On the Theory of Hydromagnetic Equilibrium

L. WOLTJER
Leiden University Observatory, Leiden, Netherlands

RECENTLY a theory of hydromagnetic equilibrium, based on a variational principle has been developed,[1] in which the invariants of the hydromagnetic equations have an important role. It has been found that in a hydromagnetic system (single-fluid theory) that is surrounded by a rigid perfectly conducting wall, on which the boundary condition $\mathbf{H}\cdot\mathbf{n}=0$ holds, the following integrals are constant during the evolution of the system:

$$I_1=\int \mathbf{A}\cdot\mathbf{H}d\tau \qquad (1)$$

and

$$I_2=\int \mathbf{H}\cdot\mathbf{v}d\tau. \qquad (2)$$

The integrals are taken over the whole volume of the configuration. The medium is assumed to be a nonviscous perfect conductor with a $P=f(\rho)$-type equation of state. In these equations $\mathbf{H}$ is the magnetic field, $\mathbf{A}$ a single-valued vector potential, and $\mathbf{v}$ the macroscopic fluid velocity. Apart from these integrals the mass and angular momentum of the system are also conserved.

If we make the additional assumption of axial symmetry, four infinite sets of integrals emerge,[2,3] which imply that the quantities $\mathbf{A}\cdot\mathbf{H}$, $\mathbf{H}\cdot\mathbf{v}$, the angular momentum, and the mass taken over a volume bounded between a pair of magnetic surfaces are constant during the evolution of the system. It can also be shown that no other independent invariants exist.

It has further been shown that in this case the states of hydromagnetic equilibrium may be defined by a variational principle which states that the equilibrium configurations are configurations of extremal energy, when the extrema are determined subject to the condition that the invariants have definite values. The invariants thus serve as the constraints in the variational principle. For static equilibria the system is stable in the usual sense if the extremum is a minimum. In a number of cases with motions it can be shown that the same statement holds.

Here we wish to investigate the limits of the applicability of this variational approach, both with respect to a generalization of the geometry and a generalization of the equations which describe the dynamical behavior of the plasma.

As to the first point, an obvious generalization of the axisymmetric case is the case in which the lines of force form nested toroidal surfaces of arbitrary shape. For this case Kruskal and Kulsrud[4] have demonstrated the existence of certain invariants which correspond to the invariance of the integrals of $\mathbf{A}\cdot\mathbf{H}$ and the mass between any two of these surfaces. It is easily shown that the same is the case for the integrals of $\mathbf{H}\cdot\mathbf{v}$, but no such integrals exist for the angular momentum. Thus, the assumption of axial symmetry seems to be essential to an approach where all constraints appear in integral form. For even more general geometries than the toroidal ones, the integrals (1) and (2), the total angular momentum, and the mass seem to be the only integral invariants, since the concept of a magnetic surface loses much of its meaning.

We next investigate whether the integrals (1) and (2) which were derived on the basis of the single-fluid equations remain applicable when more general plasmas are considered. We start from the two-fluid equations for a gas consisting of protons and electrons. If we restrict ourselves for a moment to cases with a scalar pressure, we have

$$m_i\left\{\frac{\partial \mathbf{v}_i}{\partial t}+(\nabla\times\mathbf{v}_i)\times\mathbf{v}_i+\tfrac{1}{2}\nabla v_i^2\right\}=e(\mathbf{E}+\mathbf{v}_i\times\mathbf{H})-\frac{1}{n_i}\nabla P_i-m_i\nabla\Phi+\frac{1}{n_i}\mathbf{\Pi}_{ie} \qquad (3)$$

and the same equation with the suffix i replaced by e and $+e$ by $-e$ for electrons. Here Φ represents a nonelectromagnetic potential, and $\prod_{ie}$ and $\prod_{ei}$ are the momentum transfer from the electrons to the protons and vice versa. We introduce the electromagnetic potentials with the gauge chosen in such a way that the scalar potential vanishes. Thus

$$\mathbf{E}=-(\partial\mathbf{A}/\partial t),\quad \mathbf{H}=\nabla\times\mathbf{A}. \qquad (4)$$

At the boundary we impose the conditions

$$\mathbf{E}\times\mathbf{n}=0,\quad \mathbf{H}\cdot\mathbf{n}=0,\quad \mathbf{v}\cdot\mathbf{n}=0. \qquad (5)$$

We now multiply Eq. (3) with $\mathbf{H}$ and integrate over the configuration to obtain

$$\int\nabla\times\mathbf{A}\cdot\frac{\partial\mathbf{A}}{\partial t}d\tau+\mu_i\int\left\{\mathbf{H}_i\cdot\frac{\partial\mathbf{v}_i}{\partial t}+\nabla\times\mathbf{v}_i\cdot\mathbf{v}_i\times\mathbf{H}\right\}d\tau+\frac{1}{e}\int\left(\frac{\mathbf{H}\cdot\nabla P_i}{n_i}-\frac{\mathbf{H}\cdot\mathbf{\Pi}_{ie}}{n_i}\right)d\tau, \qquad (6)$$

[1] L. Woltjer, Proc. Natl. Acad. Sci. U. S. 44, 833 (1958); 45, 769 (1959).
[2] S. Chandrasekhar, Proc. Natl. Acad. Sci. U. S. 44, 842 (1958).
[3] L. Woltjer, Astrophys. J. 130, 400, 405 (1959).
[4] M. D. Kruskal and R. M. Kulsrud, Phys. Fluids 1, 265 (1958).

where the terms which can be written as divergences have disappeared in view of the boundary conditions and where $\mu_i = m_i/e$. On integrating by parts and making use of Maxwell's equation $\nabla\times\mathbf{E} = -\partial\mathbf{H}/\partial t$, we obtain

$$2\int\nabla\times\mathbf{A}\cdot\frac{\partial\mathbf{A}}{\partial t}d\tau = \int\nabla\times\mathbf{A}\cdot\frac{\partial\mathbf{A}}{\partial t}d\tau + \int\mathbf{A}\cdot\nabla\times\frac{\partial\mathbf{A}}{\partial t}d\tau + \int_s\mathbf{E}\times\mathbf{A}\cdot d\mathbf{S} = \frac{d}{dt}\int\mathbf{A}\cdot\mathbf{H}d\tau, \quad (7)$$

where the surface integral vanishes because of the boundary conditions, and

$$\int\nabla\times\mathbf{v}_i\cdot\mathbf{v}_i\times\mathbf{H}d\tau = \int\mathbf{v}_i\cdot\nabla\times(\mathbf{v}_i\times\mathbf{H})d\tau = \int\left\{\mathbf{v}_i\cdot\frac{\partial\mathbf{H}}{\partial t}+\mu_i\mathbf{v}_i\cdot\nabla\times\frac{\partial\mathbf{v}_i}{\partial t}\right\}d\tau + \frac{1}{e}\int\left\{\mathbf{v}_i\cdot\nabla\frac{1}{n_i}\times\nabla P+\mathbf{v}_i\cdot\nabla\times\left(\frac{\Pi_{ie}}{n_i}\right)\right\}d\tau. \quad (8)$$

On inserting this in Eq. (6), we have

$$\frac{d}{dt}\left\{\frac{1}{2}\int\mathbf{A}\cdot\mathbf{H}d\tau+\mu_i\int\mathbf{H}\cdot\mathbf{v}_i d\tau\right\} = \frac{1}{e}\int\left(\frac{\mathbf{H}}{n_i}\cdot\nabla P_i - \frac{\mathbf{H}\cdot\mathbf{\Pi}_{ie}}{n_i}\right)d\tau + \frac{\mu_i}{e}\int\left\{\mathbf{v}_i\cdot\nabla\frac{1}{n_i}\times\nabla P_i + \mathbf{v}_i\cdot\nabla\times\left(\frac{\mathbf{\Pi}_{ie}}{n_i}\right)\right\}d\tau + \{\mu_i^2\}, \quad (9)$$

where $\{\mu_i^2\}$ denotes the terms which are multiplied by μ_i^2. Since μ_i is small, these terms are usually negligible.

Let us consider the terms which involve the pressure. If the pressure is a function of the density only, the integrals over the pressure terms disappear. If this is not the case, the order of magnitude of the $(1/e)\times(\mathbf{H}/n_i)\cdot\nabla P_i$ term is $\mu_i H(kT_i/Lm_i)$, where L represents a characteristic length which is about $\mu_i H/L$ times the thermal energy per unit mass in the medium. The term $\mu_i d\mathbf{H}\cdot\mathbf{v}_i/dt$ is of the order $\mu_i H v_i^2/L$, which is of the order of the kinetic energy of the medium per unit mass multiplied by $\mu_i H/L$. Thus the lowest-order pressure term is usually of the first order in μ_i in the equation for ions and in μ_e for the electrons. This result depends not very much on the pressure being a scalar. Thus, we have to the first order in μ_i or μ_e:

$$\frac{d}{dt}\int\mathbf{A}\cdot\mathbf{H}d\tau = -\int\frac{\mathbf{H}\cdot\Pi_{ie}}{en_i}d\tau \simeq -2\int\eta\mathbf{H}\cdot\mathbf{j}d\tau. \quad (10)$$

In view of the boundary condition $\mathbf{H}\cdot\mathbf{n}=0$ at the surface, this result is gauge independent. If we drop the boundary condition on $\mathbf{H}$ but retain the gauge of $\mathbf{A}$ (scalar potential=0), we have

$$\frac{d}{dt}\int\mathbf{A}\cdot\mathbf{H}d\tau = -2\int\eta\mathbf{H}\cdot\mathbf{j}d\tau + \int\mathbf{A}\times\mathbf{E}\cdot d\mathbf{S}, \quad (11)$$

a result which shows a remarkable resemblance to Poyntings theorem; it can also be written in a gauge invariant form.

On returning to Eq. (9) we note that the first pressure term which is of order μ vanishes if $(\mathbf{H}\cdot\nabla P/n) = \mathbf{H}\cdot\nabla f(n)$, i.e., if the medium behaves polytropically along the lines of force. On subtracting the equations for the ions and that for the electrons, we have

$$\frac{d}{dt}\int\mathbf{H}\cdot(\mu_i\cdot\mathbf{v}_i+\mu_e\mathbf{v}_e)d\tau = -\int\frac{\mathbf{H}}{e}\left(\frac{\mathbf{\Pi}_{ie}}{n_i}+\frac{\mathbf{\Pi}_e'}{n_e}\right)d\tau + \int\left\{\frac{\mu_i}{e}\mathbf{v}_i\cdot\nabla\times\left(\frac{\mathbf{\Pi}_{ie}}{n_i}\right)+\frac{\mu_e}{e}\mathbf{v}_e\cdot\nabla\times\left(\frac{\mathbf{\Pi}_{ei}}{n_e}\right)\right\}d\tau. \quad (12)$$

In the case of charge neutrality this becomes approximately

$$\frac{d}{dt}\int\mathbf{H}\cdot\mathbf{v}d\tau \simeq \int\eta\mathbf{v}\cdot\nabla\times\mathbf{j}d\tau, \quad (13)$$

which vanishes for a perfect conductor $(\eta=0)$. It thus appears that the hydromagnetic integral $\mathbf{A}\cdot\mathbf{H}$ is generally constant for a perfectly conducting plasma to the first order in μ_e. If the plasma is also everywhere neutral and in addition the electrons and protons have a polytropic equation of state along the lines of force, the $\mathbf{H}\cdot\mathbf{v}$ integral is also approximately constant during the evolution of the system.