and from solutions of particular problems. In the latter, the fluid considered may be idealized in some respects. For application to real fluids, the consequences of the neglected properties must later be considered. Ferraro and the author in 1940,[1] and Ferraro in 1952,[2] solved by "particle" methods two particular problems of plasma dynamics that bear on the first phase of geomagnetic storms. The main features of these problems and of their solutions are described, and some parts of the solutions are illustrated more fully. The changes in the solutions if account is taken of some neglected factors are discussed, particularly as regards the thermal motions and collisions of the particles, and as regards the oblique impact of some of the particles on the magnetic field. Some further idealized problems are formulated, which are probably soluble by similar methods, and which may increase our understanding of plasma dynamics and of geomagnetic storms.

2. SOLAR PLASMA STREAMS AND MAGNETIC STORMS

For some decades the general opinion has been that geomagnetic storms are caused by streams of neutral ionized gas ejected from the sun. The gas consists mainly of protons and electrons. The speed is estimated to be of order 1000 km/sec. This corresponds to a kinetic energy of 5.2 kev for protons and 2.8 ev for electrons. Such a stream, when directed towards the earth, is affected by the geomagnetic field.

During the years 1929–1940 Ferraro and the author tried to infer how the stream and the geomagnetic field would be altered by their interaction.[1,3,4] At the same time Langmuir[5] was studying neutral ionized gases under laboratory conditions. There electrodes and other solid boundaries are important. He gave the name *plasma* to such gases. Some of the results he and we obtained, independently and unaware of each other's work, were similar. In general, however, the problems considered differed greatly in scale and nature. Ferraro and the author considered the simplest type of plasma, in which the positive ions, equal in number to the electrons, are all alike. The stream until its arrival near the earth was regarded as traversing space free from magnetic and electric fields. This space, and also the outer regions of the geomagnetic field, were treated as being empty until the advent of the stream. Thus the interplanetary gas and the earth's atmosphere above a few hundred km were ignored.

From about 1937 Alfvén[6,7] studied solar streams and magnetic storms. His approach differed from ours in two ways. He attached importance to the solar magnetic field in interplanetary space, even at the earth's distance. Also he envisaged a stream containing electrons with energies up to 100 Mev. They were regarded as spiraling in the magnetic field with great speed, while also traveling away from the sun with a speed of order 1000 km/sec. His later studies dealt with a simpler plasma stream without such high energy electrons.

3. ENERGETIC ELECTRONS AND PROTONS

Recent observations with earth satellites and far ranging rockets have revealed the presence around the earth of "radiation belts." [8–11] These contain electrons with energy up to 100 kev or more, and protons with energy up to at least 100 Mev. Rockoon observations by Van Allen and his colleagues had previously revealed a continual influx of electrons into the earth's atmosphere in auroral latitudes (from about 65° to 75° geomagnetic latitude). Their energies ranged up to about 100 kev. Winckler *et al.*[12,13] found a similar in-

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[1] S. Chapman and V. C. A. Ferraro, Terrestrial Magnetism and Atmospheric Elec. **45**, 245 (1940).

[2] V. C. A. Ferraro, J. Geophys. Research **57**, 15 (1952).

[3] S. Chapman and V. C. A. Ferraro, Terrestrial Magnetism and Atmospheric Elec. **36**, 77 (1931a); 171 (1931b).

[4] S. Chapman and V. C. A. Ferraro, Terrestrial Magnetism and Atmospheric Elec. **37**, 147, 421 (1932); **38**, 175 (1933).

[5] I. Langmuir, Phys. Rev. **33**, 954 (1929); cf. also L. Tonks and I. Langmuir, *ibid.* **33**, 195 (1929).

[6] H. Alfvén, Kgl. Svenska Vetenskapsakad. Handl. (3) **18**, No. 3 (1939); No. 9 (1940); *Cosmical Electrodynamics* (Clarendon Press, Oxford, England, 1950).

[7] H. Alfvén, Arkiv Mat. Astron. Fysik **B29**, No. 2 (1942); *Cosmical Electrodynamics* (Clarendon Press, Oxford, England, 1950).

[8] J. A. Van Allen and L. A. Frank, Nature **183**, 430 (1959).

[9] J. A. Van Allen, C. E. McIlwain, and G. H. Ludwig, J. Geophys. Research **64**, 271 (1959).

[10] S. N. Vernov, A. E. Chudakov, E. V. Gorchakov, J. L. Logadiv, and P. V. Vakulov, Planet. Space Sci **1**, 86 (1959).

[11] S. N. Vernov, A. E. Chudakov, P. V. Vakulov, and Yu. I. Logachev, Soviet Phys. (Doklady) **4**, 338 (1959).

[12] J. R. Winckler, L. Peterson, R. Arnoldy, and R. Hoffman, Phys. Rev. **110**, 1221 (1959).

[13] J. R. Winckler, L. Peterson, R. Hoffman, and R. Arnoldy, J. Geophys. Research **64**, 597 (1959).

flux during great magnetic storms in the lower latitudes in which auroras then appear. Their observations were made with balloon-borne instruments. They found also that at times protons with energies up to 100 Mev also enter the earth's atmosphere.

These discoveries prompted the suggestion that sometimes the solar streams may carry such abnormally energetic particles with them, trapped by a magnetic field within the stream. This magnetic field may have been transported away from the sun by the ejected stream of plasma. Even without this complication, the interaction of the stream and the geomagnetic field presents a difficult problem. As yet it has not been possible to determine the interaction experimentally, using proved scale relationships. As a theoretical problem it has so far received only preliminary treatment.

4. HYDROMAGNETICS AND HYDROMAGNETIC INFERENCES

This problem is one of importance in cosmic magneto-fluid dynamics, or magnetohydrodynamics, or, as the author prefers to say (following Cowling), in hydromagnetics.

Hydromagnetics is a kind of superhydrodynamics, much more difficult than classical hydrodynamics. In cosmic hydromagnetics, the medium, being gaseous, is compressible, and has a greater number of significant properties than have the fluids usually considered in hydrodynamics.

Classical hydrodynamics developed during the nineteenth century along two main lines. The general equations were formulated by Euler and Lagrange and others, and some general consequences were inferred from them. Also many special problems of fluid motion were solved. Besides their individual interest, these solutions increased our understanding of fluid motions in general, with and without boundaries. Still later the special features of turbulent motion were studied.

Hydromagnetics is still in its earliest stages. The general equations have been formulated and some general inferences have been drawn from them; among these, Alfvén's discovery of hydromagnetic waves in 1942 is outstanding.[7] Other inferences include some that deserve critical examination in particular applications.

One such inference is that magnetic fields in a plasma are "frozen" into it, and tend to move with it; the more completely so, the greater is the electrical conductivity. Another is that in a turbulent plasma containing a magnetic field, the field becomes tangled, and the magnetic energy, if initially small, may increase to approximate equality with the kinetic energy of turbulence.

Few special problems of hydromagnetics have been solved, and these are too simple to approximate closely to the magnetic storm problem here discussed. At best they can throw light on particular aspects of this problem.

5. HYDROMAGNETICS AND PLASMA KINETICS

A few comments on terminology may be appropriate. In 1956, at a Symposium of the International Astronomical Union, van de Hulst[14] remarked that the term *hydromagnetics* is now often reserved for *fluid* dynamics involving magnetic fields, and the term *plasma dynamics* for the dynamics of *ionized gases* in magnetic fields. Elsasser[15] suggested that *hydromagnetics* should have reference to phenomena that can be described in terms of purely classical theory, electromagnetic and hydrodynamic. "On the other hand, there are classes of phenomena, particularly those of interest to engineers, where the classical description breaks down and it is necessary to go into what is conventionally known as kinetic theory." He cited the dynamo theory of the main geomagnetic field, originating in the deep interior of the earth, as an example of hydromagnetics. *Plasma mechanics* (statics and dynamics) would refer to problems of highly ionized gases.

Some plasma phenomena may be considered as being also hydromagnetic, for example, hydromagnetic turbulence in plasmas. Other phenomena may be considered separately under one or other aspect; this is so also in aerodynamics, where classical methods can be applied in problems involving boundary layers and shock waves by considering discontinuities at certain surfaces. Closer study of these "discontinuities" must be sought by kinetic theory methods.

Kantrowitz and Petschek[16] have given an outline analysis of the domains of gaseous hydromagnetics. They indicate a number of lengths of importance for ionized gases in magnetic fields, including the mean free path, the Debye length, and a particular case of the Larmor radius, for equal gas pressure and magnetic pressure. It seems appropriate to use the name *plasma kinetics* for studies which involve detailed examination of the particle motions, where these change materially within a distance equal to the shortest of these lengths. The same phenomena, however, may in some respects also be viewed hydromagnetically.

6. PROBLEMS OF UNIFORM MOTION THROUGH A UNIFORM FIELD

One primitive hydromagnetic problem that has been solved concerns an "infinite plane slab" of uniform plasma moving with uniform velocity $\mathbf{v}$ in an all-pervading uniform magnetic field $\mathbf{H}$ which is perpen-

[14] H. C. van de Hulst, *Electromagnetic Phenomena in Cosmical Physics*, International Astronomical Union Symposium No. 6 (Cambridge University Press, New York, 1958).
[15] W. M. Elsasser, *Magnetohydrodynamics* (Stanford University Press, Stanford, California, 1957), p. 16.
[16] A. R. Kantrowitz and H. E. Petschek, *Magnetohydrodynamics* (Stanford University Press, Stanford, California, 1957), p. 3.

dicular to **v**. The plasma lies between planes perpendicular to both **v** and **H**. Except near the boundaries, the plasma particles move steadily. The electromagnetic force $e\mathbf{v}\times\mathbf{H}$ acting on them is neutralized by an electric field. This arises from a uniform polarization of the "slab," which acquires oppositely charged layers at the two boundaries. In these transition layers, the force $e\mathbf{v}\times\mathbf{H}$ is only partly neutralized, and the particles oscillate with the gyrofrequency.

If the plasma boundary is less simple, a steady state may not be possible. An example is any cylindrical boundary with its generators parallel to **v**. The polarization and surface charge distribution neutralize the force $e\mathbf{v}\times\mathbf{H}$ within the plasma, but the particles in the surface transition layer are only in quasi-equilibrium. They leak away, along paths that converge to the lines of magnetic force. The plasma stream thus gradually disperses.

In these cases there is no apparent transport of the magnetic field with the plasma. When first set in motion, a polarizing cross current flows. This reduces the original speed. But afterwards the main body of the plasma moves freely and steadily through and across the field.

As here presented, with its particles all in perfectly uniform motion, the plasma has zero temperature. Also it has no electrical conductivity in the usual sense, depending on "peculiar" motions of the particles superposed on their mean motion.

In other respects also these problems are very special —in their infinite extent and the complete uniformity of the magnetic field; also in the absence of any ionized background gas within or through which the plasma moves. The presence of such gas tends to neutralize and to reduce the polarization surface charges of the moving plasma.

7. APPROACH OF A SOLAR STREAM INTO THE GEOMAGNETIC FIELD

The earth's field is by no means uniform. When in 1931 we first considered its interaction with a solar stream, Ferraro and the author simplified the problem in various ways, and our discussion was partly intuitive, only partly quantitative. We treated the stream front as plane, normal to the stream velocity v and of practically infinite extent. We regarded the plasma as a medium of high electrical conductivity (reference 3, pp. 175–180) and pictured the induction of electrical currents in its surface layer. These would wholly or partly shield the interior from the geomagnetic field. They would also modify this field on the earthward side of the stream surface. The interaction between the surface-layer currents and the magnetic field would retard the advance of the surface. The retardation would be nonuniform, and greatest along the sun-earth line. Thus the surface would become distorted and the geomagnetic field would "carve out a hollow" in the stream. So long as the distortion was small, the magnetic field of the surface currents, in the space outside the plasma, would approximate to that of an image dipole. Here we used Maxwell's solution[17] of the problem of the advance of a magnetic dipole towards a thin conducting plane sheet. At the point midway between the dipole and its image, the induced currents would double the normal field at the surface, adding to it an amount there equal to the normal field.

These conceptions, and this partly intuitive, partly quantitative treatment, seemed to have some useful bearing on the initial interaction between a solar stream and the geomagnetic field. A magnetic storm typically increases the horizontal field at the earth's surface during the first phase.

Later we attempted to illustrate some features of this initial interaction in quite a different way. We considered an idealized case which removed some of the complicating features of the actual problem. The problem we treated was soluble, and its solution (we believe) throws light on some important aspects of the actual problem.

8. CYLINDRICAL SHEET PROBLEM

As in the idealized problems already mentioned (Sec. 6), we considered a plasma whose particles have completely ordered motion without superposed peculiar motions. We followed the paths and motions of the individual particles almost without reference to the quasi-continuous aspects of the plasma. Instead of a volume distribution we considered a surface distribution, in the form of an infinitely long (circular) cylindrical sheet. Positive ions and electrons are distributed over this sheet, in a random manner. Everywhere there is effective electrical neutrality.

Initially the sheet is at rest in the presence of a magnetic field everywhere parallel to the axis of the cylinder. This "permanent" field has intensity H_p expressed by

$$H_p = H_0(a/r)^n = M/r^n, \tag{1}$$

where r denotes the distance from the axis, and $n>2$. In our numerical illustrations we took $n=3$, so that the field distribution over any plane normal to the axis is identical with that in the equatorial plane of the earth's dipole field. Such a variation in a magnetic field with straight parallel lines of force would require the presence of a distribution of magnetic matter or electric currents. We ignored such presence, postulating that the cause of the field is constant (uninfluenced by the sheet) and completely permeable, and that it influences the plasma motion only through the magnetic field.

At an initial instant every particle in the cylindrical sheet is given the same speed U radially inward towards the axis and at right angles to it. Suppose we look towards the axis from any particle, so as to see the lines

[17] J. C. Maxwell, *Electricity and Magnetism* (Oxford University Press, New York, 1881), 2nd ed.

of force directed upward. Let this direction be reckoned as northward.

The magnetic field deflects the ions to the right, the electrons to the left. These motions constitute an eastward electric current round the cylinder. This current is solenoidal. It produces no magnetic field outside the cylinder. If the speed of propagation of the field changes is treated as infinite, the current produces a uniform increase of the magnetic field within the cylinder.

The lateral speed of the electrons exceeds that of the ions in the inverse ratio m_i/m_e of their masses. The magnetic field acting on these lateral motions produces a radial outward force on the particles. This force on the electrons is m_i/m_e times that on the ions. Thus the plasma distribution tends to break up into two coaxial cylinders, uniformly and oppositely charged. Such a separation would produce a radial electric field between the two sheets, and none elsewhere. The intensity of this electric field at the surface of each sheet is $4\pi\sigma$, where σ denotes the charge density of the sheet (it is therefore less for the outer sheet than for the inner one). If Q and $-Q$ are the total amounts of positive and negative charge per unit length of the cylinder, σ is numerically equal to $Q/2\pi r$ when the sheet has radius r.

The lateral speeds and the current(s) in the sheet(s) grow as the motion proceeds, until the radial retardation halts the sheet(s), at some radii r_e, r_i. These radii differ, and r_e exceeds r_i, if the sheet separates into two. But if Q is sufficiently great, the sheets never separate, because the difference between the radial forces on the electrons and ions may always remain less than $4\pi\sigma$. Then the sheet keeps together and remains always neutral.

Whether or not the sheet divides into two, after the radial inward motion is halted, the sheet or sheets are repelled away again. Ignoring any loss by radiation, from the slightly deflected and retarded or accelerated particles, the return speeds of the sheets are the same, at corresponding radii, as during the inward motion. But each particle of the plasma has been laterally displaced. It has followed an orbit symmetrical relative to the radius through the apsidal point at which its radial speed has been reduced to zero.

8.1. Some Main Results

The equations of motion of the particles were solved, and the solution was discussed.[1] Later Ferraro[2] gave the solution of an analogous problem, in some respects simpler, in others more complex. It dealt with a neutral ionized stream with infinite plane boundary projected normal to itself in the z direction towards the stronger part of a magnetic field. The lines of force of this field are parallel to the plane and its intensity is $H_0(a/z)^n$.

Some main results of the 1940 paper,[1] on the cylindrical problem, are quoted here and further consequences are derived. That paper gives some of the results in a more detailed and accurate form, but here only the simplified results are given. They correspond to the projection of the sheet from an infinite distance, with speed U; and they involve the approximation that m_e/m_i is neglected when it is one term in a sum whose magnitude is of order unity.

(a) The condition for no separation of the sheet into two is

$$Q > (n-1)m_iU^2/e, \tag{2}$$

where e is expressed in esu. This condition depends on the ratio energy/charge for an ion, but not on the intensity of the permanent field—only on its law of variation with distance.

(b) If the sheets separate, they do so at the radius r_s given by

$$\left(\frac{r_s}{a}\right)^{n-1} = \left(\frac{n-1}{\alpha_e}\right)^{\frac{1}{2}} \frac{aH_0}{(n-2)Q}, \tag{3}$$

where

$$\alpha_e = 1 + m_ec^2/eQ = 1 + 1700/Q. \tag{4}$$

(c) If the sheet remains intact, it attains the minimum radius r_0 given by

$$\left(\frac{r_0}{a}\right)^{n-1} = \left(\frac{e}{\alpha_e m_i Q}\right)^{\frac{1}{2}} \frac{aH_0}{(n-2)U}. \tag{5}$$

(d) In the latter case, the radial speed u at any radius r is given by

$$u^2/U^2 = 1 - (r_0/r)^{2n-2} \quad \text{(no separation)}. \tag{6}$$

(e) For any electron or ion, let ϕ_e or ϕ_i denote the cylindrical polar angular coordinate. For definiteness let ϕ be measured for any particle in the eastward direction from that of the radius through its position of minimum distance. Then at distance r the rate of change of ϕ is given by

$$m_e\dot{\phi}_e = -m_i\dot{\phi}_i = -[cm_ea^nH_0/(n-2)\alpha_eQr^n]. \tag{7}$$

(f) Let ΔH denote the uniform intensity of the magnetic field of the sheet currents, in the space enclosed by the sheet; let ΔH_0 denote its maximum value, corresponding to the minimum radius r_0. Let H_{ps} denote the maximum intensity of the permanent field just inside the sheet, when this has radius r; the subscript s in H_{ps} denotes *surface*; $H_{ps} = a^nH_0/r^n$. The ratio $\Delta H/H_{ps}$ has at all times the *constant* value given by

$$\Delta H/H_{ps} = 2/(n-2)\alpha_e. \tag{8}$$

This applies to any instant, so long as the sheet is not separated into two.

(g) The eastward electric current J in the sheet, per unit length along the axis, is given by

$$J = H_{ps}/2\pi(n-2)\alpha_e \text{ emu}. \tag{9}$$

[Equations (2)–(8) correspond to the Eqs. (8), (5), (2), (10), (1), and (6) of Sec. 3.5 in reference 1.] When $n=3$, Eqs. (3) and (5)–(9) become

$$\frac{r_s}{a}=\left(\frac{2}{\alpha_e}\right)^{\frac{1}{4}}\left(\frac{aH_0}{Q}\right)^{\frac{1}{2}},\quad r_s=\left[\frac{2}{\alpha_e}\right]^{\frac{1}{4}}\left[\frac{M}{Q}\right]^{\frac{1}{2}},\tag{3a}$$

$$\frac{r_0}{a}=\left(\frac{e}{\alpha_e m_i Q}\right)^{\frac{1}{4}}\left(\frac{aH_0}{U}\right)^{\frac{1}{2}},\quad r_0=\left[\frac{e}{\alpha_e m_i Q}\right]^{\frac{1}{4}}\left[\frac{M}{U}\right]^{\frac{1}{2}},\tag{5a}$$

$$u^2/U^2=1-(r_0/r)^4,\tag{6a}$$

$$m_e\phi_e=-m_i\phi_i=-cm_eH_0a^3/\alpha_eQr^3,\tag{7a}$$

$$\Delta H/H_{ps}=2/\alpha_e;\tag{8a}$$

$$J=H_{ps}/2\pi\alpha_e,\tag{9a}$$

where

$$M=H_0a^3,\quad N=Q/e.\tag{10}$$

In the case of the earth, H_0a^3 is its magnetic moment. Here N denotes the number of ions (or electrons)/cm length of the ionized sheet.

8.2. Numerical Discussion

The preceding results for $n=3$ were discussed in relation to the problem of magnetic storms as follows. The values of H_0 and a (and M) were taken to correspond with the equatorial field intensity and radius, and magnetic moment of the earth,[18] namely, 0.312 gauss, 6.37×10^8 cm, 8.06×10^{25} gauss cm^3. The value of U was taken to be 10^8 cm/sec. The ions were taken to be protons ($m_i=1.67\times10^{-24}$ g), or alternatively calcium ions. This latter alternative is not considered here.

The least value of Q that just prevents separation of the sheet into two is given by $Q=2m_iU^2/e=67$ esu. This corresponds to $\Delta H_0=3\times10^{-11}$ gauss—utterly negligible. The minimum distance r_0 attained by the sheet is 5.1×10^{11} cm or about $800a$. At this distance there is about one proton and one electron for each 22 cm^2.

This result may be compared with the distance attained by a solitary charged particle projected radially from infinity into the permanent field given by (1), for $n=3$. This is a special case of the problem considered by Störmer,[19] and corresponds to motion in the plane of the geomagnetic equator. Referring to his book *The Polar Aurora* (reference 19, p. 219), his Eq. (7.4) contains an integration constant γ: for the equatorial plane the form of (7.4) is

$$r^2d\phi/ds=2\gamma+r^{-1}.\tag{11}$$

The left-hand side is equal to the ratio $p_\theta/2p$, where p denotes the magnitude of the linear momentum of the particle and p_θ its angular momentum about the axis. In Störmer's problem p is constant throughout the motion. Thus, for a path that extends to or from infinity, γ denotes the value of $p_\theta/2p$ at infinity. (It is easily seen that this is also the physical interpretation of γ for paths to or from infinity, not confined to the equatorial plane.) For particles moving in the equatorial plane, γ is half the distance between the line of projection and the center, in the Störmer unit. For a particle projected radially, $p_\theta=0$, hence also $\gamma=0$. Störmer showed (reference 19, p. 222) that in this case the minimum distance attained is his "unit" length

$$C_{\mathrm{St}}=(Me/mvc)^{\frac{1}{2}}.\tag{12}$$

For a proton for which $v=U=10^8$ cm/sec, the motion being in the geomagnetic equatorial plane,

$$C_{\mathrm{St}}=8.8\times10^{10}\ \mathrm{cm}=r_0/5.8=138a,$$

and for an electron

$$C_{\mathrm{St}}=3.8\times10^{12}\ \mathrm{cm}=7.4r_0=5900\mathrm{a}.$$

Here r_0 refers to the minimum distance for the sheet just discussed, which just avoids separation into two. In such a sheet, the mutual influence between the particles reduces the penetration of the protons into the field by a factor 5.8 as compared with their penetration if solitary. The electrons, conversely, are carried 7.4 times closer in than they would go if solitary. Even for such a stream, so weak as to have an utterly insignificant magnetic effect, the mutual influence of the particles thus powerfully affects their motions.

One way in which this happens, in the plasma sheet problem, is that self-induction acts to increase the virtual masses of the electrons and ions for changes of lateral motion. Thus the virtual mass of an ion is increased from m_i to m_i+eQ/c^2, and there is the same increase, eQ/c^2, for an electron—representing in this case a far greater proportionate change of mass, namely, by eQ/m_ec^2, or $Q/1700$. In the foregoing case this is small, about 4%, yet this small change suffices to change the character of the motion completely.

8.3. Initial Field Increase in a Magnetic Storm

Ferraro and the author associated the increase of field within the sheet with the initial increase in the horizontal component of magnetic force observed at the earth's surface in low latitudes during a magnetic storm. This increase may range up to 200γ (or 2×10^{-3} gauss), but values of 20γ to 40γ are more usual.

Let ΔH_0 denote the maximum field of the sheet currents, given by (9a). The following results are calculated for two cases, $\Delta H_0=20\gamma$ and $H_0=200\gamma$, adopt-

[18] E. H. Vestine *et al.*, Carnegie Inst. Washington, Publ. No. 580, 4 (1947).

[19] C. Störmer, *The Polar Aurora* (Clarendon Press, Oxford, England, 1955).

ing the value 10^8 cm/sec for U:

	r_0/a	Q	N	r_0/U	n_0
$\Delta H_0=20\gamma$	14.6	2.5×10^{10}/cm	5.2×10^{19}/cm	93 sec	8.9×10^{8}/cm^2
$\Delta H_0=200\gamma$	6.8	5.35×10^{11}/cm	1.1×10^{21}/cm	43 sec	4.1×10^{10}/cm^2.

The last quantity here given, n_0, is the surface number density of protons (and electrons) when the sheet has its minimum radius ($n_0=N/2\pi r_0$).

Corresponding values of the electric current density in the sheet at its minimum radius are as follows: the current aJ in a band of the cylinder of breadth equal to the earth's radius a is also given.

	J	aJ
$\Delta H_0=20\gamma$	1.6×10^{-4} amp/cm	100 000 amp
$\Delta H_0=200\gamma$	1.6×10^{-3} amp/cm	1 000 000 amp.

Thus the population of the sheet and its $+$ or $-$ charge Q, required to produce even the very moderate change 20γ, is more than 10^8 times what suffices to prevent any separation of the ions and electrons into two sheets. In this case the protons as well as the electrons penetrate much more deeply into the field than could a solitary proton with the same radially directed initial speed.

The value of Q for any case in which ΔH_0 is geomagnetically significant (e.g., at least one gamma) is such that α_e differs negligibly from unity. Consequently (8a) implies that throughout the approach and recession of the sheet, ΔH has twice the intensity of the permanent field just outside the sheet. Hence also the total field intensity just inside the sheet has three times its normal value.

In Sec. 6.1 of our 1940 paper we discussed the *volume* density that might correspond to the preceding values of Q and N, interpreting the ionized sheet as an idealization of a cylindrical *shell* of gas. We suggested that the appropriate thickness of the shell might be of order $5a$, giving a volume density, before the gas is much affected by the field, of order 0.14 proton-electron pair/cm^3.

8.4. Energy

The energy density of the uniform field ΔH produced by the sheet current in the space enclosed by the sheet is $(\Delta H)^2/8\pi$. Its amount E per unit length of the cylinder, at any instant when the sheet has radius r, is $(r\Delta H)^2/8$. On using (8), we obtain

$$E=a^{2n}H_0^2/\{2(n-2)^2\alpha_e^2r^{2n-2}\}. \tag{13}$$

The loss of kinetic energy of radial motion of the sheet, namely, $\frac{1}{2}N(m_e+m_i)(U^2-u^2)$, is expressible (making the approximations mentioned at the outset) as α_eE, using (5) and (6). This exceeds E because α_e exceeds unity. It is easy to show, using (7), that the difference, m_ec^2E/eQ, is equal to the kinetic energy E' of the lateral motion of the particles.

Thus the ratio E'/E is m_ec^2/eQ or $1700/Q$—independent of the initial speed of the sheet, and dependent only on its total positive and negative charge per unit length. This ratio is constant throughout the approach and recession of the sheet. It is extremely small for sheets that are able to produce an appreciable magnetic field ΔH_0, e.g., for the case first considered, where $\Delta H_0=20$ gammas (and $U=10^8$), $Q=2.5\times10^{10}$. Hence for such streams the conversion of kinetic energy of radial motion into field energy is extraordinarily efficient. When the radial motion ceases, at the minimum distance of the sheet, the initial kinetic energy is almost all transformed into field energy. The remaining energy is almost all possessed by the electrons. That of the ions is smaller in the ratio m_e/m_i. Even for the electrons the energy is extremely small, namely, 3.6×10^{-4} ev or about $\frac{1}{3}\,\mu$ev, in the case of $\Delta H_0=20\gamma$, $U=10^8$ cm/sec. The corresponding proton energy is only 0.15 μev.

However, for streams in which Q is comparable with or less than 1700, the interaction with the permanent field only slightly changes the total kinetic energy (or radial and lateral motion). The proportion transformed into field energy is then only $1/\alpha_e$. For example, if $Q=67$ (just sufficient to prevent breakup of the sheet into two, if $U=10^8$), the fraction of the kinetic energy transformed into field energy is only 3.8%.

The field of the sheet currents is superposed on the permanent field. As the two fields are in the same direction at every point, the total field is $H_p+\Delta H$. Thus the magnetic energy density in the space enclosed by the sheet is $(H_p+\Delta H)^2/8\pi$ and the change of energy density produced by the sheet is $(\Delta H)^2/8\pi+H_p\Delta H/4\pi$. The conservation of energy described previously makes no allowance for the latter term. The motions of the particles are influenced only by the field they transverse. They cannot depend on the nature or presence of any "permanent" field within the minimum radius r_0. Thus the integral of $H_p\Delta H/4\pi$ within this distance might have any value so far as concerns the rest of the problem. There is here an apparent paradox.

The following explanation of the paradox results from discussion with S.-I. Akasofu.

In the case of two current circuits 1, 2, carrying current J_1, J_2, the field energy can be expressed as the sum of volume integrals of $H_1^2/8\pi$, $H_2^2/8\pi$ and $\mathbf{H}_1\mathbf{H}_2/4\pi$, where $\mathbf{H}_1$ and $\mathbf{H}_2$ denote the vectors of the fields of J_1 and J_2. These integrals are, respectively, equal to $\frac{1}{2}L_{11}J_1^2$, $\frac{1}{2}L_{22}J_2^2$, and $L_{12}J_1J_2$, where L_{11} and L_{22} denote the self-inductances of the two circuits, and L_{12} their mutual inductance. In the cylindrical sheet problem here considered, L_{12} must be reckoned zero. This is because of the initial postulate that the system of currents (or magnetic matter) that produces the per-

TABLE I. Cylindrical sheet problem.

r/r_0	1	1.1	1.2	1.3	1.4	1.5	1.75	2	2.5	5
ϕ/ϕ_0	0	0.381	0.511	0.597	0.659	0.707	0.788	0.839	0.898	0.975
u/U	0	0.563	0.720	0.806	0.860	0.896	0.945	0.968	0.987	0.999
$v/v_0=(r_0/r)^2$	1	0.826	0.694	0.592	0.510	0.444	0.327	0.250	0.160	0.040
$\Delta H/\Delta H_0$	1	0.751	0.579	0.455	0.364	0.296	0.187	0.125	0.064	0.008
Ut/r_0	0	0.329	0.484	0.614	0.733	0.847	1.118	1.380	1.890	4.400

manent field is uninfluenced by the field changes due to the moving sheet.

8.5. Orbits of the Particles

The shape of the orbits of the particles was not discussed in the original paper. The plane polar equation is obtained by eliminating the time in (7) by means of the expression for u (or $\dot{r}$) in (6).

On taking $n=3$, and using (6a) and (7a), we obtain, for a nonseparating sheet,

$$\frac{d\phi_e}{dr}=-\frac{ca^3H_0}{\alpha_eQUr^3}\frac{1}{\{1-(r_0/r)^4\}^{\frac{1}{2}}} \tag{14}$$

and

$$m_ed\phi_e/dr=-m_id\phi_i/dr. \tag{15}$$

Write

$$\sin\theta=(r_0/r)^2,$$

so that $\theta=0$ when $r=\infty$ and $\theta=\frac{1}{2}\pi$ when $r=r_0$. Then (14) has the integral

$$\phi_e=A\theta, \tag{16}$$

where

$$A=\frac{ca^3H_0}{2\alpha_eQUr_0^2}=\frac{cm_iU}{ea(2H_0)^{\frac{1}{3}}(\Delta H_0)^{\frac{2}{3}}}. \tag{17}$$

Thus the equation of the electron orbits is

$$\phi_e=A\sin^{-1}(r_0/r)^2. \tag{18}$$

The radius from the axis to an electron turns through an angle $\phi_0=\pi A/2$ during the approach from infinity to the minimum distance r_0, and through an angle $2\phi_0$ or πA during the complete path from and back to infinity. This angle is proportional to m_iU, the original momentum of the ions, and inversely proportional to $(\Delta H_0)^{\frac{2}{3}}$.

For the particular cases previously considered (Sec. 8.3), the following results are obtained:

	A	$2\phi_{e0}$	$2\phi_{i0}$
$\Delta H_0=20\gamma$	5.6×10^{-3}	$-1.01°$	5.5×10^{-3} deg
$\Delta H_0=200\gamma$	1.2×10^{-3}	$-0.22°$	1.2×10^{-3} deg.

Thus the angle between the initial and final asymptotes of the path of an electron in these cases is of order 1° or less, and for a proton, far smaller still. Thus the particles are turned back almost along their inward tracks.

It is of interest to consider over what range of distance most of this deflection occurs. This is shown in the second row of Table I, which gives ϕ/ϕ_0. This changes between -1 at the initial to $+1$ at the final asymptote of the path of any particle. More than four-fifths of the deflection occurs between the distances $2r_0$ and r_0. During the small distance range $1.1r_0$ to r_0 the deflection is more than a third of the whole. The results apply to any sheet that does not separate; differences of ionic mass, initial speed, or mass per unit length of the sheet, only affect A and ϕ_0, but not the ratio ϕ/ϕ_0.

The third row of Table I shows the radial speed of the sheet at each distance, as a fraction of the original speed, $u/U=\cos\theta$. More than half the speed is lost in the final radial distance $0.1r_0$.

The fourth row of Table I gives the fraction v/v_0. Here v_0 denotes the maximum lateral speed, attained at the minimum distance r_0, and v is the lateral speed at distance r. From (7a) it is clear that

$$v/v_0=(r_0/r)^2 \tag{19}$$

equally for the ions and the electrons.

The ratio of the maximum lateral speed v_{e0} for the electrons to their initial speed U is given by

$$v_{e0}/U=2A. \tag{20}$$

Table II gives, for various values of ΔH_0, values of this ratio, and of v_{e0} for a proton-electron sheet, for $U=10^8$ cm/sec. The last line gives the electron energy at minimum distance r_0, in microelectron volts (μev). For any other value of U the values of v_{e0} are proportionately changed.

8.6. Time of Approach and Recession

It is of interest to consider the time of approach and recession of the sheet. A convenient time origin for this purpose is the epoch of maximum field or minimum distance. As $u=dr/dt$, Eq. (6a) may be written

$$dt=dr/u=dr/U(1-y^4)^{\frac{1}{2}},$$

where

$$y=r_0/r.$$

It is convenient to change the variable to x given by

$$2x=1-(1-y^4)^{\frac{1}{2}}, \tag{21}$$

so that $x=\frac{1}{2}$ when $r=r_0$, and $x=0$ when r is infinite. In terms of x we obtain the following expression for the

TABLE II.

H_0 (in γ)	20	50	100	200
v_{e0}/U	1.12×10^{-3}	0.61×10^{-3}	0.38×10^{-3}	0.24×10^{-3}
v_{e0} for $U=1000$ (unit is km/sec)	11.2	6.1	3.8	2.4
Electron energy in μev (for $U=1000$ km/sec)	359	105	42	17

time of passage from r to r_0 or from r_0 to r:

$$t=\frac{r-r_0}{U}+\frac{r_0}{2^{\frac{3}{4}}U}\int_x^{\frac{1}{2}}\frac{dx}{x^{\frac{1}{4}}(1-x)^{5/4}}. \tag{22}$$

The integral can be calculated by expanding the integrand as a series of powers of x and integrating term by term. The value of the second term when $x=0$ is $0.401(r_0/U)$. The times calculated for various values of r are given in Table I.

These and other features of the problem are illustrated in Fig. 1.

9. OBLIQUE PROJECTION

The solution of the cylindrical sheet problem indicates that the particles of a solar stream that are directed towards the earth's dipole axis are likely to be turned back again with very small deflection, almost as if they were reflected normally from a cylindrical (or plane) mirror. This applies most clearly to the particles at the front of a stream.

In a solar stream most of the particles are not directed toward the axis. If the motions are regarded as parallel, the particles to the right and left of the axial plane of the earth dipole that passes through the sun have angular momentum round the axis, in opposite directions. Intuitively, it seems likely that they are also turned back as by a kind of reflection. The distance from the earth at which this occurs may not be the same as for the particles projected towards the axis.

Recently this author noticed that the cylindrical sheet problem studied and solved in 1940 can be generalized to throw light on this situation. The extension consists in supposing that at initial projection all the particles have a lateral component of velocity V as well as the radial component U. (The addition of a third component W parallel to the lines of forces may also be considered, but has little interest, because this component remains constant throughout.) The author has little doubt that this problem can be solved as completely as was the original "radial projection" problem of 1940. P. C. Kendall will cooperate with the author in its solution and discussion. The infinite plane form of the problem can be similarly generalized (see also reference 2, p. 48).

Störmer considered the case of a single particle with oblique projection, and Fig. 2 reproduces some of the paths he calculated for the motion in the dipole equatorial plane. The particle with radial projection just reaches the Störmerian distance C_{St} given by (12), though only after considerable deflection. (This is shown in Fig. 2 for the case of a negatively charged particle, if the equatorial plane is supposed to be viewed from the northward side; a positively charged particle would be deflected to the right.) Some of the particles obliquely projected with angular momentum in the same sense as this deflection penetrate somewhat more deeply into the field; their path may be simple or looped, and the total deflection can be through less or more than 180°, or even more than 360°.

In the collective problem of the obliquely projected cylindrical sheet (and in the analogous plane problem), the motions of the particles are much simpler—at least for a sheet populated sufficiently to produce an appreciable magnetic effect. Further discussion of this extended problem must await the detailed solution. It will have some bearing on the shape of the hollow carved by the geomagnetic field in a solar stream—both as regards its section by the dipole equatorial plane, and by the dipole axial plane through the sun.

10. EXTENSION TO A VOLUME DISTRIBUTION AND PLASMA

The surface distribution of plasma considered in the cylindrical sheet problem is a drastic idealization of a solar stream. It may be regarded as merely a first step in the study of the interaction between a solar stream and the geomagnetic field.

A less drastic change in the problem is to replace the surface distribution by a volume distribution within a thin or thick cylindrical shell. Such a shell, initially projected radially toward the axis with speed U, could to some extent be treated by the general hydromagnetic equations (but see Sec. 15). This has not been attempted. An alternative approach, suggested in the 1940 paper (reference 1, p. 266), is to represent the solar stream by a series of closely spaced parallel surface distributions of plasma.

This plan was successfully carried out by Ferraro[2] considering a "plane" analog of the 1940 problem. This contained many results similar to those for the cylindrical sheet problem, but it also made important advances on the 1940 paper. Both papers gave general confirmation of most of the chief conclusions reached by us in 1931–3, when we treated the solar stream as a continuous medium, and did not try to follow the motions of the individual particles.

The main results of Ferraro's paper[2] are described briefly here (with page references) for a stream dense enough to produce a magnetic field of intensity 20γ or more.

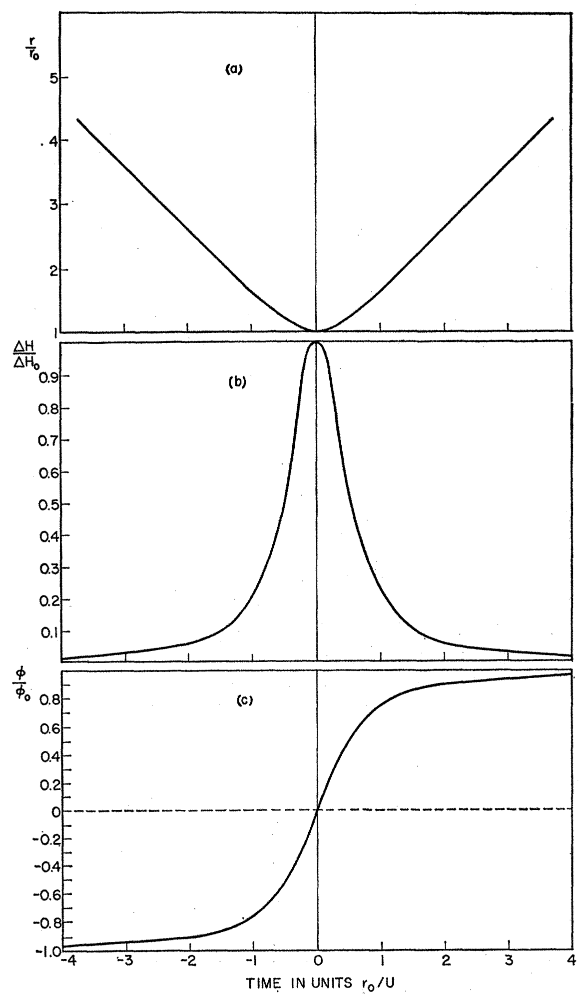

FIG. 1. Protons and electrons are uniformly distributed over an infinitely long cylindrical surface, in the presence of a magnetic field everywhere parallel to the axis, and of intensity $H=M/r^3$ at radius r. At time $t=-\infty$, every particle, at $r=\infty$, is projected radially inward with speed U. The oppositely charged particles are oppositely deflected, generating an electric current around the cylindrical surface, and producing a uniform magnetic disturbing field ΔH within the cylinder. The inward motion of the plasma sheet, retarded by the action of the magnetic field on the electric current, is halted at radius r_0, when $\Delta H=\Delta H_0$, at an instant which is taken as the time origin $t=0$. (a) Radius r as a function of t, taking r_0 as the unit distance and r_0/U as the time unit. (b) Change with time of $\Delta H/\Delta H_0$. (c) Change of azimuth of a typical particle, relative to its azimuth at $r=r_0$, $t=0$, between the asymptotic values $-\phi_0$ and ϕ_0. In each case most of the change of radial speed, lateral motion, and magnetic disturbance occurs in the two central time units. On taking $M=8\times10^{25}$ (the earth's dipole moment), $U=1000$ km/sec, then if $\Delta H_0=20\gamma$ or 215γ, r_0 is 14.6 or 6.8 earth radii, and ϕ_0 for the electrons is 1° or 0.22°; for the protons it is less in the mass ratio m_e/m_p. The two cases differ in the amount of positive (or negative) charge/cm length of the cylinder; the values are 2.5×10^{10} and 5.35×10^{11} esu/cm.

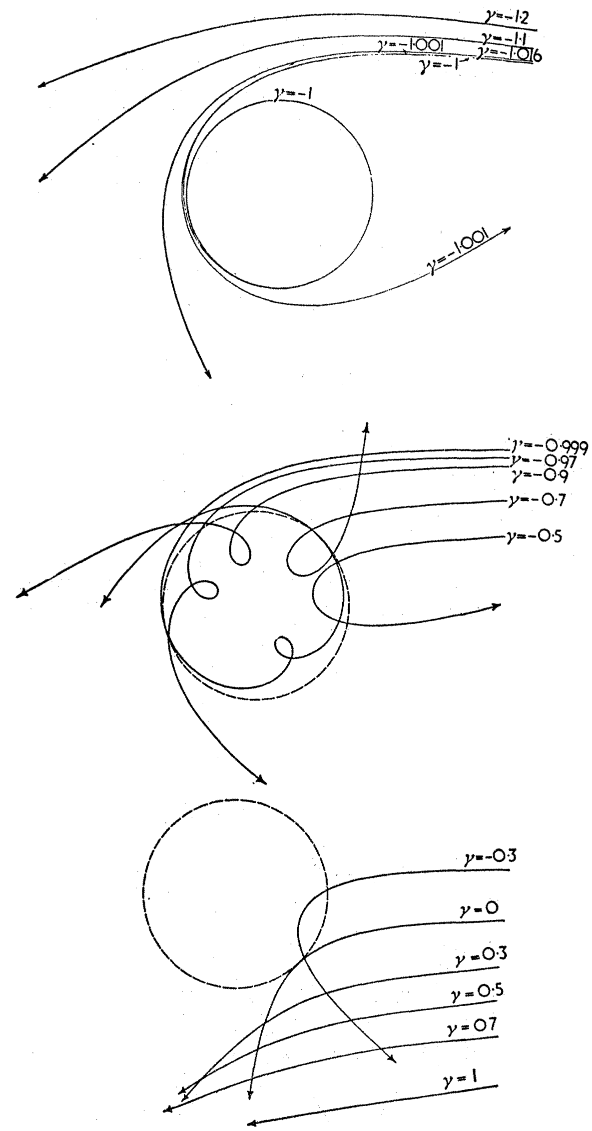

FIG. 2. Orbits determined by Störmer, for a solitary charged particle of mass m and charge e (emu), moving from infinity with speed v in the equatorial plane of a magnetic dipole of moment M. The distance between the line of projection and the dipole is 2γ. The circles indicate the scale; their radius is one Störmer unit of length, $(Me/mv)^{\frac{1}{2}}$. In the geomagnetic case, the orbits are those of negative particles as seen from the north side of the plane, or those of positive particles as seen from the south side. With direct projection ($\gamma=0$) the particle just reaches unit distance from the dipole. Particles projected so that γ lies between 0 and -1 attain distances less than 1 unit. The unit distance for $v=1000$ km/sec is $138a$ for a proton and $5900a$ for an electron (a=earth radius).

11. ADVANCE OF A SOLAR STREAM INTO A "PLANE MODEL" MAGNETIC FIELD

The "plane model" magnetic field considered in this problem is unidirectional, along the y direction, and its intensity H_p is given by

$$H_p=H_0(a/z)^n \quad n>2. \tag{23}$$

Thus the field varies only in one direction, instead of in two, as in the cylindrical problem. Hence it is a step further away from reality, as the geomagnetic field varies in three directions (and is not unidirectional).

The solar stream is supposed to be initially uniform and in uniform motion, with speed W in the $-z$ direction. It is bounded only by a plane face normal to this direction. It starts with its front at a great distance, where the field H_p is negligible, and advances into the stronger part of the field. As before, in numerical illustrations W is taken to be 10^8 cm/sec, and the values given to H_0, a, and n are the same as before. Collisions between the particles are ignored (and also any background interplanetary or terrestrial atmospheric gas).

The greater mass of the ions (here taken to be protons) carries them further into the field than the electrons. Thus the fore part of the stream is polarized, its front layer being positively charged. The interaction between the field H_p and the gas induces current flow in the stream in the x direction. The field ΔH of these currents is uniform outside the stream; there (reference 2, p. 35)

$$\Delta H = \tfrac{1}{2} H_{ps}, \tag{24}$$

where H_{ps} denotes the value of H_p at the front surface. Within the stream the total field decreases rapidly (and approximately exponentially) inwards to zero (reference 2, p. 42). Thus the induced currents in this case (differing from the cylindrical case) shield all but a thin front layer of the stream from the permanent field. That is, their own field annuls H_p in the body of the stream.

In the fore part of the stream (and nowhere else) there is an electrostatic field in the z direction. It also decreases exponentially inwards, twice as fast as the magnetic field. Their rates of decrease vary as $e^{-2z/l}$, $e^{-z/l}$, respectively, where (reference 2, p. 42)

$$l = (m_e c^2/4\pi n_e e^2)^{\frac{1}{2}} = 5.3\times 10^5/n_e^{\frac{1}{2}}\ \text{cm}. \tag{25}$$

As n_e ranges from 1 to 10^5 cm^3, l ranges from 5.3 km to 17 m. Thus the layer near the surface affected by these fields is very thin compared with the earth's radius. The front of the stream comes to rest at a distance z_0, where (reference 2, p. 35)

$$z_0 = (3H_0/2)^{1/3}(8\pi n_i m_i W^2)^{-1/6} a. \tag{26}$$

With the stated values of H_0, m_i, and W, this is equivalent (reference 2, p. 39) to

$$z_0/a = 8.86 n_i^{-1/6}. \tag{27}$$

The forward motion of the ions (protons) is stopped almost entirely by the action of the electrostatic field. This field draws the electrons onward, almost but not quite up to the same point (reference 2, p. 33). The motion of an ion-electron pair is the same (reference 2, p. 33) as if both + and − particles had each the same mass, $(m_e m_i)^{\frac{1}{2}}$.

The maximum magnetic disturbance ΔH_0 caused by the solar stream is obtained by combining (23), (26), and (27); it is given by

$$\Delta H_0 = \tfrac{1}{3}(8\pi n_i m_i W^2)^{\frac{1}{2}}. \tag{28}$$

For a proton-electron stream, taking $W = 10^8$ cm/sec, this gives

$$\Delta H_0\ (\text{in}\ \gamma) = 21.5\ (n_i)^{\frac{1}{2}}. \tag{28a}$$

Figure 3 shows a small table giving values of ΔH_0 and z_0 for three values of n_i (or n_H, the number of dissociated hydrogen atoms) per cc. For $n_i = 100$, $\Delta H_0 = 215\gamma$. This is a rather high value, not often recorded. It should be remarked that ΔH_0 represents the field of the solar stream in this "plane model" case. The actual solar stream folds somewhat round the earth on both sides, giving a greater disturbing field for the same value of z_0 (though not so great as for a cylindrical stream, that encloses the permanent field on all sides). Moreover, the observed disturbing field at the sudden commencement of a magnetic storm is partly due to suddenly induced currents within the earth. An external disturbing field ΔH_0 of 215γ would correspond to an observed disturbance of at least 320γ in low latitudes (but outside the electrojet belt near the magnetic equator). (Even greater values of the initial disturbance have been observed, as in the storm of February 11, 1958, but mainly in the higher latitudes, where much of the initial disturbance may come from electric currents in the auroral ionosphere.) Hence the author thinks it likely that the solar stream has a density of 100/cm^3 at most.

The magnetic energy density of the permanent field at the face of the stream, when this has reached a nearly steady condition, corresponds to

$$H_{ps}^2/8\pi = n_i m_i W^2. \tag{29}$$

This result indicates that the stream is stopped and turned back in a region where the magnetic energy density of the permanent field is comparable with the initial kinetic energy density of the stream. Also, as Martyn[20] first indicated, (29) can be interpreted in terms of the magnetic pressure on the front face of the stream, normal to the field lines. The pressure is absent at the rear of the thin front layer wherein the electric currents flow that shield the stream interior from the permanent field. It is in this layer that the forward momentum of the stream is reversed; the magnetic pressure balances half the rate of change of momentum $2n_i m_i W^2$/cm^2 sec. The other half is taken up by the retarding electrostatic field in the front charged layer. Equation (29) may be regarded (Sec. 5) as hydromagnetic, dealing with the larger aspects of the present problem, although it has been derived by a plasma kinetic treatment.

The motion of the particles during the brief interval of their reversal near the front of the stream is difficult to follow. This is so (reference 2, p. 30) even in the simplest case when the positive and negative particles have equal mass (in which case there is no charge sepa-

[20] D. F. Martyn, Nature 167, 92 (1951).

ration or polarization of the stream, even near the front surface). Ferraro discussed this case first. Then he showed (reference 2, p. 33) that when the masses are unequal, the motion of an ion-electron pair in the z direction is the same as if the masses were equal, with the value $(m_e m_i)^{\frac{1}{2}}$. The retardation when the masses are unequal exceeds what it would be if both were equal to m_i. This is because of the extra retardation caused by the polarization electric field.

As stated, Ferraro treated the stream as a succession of plane sheets—surface distributions of plasma. Advancing into the permanent field, those ahead are most retarded, those behind overtake them, and pass through them. He showed (reference 2, pp. 28, 38) that this

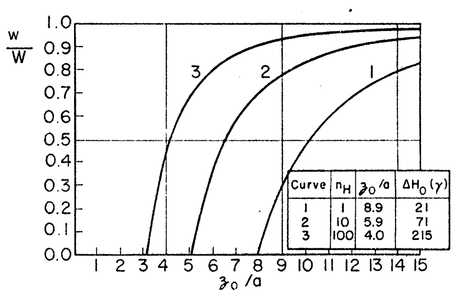

FIG. 3. Speed-distance relation for the stream front for three stream densities $n=1$, 10, 100/cm^3, taking $W=1000$ km/sec, $M=8.06\times10^{25}$ (the earth's dipole moment). The distance unit is the earth radius a.

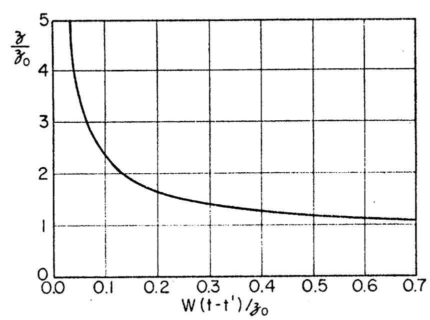

FIG. 4. Distance-time relation for the front surface; the time unit, z_0/W, has the values 57, 38, 25 sec in the respective three cases of stream density.

overtaking first occurs at a distance

$$3KC_{\mathrm{St}}/2^{\frac{1}{2}},$$

where $K(=2.781)$ is the first zero of the Bessel function $J_{1/4}(x)$, and C_{St} is the Störmer unit—see (12)—for the case $m=(m_e m_i)^{\frac{1}{2}}$. For $W=10^8$ and ions that are protons, this distance is 27 000 earth radii!

The speed w of initial advance of the stream surface towards its maximum penetration into the field is expressible thus (reference 2, p. 36):

$$w/W=1-(z_0/z)^3, \tag{30}$$

somewhat analogous to (6) in the cylindrical sheet

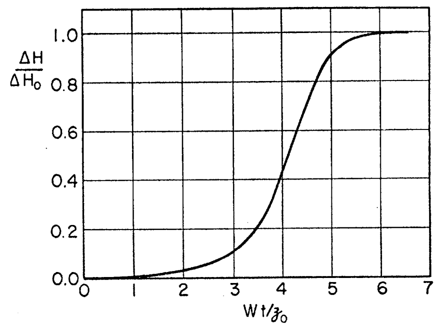

FIG. 5. Rate of rise of the magnetic field of the currents, in the same time units as used in Fig. 4.

problem. The variation of forward speed with distance is illustrated in Fig. 3 for various stream densities. This equation (as $w=-dz/dt$) has the inverse solution (reference 2, p. 27)

$$\frac{Wt}{z_0}=-Z+\frac{1}{6}\ln\frac{Z^3-1}{(Z-1)^3}-\frac{1}{\sqrt{3}}\cot^{-1}\left(\frac{2Z+1}{\sqrt{3}}\right), \tag{31}$$

where Z denotes z/z_0.

If t' denotes what would be the time of arrival of the stream at z_0 if the magnetic field were absent, $W(t-t')/z_0$ is given by the right-hand side of (31) with the first term $(-Z)$ omitted. It represents the lag of the real stream behind the unretarded stream; it is illustrated in Fig. 4. The minimum distance is approached asymptotically, only after a (logarithmically) infinite time.

The time variation of ΔH can be inferred from (24), (23), and (31). It is illustrated in Fig. 5, which shows that most of the change occurs rapidly, in a time interval of about $2z_0/W$. For a stream of density $n_i=1$, that produces a field of 21.5γ [cf. (28a)], $z_0=8.86a$, and $2z_0/W=113$ sec.

Figure 6 illustrates the distribution of the electric potential in the stream. Behind the front positively charged layer, on the left, is a layer of equivalent negative charge. The electric potential rises, as shown, towards the front surface. The ions are retarded in this region, but the electrons are accelerated. Since their forward motion conforms to that of the positive ions,

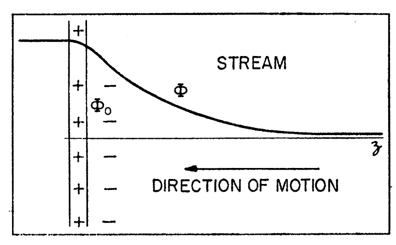

FIG. 6. Charge separation and electrostatic potential near the stream front, caused by the proton-electron mass difference. Particles continually pour into the front layer and are turned back (with a side step). The space behind the front layer is shielded from the permanent field.

their gain of energy goes into their lateral motion, in the x direction.

In Figs. 3–6 a magnetic field of intensity M/z^3 is parallel to Oy. A uniform plasma stream with a plane front $z=Z$ has n electrons and n protons per cm^3 on the side $z>Z$. All are moving with speed W in the $-z$ direction. Electric current is induced in the front layer. The magnetic field acts on it, and halts the advance of the front at a distance z_0.

The approximations made in the solution begin to fail near the positively charged layer (reference 2, p. 45), but Ferraro calculated that the ions lose approximately half their energy $\frac{1}{2}m_iW^2$ and the electrons gain the same amount of energy in this region. The other half of the energy of the ions supplies the energy of the magnetic field (reference 2, p. 45). Thus, in this case the electron energy and speed increase greatly, in the ratio $(m_i/2m_e)$.

Near the front of the stream the successive sheets may be somewhat more crowded together than elsewhere, so that the stream density is increased. "The increase is unlikely to be large" (reference 2, p. 30), but its amount was not calculated (reference 2, p. 47).

In our first studies (1931–3) Ferraro and the author surmised that there would be a great increase of density, as the gas pours onward towards the nearly stationary stream front. At that time we had not realized the relief to this situation, provided by the reversal of the forward motion.

The reversal leads to final speeds $-W$ for the ions and electrons, and paths asymptotic to lines parallel to but displaced from the original asymptotes. The displacements are inversely as the masses. The value is $\int u dt$ integrated throughout the motion, or $\int (u/w)dz$. Only the part of the path in the region near the stream surface contributes much to the integral; there u can be rather simply expressed, but the complexity of the expression for w makes the calculation difficult. Values of these "sidesteps" are not given by Ferraro.

12. THERMAL AND OTHER RELATIVE MOTIONS OF THE PARTICLES

The complete regularity of the initial motions in the two problems here described corresponds to the absence of random thermal motions, or zero temperature. The temperature T of the gas in the solar streams that produce magnetic storms is not yet known. The possible range may be from 10^4 to 10^6 °K. The following are some values of the mean speeds $(8kT/\pi m)^{\frac{1}{2}}$ for protons and electrons, at temperatures over this range:

T	10^4	10^5	10^6
Mean random speed: proton	1.44×10^6	4.56×10^6	1.44×10^7
Mean random speed: electron	$6.2\ \times10^7$	1.97×10^8	$6.2\ \times10^8$.

The numerical mean component of random speed along any direction is half this mean speed. Thus if the forward mean speed of a solar stream is 10^8 cm/sec, the electron speeds in a gas at these temperatures are comparable with or greater than the mean motion. For the protons, however, the random speed is at most only a minor fraction of the mean speed.

In the body of a solar stream whose lateral dimensions much exceed the free paths, the electrons travel with such speeds. A few electrons that cross the boundaries escape, until they leave behind, near the surface, a residual positive charge. If there is no background interplanetary gas, this surface charge accumulates until it is able to equalize the escape of positive and negative charge. The stream then freely expands laterally, with a speed similar to the foregoing proton speeds. This broadening is probably less important than that due to the initial dispersion of the directions of ejection of the gas from the emitting area on the sun.

If such a stream advances into a unidirectional magnetic field, the motion along the direction of the field remains unaltered. The lateral random motions also seem likely to be little affected by impact with the field. The solution of the problem described in Sec. 9 may throw some light on this point.

The random component motions along the direction of travel of the stream enable some ions to arrive earlier than others; the inability of the electrons to separate from the ions renders the ionic speeds the determining factor. On taking $T=10^5$ as an illustration, half the ions, with forward random speeds, have a mean speed of 1023 km/sec; for the other half it is 977 km/sec.

The time of travel from sun to earth with speed 1000 km/sec is about 150 000 sec or $1\frac{3}{4}$ days. A speed difference of 46 km/sec over this interval would separate the two groups of ions by about 7 000 000 km. The faster group would enter the geomagnetic field about two hours before the slower group. If the stream is of limited length along its direction of motion, it may be appreciably lengthened in the mean direction of travel during its journey from sun to earth. The time is sufficient to sort out the particles of the gas to some extent according to their radial speed. Those arriving first at the earth would be the faster moving particles, and the slower ones would follow later. In actual streams the speed and density on arrival near the earth may vary still more because of irregularities in the emission process. This might be notably the case for a limited burst of gas associated with a solar flare, lasting for a time of order one hour. The consequences in this case have been considered by Kahn.[21]

Near the front of the stream the distribution of the velocities of the particles is highly anisotropic (Sec. 15) because of the mixture of advancing and retreating particles. The spread of the radial speeds on this account is likely to be much greater than that due to the thermal motions, except in the few km at the very front of the stream.

[21] F. D. Kahn, Monthly Notices Roy. Astron. Soc. **109**, 324 (1949).

13. COLLISIONS BETWEEN THE PARTICLES

In a fully ionized gas at temperature T, the frequency of collisions between ions and electrons is given[22] by[22a]

$$\{34+4.18\log_{10}(T^3/n_e)\}n_e/T^{\frac{3}{2}}, \tag{32}$$

and the collision interval is given by the reciprocal of (32). The gas at the front of a solar stream that has attained its maximum penetration into the earth's field is unlikely to be in thermal equilibrium. But if the order of magnitude V of the relative velocities at collisions between protons and electrons there can be estimated, the preceding formulas can be used with rough validity, taking T to be given by

$$\tfrac{3}{2}kT=\tfrac{1}{2}m_eV^2. \tag{33}$$

(Owing to the small value of m_e/m_i, the electron speed relative to the mass center of a colliding ion-electron pair is almost equal to the whole relative speed V.)

According to Ferraro[2] the electrons in the shielding-current layer just behind the front of a solar stream have speeds corresponding to an energy of approximately $\frac{1}{4}m_iU^2$. This corresponds to $T=2\times10^7$ (T in this nonthermal problem is used merely as a convenient intermediary in getting the order of magnitude of the collision cross section, frequency, and interval). For this value of T, the values of the collision interval for a few typical values of n_e are as follows:

n_e/cm^3	1	10	100	1000
Collision interval (sec)	7.2×10^8	7.5×10^7	7.8×10^6	8.1×10^5.

These intervals are long—several days or more—and much greater than the time spent by the electrons in the front of the stream. Thus it may be concluded that it is justifiable to neglect collisions in this treatment of the entry of a solar stream into the geomagnetic field.

The foregoing calculation bears also on the question of the passage of the solar stream through the interplanetary gas—a subject considered at the 1956 Symposium already quoted (reference 14, pp. 12–14). Suppose this gas is fully ionized hydrogen up to the earth's distance, with a number density $n_i=n_e=200$/cm^3 near the earth. If it has no radial outward velocity from the sun, the relative velocity with which the stream protons and electrons collide with it is of order 10^8 cm/sec. The collision interval for such impact between a stream particle and the interplanetary particles is therefore of order 80 days. Thus the stream would seem to be able to traverse the interplanetary gas near the earth (or for the last one-third of the distance from sun to earth).[23] The solar corona offers more resistance, requiring an initial stream speed greater than 10^8 cm/sec.

In the cylindrical *sheet* problem the particle speeds at minimum distance are much less than the V here used. The collision interval for such low relative speeds would be materially less than those just given. But if account is taken of random thermal speeds corresponding to T no higher than 10^4—preserved throughout the motion —the collision interval, for a *volume* number density (Sec. 8.3) of 1 or less, is of order 10^4 sec. This is very long compared with the time spent by the electrons of the sheet near the minimum distance. Thus again the neglect of collisions appears to be well justified.

14. FURTHER IDEALIZED PLASMA PROBLEMS

It seems to me likely that the theory of magnetic storms—including the main phase as well as the first phase—can be advanced through the conception, formulation, and solution of further appropriate idealized problems. Moreover, some help may be gained in this way for our understanding of plasma problems in general, as was the case in hydrodynamics (Sec. 4). In the latter attempt there need not be any association with the features of geomagnetic storms.

The final section of the 1940 paper by Ferraro and the author discusses another type of cylindrical sheet problem which at that time we had considered and at least partly solved. Some other idealized problems are here suggested.

(a) A suggestion has been made in Sec. 9 for an extension of the cylindrical sheet problem by making the initial projection oblique. A single plane sheet obliquely projected into the type of field considered in Sec. 11 is a naturally associated problem.

(b) An interesting but more difficult problem is to extend the methods used by Ferraro[2] for a succession of plane plasma sheets to a succession of cylindrical plasma sheets.

As this author suggested in 1951 at the Auroral Conference in London, Ontario, the permanent presence of a ring current of fluctuating magnitude and radius may be a feature that affects the onset and development of a magnetic storm.[27] Since then the existence of the ring current, in a form rather different from what this author had envisaged, has been discovered by the I.G.Y. earth satellites and moon rockets.[8–11] As regards the first phase of a magnetic storm, this suggests the following idealized problems for consideration.

(c) The single cylindrical sheet problem can be

[22] S. Chapman, Nuovo cimento Suppl. 4, 1385 (1956); (a) Note corrections: p. 24 (75), $J_y=-k_{xy}E_x+k_{yy}E_y$; p. 25, Sec. 6, lines 3–5, for *reduces, reduction, rise* read *increases, increase, fall*; p. 26, 2nd and 3rd formulas, and last line of Table V, reduce by factor $\sqrt{2}$.

[23] Kahn (reference 24), Parker (reference 25), and Piddington (reference 26) have discussed the motion of one rare ionized gas through another. They infer that in some cases spontaneous irregularities of electron density may produce plasma oscillations and lead to a rise in the electron temperature at a small expense to the energy of the ions. Piddington (reference 26, Appendix) concludes that neutral ionized clouds may interpenetrate freely once the electron thermal speed has risen to equality with the ionic stream speed. This condition is already fulfilled when a solar stream leaves the sun, if its temperature is about 5×10^4 or more.

[24] F. D. Kahn, J. Fluid Mech. 2, 601 (1957).

[25] E. N. Parker, Phys. Rev. 112, 1429 (1958).

[26] J. H. Piddington, J. Geophys. Research 65, 93 (1960).

[27] S. Chapman, Ann. géophys. 8, 209 (1952).

extended by adding to the initial conditions another cylindrical plasma solenoidal sheet (of radius a few times a). The inner sheet initially carries a lateral electric current, and encloses its own uniform field. As it has no outside field it will not affect the history of the projected outer sheet (unless the projected sheet is able to penetrate inside the initially inner sheet) but the changing magnetic field of the latter affects the radius and current of the inner solenoid.

(d) The following rather different treatment of the first phase, including the influence of the prior presence of the ring current, allows the ring current to react on the projected stream. In the presence of the permanent magnetic field described in Sec. 8, the ring current is represented not by a cylindrical sheet but by a true ring of small cross section. Thus its magnetic field extends also to greater distances than its own radius. The projected stream is in this case to be represented by a coaxial line plasma (or ring plasma of small cross section) in the same plane as the "permanent" ring current. So long as distortions of the sections of the two plasma rings are left out of account, this problem seems likely to be soluble.

The following are some interesting problems without any necessary connection with cosmic phenomena.

(e) A cylindrical plasma sheet is situated, initially at rest, in a cylindrical undirectional magnetic field H_p of the form defined in Sec. 8. No current flows initially in the sheet. At zero time all its particles are projected outwards, radially or obliquely, with the same speed; currents are induced in the sheet and its radial motion retarded. It may be possible to follow the subsequent course of events.

(f) A cylindrical plasma surface distribution of radius a is initially at rest in the field of a line magnetic pole lying along its axis. Thus there is a permanent magnetic field radiating from the axis. At zero time the plasma sheet is given an angular velocity as a whole, about the axis. This causes a current to flow along the generators of the cylinder. The current flow produces a magnetic field round the outside of the cylinder, but no field inside it. This field pinches the cylinder. It seems likely that at least the initial changes set up in the system by the rotation can be determined.

All these problems in one way or another involve infinities—of length or area. A completely finite analog of the problem (f) is as follows.

(g) A circular ring magnetic pole lies along the circular axis of a toroidal surface whose section by a plane through the axis Oz of the ring is circular. There is a uniform plasma distribution over the toroidal surface, initially at rest. At zero time the particles in all sections by planes containing Oz are all given a uniform velocity round the section of the toroid. This generates current flow in the surface, round Oz. The magnetic field of this current flow is linked with the toroid and pinches it. This problem is unlikely to have a simple solution.

The solution of some of these problems is now under consideration, e.g., for problem (f) by V. C. Liu, and for problem (c) by S.-I. Akasofu.

There is another class of idealized plasma problem of much interest but much greater difficulty, in which the course of events includes deformation of a moving-plasma sheet. So far as this author knows, no problem of this kind has yet been solved. The following are some of the probably less difficult cases, which have some interest in connection with magnetic storms.

(h) Along an axis Ox flows a line current J. The lines of force of its magnetic field are circles in planes normal to Ox and centered on Ox. At zero time a plane plasma sheet $y=C$, where C is large or infinite, is projected normally to itself with uniform speed $-v$ towards the plane $y=0$.

The sheet is most retarded in the plane $z=0$, and becomes bent round the magnetic field. Later it is reflected away again, the particles with different z coordinates suffering deflections increasing with the (numerical) z coordinate. In this problem the permanent magnetic intensity varies with distance r from the x axis as $2J/r$. The problem could be attacked also for other laws of variation of intensity with r (the field lines still being circles round the axis). As in Secs. 8 and 11, in this case it would be supposed that the current system or magnetized matter that produced the field was completely pervious and uninfluenced by the changing field of the moving plasma sheet.

(i) A similar problem, perhaps of the same order of difficulty as (g), involves the same projected plasma sheet, in the presence of a permanent magnetic field provided by a line magnetic dipole along Ox, with its magnetic moment in the plane $y=0$.

(j) Another such problem concerns a plane plasma sheet projected normal to itself, in a permanent field like that of Sec. 8, with its axis parallel to the sheet. The particles describe plane curves in planes normal to the field; the sheet folds partly round the axis and then retreats and disperses.

Such problems as (h)–(j), involving distortion of plasma sheets, can be generalized to deal with a succession of sheets, initially parallel and closely spaced. These extended problems are of a still higher order of difficulty.

15. VELOCITY DISTRIBUTION FUNCTION

At first sight it might seem that a problem like that of the impact of a solar stream of gas upon the earth's field could be treated hydromagnetically, by the hydrodynamic equations for a compressible fluid—a gas—combined with Maxwell's electromagnetic equations. In seeking problems soluble by considering the motions of the actual particles (plasma kinetics, cf. Sec. 5), this author's original feeling was that this was a move from weakness—inability to deal readily with interlinked sets of partial differential equations. On further reflection, however, the situation seems otherwise. The usual hydrodynamic, viscosity, and heat conduction equations can be derived from the equations of the kinetic theory of matter, so long as the departure from the Maxwellian equilibrium state and velocity distribution is small. But in the case of the solar stream this is far from true. The mean motion is highly supersonic, and for example, in the case of the plane problem of Sec. 11, the velocity distribution is highly non-Maxwellian. In the body of the stream, where it is shielded from the permanent magnetic field, the gas includes two interpenetrating streams with mean motion $\pm W$, together with thermal distributions of peculiar motions relative to these. For the electrons the magnitude of the peculiar motions may be comparable with or exceed W, so that the resultant isosurfaces in the velocity space may not greatly differ from spheres. But for the ions the random speeds are much less; the ionic velocity distribution isosurfaces more resemble a dumbbell.

Near the front of the stream, also, the velocity distribution function is non-Maxwellian, with a decided difference between the mean motions of the ions and electrons along the direction of current flow parallel to the stream surface. This difference of mean motion is large compared with the random ionic motions.

In these circumstances the hydromagnetic equations are inadequate. The important changes take place in a distance of the order of the Larmor radius for the particular case considered by Kantrowitz and Petschek,[16] namely, when the magnetic and gas pressures are equal [cf. Eq. (29)]. The problem is essentially one of plasma kinetics, as defined in Sec. 5.

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