Pinch Buckling*

K. O. FRIEDRICHS

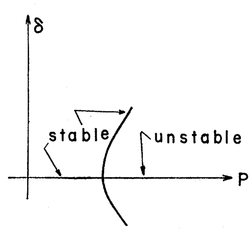
New York University, New York, New York

IT is well known[1] that plasma compressed by the pinching action of a magnetic field may be unstable, depending on the magnitude of the fields present and the nature of the region it occupies. The question has on occasion been asked[2] whether or not there exist equilibrium states of a pinched plasma with large (or rather noninfinitesimal) deflections and, if so, whether or not these states are stable? That such states may exist is suggested by analogy with the instability of an elastic structure under the action of a compressive force. Such an elastic structure buckles when the critical compression is exceeded and may assume a stable buckled state with "large" deflections.

There are different possibilities, however. While a thin rod assumes a stable buckled equilibrium when the axial compressive force P exceeds a critical value (see Fig. 1), the situation may be different for other structures. For example, as von Kármán and Tsien (see references 2, 5, and 7 of Friedrichs)[3] have observed, a thin elastic shell subjected to a constant external pressure possesses an unstable buckled state and in addition a stable buckled state with a still larger deflection, even before the critical pressure is reached (see Fig. 2). Naturally, one asks whether a pinched plasma behaves as any of the elastic structures named if an appropriate quantity—in place of pressure—is varied. It is shown that to a certain degree this is the case.

The primary cause of instability of a pinched plasma —assumed to be cylindrical—is the fact that the circumferential magnetic field strength B_θ in the vacuum outside of the plasma decreases away from the

FIG. 1. Deflection δ of a rod under the influence of an axial compression force P.



plasma. If some of the plasma enters the vacuum region, it experiences a reduced magnetic pressure there and hence tends to move further into the vacuum; in doing so, however, it pushes out the magnetic field which therefore becomes stronger, causing the magnetic pressure to rise again. If at the beginning the first effect dominates the second one, the cylindrical plasma is unstable. Eventually, though, the second effect may balance the first one so that a buckled equilibrium state may be attained.

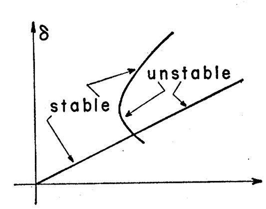
The "buckled" states of plasma are exhibited with a satisfactory degree of approximation. These equilibrium states involve a wavy interface whose amplitude grows if the data of the problem are changed away from those values at which transition from stability to instability takes place. It is remarkable that these buckled states exist for values of the data for which the unbuckled state is stable. It is a consequence of this fact that *these buckled states are unstable* (see Fig. 3).

On the other hand, it is indicated that stable buckled states with larger amplitudes may exist provided the outer cylinder is near enough to the interface. This question is not discussed here.

The unbuckled plasma is considered to form a cylinder $r\leq R_0$, of length l. It carries an axial field B_z^P, and is under a constant pressure. It is furthermore surrounded by a vacuum which carries a magnetic field with a circumferential component $B_\theta={}^0B_\theta R_0/r$ and possibly with an axial component B_z^V. (In general, we write simply B_θ instead of B_θ^V.) The vacuum is bounded by a cylindrical conductor $r=R_1$.

The buckled plasma states are assumed to have a "corrugated" surface, $r=R(\tau)$, where $R(\tau)$ is a function of the combination $\tau=kz+m\theta$ involving the axial and circumferential wave numbers k and m with integral m. The buckled field is assumed to be "helical," i.e., its components B_r, B_θ, and B_z should depend only upon the combination $\tau=kz+m\theta$, in addition to r. The buckled plasma is assumed to be at rest and to be under constant pressure. For simplicity the plasma is assumed to be incompressible so that it occupies the same volume, $\pi R_0^2 l$, as in the unbuckled state. It is

FIG. 2. Contraction δ of a thin spherical shell under the influence of a uniform external pressure p.



* The work presented in this paper was done partly while the author was a consultant at the Los Alamos Scientific Laboratory, partly at New York University, supported by the U. S. Atomic Energy Commission.

[1] M. Rosenbluth, Los Alamos Rept. No. 2030 (1956); and Third International Conference on Ionization Phenomena in Gases, 1957, Venice.

[2] R. J. Tayler, Proc. Phys. Soc. (London) **B70**, 31, 1049 (1957); *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, New York, 1958), Vol. 31, p. 160.

[3] K. O. Friedrichs, "The edge effect in bending and buckling with large deflections," *Proceedings of Symposia in Applied Mathematics* (American Mathematical Society, 1949), Vol. I, p. 188.

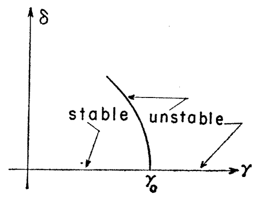


FIG. 3. Amplitude δ of the buckled interface as function of the data in the case of small wavelength ($kR_0 \sim \infty$).

because of this assumption that the case $m=0$ is excluded.

Most of the results are derived on the basis of a simplifying assumption, namely, the *assumption that the wave length* $2\pi/k$ *of the corrugation is* relatively *small* compared with the perimeter $2\pi R_0$ of the plasma cylinder. The simplification is obtained by letting kR_0 tend to infinity in the equations, while letting the prescribed values of the fluxes depend on kR_0 in a proper way. Denoting the corresponding mean values of B_z and B_θ by $\bar{B}_z$ and $\bar{B}_\theta$, we keep $\bar{B}_z^P$ and $\bar{B}_z^V$ fixed and let $\bar{B}_\theta$ grow such that $\bar{B}_\theta(kR_0)^{\frac{1}{2}}$ remains unchanged. At the same time the wave number m is allowed to tend to infinity but such that $m/(kR_0)^{\frac{1}{2}}$ remains finite.

The problem of small wavelength resulting from this limit process is described by very simple equations. The solutions of this simplified problem give a good approximation to the solutions of the original problem roughly for $kR_0 \geq 3$ provided $|\bar{B}_\theta^V|(kR_0)^{\frac{1}{2}}$ is the same order of magnitude as $|\bar{B}_z^P|$, while $|\bar{B}_z^V|$ is relatively small. This follows from an analysis of the unsimplified problem, to be presented in a later publication.

The basic quantities entering the simplified problem are the field components

$$\bar{B}_z^P \quad \text{and} \quad \bar{B}_z^V + (m/kR_0)\bar{B}_\theta^V$$

and the quantities

$$\gamma = (1/R_0)|\bar{B}_\theta^V|^2$$

and

$$\gamma_0 = k|\bar{B}_z^P|^2 + k|B_z^V + (m/kR_0)\bar{B}_\theta^V|^2.$$

First of all, it depends on these quantities whether the unbuckled plasma is stable or unstable. In this context the notion of "stability relative to a mode (m,k)" must be employed. By this relative stability is meant stability with respect to deviations for which the disturbed field components B_r, B_θ, B_z depend only on the combination $\tau = kr + m\theta$ (with unchanged k and m) in addition to r. In this relative sense then, *the unbuckled state is stable for* $\gamma < \gamma_0$ *and unstable for* $\gamma > \gamma_0$.

Values of the field data $\bar{B}_z^P$, $\bar{B}_z^V$, $\bar{B}_\theta^V$, together with values of kR_0 and m, for which the "transition equation" $\gamma = \gamma_0$ is satisfied, are called "transition data."

Suppose that $\bar{B}_z^V$ is zero or relatively small. Then the possible transition values of kR_0 are bounded below. Specifically, as numerical computation shows, we have

$$kR_0 \geq m^2 - 1 \quad (\text{if} \quad \bar{B}_z^V = 0)$$

for these values. From the statement made previously that this simplification is acceptable for about $kR_0 \geq 3$, we conclude that *this simplification is acceptable for data in the neighborhood of all transition data provided* $m \geq 2$. For the simplified transition equation itself, this statement can be verified by comparison with the results of earlier workers.[1,2,4]

Transition data are those data for which buckled states with an infinitesimal amplitude exist. Buckled states with a noninfinitesimal amplitude are shown to exist in the neighborhood of transition data. This analysis shows that the amplitude δ of the wavy interface of such buckled states is connected with the data of the problem to second order through the relation

$$\gamma = \gamma_0[1 - \alpha k^2 \delta^2],$$

where the constant α depends on the data, but varies only between the values $\frac{1}{2}$ and 1. It is then seen that buckled states exist for $\gamma < \gamma_0$ and not for $\gamma > \gamma_0$. In this way we are led to the statement made previously that *buckled equilibrium states exist for data near transition data if and only if the unbuckled state is stable for these values.* This statement, derived here only for the case of small wavelength, is valid also for $m=1$ and large wavelength, as is shown elsewhere.

In the situation just described *the buckled states are unstable*, as is proved later. This should be expected by analogy with the instability of an elastic structure in the analogous situation.

We make a few remarks about the results of this analysis concerning the shape of the buckled interface. Evidently, the shape of this interface is approximately sinusoidal if the deviation from equilibrium is small. This analysis shows that for larger wave amplitudes the buckled interface flattens out toward the plasma and steepens toward the field—or the other way around—according as

$$|(m/kR_0)\bar{B}_\theta^V + \bar{B}_z^V| < |\bar{B}_z^P| \text{ or } > |\bar{B}_z^P|.$$

In case one of these two quantities vanishes, there is no doubt that the growth of the amplitude is limited; specifically, it is to be expected that on increasing the ratio γ_0/γ (to about 1.2 [5]), an extreme situation is reached in which the crest of the interface has developed a sharp corner towards the vacuum enclosing an angle of 120°. There does not seem to be an equilibrium state with an amplitude greater than that in this extreme case.

The facts just stated are easily established simply because the mathematical description of the buckled plasma in the case of small wavelength precisely agrees with the mathematical description of *steady water waves* under the influence of *gravity*. The prescribed values of

$$\bar{B}_z^P \quad \text{and} \quad \bar{B}_z^V + (m/kR_0)\bar{B}_\theta^V$$

[4] M. Kruskal and J. L. Tuck, Proc. Roy. Soc. (London) **A245**, 222 (1958).

[5] H. Yamada, Repts. Research Inst. Appl. Mech. **5**, No. 18, 37 (1957).

correspond to the prescribed values of the horizontal velocity component of the two fluids at whose interface waves are formed. [If one of these quantities vanishes the existence of the corresponding nonlinear mathematical problem has been proved rigorously by Levi-Civita and Struik[6] (see references therein).] It may be supposed that the heavier fluid, the water, takes the place of the plasma and the lighter fluid, the air, takes the place of the vacuum, and that gravity acts in the direction from air towards water. Letting g be the acceleration of gravity and ρ^W, ρ^A be the density of water and air, we find that the difference of specific gravities is given by

$$(\rho^W-\rho^A)g=\gamma/\mu,$$

where μ is the diamagnetic constant in empty space.

To derive this formula one may roughly argue as follows: If the wavelength of the corrugation is small, the deviation from the undisturbed state is confined to a small neighborhood of the interface whose thickness is comparable with the wavelength. The drop of $|B_\theta|$ away from the unbuckled interface then produces a drop of magnetic pressure at the new position $R_0+\delta R$ of the interface. Approximately, this drop is given by

$$-\mu^{-1}\bar{B}_\theta{}^V\delta B_\theta=(|\bar{B}_\theta{}^V|^2/\mu R_0)\delta R=(\gamma/\mu)\delta R.$$

The latter expression should therefore be the reduction in pressure which in a water wave with the elevation δR is to be balanced by a reduction in flow speed—owing to gravity.

The water-wave analogy breaks down if the wavelength of the corrugation is comparable with the diameter of the cylinder. For, since $B_\theta=\bar{B}_\theta{}^V R_0/r$ varies as $1/r$, the rate of reduction of magnetic pressure falls off as the wave crest moves into the vacuum. The destabilizing effect of the decrease of magnetic pressure is thus less than for water waves. Therefore, the stabilizing effect of the presence of the conducting outer cylinder may become effective and may produce a stable buckled state.

BUCKLED PINCHED PLASMA CYLINDER

We first formulate the basic equation for the magnetic fields $B=\mathbf{B}$ in plasma and vacuum in general terms. The differential equation

$$\nabla\cdot B=0,\quad \nabla\times B=\theta,$$

in which $\nabla\cdot$ and $\times$ signify divergence and curl, certainly hold in the vacuum. The buckled plasma not only is assumed to be at rest but moreover to be "force free," so that the same differential equations are satisfied in it, while the fluid pressure in the plasma remains constant. At the interface and at the outer cylinder, the field should be tangential, i.e., the boundary conditions $B_n=n\cdot B=0$ at $\mathcal{S}$ and at $\mathcal{S}_1$ should be satisfied. Here $n=\mathbf{n}$ is the outer normal vector at $\mathcal{S}$ and at $\mathcal{S}_1$. These surfaces are assumed to be given by equations

$$r=R(\theta,z)\quad \text{for}\quad \mathcal{S},$$
$$r=R_1=\text{const}\quad \text{for}\quad \mathcal{S}_1.$$

The condition of pressure balance at the interface $\mathcal{S}$ is

$$[(1/2\mu)|B|^2]_P{}^P=p,$$

where p is the fluid pressure in this plasma, while $|B^p|$ and $|B^V|$ stand for the magnitude of the magnetic fields on both sides of the interface. The symbol $[Q]_P{}^V$ stands for the jump

$$[Q]_P{}^V=Q^V-Q^P$$

at the interface.

At the ends $z=0$ and $z=l$ of the cylinder, we should[7] impose proper conditions to make sure that the buckled state could have developed from the unbuckled one by a motion. Instead, we require the field to be periodic,

$$B^{z=l}=B^{z=0},$$

and prescribe the axial fluxes

$$\Phi_z{}^P=\int_0^{2\pi}\int_0^R B_z r dr dt\,|^{z=\text{const}}$$

$$\Phi_z{}^V=\int_0^{2\pi}\int^{R_1} B_z r dr d\theta\,|^{z=\text{const}}$$

in plasma and vacuum.

It is because of this simplification that we are permitted to assume the field to be force free. It is likely that the conditions of periodicity and force-freeness are sufficiently well satisfied for the solution of the problem in which the correct boundary conditions are imposed at the end plates.

In addition to prescribing the fluxes Φ_z the circumferential flux in the vacuum must be prescribed:

$$\Phi_\theta=\int_0^l\int_R^{R_1} B_\theta dr dz\,|^{\theta=\text{const}}.$$

Finally, since the plasma is assumed to be incompressible, we prescribe its volume

$$\int_0^l\int_0^{2\pi}\int_0^R r dr d\theta dz=\pi R_0^2 l.$$

In the *unbuckled state* we have $B_r=0$; further,

$$B_z={}^0B_z{}^P=\Phi_z{}^P/\pi R_0^2,\quad B_\theta=0 \text{ in the plasma,}$$
$$B_z={}^0B_z{}^V=\Phi_z{}^V/\pi(R_1^2-R_0^2),\text{ and}$$
$$rB_\theta=R_0{}^0B_\theta{}^V=\Phi_\theta{}^V/l\log(R_1/R_0)\text{ in the vacuum.}$$

The stability or instability of this state can be ascertained after those values of ${}^0B_z{}^P$, ${}^0B_z{}^V$, ${}^0B_\theta{}^V$ have been determined for each mode (k,m) for which transition

[6] J. J. Stoker, *Water Waves* (Interscience Publishers, Inc., New York, 1957).

[7] This was pointed out to me by H. Grad.

from relative stability to instability takes place. This has been done by various authors.[1,2,4]

HELICAL FIELD

In order to describe buckled states it is assumed that the field components B_r, B_θ, B_z depend only on the combination

$$\tau = kz + m\theta$$

(in addition to r) with the period 2π in τ. It is assumed that the wavelength $2\pi/k$ in the $z-$ direction is an integral fraction l/n of the length l of the cylinder. Instead of introducing a potential function to satisfy the condition $\nabla \cdot B = 0$—as is done in most linearizing treatments—we prefer to introduce a function $\chi(r,\tau)$ to satisfy the condition $\nabla \cdot B = 0$ by setting

$$\chi_{/\tau} = -rB_r, \quad \chi_{/r} = mB_\theta + krB_z.$$

We also introduce the quantity

$$\beta = mB_z - krB_\theta,$$

so that

$$B_z = \sigma^{-2}(kr\chi_{/r} + m\beta)$$
$$B_\theta = \sigma^{-2}(m\chi_{/r} - kr\beta),$$

where

$$\sigma^2 = k^2r^2 + m^2.$$

The condition $\nabla \times B = 0$ then leads, on the one hand, to the differential equation

$$\chi_{/\tau\tau} + r(\sigma^{-2}r\chi_{/r/r}) = \beta r g(r)$$

with

$$g(r) = 2mkr\sigma^{-4},$$

and, on the other hand, to the condition

$$\beta = \text{const.}$$

We note that $\sigma^{-1}\beta$ is the component of the magnetic field B in the direction of the screw lines $\tau = \text{constant}$. Also, the quantity $\chi_{/\sigma}$ can be taken as the component of a vector potential in this direction.

We assume that the interface $\mathcal{S}$ can be described by an equation

$$r = R(\tau)$$

in terms of a function $R(\tau)$ having the period 2π. The outer conductor is given by $r = R_1 = \text{constant}$, as before.

To express the pressure balance condition

$$[(1/2\mu)|B|^2]_P{}^V = p$$

on the interface in terms of the function χ and the constant β one must use the expression

$$|B|^2 = r^{-2}\chi_{/\tau}{}^2 + \sigma^{-2}(\chi_{/r}{}^2 + \beta^2)$$

for the magnitude $|B|$ of the magnetic field.

The condition that the field B is tangential at the conductor $\mathcal{S}_1$ and the interface $\mathcal{S}$ can be expressed in a simple manner in terms of the function $\chi(r,\tau)$, namely, as the condition that $\chi(r,\tau) = \text{constant}$ on $\mathcal{S}_1$ and $\mathcal{S}$. We choose $\chi = 0$ on $\mathcal{S}$, $\chi = \chi_1$ on $\mathcal{S}_1$, where the constant χ_1 is given in terms of the fluxes by

$$\chi_1 = (m/l)\Phi_\theta{}^V + (n/l)\Phi_z{}^V$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\int_R^{R_1} mB_\theta dr d\tau + \frac{1}{2\pi}\int_0^{2\pi}\int_R^{R_1} kB_z r dr d\tau.$$

The constant β can be determined from the function χ with the aid of the fluxes Φ_z, viz.,

$$\beta_V m \int_0^{2\pi}\int_R^{R_1} \sigma^{-2} r dr d\tau + k\int_0^{2\pi}\int_R^{R_1} \sigma^{-2}r^2\chi_{/r} dr d\tau = \Phi_z{}^V,$$

$$\beta_P m \int_0^{2\pi}\int_0^{R} \sigma^{-2} r dr d\tau + k\int_0^{2\pi}\int_0^{R} \sigma^{-2}r^2\chi_{/r} dr d\tau = \Phi_z{}^P.$$

The integral factors of β can be evaluated explicitly and through integration by parts the other integrals could be replaced by expressions containing the function χ but not its derivative $\chi_{/r}$.

The condition that the plasma volume has the given value $\pi R_0{}^2 l$ reduces to

$$\frac{1}{2}\int_0^{2\pi} R^2(\tau)d\tau = \pi R_0{}^2.$$

It should be mentioned that the problem thus formulated is associated with a variational problem, namely, the problem of making the energy

$$E = \frac{1}{2\mu}\int^l \int_0^{2\pi}\int_0^{R_1} |B|^2 r dr d\theta dz$$

stationary among all fields B which are divergence free, $\nabla \cdot B = 0$, are tangential at interface and conductor, and yield the prescribed fluxes. For the helical field the energy E is given by

$$l^{-1}\mu E = \frac{1}{3}\int_0^{2\pi}\int_0^{R_1} [r^{-2}\chi_{/\tau}{}^2 + \sigma^{-2}\chi_{/r}{}^2 + \sigma^{-2}\beta^2] r dr d\tau,$$

where all functions $\chi(r,\tau)$ are admitted which have in τ the period 2π and satisfy the condition

$$\chi = 0 \text{ on } \mathcal{S}, \quad \chi = \chi_1 \text{ on } \mathcal{S}_1.$$

Here $\mathcal{S}$ is the interface $r = R(\tau)$ with periodic $R(\tau)$. Derivatives of the function $\chi(r,\tau)$ are permitted to jump at $\mathcal{S}$. The two constants β are supposed to be given in terms of the function χ and the two fluxes Φ_z through the two relations formulated previously. [In fact, it would not even be necessary to require the β to be constant for admission; then the functions $\beta^{V,P}(r,\tau)$ would have to be placed inside the integral in the two preceding formulas.] In a routine manner one verifies that every solution of the boundary value problem formulated makes this energy functional stationary.

It is necessary to describe the *unbuckled state* in terms of functions $\chi(r,\tau)$ and constants β, in spite of the fact that this state does not depend on the choice of the wave numbers k and m. In terms of the constants ${}^0B_z{}^P$, ${}^0B_z{}^V$, ${}^0B_\theta{}^V$, connected with the fluxes through the relations given previously, the constants β are

$$ {}^0\beta_P = m\,{}^0B_z{}^P, \quad {}^0\beta_V = m\,{}^0B_z{}^V - kR_0\,{}^0B_\theta{}^V, $$

and the function $\chi(r,\tau)=\chi^0(r)$ is given by

$$ \begin{aligned} {}^0\psi(r) &= \tfrac{1}{2}k(r^2-R_0{}^2)\,{}^0B_z{}^P \quad \text{for} \quad 0\leqq r\leq R_0, \\ &= \tfrac{1}{2}k(r^2-R_0{}^2)\,{}^0B_z{}^V + m\log|r/R_0|\,R_0\,{}^0B_\theta{}^V \\ &\qquad \text{for} \quad R_0\leqq r\leq R_1. \end{aligned} $$

BUCKLING WITH SMALL WAVELENGTH

As explained in the Introduction a limiting case is considered; viz., the case which arises if we let the number kR_0 tend to infinity. Keeping R_0 fixed, the limiting case may be regarded as one in which the wavelength $2\pi/k$ is infinitesimally small. For the sake of description it is a little more convenient to keep k fixed and to let R_0 tend to infinity. At the same time R_1 is allowed to tend to infinity such that the distance R_1-R_0 is fixed; eventually R_1-R_0 may also tend to infinity.

Also the wave number m is allowed to tend to infinity; specifically, it is required that m^2/kR_0 tend to a finite limit. If m is kept fixed we would have $m^2/kR_0\to 0$ and the effect of the wave number m would be wiped out in the limit. Actually, the results of assuming that m^2 grows like kR_0 turn out to be valid in good approximation for finite m provided $m\geq 2$, as explained in the Introduction.

In this limiting process the fluxes must be allowed to grow in an appropriate manner with R_0. It is required that the undisturbed axial fields

$$ \bar{B}_z{}^P = \Phi_z{}^P/\pi R_0{}^2, \quad \bar{B}_z{}^V = \Phi_z{}^V/\pi(R_1{}^2-R_0{}^2) $$

remain fixed as $R_0\to\infty$. Furthermore it is allowed that the circumferential "mean flux density"

$$ \bar{B}_\theta{}^V = \bar{B}_\theta = l^{-1}(R_1-R_0)^{-1}\Phi_\theta{}^V $$

grows as $(R_0)^{\frac{1}{2}}$ so that the ratio

$$ \gamma = (\bar{B}_\theta)^2/R_0 $$

remains fixed.

We introduce

$$ x = R-R_0, \quad y=\tau/k $$

as new independent variables to run from $-R_0$ to $X_1=R_1-R_0$ and 0 to $2\pi/k=l/n$, respectively. It is then postulated, without giving a detailed justification, that, as R_0 tends to infinity, the functions

$$ (kR_0)^{-1}\chi(R_0+x,ky) \quad \text{and} \quad R(ky)-R_0 $$

associated with the assumed solution tend to limit functions

$$ \psi(x,y) \quad \text{and} \quad X(y) $$

defined for $-\infty<x\leq X_1$ and $0\leq y\leq 2\pi/k$.

In a formal manner it is then concluded that the limit function χ satisfies the differential equation

$$ \psi_{/xx}+\psi_{/yy}=0 \quad \text{for} \quad -\infty<x<X_1 $$

and the boundary conditions

$$ \begin{aligned} \psi &= 0 && \text{for} \quad x=X(y), \\ \psi &= \psi_1 = b_+X_1 && \text{for} \quad x=X_1, \end{aligned} $$

where

$$ b_+ = \bar{B}_z{}^V + (m/kR_0)\bar{B}_\theta. $$

Note that the two terms occurring in this expression are finite in the limit, according to our assumptions.

The condition that the flux $\Phi_z{}^P$ assumes the prescribed value yields, as a simple formal consideration would show, in the limit the condition that the function $\psi(x,y)$ behaves as

$$ \psi(x,y)\sim b_-x \quad \text{as} \quad x\to-\infty, $$

where

$$ b_- = \bar{B}_z{}^P. $$

The quantities β behave in the limit process as

$$ \beta_P\sim m\bar{B}_z,\ \beta_V\sim m\bar{B}_z{}^V - kR_0\,\bar{B}_\theta \sim -kR_0B_\theta. $$

(In effect this behavior was used in deriving the limit differential equation.) Since β_V grows as $(kR_0)^{\frac{3}{2}}$ with R_0, the contribution from $\sigma^{-2}\beta_V{}^2$ in the jump condition $[(1/2\mu)|B|^2]=p$ grows like kR_0 and thus dominates the other terms, which tend to finite limits. Since the leading term $(kR_0)^{-2}\beta_V{}^2$ of $\sigma^{-2}\beta_V{}^2$ is constant, it is balanced by the constant pressure p, which is thus seen to grow as kR_0. The term of next order coming from

$$ \sigma^{-2}\beta_V{}^2 = \{k^2(R_0+x)^2+m^2\}^{-1}\beta_V{}^2, $$

which equals $-2(\bar{B}_\theta)^2R_0{}^{-1}x=-2\gamma x$, except for a constant, then balances the other terms in the jump condition. In the limit this condition thus becomes

$$ \tfrac{1}{2}[\psi_{/x}{}^2+\psi_{/y}{}^2]_P{}^V = \gamma x + \text{const} \quad \text{at} \quad x=X(y) $$

with

$$ \gamma = (\bar{B}_\theta)^2/R_0. $$

Thus we have formulated the limit problem that corresponds to large values of the radius R_0 but finite value of R_1-R_0, or, which is equivalent, to large values of k but finite values of R_0 and $k(R_1-R_0)$. If the value of $k(R_1-R_0)$ is large the domain of x may be extended to infinity setting $X_1=\infty$, or replacing the boundary condition on $\mathcal{S}_1$ there by the condition that $\psi(x,y)$ behaves as

$$ \psi\sim b_+x \quad \text{at} \quad x\sim\infty. $$

Condition

$$ \int_0^{2\pi} R^2(\tau)d\tau = 2\pi R_0{}^2 $$

goes over into the condition

$$ \int_0^{2\pi} X(y)d\tau = 0, \quad \tau = ky. $$

The problem thus formulated is mathematically equivalent with the *problems* of *gravity waves* in *water*. The corresponding water flow takes place in the xy plane. There are two fluids covering the regions $-\infty<x<X(y)$ and $X(y)<x<X_1$. The quantity b_+ is the mean velocity in the y direction in the vacuum, while b_- is the velocity in the y direction at $x=-\infty$ in the plasma. The quantity $\gamma=|\bar{B}_\theta|^2/R_0$ corresponds to the difference $|\rho^P-\rho^V|g$ of gravity per unit volume of the two fluids. Gravity is directed from the thinner to the denser fluid, irrespective of which is which.

In case one of the two fluids is absent ($b_+=0$ or $b_-=0$) the shape of water waves of noninfinitesimal amplitude was described by Stokes and others; their mathematical existence was proved rigorously by Levi-Cività and Struik.

Although in case of "wind," $b_+b_-\neq0$, a rigorous existence proof has apparently not been given, there is no doubt that a solution exists which possesses the series expansion derived from the standard bifurcation analysis. We determine the terms of lowest order of this expansion for the limiting case $X_1=\infty$.

BIFURCATION ANALYSIS FOR THE CASE OF SMALL WAVELENGTH

The aim is to obtain a series expansion

$$\psi(x,y)={}^0\psi(x)+\delta\psi(x,y)+\delta^2\psi(x,y)+\cdots$$
$$X(y)=\delta X(y)+\delta^2X(y)+\cdots$$

of $\psi(x,y)$ and $X(y)$, in powers of an appropriate parameter. It is essential that the data of the problem are not all kept fixed in this expansion. It is convenient to keep the quantities $b_\pm$ fixed and to let the quantity γ vary. Specifically, we assume that γ is also given as a series:

$$\gamma=\gamma_0+\delta\gamma+\delta^2\gamma+\cdots,$$

where the value γ_0 is still to be chosen.

The most convenient expansion parameter is the first Fourier coefficient

$$\delta_1=\frac{1}{\pi}\int_0^{2\pi}X(y)\cos\tau d\tau;\quad \tau=ky$$

of the function $X(y)$ describing the interface. We call δ_1 the "amplitude" of this interface. The symbol δ has thus two different meanings; but there should be no confusion. The terms qualified by the "variation" δ and δ^2 are supposed to be proportional to the "parameters" δ_1 and δ_1^2, respectively.

On inserting such expansions into the differential equation, the boundary and jump conditions, one finds first the terms of *order 0*: ${}^0\psi=b_\pm x$ for $x\gtrless0$, and then the relations of *order 1*: $\Delta\delta\psi=0$, and

$$\text{J:}\quad [{}^0\psi_{/x}\delta\psi_{/x}]_-^+=\gamma_0\delta X+\text{const},$$
$$\text{A:}\quad \delta\psi+{}^0\psi_{/x}\delta X=0\quad\text{at}\quad x=0.$$

The last equation stands for two equations, one on each side of $x=0$. On using

$${}^0\psi_{/x}=b_\pm\quad\text{for}\quad x\gtrless0,$$

equations J and A may be written as

$$\text{J:}\quad [b\delta\psi_{/x}]_-^+=\gamma_0\delta X+\text{const},$$
$$\text{A:}\quad \delta\psi+b\delta X=0\quad\text{at}\quad X=0.$$

By assumption we have

$$\delta X=\delta_1\cos ky,$$

and we assume $\delta_1\neq0$ in the following. (Additional terms involving multiples of k are not excluded at this stage; but they would be found to equal zero after the jump condition J was used.) On using A and $\delta\psi\to0$ as $|x|\to\infty$, it follows that $\delta\psi$ is given by

$$\delta\psi(x,y)=-\delta_1be^{\mp kx}\cos ky.$$

Insertion into the jump condition J yields the relation

$$\gamma_0=k[\pm b^2]=k(b_+^2+b_-^2)$$

for the value γ_0 of γ. According to the standard theory of stability, it is this value γ_0 of γ at which transition from stability to instability takes place. In terms of the original quantities this "bifurcation equation" has the form

$$(1/kR_0)\bar{B}_\theta^2=[(m/kR_0)\bar{B}_\theta+\bar{B}_z^V]^2+(\bar{B}_z^P)^2.$$

We discuss this relation by plotting its graph in the $(kR_0,\bar{B}_z^P)$ plane for fixed values of $m_1\bar{B}_\theta$ and $\bar{B}_z^V$. We notice that, for given values of the field components $\bar{B}_\theta$ and $\bar{B}_z^V$, the values of kR_0 for which instability can occur are bounded below. (The upper bound is sensitive to changes in $\bar{B}_z^V$; it is infinite for $\bar{B}_z^V=0$.) In case $m\bar{B}_z^V/\bar{B}_\theta>0$; for example, the value of kR_0 is restricted to lie above m^2, so that $m^2/kR_0<1$. This is consistent with the requirement that m^2/kr_0 be finite.

A closer examination of the bifurcation equation derived without going to the limit $kR_0\to\infty$ would show that the results of this limit process is valid in good approximation for the whole range of data for which instability may occur provided $m\geq2$. For $m=1$ the bifurcation relation differs essentially from the one just discussed for small values of kR_0; in fact, it has a solution $|\bar{B}_z^P|$ for all values of kR_0, and these values of $|\bar{B}_z^P|$ tend to infinity as $kR_0\to0$. This case is not described here.

At present the bifurcation analysis is continued in the limiting case $kR_0\to\infty$. For the quantities $\delta^2\psi,\delta^2X$ of *order 2*: $\Delta\delta^2\psi=0$, and

$$\text{A:}\quad \delta^2\psi+\delta\psi_{/x}\delta X+{}^0\psi_{/x}\delta^2X=0\quad\text{at}\quad x=0$$

$$\text{J:}\quad [{}^0\psi_{/x}\delta^2\psi_{/x}+\tfrac{1}{2}(\delta\psi_{/x})^2+\tfrac{1}{2}(\delta\psi_{/y})^2+{}^0\psi_{/x}\delta\psi_{/xx}\delta X]$$
$$=\gamma_0\delta^2X+\delta\gamma\delta X+\text{const}.$$

[Note that in the expansion of $\psi[X(y),y]$, also the argument $x=X(y)$ must be developed.]

The choice of δ_1 as the first Fourier coefficient of $X(\tau)$ implies that the first coefficient of $\delta^2 X$ vanishes. From the vanishing of the first Fourier coefficient of the combination $J+[\pm k^0\psi_{/x}A]=0$, in obvious notation, one then derives the relation

$$\delta\gamma=0.$$

From the vanishing of the second Fourier coefficient of the combination $J+[\pm 2^0\psi_{/x}A]=0$ one then derives the result

$$\delta^2 X=\delta_2 \cos 2ky$$

with

$$\delta_2=\tfrac{1}{2}\kappa k\delta_1^2$$

where

$$\kappa=-[b^2]/[\pm b^2]=(b_-^2-b_+^2)/(b_-^2+b_+^2).$$

From relation $A=0$ one then obtains

$$\delta^2\psi=-\tfrac{1}{2}(\pm 1+x)k\delta_1^2 b_\pm e^{\mp 2kx}\cos 2ky+\text{const.}$$

The expression for δ_2 obtained enables us to describe the shape of the interface in its dependence on the amplitude δ_1. The equation

$$X=\delta_1 \cos ky+\tfrac{1}{2}\kappa k\delta_1^2 \cos 2ky$$

shows that, relative to a sinusoidal wave the waveform is flattened toward the plasma and steepened toward the vacuum, provided $|b_-|>|b_+|$, while it is steepened toward the plasma and flattened toward the vacuum of $|b_-|<|b_+|$ (see Fig. 4).

The first case arises, for example, if the B_z component in the vacuum is absent or at least is dominated by that in the plasma, while at the same time m is finite or at least relatively small. The second case arises if either m or B_z^P is relatively small.

Since $\delta\gamma=0$ was found from the terms of second order, it is necessary to go to the terms of the third order to find out how the quantity γ changes if δ_1 varies. *Order* 3: $\Delta\delta^3\psi=0$, and

$$\text{A:}\quad \delta^3\psi+\delta^2\psi_{/x}\delta X+\delta\psi_{/x}\delta^2 X+\tfrac{1}{2}\delta\psi_{/xx}(\delta X)^2+{}^0\psi_{/x}\delta^3 X=0,$$

$$\text{J:}\quad [{}^0\psi_{/x}\{\delta^3\psi_{/x}+\delta^2\psi_{/xx}\delta X+\delta\psi_{/xx}\delta^2 X+\tfrac{1}{2}\delta\psi_{/xxx}(\delta X)^2\} +\delta\psi_{/x}\delta^2\psi_{/x}+\delta\psi_{/y}\delta^2\psi_{/y}+(\delta\psi_{/y}\delta\psi_{/xx} +\delta\psi_{/y}\delta\psi_{/yy})\delta X]=\gamma_0\delta^3 X+\delta^2\gamma\delta X.$$

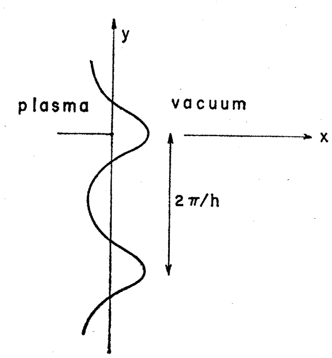


FIG. 4. Waveform $x=X(y)$ of the buckled interface between plasma and vacuum in case $|\bar{B}_z^V+(m/kR_0)\bar{B}_\theta|\ll|\bar{B}_z^P|$ for $kR_0\sim\infty$.

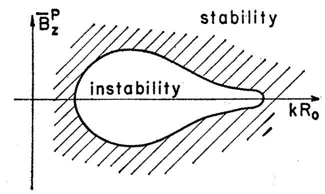


FIG. 5. Graph of the bifurcation equation for fixed value of m, $\bar{B}_\theta^V$, and $\bar{B}_z^V$ in case $kR\sim\infty$. The shaded area indicates the values of kR_0 and $\bar{B}_z^P$ for which buckled states exist.

In order to determine $\delta^2\gamma$ it is sufficient to consider the coefficient of $\cos\tau$ in the combination $J+[\pm{}^0\psi_{/x}A]=0$. The result is

$$\delta^2\gamma=-\tfrac{1}{2}(1+\kappa^2)k^2\delta_1^2\gamma_0.$$

This result, which may be put into the form

$$\gamma\sim\gamma_0[1-\tfrac{1}{2}(1+\kappa^2)k^2\delta_1^2],$$

is of particular significance since it shows that the buckled state exist for $\gamma<\gamma_0$. In terms of the original quantities this relation assumes the form

$$(1/kR_0)\bar{B}_\theta^2<[(m/kR_0)\bar{B}_\theta+\bar{B}_z^V]^2+(\bar{B}_z^P)^2.$$

In the $(kR_0,\bar{B}_z^P)$ plane it corresponds to the shaded side of the curve shown in Fig. 5.

Now, as is shown in the earlier work quoted previously,[1,2,4] for values $\gamma<\gamma_0$ the unbuckled state is stable relative to the considered mode (m,k). Thus we have found that the buckled state exists for values of the data for which the unbuckled state is stable.

INSTABILITY OF THE BUCKLED PLASMA WITH SMALL WAVELENGTH

The phenomenon described—that the buckled states exist for values of the data for which the unbuckled state is stable—has its counterpart in the buckling of a shell. There the buckled states which exist close to the unbuckled states are unstable. We show that also in our case *the buckled states of the plasma described by the results of our bifurcation analysis are unstable.* In fact, we establish slightly more; namely, the *relative* instability of these states. We recall that "relative stability" refers to only those disturbances which belong to a definite mode (k,m).

We first write down the second variation of the energy expansion, which is stationary for the solution. Then we show that this second variation can be made negative, and hence is not positive definite. Finally, we show that this fact implies the stated instability.

Instead of the energy E we consider an expression proportional to E, namely, $E^*=E\mu/kR_0l$, or

$$E^*=\tfrac{1}{2}\int_0^{2\pi}\int_{-\infty}^{X_1}(\psi_{/x}^2+\psi_{/y}^2)dxd\tau-\tfrac{1}{2}\gamma\int_0^{2\pi}X^2(y)d\tau.$$

Its first variation is

$$\delta E^*=\int_0^{2\pi}\int_{-00}^{X_1}(\psi_{/x}\delta\psi_{/x}+\psi_{/y}\delta\psi_{/y})dxd\tau -\int_0^{2\pi}\{[\tfrac{1}{2}(\psi_{/x}^2+\psi_{/y}^2)]^{x=X(y)}+\gamma X\}\delta X d\tau,$$

taken with functions $\delta\psi(x,y)$ and $\delta X(y)$ which satisfy the condition

$$\delta\psi+\psi_{/x}\delta X=0 \text{ on } x=X(y), \quad \delta\psi=0 \text{ on } x=X_1,$$

called "admissible." It vanishes for the solution $\{\psi(x,y),X(y)\}$. The second variation of E^* can be shown to equal

$$\delta^2E^*=\int_0^{2\pi}\int_{-00}^{X_1}\{(\delta\psi_{/x}-\psi_{/x}{}^{-1}\psi_{/xx}\delta\psi)^2$$

$$+(\delta\psi_{/y}-\psi_{/x}{}^{-1}\psi_{/xx}\delta\psi)^2dxd\tau-\gamma\int_0^{2\pi}(\delta X)^2d\tau$$

$$=\int_0^{2\pi}\int_{-00}^{X_1}(\delta\psi_{/x}{}^2+\delta\psi_{/y}{}^2)dxd\tau$$

$$+\int_0^{2\pi}\{[\tfrac{1}{2}(\psi_{/x}{}^2+\psi_{/y}{}^2)/x]^{x=X(y)}-\gamma\}(\delta X)^2d\tau.$$

It is annihilated by a solution $\{\delta\psi,\delta X\}$ of the variational equations

$$\Delta\delta\psi=0, \quad \delta\psi+\psi_x\delta_/X=0 \text{ on } x=X(y), \quad \delta\psi=0 \text{ on } x=X_1,$$
$$[\psi_{/x}\delta\psi_{/x}+\psi_{/y}\delta\psi_{/y}+\tfrac{1}{2}(\psi_{/x}{}^2+\psi_{/y}{}^2)_{/x}\delta X]^{x=X(y)}=\gamma\delta X.$$

Suppose E^* were a potential energy in the sense of Mechanics, so that stability of the equilibrium state could be characterized by the condition $\delta^2E^*>0$ for all admissible $\{\delta\psi\cdot\delta X\}=0$. Then instability could be established by exhibiting a particular admissible $\{\delta\psi,\delta X\}\neq 0$ for which $\delta^2E^*<0$.

Later on we show that the same conclusion can be drawn for our fluid-magnetic problem. At first, however, we exhibit such a pair $\{\delta\psi,\delta X\}$ for the buckled equilibrium states determined by our bifurcation analysis.

To this end we consider the solutions $\psi(x,y)$ and $X(y)$ as functions of the quantity γ. We select a value $\gamma<\gamma_0$ for which such solutions are established and set

$$\delta\psi=\delta\gamma\partial\psi/\partial\gamma, \quad \delta X=\delta\gamma\partial X/\partial\gamma$$

with arbitrary $\delta\gamma>0$. These functions evidently satisfy the equations

$$\Delta\delta\psi=0, \quad \delta\psi+\psi_{/x}\delta X=0 \text{ on } \mathcal{S}, \quad \delta\psi=0 \text{ on } \mathcal{S}_1,$$
$$[\psi_{/x}\delta\psi_{/x}+\psi_{/y}\delta\psi_{/y}]+[\tfrac{1}{2}(\psi_{/x}{}^2+\psi_{/y}{}^2)_{/x}]-\gamma\delta X=X\delta\gamma \text{ on } \mathcal{S}.$$

These equations differ only in the last term from the "variational equations" mentioned previously. The first two equations ensure that this pair $\{\delta\psi,\delta X\}$ is admissible. Insertion in δ^2E^* and integration by parts yields

$$\delta^2E^*=\delta\gamma\int_0^{2\pi}X\delta Xd\tau=\tfrac{1}{2}\delta\gamma\,\delta\int_0^{2\pi}X^2d\tau.$$

Now, for our solutions $\{\delta\psi,\delta X\}$ the quantity

$$\delta^2=\frac{1}{\pi}\int_0^{2\pi}X^2(y)d\tau$$

is related to the "amplitude" δ_1 by

$$\bar{\delta}^2=\frac{1}{\pi}\int_0^{2\pi}(\delta X)^2d\tau=\frac{1}{\pi}\int_0^{2\pi}\{\delta_1\cos\tau+\delta_2\cos2\tau\}^2d\tau$$
$$=(\delta_1{}^2+\delta_2{}^2)=\delta_1{}^2(1+\tfrac{1}{4}\kappa^2k^2\delta_1{}^2)$$

in second order. From our expression for γ in second order we have

$$d\gamma/d(\delta_1{}^2)=-\tfrac{1}{2}\gamma_0(1+\kappa^2)k^2<0.$$

Consequently, $d\gamma/d\bar{\delta}^2<0$, as long as our second-order approximation is valid, so that

$$\delta^2E^*<0$$

for the solution $\{\psi,X\}$ and the special pair $\{\delta\psi,\delta X\}$ selected.

The fact that the second variation can be made negative would imply instability if the variations $\{\delta\psi,\delta X\}$ entering it were such that they could be attained by a motion of the system from the undisturbed one under the influence of additional forces. This requirement imposes additional constraints on the possible variation δB of the magnetic field. Such constraints were formulated by Bernstein, Frieman, Kruskal, and Kulsrud.[8] The time rate of change $\dot{B}$ of the magnetic field is given by $\dot{B}=\nabla\times E$ or, by virtue of infinite conductivity, by $\dot{B}=\nabla\times[u\times B]$ where u is the flow velocity. For the variations δB and δu of B and u at the undisturbed situation, where $u=0$, we have $\delta\dot{B}=\nabla\times[\delta u\times B]$. Bernstein *et al.* introduce a vector $\delta\xi=\delta\xi(t)$ such that $\dot{\xi}=u$ and then satisfy the last equation by requiring

$$\delta B\nabla\times[\delta\xi\times B].$$

The vector $\delta\xi$ at the interface $\mathcal{S}$ describes its displacement; also $\delta\xi=0$ at $\mathcal{S}_1$. Furthermore, $\nabla\cdot\delta\xi$ is required in accordance with $\nabla\cdot u=0$.

Now the constraint imposed on δB in the plasma is that this quantity can be derived by the preceding formula from a vector $\delta\xi$, as a function of space and no longer of time, with $\nabla\cdot\xi=0$, $\xi=0$ on $\mathcal{S}_1$, and at $\mathcal{S}$ describing the displacement of this surface. The condition that the disturbed field be tangential at the disturbed interface is then automatically satisfied. Bernstein *et al.* now show that stability is equivalent with the condition that the second variation of the

[8] I. Bernstein, E. Frieman, M. Kruskal, and R. Kulsrud, Project Matterhorn Rept. No. S-25; Proc. Roy. Soc. (London) **A244**, 17 (1958); see also papers by J. Berkowitz *et al.*, H. Grad and H. Rubin, M. Kruskal, M. Rosenbluth, B. R. Suydam, R. J. Tayler, and others, in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, New York, 1958), Vol. 31.

energy integral be positive definite for disturbing fields δB which in the plasma obey the constraints as now described. In the vacuum there are no additional constraints.

In our simplified problem of small wavelength we have $\delta B=\{-\delta\psi_{/y},\delta\psi_{/x}\}$, so that the constraint condition reduces to

$$\delta\psi=B_x\delta\xi_y-B_y\delta\xi_x=-(\psi_{/x}\delta\xi_x+\psi_{/y}\delta\xi_y).$$

At the interface $\mathcal{S}$ where $\psi_{/y}=-\psi_{/x}X_{/y}$, this condition reduces to

$$\delta\psi=-\psi_{/x}\delta X \quad \text{with} \quad \delta X=\delta\xi_x-X_{/y}\delta\xi_y.$$

A pair of functions $\delta\xi_x$, $\delta\xi_y$, satisfying $\delta\xi_{x/x}+\delta\xi_{y/y}=0$ and the constraint conditions, can easily be given. We introduce the potential function $\phi(x,y)$ in addition to the "stream function" $\psi(x,y)$ for each solution, so that

$$B_x=-\psi_{/y}=-\phi_{/x}, \quad B_y=\psi_{/x}=-\phi_{/y}$$

and consider x and y as functions of ϕ, ψ, and γ. The derivatives of these functions with respect to the parameter γ, or rather the expressions

$$\delta\xi_x=\delta\gamma x_{/\gamma}, \quad \delta\xi_y=\delta\gamma y_{/\gamma},$$

have the desired properties. Evidently, $\delta\xi_x-i\delta\xi_y$ is an analytic function of $\phi+i\psi$ and hence of $x+iy$; therefore $\delta\xi_{x/x}+\delta\xi_{y/y}=0$ holds. Since the interface $\mathcal{S}$ is described by x and y as functions of ϕ with $\psi=0$, we have

$$X[y(\phi,0)]=X(\phi,0) \text{ and hence } \delta X+X_{/y}\delta\xi_y=\delta\xi_x \text{ on } \mathcal{S}.$$

From $\psi[x(\phi,\psi), y(\phi,\psi)]=\psi$ we finally derive $\delta\psi+\psi_{/x}\delta\xi_x+\psi_{/y}\delta\xi_y=0$.

Thus we see that our choice $\delta\psi=\delta\gamma\delta\psi/\delta\gamma$ automatically satisfies the constraint to be imposed. *Our conclusion as to the instability of the buckled states therefore is valid.*

DISCUSSION

Session Reporter: Y. Nakagawa

S. A. Colgate, *Lawrence Radiation Laboratory, University of California, Livermore, California:* In support of this extremely interesting paper, I would like to comment that we have observed axially symmetric, helical deformations of the B_z stablized pinch which may correspond to these continuously connected stable solutions. These observed helical deformations are reversible, in the sense that the helix winds or unwinds itself in a continuous fashion depending upon the external constraints.

R. J. Tayler, *Atomic Energy Research Establishment, Harwell, Berkshire, England:* There is one question connected with comparisons of theory and experiment. If a stable equilibrium (of the type you have considered) can exist, then it is a stable perturbed state; however, in many experiments what is observed is not a perturbed equilibrium but steady motions of a perturbed plasma. You did not consider the possibility of such an equilibrium state, namely, with steadily moving helical perturbations.

H. Grad, *Institute of Mathematical Sciences, New York University, New York, New York:* There is one trivial motion which can be included in this treatment, that is a translation along the axis. Then the present study can be regarded as including moving helical perturbations.

W. B. Thompson, *Atomic Energy Research Establishment, Harwell, Berkshire, England:* One can look at this problem in the other limit, as $kR_0\to 0$. Then, one might believe that internal motion and changes of cross section of the current channel are unimportant, and the problem resembles that of an extensible wire. Here one finds a more or less stable helical configuration with a long wavelength. The apparent stability probably disappears if disturbances which change the cross section are considered.

H. Grad: This problem of the longer wavelengths can be treated by more rigorous mathematical approaches using Bessel functions, but this has not yet been completed. As indicated in the paper, the helical equilibria (which are only slightly perturbed from a cyclinder) are all unstable even for small kR_0.

J. L. Neuringer, *Republic Aviation Corporation, Farmingdale, Long Island, New York:* Is this treatment a small perturbation theory or does the plasma actually become unstable in a physical sense?

H. Grad: This is a linear stability theory applied to a nonlinear equilibrium theory. There are known examples in which apparent transition from stability to instability as a parameter is varied implies only a transfer of stability from a sinple equilibrium to a more complicated one. Thus the nonlinear equilibrium study increases the applicability of the linear stability analysis, but it does not refer to finite amplitude stability.