Wave Motions of Small Amplitude in a Fully Ionized Plasma without External Magnetic Field*

S. I. PAI

Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland

I. INTRODUCTION

WAVE propagation through an ionized gas is of great interest in many problems such as space communication and astrophysical phenomena. Previous analyses of this problem have been carried out either from a microscopic point of view by means of Boltzmann equations,[1–4] or from a macroscopic point of view by means of transport equations.[5–9] Even though the treatment with the aid of Boltzmann equations is more fundamental and detailed, its mathematical difficulty is very great, hence one has to limit onself to rather simple cases. For instance, it is sometimes proposed to neglect the motion of the heavy particles in comparison with that of the electrons,[3] or to neglect the collisions between the particles.[4] In the macroscopic treatment, most of the analyses are based on a single-fluid theory.[5–7] Hence the effects of plasma oscillations have dropped out at the beginning. In order to find out the interaction between the sound waves and the plasma oscillations, one has to use the multifluid theory in the macroscopic treatment. Some approximate two-fluid theories for this problem have been reported,[8,9] but these treatments did not use the complete system of equations for all fluid dynamic variables. For instance, the pressure and the temperature of both ions and electrons were not used as dependent variables. This entailed that no use was made of the equations of energy and of state in the analysis and the relations between the pressure gradient and the density changes are approximated by some simple expression. Here we intend to use a complete two-fluid theory to investigate the infinitesimal wave motion in a fully ionized plasma, so that all the fluid dynamic variables, i.e., velocity vector, pressure, temperature, and density, for both ions and electrons are considered as unknowns in this treatment.

We consider the plasma as a mixture of singly charged ions and electrons. We assume that originally the plasma is at rest with a pressure $p=2p_0$, temperature T_0, and number density $2\nu_0$. Hence the original partial pressure for ions is p_0 and that for electrons is also p_0. The original number densities for ions and for electrons are both ν_0. We assume that there is no externally applied electromagnetic field. Hence there is no electric current nor an excess electric charge in the unperturbed state of the field. The plasma is perturbed by a small disturbance, so that in the resultant motion we have

$$
\begin{gathered}
u_1=u_1(x,t), \quad v_1=v_1(x,t), \quad w_1=w_1(x,t), \\
u_2=u_2(x,t), \quad v_2=v_2(x,t), \quad w_2=w_2(x,t), \\
p_1=p_0+p_1'(x,t), \quad T_1=T_0+T_1'(x,t), \quad \nu_1=\nu_0+\nu_1'(x,t), \\
p_2=p_0+p_2'(x,t), \quad T_2=T_0+T_2'(x,t), \quad \nu_2=\nu_0+\nu_2'(x,t), \\
\mathbf{E}=\mathbf{E}(x,t), \quad \mathbf{H}=\mathbf{h}(x,t), \qquad (1)
\end{gathered}
$$

where subscript 1 refers to the values for ions and subscript 2 refer to the values for electrons. u, v, and w are the x-, y-, and z-velocity components, respectively; p' is the perturbed partial pressure; T', the perturbed partial temperature; ν', the perturbed partial number density; $\mathbf{E}=iE_x+jE_y+kE_z$, the perturbed electric field and $\mathbf{h}=ih_x+jh_y+kh_z$, the perturbed magnetic field. (i, j, k are the x-, y-, and z-wise unit vectors, respectively). We assume that all the perturbed quantities are small, so that second- and higher-order terms in these quantities are negligible. For simplicity we assume that all perturbed quantities are functions of one space coordinate x and the time t only. Thus we discuss only wave propagation in the direction of the x axis. It is a straightforward process to generalize our results to the three-dimensional case in which the perturbed quantities are functions of x, y, z, and t.

II. LINEARIZED FUNDAMENTAL EQUATIONS

We make the following assumptions in one fundamental dynamical equation:

(i) Both the ions and the electrons may be considered as an inviscid and nonheat-conducting gas.

(ii) Perfect gas laws may be applied to both ions and electrons in the plasma.

(iii) The interaction forces between ions and electrons are proportional to the difference between their mean flow velocities

$$\mathbf{F}_{12}=\alpha_{12}(\mathbf{u}_1-\mathbf{u}_2)=-\mathbf{F}_{21}, \qquad (2)$$

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where $\alpha_{12}=\alpha_{21}$ is the friction coefficient. In first approximation the relation between the electrical conductivity σ and the friction coefficient α_{12} is[10]

$$\sigma=e^2\nu_{02}{}^2/\alpha_{12}. \tag{3}$$

Since we have 18 perturbed quantities, we have also 18 fundamental equations, which consists of six electromagnetic field equations, six gas-dynamic equations for the ions, and six gas-dynamic equations for the electrons. On neglecting the higher-order terms in the perturbed quantities, we obtain the following linear equation in mks units for our problem.

(a) Maxwells' equations for the electromagnetic field:

$$\partial\mu_e h_x/\partial t=0, \tag{4}$$

$$\partial\mu_e h_y/\partial t=\partial E_z/\partial x, \tag{5}$$

$$\partial\mu_e h_z/\partial t=-\partial E_y/\partial x, \tag{6}$$

$$(\partial\epsilon E_x/\partial t)+e\nu_0(u_1-u_2)=0, \tag{7}$$

$$(\partial\epsilon E_y/\partial t)+e\nu_0(v_1-v_2)=-\partial h_z/\partial x, \tag{8}$$

$$(\partial\epsilon E_z/\partial t)+e\nu_0(w_1-w_2)=\partial h_y/\partial x, \tag{9}$$

where μ_e is the magnetic permeability, ϵ is the inductive capacity and e is the absolute electric charge (the charge of an ion is e while that of an election is $-e$).

(b) The equations of state for each species in the plasma are, respectively,

$$p_1'/p_0=\nu_1'/\nu_0+T_1'/T_0, \tag{10}$$

$$p_2'/p_0=\nu_2'/\nu_0+T_2'/T_0. \tag{11}$$

(c) The equation for the conservation of mass for each species in the plasma are, respectively,

$$\partial\nu_1'/\partial t+\nu_0\partial u_1/\partial x=0, \tag{12}$$

$$\partial\nu_2/\partial t+\nu_0\partial u_2'/\partial x=0. \tag{13}$$

Here we assume that there is no mass source for ions or electrons.

When Eq. (7) is differentiated with respect to x, and use is made of Eqs. (12) and (13), we arrive at

$$(\partial/\partial t)[(\partial\epsilon E_x/\partial x)-e(\nu_1'-\nu_2')]=0, \tag{7a}$$

which is Poisson's equation for the relation between electric field and charge density.

(d) The equations of motion for each species in the plasma are

$$m_1\nu_0\partial u_1/\partial t=-(\partial p_1'/\partial x)+e\nu_0E_x+\alpha_{12}(u_1-u_2), \tag{14}$$

$$m_1\nu_0\partial v_1/\partial t=e\nu_0E_y+\alpha_{12}(v_1-v_2), \tag{15}$$

$$m_1\nu_0\partial w_1/\partial t=e\nu_0E_z+\alpha_{12}(w_1-w_2), \tag{16}$$

$$m_2\nu_0\partial u_2/\partial t=-(\partial p_2'/\partial x)-e\nu_0E_x+\alpha_{12}(u_2-u_1), \tag{17}$$

$$m_2\nu_0\partial v_2/\partial t=-e\nu_0E_y+\alpha_{12}(v_2-v_1), \tag{18}$$

$$m_2\nu_0\partial w_2/\partial t=-e\nu_0E_z+\alpha_{12}(w_2-w_1), \tag{19}$$

where m_1 is the mass of an ion, while m_2 is the mass of an electron. We have $m_1\gg m_2$.

(e) The energy equations for each species in the plasma are

$$m_1\nu_0C_{p1}\partial T_1'/\partial t=\partial p_1'/\partial t, \tag{20}$$

$$m_2\nu_0C_{p2}\partial T_2'/\partial t=\partial p_2'/\partial t, \tag{21}$$

where $C_{p1}=\frac{5}{2}(k/m_1)$, and $C_{p2}=\frac{5}{2}(k/m_2)$. C_p is the specific heat at constant pressure per unit mass for each species and k is the Boltzmann constant. Because of the great difference of masses between ions and electrons, there are no first-order term of energy exchange between these two species.[11]

On examining the fundamental equations (4) to (21), we see that the equations and the variables may be separated into four independent groups as follows.

(i) The quantity h_x is given by Eq. (4) alone, and is independent of all other quantities. Furthermore, since the divergence of h is zero, we conclude that h_x=constant, which may be put equal to zero.

(ii) The second group consists of v_1, v_2, E_y, and h_z, which are governed by Eqs. (6), (8), (15), and (18).

(iii) The third group consist of w_1, w_2, E_z, and h_y, which are governed by Eqs. (5), (9), (16), and (19).

Both the second and the third groups may be regarded as describing transverse waves, as they deal with components of the velocity and of the electromagnetic field perpendicular to the direction of wave propagation. The determinantal equations for these two groups of variables are identical. Hence, the modes of propagation for these transverse waves are identical, and as far as wave propagation is concerned, there is no distinction between these two groups.

(iv) The last group consists of u_1, u_2, p_1', p_2', ν_1', ν_2', T_1', T_2', and E_x which characterize a longitudinal wave. In the absence of an external magnetic field, the longitudinal wave does not involve magnetic force.

We are looking for periodic solutions in which all the perturbed quantities are proportional to

$$\begin{aligned}\exp i(\omega t-\lambda x)&=\exp i(\omega t-\lambda_R x)\exp(\lambda_i x)\\&=\exp[-i\lambda_R(x-Vt)]\exp(\lambda_i x),\end{aligned} \tag{22}$$

where ω is a given real frequency, λ is the wave number which may be complex, i.e., $\lambda=\lambda_R+i\lambda_i$ $[i=(-1)^{\frac{1}{2}}]$. The velocity of wave propagation is $V=\omega/\lambda_R$. On substituting these variables into the fundamental equations for the perturbed quantities, we obtain one determinantal equation for each group of variables. The eigenvalues λ of these determinantal equations give the different modes of wave propagation through the plasma.

In analyzing the determinantal equations, the fol-

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lowing two characteristic frequencies play important roles:

$$\omega_{p1}=\omega_i=e[\nu_0/\epsilon m_1]^{\frac{1}{2}}=\text{ion plasma frequency}, \tag{23}$$

$$\omega_{p2}=\omega_e=e(\nu_0/\epsilon m_2)^{\frac{1}{2}}=\text{electron plasma frequency}. \tag{24}$$

Because $m_1 \gg m_2$, we use the approximations

$$e[\nu_0/\epsilon(m_1+m_2)]^{\frac{1}{2}}\cong e(\nu_0/\epsilon m_1)^{\frac{1}{2}}=\omega_i \tag{23a}$$

and

$$e[\nu_0(m_1+m_2)/\epsilon m_1 m_2]^{\frac{1}{2}}\cong e(\nu_0/\epsilon m_2)^{\frac{1}{2}}=\omega_e \tag{24a}$$

in evaluating the value λ in our determinantal equations.

III. TRANSVERSE WAVE

As mentioned before, the determinantal equations for the second and the third group are identical. They give the following relation for transverse waves:

$$c^2\lambda^2=\omega^2-\omega_e^2[(1-i\alpha_{12}^*)/(1+\alpha_{12}^{*2})], \tag{25}$$

where $c=(\epsilon\mu_e)^{-\frac{1}{2}}=$velocity of light;

$$\alpha_{12}^*=\alpha_{12}/m_2\omega\nu_0. \tag{26}$$

The friction coefficient α_{12} is closely related to the electrical conductivity σ as given in Eq. (3). Hence the case $\alpha_{12}=0$ corresponds to infinite conductivity $\sigma=\infty$. For this case, $\alpha_{12}^*=0$, Eq. (25) reduces to

$$\lambda^2=(\omega^2/c^2)[1-(\omega_e^2/\omega^2)], \tag{27}$$

or

$$V=\omega/\lambda=c/[1-(\omega_e^2/\omega^2)]^{\frac{1}{2}}. \tag{27a}$$

Equations (27) and (27a) give the well-known result of the elementary theory of plasma oscillations.[12] Disturbances of frequency ω larger than ω_e propagate with the speed V of Eq. (27a). If the frequency ω of the disturbance is smaller than ω_e, λ is pure imaginary. Equation (22) gives

$$Q\sim\exp(i\omega t)\exp(\lambda_i x), \tag{28}$$

where Q is any one of the perturbed quantities. Hence the perturbed quantity Q is an exponential function of x. From physical consideration, we should take λ_i as a negative quantity. Hence the wave does not propagate in space but its amplitude decreases as the distance from the center of disturbance ($x=0$) increases. The distance through which the amplitude of the wave will decrease by a factor $1/e$ (where e is the base of the natural logarithms) is

$$x_d=\frac{1}{|\lambda_i|}=\frac{c}{\omega_e}\frac{1}{[1-(\omega^2/\omega_e^2)]^{\frac{1}{2}}}. \tag{29}$$

For finite electrical conductivity, $\alpha_{12}\neq 0$, the value of λ given by Eq. (25) is complex. Thus we have damped waves because both λ_R and λ_i are different from zero.

[12] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), Chap. 4.

Equation (25) may be rewritten as follows:

$$\lambda^2=(\lambda_R+i\lambda_i)^2=\frac{\omega^2}{c^2}\left[1-\frac{\omega_e^2}{\omega^2(1+\alpha_{12}^{*2})}\right]+i\frac{\omega_e^2}{c^2}\frac{\alpha_{12}^*}{(1+\alpha_{12}^{*2})}$$
$$=K_1+iK_2,\quad\text{say}, \tag{30}$$

where λ_R and λ_i are real numbers. K_1 may be a positive number or a negative number but K_2 is always a positive number. From Eq. (30), we have

$$\lambda_R^2-\lambda_i^2=K_1,\quad 2\lambda_R\lambda_i=K_2. \tag{31}$$

Hence, we obtain

$$\lambda_R^2=\tfrac{1}{2}[K_1+(K_1^2+K_2^2)^{\frac{1}{2}}]$$

and

$$\lambda_i^2=\tfrac{1}{2}[-K_1+(K_1^2+K_2^2)^{\frac{1}{2}}]. \tag{32}$$

The positive sign before the radical is taken because both λ_R^2 and λ_i^2 must be a positive number.

When $\alpha_{12}\to 0$, $K_2\to 0$, Eq. (32) then gives the result of Eq. (27). In other words, in the case $K_1>0(\omega>\omega_e)$, we obtain $\lambda_R^2=K_1$, $\lambda_i=0$; while if $K_1<0(\omega_e>\omega)$, we find $\lambda_R=0$, $\lambda_i^2=|K_1|$.

For large but finite electrical conductivity, $K_2\ll K_1$, we have

$$\begin{aligned}\lambda_R^2&\cong\tfrac{1}{2}(K_1+|K_1|)+\tfrac{1}{4}(K_2^2/|K_1|),\\ \lambda_i^2&=\tfrac{1}{2}(-K_1+|K_1|)+\tfrac{1}{4}(K_2^2/|K_1|).\end{aligned} \tag{33}$$

If $\omega>\omega_e$ such that $K_1>0$, Eq. (33) gives

$$\lambda_R^2\cong K_1,\quad \lambda_i^2=\tfrac{1}{4}(K_2^2/|K_1|). \tag{34}$$

This is a damped transverse wave with a speed of propagation the same as that of undamped waves $[V=\omega/(K_1)^{\frac{1}{2}}]$ but with a damping factor λ_i

$$\lambda_i=\frac{1}{2}\frac{K_2}{(K_1)^{\frac{1}{2}}}=\frac{1}{2}\frac{\omega_e^2}{\omega^2}\frac{\alpha_{12}}{cm_2\nu_0}\frac{1}{[1-(\omega_e^2/\omega^2)+\alpha_{12}^{*2}]^{\frac{1}{2}}}. \tag{35}$$

The damping factor increases as α_{12} increases; and decreases as ω^2 increases.

If $\omega<\omega_e$, so that $K_1<0$, Eq. (33) gives

$$\lambda_R^2=\tfrac{1}{4}(K_2^2/|K_1|),\quad \lambda_i^2\cong K_1. \tag{36}$$

Hence, the wave number λ_R becomes very small. This leads to a very large wavelength which may be interpreted as a very large speed of propagation, but the damping factor λ_i is also very large, and is about the same as that of the exponential function for the case of infinite conductivity. We thus have highly damped waves.

In conclusion, for finite electrical conductivity we always have damped transverse waves. It should be noticed that for very large speeds of propagation ($V=\omega/\lambda_R$) we have very small values of λ_R. This means that the dependence of the sinusoidal part of the disturbance on x is very small [i.e., $\exp i(\omega t-\lambda_R x)$]. We may also interpret this situation by saying that the wave *does not* propagate in space, as $\lambda_R\to 0$.

IV. LONGITUDINAL WAVE

The determinantal equation for the longitudinal wave is as follows:

$$\lambda^4-\frac{\lambda^2\omega^2}{a_1^2}\left\{\left(1+2i\alpha_{12}^*\frac{m_2}{m_1}\right)-2\frac{\omega_i^2}{\omega^2}\right\}+\frac{\omega^4}{a_1^2a_2^2}\left\{(1+i\alpha_{12}^*)-\frac{\omega_e^2}{\omega^2}\right\}=0, \quad (37)$$

where

$$\begin{aligned} a_1&=(\gamma p_0/m_1\nu_0)^{\frac{1}{2}}=\text{ion sound speed},\\ a_2&=(\gamma p_0/m_2\nu_0)^{\frac{1}{2}}=\text{electron sound speed}, \quad (38)\\ a_p&=(2\gamma p_0/m_1\nu_0)^{\frac{1}{2}}=\text{plasma sound speed}=a_1\sqrt{2}. \end{aligned}$$

In these expressions $\gamma=5/3$.

A limiting case may be considered first, in which both the ion plasma frequency ω_i and the electron plasma frequency ω_e are much larger than ω. Equation (37) becomes

$$\lambda^4+2\lambda^2(\omega_i^2/a_1^2)-(\omega^2\cdot\omega_e^2/a_1^2a_2^2)=0. \quad (39)$$

The two roots of λ^2 of Eq. (39) are

$$\lambda_1^2=\omega^2/2a_1^2=\omega^2/a_p^2 \quad (40a)$$

and

$$\lambda_2^2=-(2\omega_i^2/a_1^2). \quad (40b)$$

The first root λ_1 gives simple sound waves for a plasma considered as a single fluid. The second root λ_2 gives an "exponential wave," in which the amplitude of the disturbance decreases from the disturbance center ($x=0$) exponentially [i.e., as $\exp-|\lambda_i|x$]. In general these two modes depend on the relative values of plasma frequencies ω_i and ω_e and the applied frequency ω.

V. LONGITUDINAL WAVES IN AN IDEAL PLASMA

For an ideal plasma, the electrical conductivity is infinite, i.e., $\alpha_{12}=0$. For this case the two solutions of λ^2 obtained from Eq. (37) are as follows:

$$\lambda^2=\frac{\omega^2}{2a_1^2}\left\{\left(1-2\frac{\omega_i^2}{\omega^2}\right)\pm\left[\left(1-2\frac{\omega_i^2}{\omega^2}\right)^2-4\frac{a_1^2}{a_2^2}\left(1-\frac{\omega_e^2}{\omega^2}\right)\right]^{\frac{1}{2}}\right\}. \quad (41)$$

The nature of these two values of λ^2 depends on the ratios of the two plasma frequencies to the applied frequency ω, i.e., ω_i/ω and ω_e/ω.

(i) If $\omega_i/\omega\gg1$, Eq. (41) may be written as

$$\lambda^2=\frac{\omega_i^2}{a_1^2}\left[-1+\frac{\omega^2}{2\omega_i^2}\pm\left(1+\frac{\omega^4}{4\omega_i^4}-\frac{m_2}{m_1}\frac{\omega^4}{\omega_i^4}\right)^{\frac{1}{2}}\right]. \quad (42)$$

As $\omega/\omega_i\rightarrow0$, the two roots of Eq. (42) reduce to those given by Eq. (40), i.e.,

$$\lambda_1^2=\omega^2/2a_1^2=\omega^2/a_p^2,\quad \lambda_2^2=-2\omega_i^2/a_1^2. \quad (40)$$

(ii) If $\omega_i/\omega\cong1$, we have

$$\lambda_1^2=(\omega^2/a_p^2)(5^{\frac{1}{2}}-1),\quad \lambda_2^2=-(\omega^2/a_p^2)(5^{\frac{1}{2}}+1). \quad (43)$$

The first root λ_1 gives an undamped wave with speed of propagation $a_p/(5^{\frac{1}{2}}-1)^{\frac{1}{2}}$ which is smaller than the sound speed of the plasma a_p. The second root λ_2 gives a damped exponential wave with a damping factor λ_2, which is smaller than the corresponding factor for the case of low frequencies.

(iii) If $\omega_e/\omega>1$, one root of λ^2 is always positive, while the other λ^2 is always negative, i.e.,

$$\lambda_1^2=\frac{\omega^2}{2a_1^2}\left\{\left(1-2\frac{\omega_i^2}{\omega^2}\right)+\left[\left(1-2\frac{\omega_i^2}{\omega^2}\right)^2+4\frac{a_1^2}{a_2^2}\left(\frac{\omega_e^2}{\omega^2}-1\right)\right]^{\frac{1}{2}}\right\}$$

$$=\text{positive quantity}, \quad (44a)$$

$$\lambda_2^2=\frac{\omega^2}{2a_1^2}\left\{\left(1-2\frac{\omega_i^2}{\omega^2}\right)-\left[\left(1-2\frac{\omega_i^2}{\omega^2}\right)^2+4\frac{a_1^2}{a_2^2}\left(\frac{\omega_e^2}{\omega^2}-1\right)\right]^{\frac{1}{2}}\right\}$$

$$=\text{negative quantity}. \quad (44b)$$

Hence when $\omega_e/\omega>1$, one mode λ_1 gives an undamped wave propagated with a finite speed, while the other mode λ_2 gives an exponential wave in which the amplitude of the wave decreases exponentially as the distance from the center of disturbance increases.

(iv) If $\omega_e/\omega=1$, Eq. (44) becomes

$$\lambda_1^2=(\omega^2/a_1^2)[1-(\omega_i^2/\omega^2)], \quad (45a)$$

$$\lambda_2^2=0. \quad (45b)$$

The first mode retains a similar behavior as in the case $\omega_e/\omega>1$, but the second mode shows motion which is independent of the space coordinate x. We may interpret it as a standing oscillation or as an undamped wave with infinite speed of propagation. This is the transition case where the second mode changes from a purely exponential damped wave to an undamped wave propagated with finite speed.

(v) If $\omega_e/\omega<1$, both λ_1^2 and λ_2^2 are positive, i.e.,

$$\lambda_1^2=\frac{\omega^2}{2a_1^2}\left\{\left(1-2\frac{\omega_i^2}{\omega^2}\right)+\left[\left(1-2\frac{\omega_i^2}{\omega^2}\right)^2-\frac{4a_1^2}{a_2^2}\left(1-\frac{\omega_e^2}{\omega^2}\right)\right]^{\frac{1}{2}}\right\}$$

$$=\text{positive quantity}, \quad (46a)$$

$$\lambda_2^2=\frac{\omega^2}{2a_1^2}\left\{\left(1-2\frac{\omega_i^2}{\omega^2}\right)-\left[\left(1-2\frac{\omega_i^2}{\omega^2}\right)^2-4\frac{a_1^2}{a_2^2}\left(1-\frac{\omega_e^2}{\omega^2}\right)\right]^{\frac{1}{2}}\right\}$$

$$=\text{positive quantity}. \quad (46b)$$

Now both modes λ_1 and λ_2 represent undamped waves traveling at finite speeds.

(vi) Finally, if $\omega\gg\omega_e$, the limiting values of Eqs. (46a) and (46b) are, respectively $\omega/\omega_e\rightarrow\infty$

$$\lambda_1^2=\omega^2/a_1^2 \quad\text{or}\quad V_1=\omega/\lambda_1=a_1, \quad (47a)$$

$$\lambda_2^2=\omega^2/a_2^2 \quad\text{or}\quad V_2=\omega/\lambda_2=a_2. \quad (47b)$$

The first mode represents a sound wave of the ions as if they exist alone in the plasma, while the second mode represents a sound wave of electrons alone.

In conclusion, the behavior of the longitudinal waves in a plasma of infinite conductivity is as follows

There are two modes of the longitudinal wave. The first mode is essentially due to ions, while the second mode is mainly due to electrons. For finite values of the frequency, there are interactions between the plasma frequencies and the sound speeds of the ions and electrons.

The first mode λ_1 is always an undamped wave propagated with a finite speed. At very low frequencies, its speed of propagation is equal to the sound speed of the plasma as a whole. As the frequency ω increases the speed of propagation decreases continuously, until it reaches the limiting value of the ion sound speed, which is $1/\sqrt{2}$ times the plasma sound speed. Figure 1 shows the variation of the speed of propagation of this first mode of the longitudinal waves in terms of the ratio of ω/ω_i.

As long as the applied frequency ω is less than the electron plasma frequency ω_e, the second mode is a damped exponential wave. The damping factor λ_i decreases as ω increases toward ω_e. At $\omega=\omega_e$, $\lambda=0$. Hence the wave is independent of the spatial coordinate x and we may say that the speed of propagation is infinite. At $\omega=\omega_e$, this mode changes from a damped exponential wave to an undamped wave in space. When $\omega/\omega_e>1$, the second mode is an undamped wave propagated at a finite speed V_2. This speed of propagation V_2 decreases continuously as ω increases toward a limiting value of the electron sound speed, as ω tends to infinity. Figure 2 shows the variation of λ_i, λ_R, and V_2 with respect to ω/ω_e for this second mode of longitudinal wave.

VI. DAMPED LONGITUDINAL WAVES

For finite electrical conductivity, $\alpha_{12}\neq 0$, the longitudinal wave is always damped. The two roots of λ^2

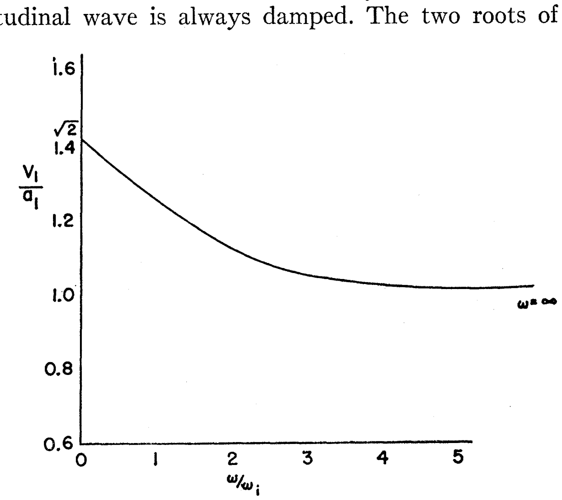

FIG. 1. Variation of speed of propagation of the first mode (ion) of the longitudinal wave in ideal plasma.

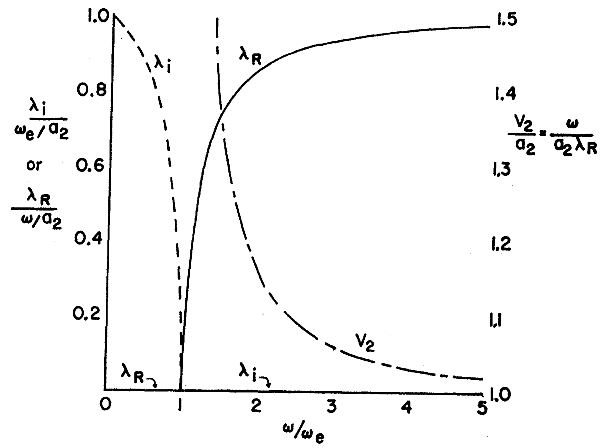

FIG. 2. Wavlength λ_R, speed of propagation V_2, and damping factor λ_i of the second mode (electron) of the longitudinal wave in an ideal plasma.

of Eq. (37) are

$$\lambda^2=\frac{\omega^2}{2a_1^2}\Bigg(\left[\left(1-2\frac{\omega_i^2}{\omega^2}\right)+2i\alpha_{12}^*\frac{m_2}{m_1}\right] \pm\left\{\left[\left(1-2\frac{\omega_i^2}{\omega^2}\right)+2i\alpha_{12}^*\frac{m_2}{m_1}\right]^2 -4\frac{a_1^2}{a_2^2}\left(1-\frac{\omega_e^2}{\omega^2}\right)+i\alpha_{12}^*\right\}^{\frac{1}{2}}\Bigg). \quad (48)$$

If $\alpha_{12}=0$, Eq. (48) reduces to Eq. (41). In order to show the first-order effect due to finite conductivity, we consider the case of large but finite electrical conductivity so that α_{12}^* is a very small quantity. We may neglect the higher-order terms of ω_{12}^* in Eq. (48) and obtain the following equation for λ^2:

$$\lambda^2=\frac{\omega^2}{2a_1^2}\left\{\left(1-2\frac{\omega_i^2}{\omega^2}\right)\pm\left[\left(1-2\frac{\omega_i^2}{\omega^2}\right)^2-4\frac{a_1^2}{a_2^2}\left(1-\frac{\omega_e^2}{\omega^2}\right)\right]^{\frac{1}{2}} +2i\alpha_{12}^*\frac{m_2}{m_1}\left(1\mp 2\frac{\omega_i^2}{\omega^2}\frac{1}{B^{\frac{1}{2}}}\right)\right\}$$
$$=\lambda_0^2+i\frac{\omega^2}{a_1^2}\alpha_{12}^*\frac{m_2}{m_1}\left(1\mp 2\frac{\omega_i^2}{\omega^2}\frac{1}{B^{\frac{1}{2}}}\right)$$
$$=\lambda_0^2+2iC_1\alpha_{12}^*,\quad \text{say,} \quad (49)$$

where λ_0^2 is the value of λ^2 when $\alpha_{12}^*=0$, which we have discussed in Sec. V and

$$B=[1-2(\omega_i^2/\omega^2)]^2-4(a_1^2/a_2^2)[1-(\omega_e^2/\omega^2)]. \quad (50)$$

If we write the complex root λ as

$$\lambda=\lambda_R+i\lambda_i, \quad (51)$$

from Eq. (49), we have

$$\lambda_R^2-\lambda_i^2=\lambda_0^2, \quad (52)$$

$$\lambda_i\lambda_R=C_1\alpha_{12}^*. \quad (53)$$

If $\alpha_{12}^*=0$, Eq. (52) gives the results of Sec. V. In this case, if λ_0^2 is positive, we have $\lambda_i=0$, $\lambda_R=\lambda_0$; on the other hand, if λ_0^2 is negative, we have $\lambda_R=0, \lambda_i=(-\lambda_0^2)^{\frac{1}{2}}$ because both λ_R and λ_i are real numbers.

If $\alpha_{12}^*\neq 0$, we have

$$\lambda_R^4-\lambda_0^2\lambda_R^2-C_1^2\alpha_{12}^{*2}=0. \tag{54}$$

We discuss Eq. (54) for the corresponding undamped and damped waves in an ideal plasma separately.

(i) For undamped waves in an ideal plasma $\lambda_0^2>0$, Eq. (54) gives

$$\lambda_R^2=\tfrac{1}{2}\lambda_0^2\{1+[1+(2C_1^2\alpha_{12}^{*2}/\lambda_0^4)]\}\cong\lambda_0^2. \tag{55}$$

For first approximation, the electrical conductivity does not affect the speed of propagation $V=\omega/\lambda_R$:

$$\lambda_i=\frac{C_1\alpha_{12}^*}{\lambda_R}=\frac{\alpha_{12}}{2m_1\nu_0\omega}\frac{1}{\lambda_0}\left(1\mp 2\frac{\omega_i^2}{\omega^2}\frac{1}{B^{\frac{1}{2}}}\right). \tag{56}$$

Equation (56) shows that the damping factor λ_i increases with α_{12}, but decreases with increase of ω.

If we substitute the values of λ_0 from Eqs. (40a), (43), (44a), (45a), (46), or (47) into Eq. (56), we have the damping factors for the corresponding waves.

(ii) For damped waves in an ideal plasma, $\lambda_0^2<0$, Eq. (54) gives

$$\lambda_R^2=\frac{\lambda_0^2}{2}\left[1-\left(1+\frac{2C_1^2\alpha_{12}^{*2}}{\lambda_0^4}\right)\right]=\frac{c_1^2\alpha_{12}^{*2}}{(-\lambda_0^2)}, \tag{57}$$

$$\lambda_i=(-\lambda_0^2)^{\frac{1}{2}}. \tag{58}$$

Equation (58) shows that the damping factor is not affected by the electrical conductivity for first approximation. Equation (57) shows that this damped wave does propagate at a very high speed which is proportional to $1/\alpha_{12}^*$.

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DISCUSSION

Session Reporter: P. S. Lykoudis

B. Lehnert, *Royal Institute of Technology, Stockholm, Sweden:* I quite agree with you that in general lines, Ohm's law is not treated the correct way and therefore some very curious results could be obtained. On the other hand, I do not see that there is a difference in the treatment where you use the mass velocity of the ions and electrons or where you treat the problem in terms of the current density and the mean motion of the plasma. This seems to be an exactly equivalent way of expression.

J. M. Burgers, *University of Maryland, College Park, Maryland:* I believe that Dr. Lehnert's equations for the one-fluid theory introduce the inertia connected to the current. Most people who have worked with the single-fluid theory do not think about the inertia contribution to the electric current.

B. Lehnert: That is right, if you write down Ohm's law, then you should do it the correct way and you should include also inertial forces, pressure gradients, and so on.

S. I. Pai: In the single-fluid theory you have to introduce the electrical current density as an independent variable. Therefore, you must introduce certain phenomenological relations. These relations are usually very simple for a generalization. If you can get a complete current equation, then you can probably get the same result. Two-fluid theory yields much more information.

B. Lehnert: Is it not, though, more difficult to have the velocity of ions and electrons in Euler's equations instead of the difference between these velocities, which enter into Maxwell's equations in the form of the current density. It might be that what you gain on one side you lose on the other. When you start writing down your equations, you should have the ion gas and the electron gas treated separately, then you lump them together. I still think that the theory where you have current density and mean motion of the plasma should be considered as a two-fluid theory.

S. I. Pai: In the single-fluid theory you use only 16 variables rather than 18. The 18 variables are: the velocity vector, two partial pressures, two partial temperatures, the density, the excess charge, and the vectors of current, electric field, and magnetic field. In the single-fluid theory you forget one of the partial pressures and you assume the temperature to be the same for both species.

B. Lehnert: I quite agree that limitations are introduced by making such assumptions; however, they are not necessary; you might as well introduce different pressures and temperatures of the ion and electron gases, and the theory then becomes as general as that which you have described.

T. Kihara, *University of Tokyo, Tokyo, Japan:* I would like to draw your attention to the fact that the fundamental equations of the two-fluid theory can be derived from the theory of thermodynamics of irreversible processes. For example, your coefficient α is part of the linear phenomenological coefficients. It is easy to prove that it is always a positive number. I would like to say that the theory of irreversible theorodynamics is much more general than Boltzmann's equation, since it is also applicable in the case of liquid plasmas.

S. I. Pai: Actually the two-fluid theory can be derived from Boltzmann's equation, but the trouble is that it is difficult to solve this equation even for the case of ordinary gas dynamics.

W. B. Thompson, *Atomic Energy Research Establishment, Harwell, Berkshire, England:* I agree that Boltzmann's equation for most problems is very difficult to solve. For this problem, however, it happens to be tolerably tractable, and you can get a solution. How do your results compare with this solution?

S. I. Pai: In the two-fluid theory, J. J. Thompson in one of his books [a] found that for isothermal conditions $v_2=(p/\rho_2)^{\frac{1}{2}}$. From Boltzmann's equation, one finds that $v_2=(3p_0/\rho_2)^{\frac{1}{2}}$. My case was an adiabatic one and I found that $v_2=(\gamma p_0/\rho_2)^{\frac{1}{2}}$. I do not see how these differences arise. It could perhaps be possible under certain conditions to obtain from Boltzmann's equation the factor γ instead of three in the expression for v_2.

[a] J. J. Thompson and G. P. Thompson, *Conduction of Electricity through Gases* (Cambridge University Press, New York, 1933), pp. 353–358.