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Stability of Twisted Magnetic Fields in a Fluid of Finite Electrical Conductivity

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1. INTRODUCTION

THE stability of uniformly twisted magnetic fields in an incompressible inviscid fluid of infinite electrical conductivity has been studied by several authors.[1–4] In addition, the problem has been solved for a fluid which is in uniform axial motion or uniform rotation or has a combination of these two motions.[5–7] Here it is shown that the stability problem can also be solved formally if the fluid has a finite scalar viscosity and electrical conductivity.[8] The perturbed hydromagnetic equations in the conducting fluid can be reduced to a single 10th-order differential equation and this can be solved in terms of Bessel functions. The solution is formal in the sense that, when boundary conditions are applied, the problem is reduced to the solution of an extremely complicated transcendental dispersion relation. In the general case it is clear that solutions of this dispersion relation can be found only by elaborate numerical techniques. The perturbation growth rate is normally complex and the problem is an eigenvalue problem in which the complex plane must be searched for the eigenvalue.

If there is no axial magnetic field and only axisymmetric perturbations are considered, the problem can be solved completely. The solution is given in Sec. 3. If the viscosity is zero, it can be shown that there is a real instability at all wave numbers and that overstability does not occur. If viscosity alone is considered, the system is stable not merely in the sense that no perturbations grow with time but also that all except trivial perturbations decay as the growth rates are real and negative. When finite values of both viscosity and electrical conductivity are considered, the system is again unstable at all wave numbers, but the viscosity causes the growth rates to be very small at large wave number.

If the magnetic field is twisted and perturbations other than axisymmetric perturbations are considered, there are always perturbation helices which exactly match the magnetic field helix. For such perturbations the problem can also be solved and these are considered in Sec. 4. The results obtained are similar to those found for the axisymmetric perturbations in the absence of an axial magnetic field. For both of these cases the system is marginally stable if the conductivity is infinite and the viscosity zero.

For the case of twisted magnetic fields and arbitrary perturbations, finite electrical conductivity alone is considered. If the electrical conductivity is low, solutions of the perturbed equations can be found as series in ascending powers of the conductivity. If only the first term in the expansion is kept, a solution of the problem can be found in the limit as the conductivity tends towards zero. As the problem has already been solved for a fluid of infinite conductivity, there is a hope that the extreme values of the perturbation growth rate have been found. In the cases that can be solved completely, the growth rates for infinite and zero conductivity are the extreme values. In all cases considered here it is found that, with a suitable normalization,[9] the growth rate for a fluid of low conductivity is greater than that for a fluid of high conductivity. In some cases a fluid of low conductivity is unstable while a fluid of infinite conductivity is stable.

The reason for the preceding result is not hard to find. For a compressible fluid both the electrical conductivity and the viscosity enter the stability problem in two places. In the equation of motion the viscous term tends to reduce velocity gradients and to increase stability, but in the magnetic-field diffusion equation the finite electrical conductivity allows the fluid to cross field lines. Both the viscosity and conductivity also occur in the energy equation where both the viscous dissipation and the Ohmic heating might be expected to damp instabilities. However, for an incompressible fluid the energy equation is not coupled with the stability problem and the only role of the conductivity is to allow the fluid to cross the field lines. It is therefore not surprising that, while the viscosity leads to enhanced stability, decreasing the electrical conductivity may lead to greater instability.[10]

The results obtained here cannot be applied to a

[1] S. Lundquist, Phys. Rev. **83**, 307 (1951).
[2] J. W. Dungey and R. E. Loughhead, Australian J. Phys. **7**, 5 (1954).
[3] P. H. Roberts, Astrophys. J. **124**, 430 (1956).
[4] R. J. Tayler, Proc. Phys. Soc. (London) **B70**, 1049 (1957).
[5] M. N. Rosenbluth, talk on stability of rotating plasma, New York, May, 1958.
[6] S. K. Trehan, Astrophys. J. **127**, 446 (1958).
[7] S. K. Trehan, Astrophys. J. **127**, 454 (1958).
[8] In the problems considered in detail here, the equilibrium fluid motions are omitted.

[9] The growth time of an instability is normalized with respect to the time it takes a hydromagnetic wave to cross the fluid.
[10] It should be noted that it is not possible to say that decreasing the conductivity always leads to greater instability. If the conductivity is altered, some of the other physical quantities must also be altered. If the current and dimensions of the system are kept constant, it becomes less stable, but this need not be so if some other parameter is kept constant.

compressible plasma, but they can be applied to a liquid conductor such as mercury or liquid sodium. Experiments have been done with both of these liquids though they do not exactly correspond with the problem considered here. One of the closest is that described by Dattner, Lehnert, and Lundquist.[11] In their experiment the mercury carrying the current is also falling under gravity and it is not clear how much this influences the results. However, the theoretical growth time for axisymmetric perturbations is in quite good agreement with the observed growth time.

Full details of the derivation and solution of the basic equations and dispersion relations are not given here. The work is described more fully in a series of A.E.R.E. Harwell reports.[12–14]

2. DESCRIPTION OF PROBLEM AND BASIC EQUATIONS

An incompressible fluid of density ρ_0, viscosity μ, and electrical conductivity σ forms a cylinder of radius r_0. The fluid has a velocity

$$\mathbf{v}_0 = v_0[0,\, v_1 r/r_0,\, v_2] \tag{1}$$

and carries a magnetic field

$$\mathbf{B}_0 = B_0[0,\, b_1 r/r_0,\, b_2]. \tag{2}$$

The fluid is surrounded by a vacuum containing a magnetic field

$$\mathbf{B}_0{}^v = B_0[0,\, b_1 r_0/r,\, b_2], \tag{3}$$

and the vacuum may or may not be surrounded by a rigid wall. In the problems considered in detail here, the wall is omitted.

The hydromagnetic equations, which have to be solved in the conducting fluid, are

$$\rho d\mathbf{v}/dt = -\operatorname{grad} p + (\operatorname{curl}\mathbf{B}\times\mathbf{B}/4\pi) - \mu \operatorname{curl}\operatorname{curl}\mathbf{v}, \tag{4}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{5}$$

and

$$\partial\mathbf{B}/\partial t + (c^2/4\pi\sigma)\operatorname{curl}\operatorname{curl}\mathbf{B} = \operatorname{curl}(\mathbf{v}\times\mathbf{B}). \tag{6}$$

In Eqs. (4)–(6) displacement currents and the electric field force term in the equation of motion have been neglected. The energy equation is required to determine the temperature variation, but it is not required for the stability problem.

Perturbations of the equilibrium are considered in which any physical variable has the form

$$q = q_0 + q_1 e^{i(m\theta + kz) + \omega t}. \tag{7}$$

Equations (4)–(6) can be linearized in terms of the perturbed variables and, using the particular forms (1) and (2) of the equilibrium quantities, all the perturbed quantities but $\mathbf{B}_1$ can be eliminated from these equations. The resulting equation for $\mathbf{B}_1$ is

$$(a_5/k^5)\operatorname{curl}\operatorname{curl}\operatorname{curl}\operatorname{curl}\operatorname{curl}\mathbf{B}_1 + (a_3/k^3)\operatorname{curl}\operatorname{curl}\operatorname{curl}\mathbf{B}_1 + (a_2/k^2)\operatorname{curl}\operatorname{curl}\mathbf{B}_1 + (a_1/k)\operatorname{curl}\mathbf{B}_1 + a_0\mathbf{B}_1 = 0, \tag{8}$$

where the $a_5 - a_0$ are all dimensionless constants and have the values

$$\begin{aligned} a_5 &= \mu c^2 r_0^2 k^4/\sigma B_0^2,\\ a_3 &= [(\rho_0 c^2 k^2 r_0 + 4\pi\mu\sigma k^2 r_0)/\sigma B_0^2][\omega r_0 + v_0(imv_1 + ikr_0v_2)],\\ a_2 &= -2i\rho_0 v_0 v_1 c^2 k^2 r_0/\sigma B_0^2,\\ a_1 &= (4\pi\rho_0/B_0^2)[\omega r_0 + v_0(imv_1+ikr_0v_2)]^2 + (mb_1+kr_0b_2)^2,\\ a_0 &= -2b_1(mb_1+kr_0b_2) - (8\pi i\rho_0 v_0 v_1/B_0^2) \times[\omega r_0 + v_0(imv_1+ikr_0v_2)]. \end{aligned} \tag{9}$$

Equation (8) can be shown to be a 10th-order differential system. It can be factorized in the form

$$\prod_{i=1}^{5}(\operatorname{curl} - \lambda_i k)\mathbf{B}_1 = 0, \tag{10}$$

where λ_i satisfies

$$a_5\lambda^5 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{11}$$

As the operators in Eq. (10) all commute, the 10 independent solutions can be found from the five second-order differential equations

$$\operatorname{curl}\mathbf{B}_1 = \lambda_i k\mathbf{B}_1. \tag{12}$$

The solution of Eq. (12), which is regular at the origin, is

$$B_{1r} = -\frac{i\alpha_i}{2(\lambda_i+1)} I_{m+1}(\alpha_i kr) + \frac{i\alpha_i}{2(\lambda_i-1)} I_{m-1}(\alpha_i kr),$$

$$B_{1\theta} = -\frac{\alpha_i}{2(\lambda_i+1)} I_{m+1}(\alpha_i kr) - \frac{\alpha_i}{2(\lambda_i-1)} I_{m-1}(\alpha_i kr),$$

and

$$B_{1z} = I_m(\alpha_i kr), \tag{13}$$

where $\alpha_i^2 = 1 - \lambda_i^2$. It can be seen from Eqs. (9) and (11) that the equation for the arguments of the Bessel functions involves the required eigenvalue ω in a complicated way. It is easy to see that, for arbitrary values of the equilibrium quantities and the perturbation wave numbers, the arguments of the Bessel functions are complex.

Once Eqs. (4)–(6) have been used to obtain expressions for the other perturbed quantities and the simpler expression for the perturbed magnetic field in the vacuum has been written down, boundary conditions must be applied on the fluid-vacuum interface,

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[11] A. Dattner, B. Lehnert, and S. Lundquist, *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, New York, 1958), Vol. 31, p. 325.

[12] R. J. Tayler, Atomic Energy Research Establ. (Gt. Brit.) T/R 2787 (1959).

[13] R. J. Tayler, Atomic Energy Research Establ. (Gt. Brit.) T/R 3100 (1960).

[14] R. J. Tayler, Atomic Energy Research Establ. (Gt. Brit.) T/R 3229 (1960).

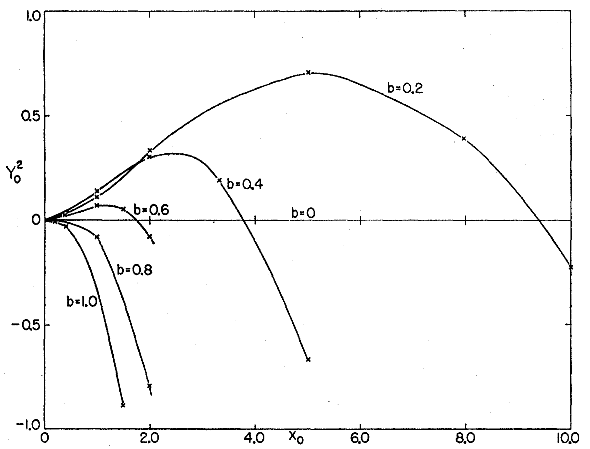

FIG. 1. Axisymmetric perturbations of an inviscid fluid of infinite conductivity. The square of the dimensionless growth rate ($Y_0^2=4\pi\rho_0\omega^2r_0^2/B_0^2$) is shown as a function of the dimensionless wave number ($X_0=kr_0$) for several values of the axial magnetic field (B_0b).

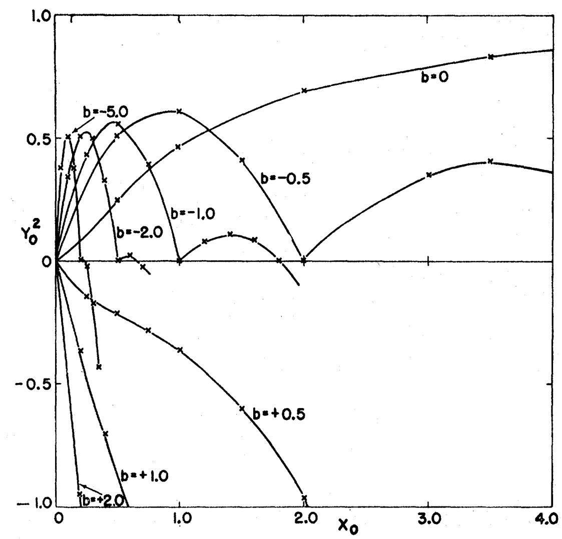

FIG. 2. $m=1$ perturbations of an inviscid fluid of infinite conductivity. The square of the dimensionless growth rate ($Y_0^2=4\pi\rho_0\omega^2r_0^2/B_0^2$) is shown as a function of the dimensionless wave number ($X_0=kr_0$) for several values of the axial magnetic field (B_0b).

and the dispersion relation can be obtained. In the general case, six boundary conditions are required, the continuity of the three components of the magnetic field and of the three components of the stress tensor containing the normal direction, and the dispersion relation can be written down as a complicated six by six determinant, all the elements of which are combinations of Bessel functions. Because of its complexity it is not written down here; instead, only certain special cases are considered.[15] For these special cases the form of the dispersion relation is stated without proof.

Before considering these special cases it is convenient to express everything in terms of dimensionless variables. Thus, first introduce a hydromagnetic velocity

$$c_H^2=B_0^2/4\pi\rho_0 \tag{14}$$

and then write

$$\begin{aligned} X_0&=kr_0,\\ Y_0&=\omega r_0/c_H,\\ V_0&=4\pi\sigma c_H r_0/c^2,\\ W_0&=\rho_0 c_H r_0/\mu. \end{aligned} \tag{15}$$

As mentioned in the Introduction, this problem has already been solved for a fluid of infinite conductivity and zero viscosity. The results obtained for the $m=0$ and $m=1$ modes are shown in Figs. 1 and 2. For axisymmetric perturbations the system is neutrally stable if there is no axial field. For small values of the field the system is unstable at small wave numbers, but for large enough values of the field all wave numbers are stable. The behavior of the growth rate for small wave numbers is given by

$$Y_0^2\approx b^2X_0^2[(2/bj_0)-1], \tag{16}$$

where j_0 is the first zero of $J_0(x)$. All $m=1$ perturbations are unstable if there is no axial field. When the axial field is nonzero, negative values of b are always less stable than positive values of b. There is always instability for small wave numbers, and the perturbation helix which coincides with the equilibrium magnetic field helix is neutrally stable.

3. AXISYMMETRIC PERTURBATIONS IN THE ABSENCE OF AN AXIAL FIELD

When $m=b=0$, the arguments of the Bessel functions become simple and the dispersion relation can be written

$$\begin{aligned} 0=&\left[\frac{I_0(X_0)}{I_1(X_0)}+\frac{2X_0^2}{Y_0W_0}\frac{I_1'(X_0)}{I_1(X_0)}\right]\left[2+\frac{Y_0W_0}{X_0^2}+\frac{4W_0}{Y_0^2V_0}\right]\\ &-\frac{4W_0(X_0^2+Y_0V_0)^{\frac{1}{2}}}{Y_0^2V_0X_0}\left[\frac{I_0(X_0^2+Y_0V_0)^{\frac{1}{2}}}{I_1(X_0^2+Y_0V_0)^{\frac{1}{2}}}\right.\\ &\left.+\frac{2X_0^2}{Y_0(W_0-V_0)}\frac{I_1'(X_0^2+Y_0V_0)^{\frac{1}{2}}}{I_1(X_0^2+Y_0V_0)^{\frac{1}{2}}}\right]\\ &-\frac{4X_0(X_0^2+Y_0W_0)^{\frac{1}{2}}}{Y_0W_0}\left[1+\frac{2W_0}{Y_0^2(V_0-W_0)}\right]\\ &\times\frac{I_1'(X_0^2+Y_0W_0)^{\frac{1}{2}}}{I_1(X_0^2+Y_0W_0)^{\frac{1}{2}}}. \end{aligned} \tag{17}$$

[15] In what follows the equilibrium fluid velocity is put equal to zero so that $v_1=v_2=0$. As the axial current does not vanish, it is convenient to take $b_1=1$ and $b_2=b$. It is also possible to take m and k to be positive if both positive and negative signs of b are considered.

Two special cases occur:

(i) conductivity only,

$$(X_0^2+Y_0V_0)^{\frac{1}{2}}\frac{I_0(X_0^2+Y_0V_0)^{\frac{1}{2}}}{I_1(X_0^2+Y_0V_0)^{\frac{1}{2}}}=\frac{X_0I_0(X_0)}{I_1(X_0)}\left[1+\frac{Y_0^3V_0}{4X_0^2}\right]; \quad (18)$$

(ii) viscosity only,

$$(X_0^2+Y_0W_0)^{\frac{1}{2}}\frac{I_1'(X_0^2+Y_0W_0)^{\frac{1}{2}}}{I_1(X_0^2+Y_0W_0)^{\frac{1}{2}}}=\left[\frac{Y_0W_0}{2X_0}+\frac{Y_0^2W_0^2}{4X_0^3}\right]\left[\frac{I_0(X_0)}{I_1(X_0)}+\frac{2X_0^2}{Y_0W_0}\frac{I_1'(X_0)}{I_1(X_0)}\right]. \quad (19)$$

When conductivity alone is present, it can be shown that Eq. (18) always has a real positive root for Y_0 for all X_0. Additionally, it can be proved that the equation has no complex roots with positive real parts. For large and small values of X_0, the positive root has approximate values

$$Y_0=\sqrt{2}+O(1/X_0) \text{ as } X_0\rightarrow\infty,$$
$$Y_0=X_0/\sqrt{2}+O(X_0^2) \text{ as } X_0\rightarrow 0. \quad (20)$$

It can be seen that in neither of these asymptotic forms is there any dependence on the value of the conductivity as long as it is finite. The positive root of Eq. (18) has been found for a set of values of V_0 and the results are shown in Fig. 3.

When viscosity alone is present, it can be seen that Y_0 and W_0 only enter Eq. (19) in the combination $U_0=Y_0W_0$. The equation can thus be solved to give U_0 as a function of X_0, and values of Y_0 are obtained once the value of W_0 is specified. Equation (19) has one root which is $U_0=0$ for all X_0, but it can be proved that all the nontrivial roots are real and negative. The solution of Eq. (19) is shown in Fig. 4.

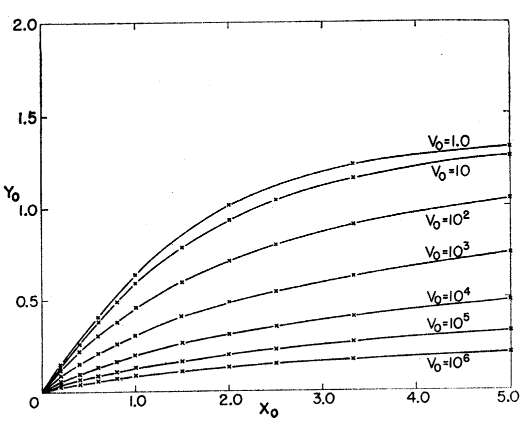

FIG. 3. Axisymmetric perturbations of an inviscid fluid of finite conductivity (no axial field). The dimensionless growth rate is shown as a function of the dimensionless wave number for several values of the dimensionless conductivity ($V_0=4\pi\sigma c_H r_0/c^2$).

When both conductivity and viscosity are present, it can be shown that there is again always a positive root for all values of X_0. The behavior of this root for large and small values of X_0 is given by

$$Y_0\approx 3W_0/2X_0^2 \text{ as } X_0\rightarrow\infty,$$
$$Y_0\approx X_0/\sqrt{2} \text{ as } X_0\rightarrow 0. \quad (21)$$

It can be seen from Eq. (21) that the viscosity does not affect the perturbations of small wave number, but it causes a considerable reduction of the growth rate of perturbations of large wave number. The real positive root of Eq. (17) has been calculated for one value of $V_0(V_0=100)$ and two values of $W_0(1, 10)$. The corresponding values of Y_0 as a function of X_0 are shown in Fig. 5. Also shown in Fig. 5 are the asymptotic solutions. The intersection of the two asymptotic solutions gives a first approximation to the wave number of maximum instability.[16] It can be seen that the true wave number of maximum instability is not very different from the approximate value, but the intersection of the asymptotic curves greatly overestimates the growth rate.

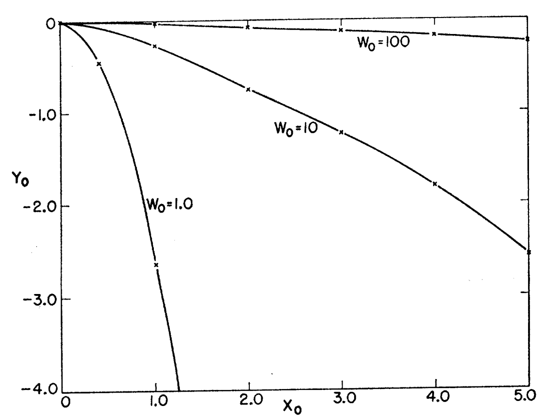

FIG. 4. Axisymmetric perturbations of a viscous fluid of infinite conductivity (no axial field). The dimensionless growth rate is shown as a function of the dimensionless wave number for several values of the dimensionless viscosity parameter ($W_0=\rho_0 c_H r_0/\mu$).

4. TWISTED FIELDS: PERTURBATION PARALLEL TO FIELD

When $m+bX_0=0$, the expressions for the arguments of the Bessel functions again become simple. The dispersion relation has the form[17]

$$Y_0^2=4F/G, \quad (22)$$

[16] Two other problems involving viscosity in which the intersection of asymptotic curves is used to estimate the wave number of maximum instability are discussed by S. Chandrasekhar [Proc. Cambridge Phil. Soc. **51**, 162 (1955)] and R. J. Taylor [Proc. Phys. Soc. (London) **B70**, 31 (1957)].

[17] In this section the viscosity is put equal to zero.

where

$$F=\frac{X_0}{Y_0V_0}\left[(X_0^2+Y_0V_0)^{\frac{1}{2}}\frac{I_m(X_0^2+Y_0V_0)^{\frac{1}{2}}}{I_m'(X_0^2+Y_0V_0)^{\frac{1}{2}}}\frac{I_m'(X_0)}{I_m(X_0)}-X_0\right]$$
$$-\frac{m}{(X_0^2+Y_0V_0)^{\frac{1}{2}}}\frac{I_m(X_0^2+Y_0V_0)^{\frac{1}{2}}}{I_m'(X_0^2+Y_0V_0)^{\frac{1}{2}}}\Phi, \quad (23)$$

$$G=1-\left[\frac{b(X_0^2+Y_0V_0)^{\frac{1}{2}}}{X_0}+\frac{m}{(X_0^2+Y_0V_0)^{\frac{1}{2}}}\right]\times\frac{I_m(X_0^2+Y_0V_0)^{\frac{1}{2}}}{I_m'(X_0^2+Y_0V_0)^{\frac{1}{2}}}\Phi \quad (24)$$

and

$$\Phi=m/(X_0^2+Y_0V_0)^{\frac{1}{2}}\times\left[\frac{I_m'(X_0^2+Y_0V_0)^{\frac{1}{2}}}{I_m(X_0^2+Y_0V_0)^{\frac{1}{2}}}-\frac{(X_0^2+Y_0V_0)^{\frac{1}{2}}}{X_0}\frac{K_m'(X_0)}{K_m(X_0)}\right]^{-1}. \quad (25)$$

Equation (22) has been solved for $m=1$ perturbations. There is a positive root for Y_0 for all values of V_0. Results have been calculated for $m=1$, $b=-1$, and a set of values of V_0, and they are shown in Fig. 7.

5. TWISTED FIELDS: LOW CONDUCTIVITY

The perturbed fluid equations, which have to be solved for an inviscid fluid, are

$$\rho_0\omega\mathbf{v}_1=-\text{grad}p_1+\frac{\text{curl}\mathbf{B}_1\times\mathbf{B}_0}{4\pi}+\frac{\text{curl}\mathbf{B}_0\times\mathbf{B}_1}{4\pi}, \quad (26)$$

$$\text{div}\mathbf{v}_1=0, \quad (27)$$

and

$$\omega\mathbf{B}_1+\frac{c^2}{4\pi\sigma}\text{curl curl}\mathbf{B}_1=\text{curl}(\mathbf{v}_1\times\mathbf{B}_0). \quad (28)$$

It is supposed that in the case of small conductivity the variables can be expanded in the form

$$\mathbf{B}_1=\mathbf{B}_{10}+\sigma\mathbf{B}_{11}+\cdots,$$
$$\mathbf{v}_1=\mathbf{v}_{10}+\sigma\mathbf{v}_{11}+\cdots, \quad (29)$$
$$p_1=p_{10}+\sigma p_{11}+\cdots.$$

It is easy to see that such an expansion does lead to a consistent series of equations, and in the problem which has been solved fully in Sec. 3, the perturbation expansion does lead to the correct solution for small values of the conductivity. Here only the first term in the expansion (29) is kept; σ is taken to be vanishingly small in the perturbed equations, though it must be finite in the equilibrium state.

The first approximation to Eqs. (26)–(28) is

$$\text{curl curl}\mathbf{B}_{10}=0, \quad (30)$$

$$\text{div}\mathbf{v}_{10}=0, \quad (31)$$

$$\rho_0\omega\mathbf{v}_{10}=-\text{grad}p_{10}+\frac{\text{curl}\mathbf{B}_{10}\times\mathbf{B}_0}{4\pi}+\frac{\text{curl}\mathbf{B}_0\times\mathbf{B}_{10}}{4\pi}. \quad (32)$$

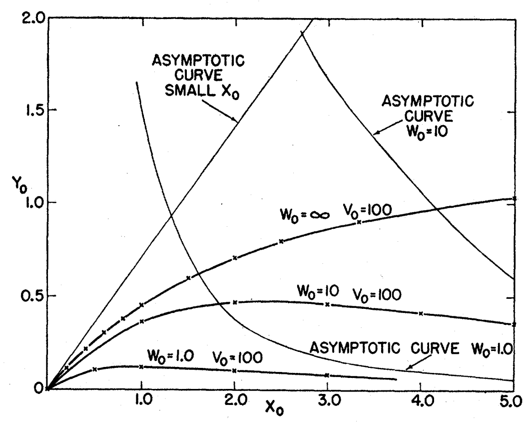

FIG. 5. Axisymmetric perturbations of a viscous fluid of finite conductivity (no axial field). The dimensionless growth rate is shown as a function of the dimensionless wave number for two pairs of values of V_0 and W_0. Also shown are the asymptotic values of the growth rate at large and small wave numbers.

Equation (30) can be solved simply to give

$$B_{10r}=\mathcal{A}I_{m+1}(kr)+\mathcal{B}I_{m-1}(kr),$$
$$B_{10\theta}=-i\mathcal{A}I_{m+1}(kr)+i\mathcal{B}I_{m-1}(kr),$$

and

$$B_{10z}=i(\mathcal{A}+\mathcal{B})I_m(kr), \quad (33)$$

and the other equations can then be solved to give expressions for $\mathbf{v}_{10}$ and p_{10}.

When boundary conditions are applied on the solutions of these equations, a dispersion relation is obtained which gives a first approximation to the value of the growth rate at low conductivity. Thus

$$Y_0^2=-2\left(mb+X_0+\frac{2m^2}{X_0}\right)\frac{I_m(X_0)}{I_m'(X_0)}+4+\frac{2X_0I_m'(X_0)}{I_m(X_0)}+\frac{4m^2}{X_0^2}\frac{I_m(X_0)}{I_m'(X_0)}\bigg/\left[\frac{K_m'(X_0)}{K_m(X_0)}-\frac{I_m'(X_0)}{I_m(X_0)}\right]. \quad (34)$$

Equation (34) has the great advantage of being an algebraic equation. It can be seen that Y_0^2 is always real. The axial magnetic field only occurs multiplied by the azimuthal wave number m, so that for axisymmetric perturbations, Eq. (34) is independent of b. Equation (34) can also be derived from Eq. (18) if it is expanded for small values of V_0, which is then allowed to tend towards zero. Thus for small enough values of the conductivity, the axial field should not be expected to affect the stability of axisymmetric perturbations. For large values of X_0, Eq. (34) becomes

$$Y_0^2\approx2(1-mb). \quad (35)$$

Thus for negative values of b, instability always occurs for large X_0, and instability occurs for positive values of b less than $1/m$.

Figure 6 shows the solution of Eq. (34) for $m=0$

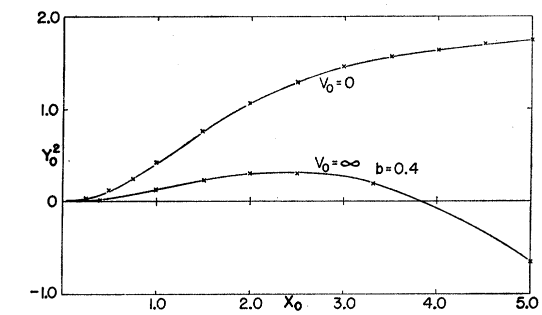

FIG. 6. Axisymmetric perturbations of a fluid of finite conductivity containing an axial magnetic field. The square of the dimensionless growth rate is shown as a function of the dimensionless wave number for a fluid of very low conductivity. The same curve applies to all values of the axial field. Also shown is the curve for a fluid of infinite conductivity for one value of the axial field.

and also shown is the infinite conductivity solution for one value of $b[b=0.4]$. Solutions of the $m=1$ equation have been found for $b=\pm1$ and 0, and these are shown in Fig. 7. Also shown in Fig. 7 are the infinite conductivity solutions for the same values of b. In the

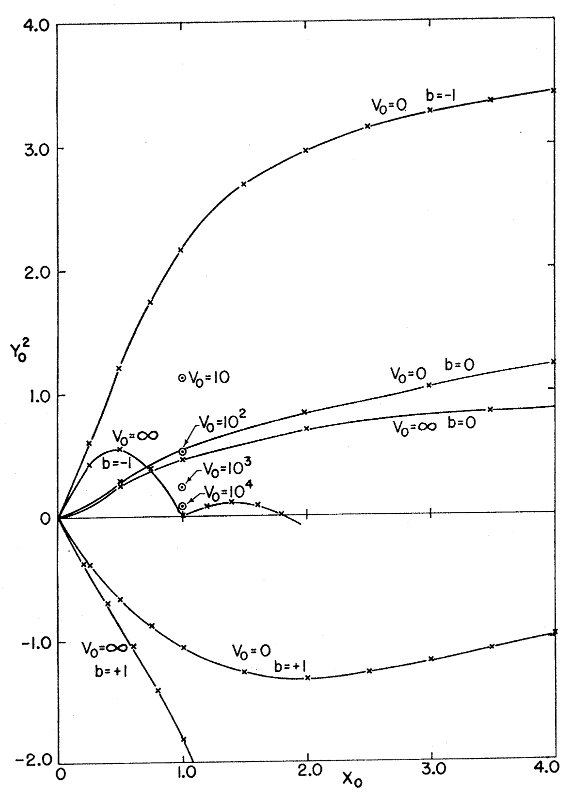

FIG. 7. $m=1$ perturbations of a fluid of finite conductivity containing an axial magnetic field. The square of the dimensionless growth rate is shown as a function of the dimensionless wave number for a fluid of very low conductivity for several values of the axial magnetic field. Similar curves are shown for a fluid of infinite conductivity for the same values of the axial field. Also shown are the growth rates for perturbations which are parallel to the magnetic field for several finite values of the conductivity.

absence of an axial field, it can be seen that for small wave numbers the results for infinite and zero conductivity are very little different. Eventually the value of Y_0^2 given by Eq. (35) is twice that obtained for infinite conductivity. The results obtained in Sec. 4, for the perturbation which is parallel to the magnetic field, vary smoothly between the results obtained for infinite and zero conductivity.

It can be seen from Eq. (35) that the worst instabilities occur for large values of m. As the conductivity approaches zero, the penetration time for field irregularities approaches zero, but only for very large values of m does the instability growth rate appear to become correspondingly large. For small values of m the growth rate is of order c_H/r_0.

6. DISCUSSION

The results given here are still far from being complete. It is hoped that the problem for arbitrary values of the conductivity will shortly be tackled using an electronic computer. The special results obtained here may then be used to give an idea of the results to be expected.

Results have been obtained for infinite and zero values of the conductivity, and it has been stated that

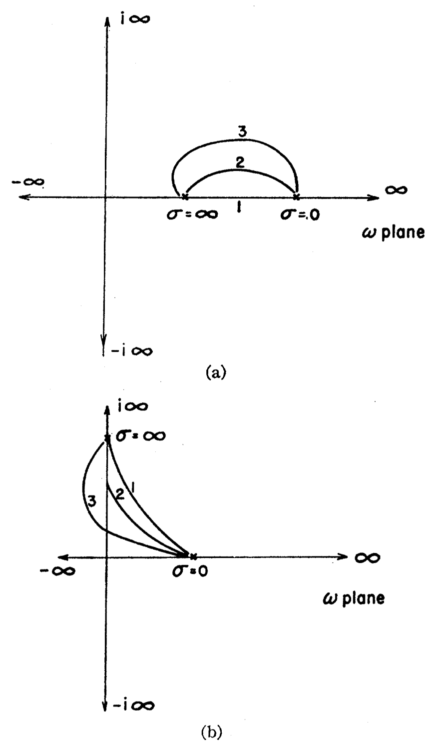

FIG. 8. Possible values of the growth rate in the complex ω plane. The growth rates for fluids of infinite and zero conductivity are known but the intermediate values are not known in general. The figure shows possible plots of these intermediate values in the complex ω plane in two cases, (a) when fluids of both infinite and zero conductivity are unstable, and (b) when the fluid of infinite conductivity is stable though that of zero conductivity is unstable.

the growth rate is normally complex for arbitrary values of the conductivity. What must be computed is the path described in the complex ω plane as σ decreases from infinity to zero. Figure 8 shows some of the possibilities. Figure 8(a) refers to a case in which instability occurs for both of the extreme values. Three possible types of curves joining the extreme values are shown. In this case it seems likely that the system is unstable for all values of the conductivity. When the growth rates for infinite and zero conductivity are of the same order of magnitude, an approximate value of the growth rate is probably known for all values of the conductivity.

The case in which the system is stable for infinite conductivity but unstable for zero conductivity is more interesting. Several possibilities for this case are shown in Fig. 8(b). Possibly the most likely case is curve 1; for this the system is unstable for all finite values of the conductivity although for large enough values the real part of ω is very small. Another possibility is curve 3 on which the real part of ω is negative for large values of σ. It is to decide between these possibilities that computation is required. In the case of axisymmetric perturbations, an attempt was made to expand the solution for large conductivity in the form

$$\omega=\omega_0+\omega_1/\sigma^{\frac{1}{2}}+\cdots. \tag{36}$$

When this was done it was found that ω_1/ω_0 was always real so that the early part of the curve appeared to look like curve 2.

The results obtained here cannot be applied directly to a plasma because of the assumption of incompressibility, but they can be applied to liquid conductors. The two obvious cases to consider are mercury and liquid sodium. In Table I the dimensionless conductivity V_0 is plotted as a function of the equilibrium magnetic field B_0 and the radius of the conductor r_0. It can be seen that, in terms of V_0, mercury and liquid sodium are both poor conductors. The viscosity parameter W_0 is also tabulated for mercury. The values of W_0 are very large and the results in Sec. 3 suggest that the influence of viscosity is not great except at very short wavelengths. It thus appears that mercury can be regarded as a poor conductor of low viscosity.

Experiments have been performed involving both liquid sodium and mercury though most of them are not directly comparable with the problem described here. Dattner *et al.* have observed the instabilities of a mercury column which is falling under gravity at the same time as it is carrying a current. When there is no axial field, they observe an $m=0$ instability. Ignoring for the moment the effect of gravity and putting Dattner's parameters $r_0=0.2$ and $B_0=300$ into the results obtained previously, the minimum e-folding time for an $m=0$ instability is about 7 msec. This compares well with Dattner's time of 30–50 msec for the channel to halve its radius. The experimental results for $m=1$ instabilities in the presence of an axial field do not agree so well with the theory; in this case the gravity forces may be more important. It is hoped to look more closely at comparisons with experiment when the problem has been solved for all values of the conductivity.

TABLE I. Properties of mercury and liquid sodium.

(a) Mercury	$\sigma\approx 10^{16}$		Gaussian units
	$\rho=13.6$		g/cc
	$\mu\approx 1.2\times10^{-2}$		poise
	Hydromagnetic velocity		
B_0	10^2	10^3	10^4
c_H	7.6	76	7.6×10^2
	Dimensionless conductivity V_0		
B_0 \\ r_0	10^{-1}	1	10
10^2	10^{-4}	10^{-3}	10^{-2}
10^3	10^{-3}	10^{-2}	10^{-1}
10^4	10^{-2}	10^{-1}	1
	Dimensionless viscosity parameter W_0		
B_0 \\ r_0	10^{-1}	1	10
10^2	9×10^2	9×10^3	9×10^4
10^3	9×10^3	9×10^4	9×10^5
10^4	9×10^4	9×10^5	9×10^6
(b) Liquid sodium	$\sigma\approx 10^{17}$		Gaussian units
	$\rho=0.9$		g/cc
	Hydromagnetic velocity		
B_0	10^2	10^3	10^4
c_H	30	3×10^2	3×10^3
	Dimensionless conductivity V_0		
B_0 \\ r_0	10^{-1}	1	10
10^2	4×10^{-3}	4×10^{-2}	4×10^{-1}
10^3	4×10^{-2}	4×10^{-1}	4
10^4	4×10^{-1}	4	40

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