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On Hydromagnetic Stability of Stationary Equilibria*

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I. INTRODUCTION

IT has been shown previously[1] that the stability of a hydromagnetic fluid in static equilibrium can be determined by an energy principle formalism. The purpose of the present investigation is to extend the methods of Bernstein *et al.*,[1] where possible, to the consideration of the stability of stationary, rather than static equilibria. As is well known, the presence of a velocity field in the equilibrium state may lead to the phenomenon of overstability. The manifestation of this in the mathematical formalism is the appearance of non-Hermitian operators. The present considerations lack the powerful theorems which are available for systems governed by Hermitian operators, nevertheless, it has been possible to obtain some general results for this case.

The plan of the paper is as follows. In Sec. II the linearized equations of motion and the boundary conditions in a Lagrangian representation are discussed. In Sec. III some properties of the equations of motion and a general sufficient condition for stability are given, and in Sec. IV a general perturbation theory for small flow velocities is presented. Appendix I presents a reformulation of the equations in Hamiltonian form. Appendix II discusses an application of the theory, calculated by A. Pytte, to a rotating "stabilized" pinch configuration.

II. EQUATIONS OF MOTION

The equations used are those which govern an ideal hydromagnetic fluid. The conditions under which they are valid are discussed in reference 1.

$$\rho(d\mathbf{v}/dt)=-\nabla p+\mathbf{j}\times\mathbf{B}, \tag{1}$$

$$(\partial\rho/\partial t)+\nabla\cdot(\rho\mathbf{v})=0, \tag{2}$$

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=0, \tag{3}$$

$$(d/dt)(p/\rho^{\gamma})=0, \tag{4}$$

$$\nabla\times\mathbf{E}=-(\partial\mathbf{B}/\partial t), \tag{5}$$

$$\nabla\times\mathbf{B}=\mathbf{j}, \tag{6}$$

$$\nabla\cdot\mathbf{B}=0. \tag{7}$$

Let ρ be the mass density, $\mathbf{v}$ the velocity, $\mathbf{B}$ the magnetic field, p the pressure, $\mathbf{j}$ the current density, $\mathbf{E}$ the electric field, and γ the ratio of specific heats.

The boundary conditions which we adopt at a fluid-vacuum interface are

$$\langle p+\tfrac{1}{2}B^2\rangle=0, \tag{8}$$

$$\mathbf{n}\times\langle\mathbf{E}\rangle=\mathbf{n}\cdot\mathbf{v}\langle\mathbf{B}\rangle, \tag{9}$$

$$\mathbf{n}\cdot\mathbf{B}=0, \tag{10}$$

$$\mathbf{n}\times\langle\mathbf{B}\rangle=\mathbf{K}, \tag{11}$$

where $\langle X\rangle=X_{\text{vac}}-X_{\text{plasma}}$, $\mathbf{n}$ denotes the unit normal to the plasma surface, and $\mathbf{K}$ is the surface current density.

For the purpose of investigating stability, it is advantageous to use a Lagrangian representation. We linearize the equations and introduce the displacement vector $\boldsymbol{\xi}$, which is considered to be a small quantity. The kinematics of the situation are shown in Fig. 1. The position vector $\mathbf{r}$ of a fluid element which at $t=0$ was at $\mathbf{r}_0$ is given by

$$\mathbf{r}=\mathbf{r}^0+\boldsymbol{\xi}(\mathbf{r}^0,t), \tag{12}$$

where $\mathbf{r}^0$ describes the equilibrium trajectory and $\boldsymbol{\xi}(\mathbf{r}^0,t)$ describes the displacement from equilibrium. We choose $\boldsymbol{\xi}$ to be a function of $\mathbf{r}^0,t$ rather than a function of $\mathbf{r}_0,t$ so that the equilibrium quantities are time independent and solutions of the form $e^{i\omega t}$ are permitted.

The perturbed quantities can then be obtained to first order in $\boldsymbol{\xi}$ from (2)–(7). The results are

$$\rho(\mathbf{r}^0+\boldsymbol{\xi})=\rho(\mathbf{r}^0)(1-\nabla^0\cdot\boldsymbol{\xi}), \tag{13}$$

$$p(\mathbf{r}^0+\boldsymbol{\xi})=p(\mathbf{r}^0)(1-\gamma\nabla^0\cdot\boldsymbol{\xi}), \tag{14}$$

$$\mathbf{B}(\mathbf{r}^0+\boldsymbol{\xi})=\mathbf{B}(\mathbf{r}^0)-\mathbf{B}\nabla^0\cdot\boldsymbol{\xi}+\mathbf{B}\cdot\nabla^0\boldsymbol{\xi}. \tag{15}$$

The further relations needed to obtain the equations of motion and boundary conditions are

$$\nabla=\nabla^0-\nabla^0\boldsymbol{\xi}\cdot\nabla^0, \tag{16}$$

$$\mathbf{v}(\mathbf{r}^0+\boldsymbol{\xi})=\mathbf{v}(\mathbf{r}^0)+\mathbf{v}^0\cdot\nabla^0\boldsymbol{\xi}+(\partial\boldsymbol{\xi}/\partial t), \tag{17}$$

$$\mathbf{n}(\mathbf{r}^0+\boldsymbol{\xi})=\mathbf{n}(\mathbf{r}^0)+\mathbf{n}\mathbf{n}\cdot\nabla^0\boldsymbol{\xi}\cdot\mathbf{n}-\nabla^0\boldsymbol{\xi}\cdot\mathbf{n}. \tag{18}$$

In addition, in the vacuum we introduce the first-order scalar and vector potentials ϕ and $\mathbf{A}$ by means of

$$\mathbf{E}_{\text{vac}}=\mathbf{E}_{\text{vac}}{}^0-\nabla\phi-(\partial\mathbf{A}/\partial t), \tag{19}$$

$$\mathbf{B}_{\text{vac}}=\mathbf{B}_{\text{vac}}{}^0+\nabla\times\mathbf{A}, \tag{20}$$

where the zero denotes a zeroth-order vacuum quantity.

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[1] I. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. (London) A244, 17 (1958).

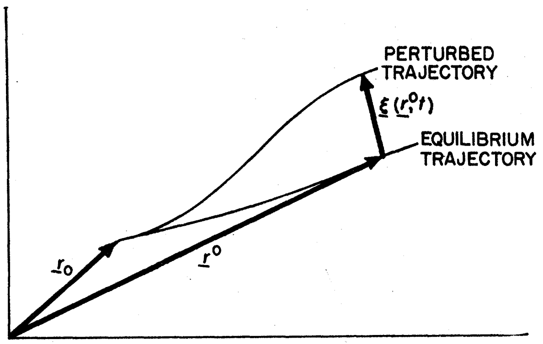

FIG. 1. Definitions of $\mathbf{r}^0$ and $\boldsymbol{\xi}(\mathbf{r}^0,t)$. The vector $\mathbf{r}_0$ represents the original position of the fluid elements. *Note.* In Figs. 1 and 2, vectors are indicated by a bar beneath the letter.

$\mathbf{A}$ must satisfy the equation

$$\nabla\times\nabla\times\mathbf{A}=0 \tag{21}$$

in the vacuum. Henceforth, the only quantities which appear are those evaluated at $\mathbf{r}^0$. Thus we can drop the superscripts with no confusion.

In terms of the potentials, the boundary conditions (8)–(10) become

$$-\gamma p\nabla\xi+\mathbf{B}\cdot(\mathbf{Q}+\xi\cdot\nabla\mathbf{B}) =\mathbf{B}_{\text{vac}}{}^0\cdot(\nabla\times\mathbf{A}+\xi\cdot\nabla\mathbf{B}_{\text{vac}}{}^0), \tag{22}$$

$$\mathbf{n}\times\mathbf{A}=-\mathbf{n}\cdot\xi\mathbf{B}_{\text{vac}}{}^0 \tag{23}$$

where

$$\mathbf{Q}=\nabla\times(\xi\times\mathbf{B}). \tag{24}$$

The freedom of the choice of gauge for the potentials has been used in obtaining (23). Finally, the linearized equation of motion which follows from (1) is

$$\rho(\partial^2\xi/\partial t^2)+2\rho\mathbf{v}\cdot\nabla(\partial\xi/\partial t)-\mathbf{F}\{\xi\}=0, \tag{25}$$

where

$$\mathbf{F}\{\xi\}=\nabla(\gamma p\nabla\cdot\xi+\xi\cdot\nabla p-\mathbf{B}\cdot\mathbf{Q})+\mathbf{B}\cdot\nabla\mathbf{Q}+\mathbf{Q}\cdot\nabla\mathbf{B} +\nabla\cdot(\rho\xi\mathbf{v}\cdot\nabla\mathbf{v}-\rho\mathbf{v}\mathbf{v}\cdot\nabla\xi). \tag{26}$$

Since the time does not appear explicitly in (21)–(25), normal mode solutions of the form

$$\xi(\mathbf{r}^0,t)=\xi(\mathbf{r}^0)e^{i\omega t} \tag{27}$$

can be sought. The equation of motion then becomes

$$-\omega^2\rho\xi+2i\omega\rho\mathbf{v}\cdot\nabla\xi-\mathbf{F}\{\xi\}=0. \tag{28}$$

III. SUFFICIENT CONDITION

A number of formal properties of (25) are now established: (a) $i\rho\mathbf{v}\cdot\nabla$ is a Hermitian operator, (b) $\mathbf{F}\{\xi\}$ is a self-adjoint operator, and (c) if $\omega=\alpha$ is an eigenvalue, $\omega=-\alpha$, $\omega=\alpha^*$, and $\omega=-\alpha^*$ are also eigenvalues. Property (a) follows immediately from an integration by parts and the use of the equilibrium equation $\nabla\cdot\rho\mathbf{v}=0$. Property (b) follows from a large number of integrations by parts and use of the equilibrium equations. Property (c) is easily demonstrated by expanding ξ in an arbitrary complete orthonormal set of functions and examining the linear equations which determine the expansion coefficients. These properties allow the demonstration of a sufficient condition for stability.[2] Multiplication of (28) by ξ^* and integration over the fluid volume lead to

$$-\omega^2\int\rho\xi^*\cdot\xi d\tau+2i\omega\int\rho\xi^*\cdot(\mathbf{v}\cdot\nabla)\xi d\tau -\int\xi^*\cdot\mathbf{F}\{\xi\}d\tau=0. \tag{29}$$

On solving for ω, we find

$$\omega=\frac{-\int i\rho\xi^*\cdot(\mathbf{v}\cdot\nabla)\xi d\tau\pm\left\{\left[\int i\rho\xi^*\cdot(\mathbf{v}\cdot\nabla)\xi d\tau\right]^2-\left[\int\rho\xi^*\cdot\xi d\tau\right]\left[\int\xi^*\cdot\mathbf{F}\{\xi\}d\tau\right]\right\}^{\frac{1}{2}}}{\int\rho\xi^*\cdot\xi d\tau}. \tag{30}$$

Note that all the integrals occurring in (30) are real by virtue of the Hermitian character of the operators. Therefore ω is real and the system stable if

$$\int\xi^*\cdot\mathbf{F}\{\xi\}d\tau<0. \tag{31}$$

If the sufficient condition (31) fails, the less stringent sufficient condition

$$\left(\int i\rho\xi^*\cdot\mathbf{v}\cdot\nabla\xi d\tau\right)^2-\left(\int\rho\xi^*\cdot\xi d\tau\right) \times\left(\int\xi^*\cdot\mathbf{F}\{\xi\}\right)d\tau>0 \tag{32}$$

may be used, although it may be more difficult to apply.

An important result following from (31) is that, if a static equilibrium state is stable, for sufficiently small flow velocities the resulting stationary equilibrium is also stable. We use this result in the perturbation theory in the next section.

It can easily be shown that (30) provides a variational principle for ω, $\delta\omega=0$, when ω is real, except for those points at which the square root in (30) vanishes. This result can be used to estimate the eigenfrequencies of the system.

A closely related principle can be derived even when

[2] This condition was found independently by I. Bernstein, R. Kulsrud, and D. Montgomery.

ω is complex. Note that the equation adjoint to (28) is

$$-\omega^2\rho\xi^A-2i\rho\omega\mathbf{v}\cdot\nabla\xi^A-\mathbf{F}\{\xi^A\}=0. \tag{33}$$

By virtue of property (c), (33) possesses the same set of eigenvalues as (28). Thus the sets of adjoint eigenfunctions and eigenvalues are the same as the sets following from (28) although the eigenfunction-eigenvalue correspondence is different. Therefore, we are guaranteed that solutions to (33) exist if solutions to (28) exist.

The variational principle which makes both the real and imaginary parts of ω stationary is just (30) with ξ^A replacing ξ^* throughout. This principle suffers from the same defect as (30) but still may be useful for some applications. A principle is given in Appendix I which holds for all ω.

IV. PERTURBATION THEORY FOR SMALL VELOCITIES

We now discuss a perturbation procedure for determining the stability of a dynamic equilibrium. The procedure consists of expanding the equilibrium equations and Eq. (28) in powers of a small parameter ϵ. ϵ represents the ratio v/v_S or v/v_A, where v_S, v_A, and v represent characteristic sound, Alfvén, and flow velocities, respectively, in the equilibrium state. To lowest order in ϵ a static equilibrium results and the stability considerations reduce to the energy principle formalism. We first illustrate the theory assuming no degeneracy.

The equation of motion (28) to lowest order in ϵ is

$$-\omega_0^2\rho_0\xi_0-\mathbf{F}_0\{\xi_0\}=0, \tag{34}$$

where the subscripts denote the order in ϵ. There are then three cases to consider:

(1) ω_0 real, $\neq 0$,
(2) ω_0 pure imaginary,
(3) $\omega_0=0$.

If ω_0 is real, the sufficient condition (31) and Eq. (34) immediately show that ω remains real for sufficiently small ϵ. If ω_0 is pure imaginary, the system is unstable in lowest order, and the small equilibrium velocity cannot stabilize it although overstability may occur in higher orders in ϵ. If, however, ω_0 vanishes, it is of interest to look at ω_1. The equations for this case are

$$\mathbf{F}_0\{\xi_0\}=0, \tag{35}$$

$$\mathbf{F}_0\{\xi_1\}=0, \tag{36}$$

$$-\omega_1^2\rho_0\xi_0+2i\omega_1\rho_0\mathbf{v}_1\cdot\nabla\xi_0-\mathbf{F}_0\{\xi_2\}-\mathbf{F}_2\{\xi_0\}=0. \tag{37}$$

Note that the equilibrium equations do not allow ρ_1 and, therefore, $\mathbf{F}_1$ to exist. Note also that ξ_0 can be found variationally from the δW formalism.[1] Further, for $\omega_0=0$, the ξ_0 found variationally is actually the normal mode displacement because the normalization does not matter.

The solution of (35) can be chosen to be of the form

$$\xi_0=\xi_0'e^{i\alpha} \tag{38}$$

where ξ_0' is real and α is merely a phase factor. Then, multiplying (37) by ξ_0^* and integrating over the fluid volume lead to

$$-\omega_1^2\int\rho_0\xi_0'^*\cdot\xi_0'd\tau-\int\xi_0'^*\cdot\mathbf{F}_2\{\xi_0'\}d\tau=0. \tag{39}$$

Thus

$$\omega_1^2=-\left(\int\xi_0'^*\cdot\mathbf{F}_2\{\xi_0'\}\Big/\int\rho_0\xi_0'^*\cdot\xi_0'\right), \tag{40}$$

from which we see that ω_1^2 is real and overstability does not occur to this order. To settle the question of stability only the integral in the numerator of (40) need be calculated. If the numerator in (40) is negative, then ω_1 is real and nonzero, and the sufficient condition can again be invoked to show that ω remains real. If ω_1 is imaginary, instability is demonstrated. But if ω_1 vanishes, the procedure must be carried to higher order. Thus the method, in general, allows the determination of stability or instability in some low order in ϵ and, in the case of stability, guarantees that ω does not become complex in some high order in ϵ.

If degeneracy occurs, some of the conclusions reached in the foregoing no longer stand. For simplicity we illustrate the procedure in the case of a twofold degeneracy. We assume that (35) has the two eigensolutions ξ_0', ξ_0'' which we choose to be orthogonal

$$\int\rho_0\xi_0'^*\cdot\xi_0''d\tau=0. \tag{41}$$

Let ξ_0 be a linear combination of the two eigenvectors

$$\xi_0=\alpha\xi_0'+\beta\xi_0''. \tag{42}$$

After substituting (42) into (37) and multiplying by $\xi_0'^*$ and $\xi_0''^*$, we arrive at two homogeneous equations for α and β. Setting the determinant of the coefficients equal to zero leads to

$$\begin{aligned}&\left[\omega_1^2+\int\xi_0'^*\cdot\mathbf{F}_2\{\xi_0'\}d\tau\right]\left[\omega_1^2+\int\xi_0''^*\cdot\mathbf{F}_2\{\xi_0''\}d\tau\right]\\&\quad+\left\{2i\omega_1\int\rho_0\xi_0'^*\cdot\mathbf{v}_2\cdot\nabla\xi_0''d\tau-\int\xi_0'^*\cdot\mathbf{F}_2\{\xi_0''\}d\tau\right\}\\&\quad\times\left\{2i\omega_1\int\rho_0\xi_0'^*\mathbf{v}_1\cdot\nabla\xi_0''d\tau\right.\\&\qquad\left.+\int\xi_0'^*\cdot\mathbf{F}_2\{\xi_0''\}d\tau\right\}=0,\end{aligned} \tag{43}$$

where

$$\int\rho_0|\xi'|^2d\tau=\int\rho_0|\xi_0''|^2d\tau=1$$

has been assumed. We see immediately that ω_1^2 need no longer be real. However, if ω_1 is real, we can again invoke (30) to prove that ω remains real. The one exception to this occurs if the square root in (30) vanishes, in which case these considerations must be carried to at least one higher order before a decisive statement can be made.

ACKNOWLEDGMENTS

We thank I. Bernstein and R. Kulsrud for much valuable discussion and A. Pytte for allowing us to use his pinch stability calculation.

APPENDIX I

It is of some interest to recast the preceding theory into a Hamiltonian form. It is easy to see that the appropriate action S for the system is

$$S=\int d\tau dt\left[\frac{1}{2}\rho\frac{(\partial\xi/\partial t)^2}{2}-\rho\xi\cdot(\mathbf{v}\cdot\nabla)\frac{\partial\xi}{\partial t}+\frac{1}{2}\xi\cdot\mathbf{F}\{\xi\}\right]. \quad \text{(I.1)}$$

From this expression a Hamiltonian $\mathcal{H}$ can be constructed in the usual way. We find

$$\mathcal{H}=(1/2\rho)[\boldsymbol{\pi}-\rho\mathbf{v}\cdot\nabla\xi]^2-\tfrac{1}{2}\xi\cdot\mathbf{F}\{\xi\}, \quad \text{(I.2)}$$

where $\boldsymbol{\pi}$ is the canonical momentum

$$\boldsymbol{\pi}=\rho(\partial\xi/\partial t)+\rho\mathbf{v}\cdot\nabla\xi. \quad \text{(I.3)}$$

The Hamilton equations which follow from (I.2) consist of a restatement of (I.3) and

$$\partial\boldsymbol{\pi}/\partial t=\mathbf{F}\{\xi\}-\rho\mathbf{v}\cdot\nabla[(\boldsymbol{\pi}/\rho)-\mathbf{v}\cdot\nabla\xi]. \quad \text{(I.4)}$$

Writing $\xi=\xi' e^{i\omega t}$, $\boldsymbol{\pi}=\boldsymbol{\pi}' e^{i\omega t}$, allows us to put the system of equations in matrix form

$$\begin{Bmatrix} -\rho\mathbf{v}\cdot\nabla\rho^{-1} & \mathbf{F}+\rho\mathbf{v}\cdot\nabla(\mathbf{v}\cdot\nabla) \\ \rho^{-1} & -\mathbf{v}\cdot\nabla \end{Bmatrix}\begin{Bmatrix}\boldsymbol{\pi}' \\ \xi'\end{Bmatrix}=i\omega\begin{Bmatrix}\boldsymbol{\pi}' \\ \xi'\end{Bmatrix}. \quad \text{(I.5)}$$

Equation (I.5) is in the standard eigenvalue equation form which can then be written in abstract operator language as

$$L\mathbf{f}=\lambda\mathbf{f}. \quad \text{(I.6)}$$

The adjoint equation is

$$L^{\dagger}\mathbf{g}=\lambda^{*}\mathbf{g}. \quad \text{(I.7)}$$

The well-known variational principle

$$\lambda=\mathbf{g}^{*}\cdot L\cdot\mathbf{f}/\mathbf{g}^{*}\cdot\mathbf{f}, \quad \delta\lambda=0 \quad \text{(I.8)}$$

leads to (I.6) and (I.7) and further makes both the real and imaginary parts of λ stationary.

APPENDIX II

To illustrate the techniques developed heretofore, the calculation of the stability of a simple model of the stabilized pinch with a small rotation has been performed by A. Pytte. By assuming that the displacement ξ is a continuous function, we disregard the surface in-

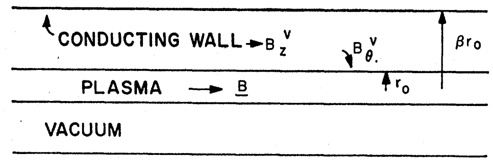

FIG. 2. Pinch effect equilibrium.

stabilities which have been discussed recently.[3,4] The stability of this situation was first considered by Rosenbluth.[5,6] The equilibrium configuration is shown in Fig. 2. A perfectly conducting plasma of radius r_0, constant density ρ_0, constant pressure p_0, and magnetic field $\mathbf{B}^p=B(r)\mathbf{e}_z$ is rotating with an angular velocity $\Omega(r)$. The rotation is caused by the imposition of a radial electric field. The momentum balance in equilibrium imposes the relation

$$\tfrac{1}{2}\nabla B^2=\rho_0 r\Omega^2\mathbf{e}_r. \quad \text{(II.1)}$$

In the absence of rotation, the displacements for the marginally stable state are[5]

$$\xi_0'=\begin{Bmatrix}\mathbf{e}_r I_m'(kr)\cos(m\theta+kz) \\ -\mathbf{e}_\theta I_m(kr)m/kr\sin(m\theta+kz) \\ -\mathbf{e}_z I_m(kr)\sin(m\theta+kz)\end{Bmatrix},$$

$$\xi_0''=\begin{Bmatrix}\mathbf{e}_r I_m'(kr)\sin(m\theta+kz) \\ \mathbf{e}_\theta m/kr I_m(kr)\cos(m\theta+kz) \\ \mathbf{e}_z I_m(kr)\cos(m\theta+kz)\end{Bmatrix}, \quad \text{(II.2)}$$

where I_m is the usual hyperbolic Bessel function well behaved at $r=0$ and $I_m'(kr)\equiv d/d(kr)I_m(kr)$. Note that $\nabla\cdot\xi_0=0$ and that ξ_0' and ξ_0'' are orthogonal in the sense of (41). In this case the operator $\mathbf{F}_2\{\xi_0\}$ reduces to

$$\mathbf{F}_2\{\xi_0\}=\rho_0\Omega^2\left[m^2\xi_0+\mathbf{e}_r\frac{\partial\xi_{0\theta}}{\partial\theta}-\mathbf{e}_\theta\frac{\partial\xi_{0r}}{\partial\theta}+\mathbf{e}_z r\frac{\partial\xi_{0r}}{\partial z}-3\mathbf{e}_r r\frac{\partial\xi_{0z}}{\partial z}\right]. \quad \text{(II.3)}$$

Let

$$f=\frac{\int d\tau\,\xi_0'^{*}\cdot\mathbf{F}_2\{\xi_0'\}}{\int d\tau\rho_0|\xi_0'|^2}=\frac{\int d\tau\,\xi_0''^{*}\cdot\mathbf{F}_2\{\xi_0''\}}{\int d\tau\rho_0|\xi_0''|^2}$$

[3] J. L. Johnson, C. R. Oberman, R. M. Kulsrud, and E. A. Frieman, Phys. Fluids **1**, 281 (1958).
[4] M. Rosenbluth, in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, New York, 1958), Vol. 31, p. 85.
[5] M. Rosenbluth, Los Alamos Rept. LA-2030 (1956).
[6] E. Gerjuoy and M. Rosenbluth, Bull. Am. Phys. Soc. Ser. II, **5**, 308 (1960).

and

$$g=\frac{\int d\tau\rho_0\Omega\xi_0'^{*}\cdot\frac{\partial\xi_0''}{\partial\theta}}{\int d\tau\rho_0|\xi_0'|^2}=-\frac{\int d\tau\rho_0\Omega\xi_0''^{*}\cdot\frac{\partial\xi_0'}{\partial\theta}}{\int d\tau\rho_0|\xi_0''|^2}. \tag{II.4}$$

By inspection from (II.2) and (II.3) we see that

$$\int\xi_0'^{*}\cdot F_2\{\xi_0''\}=0. \tag{II.5}$$

Equation (43) then reduces to

$$\omega_1=\pm[g\pm(g^2-f)^{\frac{1}{2}}], \tag{II.6}$$

which implies instability if $g^2-f<0$. It is an easy task to evaluate g and f when Ω=constant corresponding to a rigid rotation. The results are

$$\begin{aligned} g&=\Omega m[1-R_m],\\ f&=\Omega^2\{m^2[1-3R_m]+2/R_m\},\\ R_m&\equiv I_m(kr_0)/kr_0I_m'(kr_0). \end{aligned} \tag{II.7}$$

The frequency shift induced by the rotation is then

$$\omega_1=\pm\Omega\{m(1-R_m)\pm[m^2(R_m+R_m^2)-2/R_m]^{\frac{1}{2}}\}, \tag{II.8}$$

which is complex for all m. When $\Omega(r)$ is an arbitrary function of r, we cannot calculate ω_1 explicitly. However, making use of Schwartz's inequality, we can again show that $g^2-f<0$ for all m. Thus we conclude that any small rotation obeying Eq. (II.1) destabilizes the marginally stable pinch.

DISCUSSION

Session Reporter: Y. NAKAGAWA

W. B. Thompson, *Atomic Energy Research Establishment, Harwell, Berkshire, England:* Can a similar treatment be used directly on the collisionless Boltzmann equation?

E. A. Frieman: Yes, by using the distribution function itself as the variational function; however, the operators are not self-adjoint in general.