REVIEWS OF MODERN PHYSICS VOLUME 32, NUMBER 4 OCTOBER, 1960

Motion of a Completely Ionized Gas across a Magnetic Field in the Presence of an Electric Force*

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1. INTRODUCTION

THE following considerations have been developed to throw some light on the problem of whether a shock can appear in a completely ionized plasma subjected to a magnetic field, if the density and temperature are such that the randomizing effect of collisions can be omitted in the domain of transition.

We assume that the field, when described with reference to a coordinate system in which the shock is at rest, is independent of the time, and is a function of the coordinate x (or x_1) only. It is supposed that there is a uniform state of flow with constant velocity U_0 (in the direction of x_1) for both positive ions and electrons in the domain $x<0$. In this domain the number density of the ions (supposed to be singly charged), as well as the number density of the electrons, has the constant value N_0; the temperatures of the ions and of the electrons have the same constant value T_0; and their partial pressures are equal to $p_0=N_0kT_0$, without any deviatoric pressure components. It is further supposed that in this domain there is a constant magnetic field directed along the x_3 axis of strength B_0, and that there is an electric field $E_2=U_0B_0/c$ in the direction of the x_2 axis. This electric field compensates the Lorentz force on the charged particles insofar as it depends upon the mean flow velocity U_0. The situation corresponds to the case where a shock wave or some other wave pattern would advance with the velocity U_0 in the negative x_1 direction into a gas at rest; the gas at rest would then be subjected to the magnetic field B_0 in the x_3 direction, but it would not experience an electric field E_2.

With m_i=ion mass and m_e=electron mass, the density of the gas in the domain $x<0$ is $\rho_0=N_0m_i$. It is supposed that the velocity U_0 is so large that $\rho_0U_0^2>p_0$; how much larger is considered later. It is further supposed that the "magnetic pressure" $B_0^2/8\pi$ is of the same order as p_0; it may be larger or smaller. The velocity U_0 is assumed to be so far below the velocity of light c that relativity corrections need not be taken into account.

It is supposed that to the right from $x=0$ the field ceases to be uniform and becomes a function of x. We intend to investigate the relations between change of motion and changes of pressure, magnetic and electric fields, including the appearance of a component E_1. It is necessary to give attention to the deviatoric pressure components, although these are not related to velocity gradients by mean of a viscosity coefficient. The treatment is based upon the flow equations that can be derived from a Boltzmann equation, in which the right-hand side, which ordinarily represents the effect of collisions, has been replaced by zero. As is well known, the equations derived from the Boltzmann equation by integration contain the moments of the distribution function, and the equation starting with the derivatives of a moment of order n also involve derivatives of certain moments of the next higher order. Hence, in order to arrive at a closed system of equations, an auxiliary assumption must be introduced, for instance, that certain moments can be neglected, or that they can be replaced by a convenient approximation.

Since the motion of the positive ions and the motion of the electrons are coupled only by the electric and magnetic fields, the equations of motion can be expressed most conveniently with the aid of the separate mean flow velocities of the two types of particles. Thus, for each species (a) of particles we introduce the mean value $\mathbf{u}_a$ of the velocities $\boldsymbol{\xi}_a$, and we call $\mathbf{u}_a$ the mean flow velocity for the species a. We then write

$$\mathbf{c}_a=\boldsymbol{\xi}_a-\mathbf{u}_a, \tag{1}$$

and call $\mathbf{c}_a$ the *random velocity of a particle of type a with reference to the mean flow of the particles of type a.* With this definition the mean value of $\mathbf{c}_a$ vanishes:

$$\langle\mathbf{c}_a\rangle_{\text{av}}=0. \tag{1a}$$

Writing n_a for the number density per unit volume of the particles of type a and m_a for the mass of a particle a, we introduce mean values of higher order

* The work reported here was supported by the U. S. Air Force through the Air Force Office of Scientific Research, ARDC.

It is related to a brief discussion given in Secs. 5–10, 5–11, and 5–12 on "Flow without collisions in a magnetic field," etc., of the chapter on statistical plasma mechanics in *Plasma Dynamics*, F. H. Clauser, Editor (Proceedings of the International Symposium on Plasma Dynamics, June, 1958) (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1960), pp. 140–148. In particular it can be considered as an investigation of some problems suggested by Eq. (5–61), p. 146.

Some problems connected with the motion of a single charged particle across a magnetic field, in the presence of an electric force, were discussed in Secs. 6 and 7 of a paper on "Magnetogasdynamic problems from the point of view of particle dynamics," presented at the Durand Centennial Conference, August, 1959, to be published in the Proceedings of the Symposium (Pergamon Press). In that treatment only the constant component E_2 of the electric field was introduced; the component E_1, which in the present case appears as a result of the net space charge density in the field, was left aside.

defined by

$$(p_a)_{hk}=n_a m_a \langle c_{ah}c_{ak}\rangle_{\text{av}}, \tag{2}$$

$$(p_a)_{hki}=n_a m_a \langle c_{ah}c_{ak}c_{ai}\rangle_{\text{av}}, \quad \text{etc.} \tag{3}$$

The $(p_a)_{hk}$ may be called the components of the partial pressure tensor for the particles a, but it should be observed that when the u_a are unequal, these partial pressures cannot be summed with respect to a in order to obtain the total pressure. To find the latter, one should introduce a different set of random velocities, defined with respect to the mean flow velocity of the gas as a whole, and then define pressure components based on those random velocities. A similar observation holds with respect to the temperatures of the various components, which in our present system are defined by the equation

$$\tfrac{3}{2}kT_a=\tfrac{1}{2}m_a\langle (c_{ah})^2\rangle_{\text{av}}, \tag{4}$$

so that

$$p_a=\tfrac{1}{3}(p_a)_{hh}=n_a kT_a. \tag{4a}$$

The components of the heat flow carried by the particles of type a with regard to the mean flow of these particles are obtained from the third-order moments $(p_a)_{hki}$ by contraction:

$$(q_a)_i=\tfrac{1}{2}(p_a)_{hhi}. \tag{3a}$$

2. EQUATIONS OF MOTION

We write

$$D_a/Dt=\partial/\partial t+u_{ai}(\partial/\partial x_i). \tag{5}$$

In the system of description indicated, the equations for the various mean values (equation of continuity, equations of motion, etc.) obtain the following forms[1]:

$$(D_a n_a/Dt)+n_a(\partial u_{ai}/\partial x_i)=0, \tag{6}$$

$$n_a m_a\frac{D_a(u_{ah})}{Dt}+\frac{\partial(p_a)_{hi}}{\partial x_i}=n_a e_a\{E_h+\delta_{hij}u_{ai}B_j/c\},^{2} \tag{7}$$

$$\frac{D_a(p_a)_{hk}}{Dt}+(p_a)_{hk}\frac{\partial u_{ai}}{\partial x_i}+(p_a)_{ki}\frac{\partial u_{ah}}{\partial x_i}+(p_a)_{hi}\frac{\partial u_{ak}}{\partial x_i}+\frac{\partial(p_a)_{hki}}{\partial x_i}-(M_a)_{hk}=0, \tag{8}$$

$$\frac{D_a(p_a)_{hki}}{Dt}+(p_a)_{hki}\frac{\partial u_{aj}}{\partial x_j}+(p_a)_{hkj}\frac{\partial u_{ai}}{\partial x_j}+(p_a)_{kij}\frac{\partial u_{ah}}{\partial x_j}+(p_a)_{hij}\frac{\partial u_{ak}}{\partial x_j}+\frac{\partial}{\partial x_j}(p_a)_{hkij}-\left[\frac{(p_a)_{hk}}{\rho_a}\frac{\partial(p_a)_{ij}}{\partial x_j}+\frac{(p_a)_{hi}}{\rho_a}\frac{\partial(p_a)_{kj}}{\partial x_j}+\frac{(p_a)_{ki}}{\rho_a}\frac{\partial(p_a)_{hj}}{\partial x_j}\right]-(M_a)_{hki}=0. \tag{9}$$

The series of equations can be continued in this way. As mentioned before, each equation starting with the derivatives of a moment of order n involves derivatives of moments of the next higher order.

The tensors $(M_a)_{hk}$, $(M_a)_{hki}$, etc., are dependent upon the components B_i of the magnetic field. They are defined by the formulas[3]

$$(M_a)_{hk}=(e_a/m_a c)\{\delta_{hij}(p_a)_{ki}B_j+\delta_{kij}(p_a)_{hi}B_j\}, \tag{10}$$

$$(M_a)_{hki}=(e_a/m_a c)\{\delta_{hmn}(p_a)_{kim}B_n+\delta_{kmn}(p_a)_{him}B_n+\delta_{imn}(p_a)_{hkm}B_n\}. \tag{11}$$

In the problem under consideration we have taken the magnetic field in the direction of the z axis, so that $B_1=B_2=0$, and we write $B_3=B$. The tensor $(M_a)_{hk}$ then reduces to

$$\begin{aligned}
(M_a)_{11}&=2\omega_a(p_a)_{12},\\
(M_a)_{22}&=-2\omega_a(p_a)_{12},\\
(M_a)_{33}&=0,\\
(M_a)_{12}&=(M_a)_{21}=-\omega_a[(p_a)_{11}-(p_a)_{22}],\\
(M_a)_{13}&=(M_a)_{31}=\omega_a(p_a)_{23},\\
(M_a)_{23}&=(M_a)_{32}=-\omega_a(p_a)_{13},
\end{aligned} \tag{10a}$$

where

$$\omega_a=e_aB/m_ac \quad (\omega_e \text{ has a negative sign}). \tag{12}$$

Corresponding expressions for some of the components of $(M_a)_{hki}$ appear in Eqs. (56).

Instead of the subscripts a, we often use subscripts i and e when we wish to distinguish between ions and electrons. We also write u_a, v_a, w_a instead of u_{a1}, u_{a2}, u_{a3}.

Since the field is assumed to be a function of the coordinate x only and is independent of the time, derivatives with respect to x_2, x_3, and t drop out. We further assume that the electric field has two components, E_1 and E_2, while the component E_3 (parallel

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[1] Equations (6)–(9) have been obtained in the following way. The collisionless Boltzmann equation is multiplied by the factors 1; $m_a\xi_{ah}$; $m_a\xi_{ah}\xi_{ak}$; $m_a\xi_{ah}\xi_{ak}\xi_{ai}$, respectively. The equation is then integrated over the full domain of values of ξ_{a1}, ξ_{a2}, ξ_{a3}. Call the results obtained (in the same order) $[0]$, I_h, II_{hk}, III_{hki}. Then Eq. (6) corresponds to $[0]$; Eq. (7) corresponds to $I_h-m_au_{ah}[0]$; Eq. (8) corresponds to $II_{hk}-u_{ah}I_k-u_{ak}I_h+m_au_{ah}u_{ak}[0]$; and Eq. (9) corresponds to

$$III_{hki}-(u_{ah}II_{ki}+u_{ak}II_{hi}+u_{ai}II_{hk})+(u_{ah}u_{ak}I_i+u_{ah}u_{ai}I_k+u_{ak}u_{ai}I_h)-(1/\rho)(p_{hk}I_i+p_{hi}I_k+p_{ki}I_h)-m_au_{ah}u_{ak}u_{ai}[0]+(m_a/\rho)(u_{ai}p_{hk}+u_{ak}p_{hi}+u_{ah}p_{ki})[0].$$

[2] The components of the third-order tensor δ_{hij} have the following values: $\delta_{hij}=+1$ when the numbers h, i, j form an even permutation of 1, 2, 3; $\delta_{hij}=-1$ when these numbers form an odd permutation of 1, 2, 3; $\delta_{hij}=0$ when two or more of the numbers h, i, j are equal.

[3] The following observation may be of interest with regard to the effect of the Larmor motion upon the behavior of the components of the pressure tensor and of the third-order tensor: In a uniform field which would be independent of all coordinates but dependent upon the time, Eqs. (8) and (9) reduce to systems of homogeneous simultaneous linear differential equations for the $(p_a)_{hk}$ or the $(p_a)_{hki}$, respectively, with constant coefficients, which can be solved in the usual way with the aid of goniometric functions.

The sets of equations thus obtained moreover satisfy the conditions

$$(\partial/\partial t)p_a=0,$$
$$(\partial/\partial t)[(p_a)_{hk}]^2=0,$$
$$(\partial/\partial t)[(p_a)_{hki}]^2=0.$$

to the magnetic field) is supposed to be zero. The component E_2 cannot be a function of x when the magnetic field is independent of the time; hence E_2 is not a variable, but a constant. As mentioned before,

$$E_2 = U_0 B_0/c. \tag{13}$$

The component E_1 is related to the charge density by the equation

$$\frac{dE_1}{dx} = 4\pi \sum_a n_a e_a = 4\pi e(n_i - n_e). \tag{14}$$

The magnetic field strength B is related to the current density in the y direction by

$$-\frac{dB}{dx} = 4\pi \sum_a \frac{n_a e_a v_a}{c} = \frac{4\pi e}{c}(n_i v_i - n_e v_e). \tag{15}$$

3. FIRST APPROXIMATION

Equation (8), determining the change of the $(p_a)_{hk}$ with time, contains the third-order moments $(p_a)_{hki}$. Equation (9) for the latter quantities involves fourth-order moments. Similarly, equations for the fourth-order moments involve fifth-order terms, etc. As mentioned before, an additional hypothesis must be introduced in order to arrive at a closed system of equations.

As a first approximation we *replace all third-order moments by zero*. This entails that heat transfer by conduction is neglected.

A second approximation, in which certain third-order moments are taken into account, is considered in Secs. 10–14.

With this simplification, and in consequence of the assumption that all variable quantities are functions of x only, the system of equations (6)–(8) reduces to

$$(d/dx)(n_a m_a u_a) = 0, \tag{16}$$

$$n_a m_a u_a (du_a/dx) + [d(p_a)_{11}/dx] = n_a e_a [E_1 + (v_a B/c)], \tag{17}$$

$$n_a m_a u_a (dv_a/dx) + [d(p_a)_{12}/dx] = n_a e_a [E_2 - (u_a B/c)], \tag{18}$$

$$u_a [d(p_a)_{11}/dx] + 3(p_a)_{11}(du_a/dx) = 2\omega_a (p_a)_{12}, \tag{19}$$

$$u_a [d(p_a)_{22}/dx] + (p_a)_{22}(du_a/dx) + 2(p_a)_{12}(dv_a/dx) = -2\omega_a (p_a)_{12}, \tag{20}$$

$$u_a [d(p_a)_{33}/dx] + (p_a)_{33}(du_a/dx) = 0, \tag{21}$$

$$u_a [d(p_a)_{12}/dx] + 2(p_a)_{12}(du_a/dx) + (p_a)_{11}(dv_a/dx) = -\omega_a [(p_a)_{11} - (p_a)_{22}]. \tag{22}$$

We have omitted equations for w_a, $(p_a)_{13}$, and $(p_a)_{23}$, since these quantities can be supposed to be zero. Hence, we have a system with 16 unknowns: n_a, u_a, v_a, $(p_a)_{11}$, $(p_a)_{22}$, $(p_a)_{33}$, $(p_a)_{12}$, E_1, and B, with a standing either for ions or for electrons. There are also 16 equations if we consider the set just given together with Eqs. (14) and (15).

In consequence of our neglect of collisions, there are no formulas that define the tensor components $(p_a)_{hk}$ in terms of the derivatives of the flow velocities and a viscosity coefficient. The tensor components $(p_a)_{hk}$ appear to grow in time when we follow the flow, a fact which is expressed by their dependence upon the coordinate x in the equations. There is no randomizing effect of collisions which tends to bring the pressure back to isotropy.

4. FIRST STEPS IN THE CONSTRUCTION OF A SOLUTION

Some of the equations can be integrated immediately. By making use of the values assumed in the domain $x<0$, the continuity equation (16) gives

$$n_i u_i = N_0 U_0; \quad n_e u_e = N_0 U_0. \tag{23}$$

This result has the consequence that the electric current strength in the x direction is everywhere zero.

In the domain $x<0$ the ions and the electrons have the same isotropic gas pressure: $(p_a)_{11} = (p_a)_{22} = (p_a)_{33} = N_0 k T_0 = p_0$; while $(p_a)_{12} = 0$. Hence Eq. (21) gives

$$u_a (p_a)_{33} = U_0 p_0 = U_0 N_0 k T_0, \tag{24}$$

holding both for ions and for electrons.

It is evident from Eqs. (19), (20), and (22) that there are two scales of length, determined by the so-called Larmor radius u_a/ω_a, for the ions and for the electrons. As there is no reason to expect that the order of magnitude of the velocities u_a, or that of the magnetic field strength B, greatly changes when we follow the flow, it is convenient to use the values of U_0 and B_0 in order to define these scales of length and we write

$$L_i \text{ (or } L) = m_i U_0 c/eB_0; \quad L_e = m_e U_0 c/eB_0. \tag{25}$$

Since L_i is large compared with L_e, the question presents itself as to which of these two quantities is characteristic for the scale of the field. Let us first assume that the scale is determined by the smaller one of the two quantities, that is, by L_e.

From Eq. (15) it then follows that the order of magnitude of v_e (which presumably is the larger one of the two transverse velocities) is determined by

$$v_e \sim \frac{c}{4\pi N_0 e} \frac{B_0}{L_e} = \frac{B_0^2}{4\pi N_0 m_e U_0}, \quad \text{so that} \quad \frac{v_e}{U_0} \sim \frac{B_0^2}{4\pi N_0 m_e U_0^2}.$$

Now $B_0^2/4\pi$ was expected to be of the order $N_0 m_i U_0^2$. Hence we must conclude that v_e becomes much larger than U_0, unless the magnetic field would be very weak and uninteresting.

We now consider Eq. (18) for the electrons. The first term is of the order

$$N_0 m_e U_0 v_e / L_e \sim (B_0^2/4\pi)(eB_0/m_e U_0 c).$$

This requires that $d(p_e)_{12}/dx$, or $N_0e(E_2-u_eB/c)$, or both, be of the same order of magnitude. The second possibility would make

$$E_2-(u_eB/c)\sim(B_0^2/4\pi N_0m_eU_0^2)(U_0B_0/c).$$

In view of Eq. (13) this looks highly improbable and it would be much more natural to suppose that $d(p_e)_{12}/dx$ is approximately equal to $n_em_eu_e(dv_e/dx)$. Under these circumstances we cannot conclude that E_2-u_eB/c is nearly zero, so that there is no reason to assume that u_eB remains approximately constant.

We now find that $(p_e)_{12}$ must be of the order $B_0^2/4\pi$. In itself this does not look impossible. We must, however, consider it in connection with Eq. (20). Since we have found that $v_e\gg U_0$, it follows that the term $2(p_e)_{12}(dv_e/dx)$ in this equation exceeds all other terms, so that it is not possible to satisfy this equation. Thus we arrive at the conclusion that the assumption that the scale of the field should be determined by L_e, leads to unacceptable results. We can see this in a slightly different way when we observe that the kinetic energy per unit volume gained by the electrons would be of the order

$$\tfrac{1}{2}N_0m_ev_e^2\sim\tfrac{1}{2}(B_0^2/4\pi N_0m_iU_0^2)(m_i/m_e)(B_0^2/4\pi),$$

which is far in excess of the other energies in the field.

We turn, therefore, to the alternative assumption that the scale of the field is determined by L_i.

5. SIMPLIFICATIONS OBTAINED IN THE EQUATIONS OF MOTION WHEN IT IS ASSUMED THAT THE SCALE OF THE FIELD IS DETERMINED BY THE ION LARMOR RADIUS L_i

When this assumption is applied to Eqs. (19), (20), and (22) for the electrons, the coefficient ω_e appearing on the right-hand sides is of a much larger order of magnitude than the operators $u(d/dx)$ occurring in the left-hand sides. We therefore must conclude that the terms multiplied by ω_e are small. This means that

$$(p_e)_{12}\quad\text{and}\quad(p_e)_{22}-(p_e)_{11}$$

must be small compared with $(p_e)_{11}$, or with $(p_e)_{22}$ or $(p_e)_{33}$.

By making use of this result, we can combine Eqs. (19) and (20) for the electrons by addition into

$$u_e[d(p_e)_{11}/dx]+2(p_e)_{11}(du_e/dx)=0,$$

while the same equation holds for $(p_e)_{22}$. Hence,

$$u_e^2(p_e)_{11}=u_e^2(p_e)_{22}=U_0^2p_0. \tag{26}$$

By subtraction we can derive

$$2(p_e)_{11}(du_e/dx)=4\omega_e(p_e)_{12}=-4(eB/m_ec)(p_e)_{12}, \tag{26a}$$

from which the value of $(p_e)_{12}$ can be obtained. Equation (22) can then be used to calculate the small difference between $(p_e)_{11}$ and $(p_e)_{22}$ (results are given in Sec. 9).

In this way we have disposed of seven of the unknowns, viz., n_i, n_e, $(p_i)_{33}$, $(p_e)_{11}$, $(p_e)_{22}$, $(p_e)_{33}$, $(p_e)_{12}$.

We now introduce the evident assumption that E_1 is not of a larger order of magnitude than E_2 [which was given by (13)]. With the assumption that the scale of the field is determined by L_i, the derivative dE_1/dx is, at most, of the order U_0B_0/cL_i, that is, of order eB_0^2/m_ic^2. Referring to Eq. (14), we conclude that $(n_i-n_e)/N_0$ can be at most of the order $B_0^2/4\pi N_0m_ic^2$. In view of the assumption $U_0\ll c$, it follows that

$$(n_i-n_e)/N_0\ll 1. \tag{27}$$

Consequently, also u_i and u_e can differ only by a small quantity. In those expressions where the difference between n_i and n_e does not appear explicitly, we may use the letter n for both of them, and similarly use a single letter u for the velocities of both types of particles in the x direction, with

$$nu=N_0U_0. \tag{23a}$$

We return to Eq. (18) applied to the electrons. We have found that $(p_e)_{12}$ is a small quantity in comparison with $(p_e)_{11}$, and in view of the small electron mass we can neglect the first term on the left-hand side. There remains

$$0=-ne(E_2-uB/c),$$

from which we obtain

$$B=cE_2/u=B_0U_0/u, \tag{28}$$

so that now B is expressed in terms of u and follows the same rule as the number density. This result is commonly expressed by saying that the magnetic lines of force are frozen in the gas. It is often deduced from the assumption that the electric conductivity is infinite.

On neglecting the small difference between n_i and n_e, in comparison with the probably much more marked difference between v_e and v_i, we can write Eq. (15)

$$-(dB/dx)=(4\pi ne/c)(v_i-v_e);$$

substituting (28) we find

$$v_i-v_e=(cU_0B_0/4\pi ne)(1/u^2)(du/dx). \tag{29}$$

We then turn to Eq. (17) applied to the electrons. Also here we neglect the first term in view of the smallness of the electron mass and we substitute the value of $(p_e)_{11}$ obtained from (26). Elimination of v_e with the aid of (29) leads to the following expression for (E_1+v_iB/c):

$$E_1+\frac{v_iB}{c}=\frac{1}{ne}\left(2p_0+\frac{B_0^2}{4\pi}\right)\frac{U_0^2}{u^3}\frac{du}{dx}. \tag{30}$$

The treatment of the electron equations given here, which started from the assumption of the existence of a rigorously steady state, evidently omits the question of whether oscillations connected with the electron inertia (so-called plasma oscillations) might become of some importance.

6. EQUATIONS OBTAINED FOR THE IONS

The result (30) can be substituted into Eq. (17) applied to the ions; actually, this means that the gradient of the electron pressure component $(p_e)_{11}$ exerts a certain influence upon the motion of the ions in consequence of the electromagnetic coupling which is present in the field.

With $\rho_0 = N_0 m_i$ as before, Eqs. (17)–(22), applied to the ions, can now be written

$$\rho_0 U_0 \frac{du}{dx} + \frac{d(p_i)_{11}}{dx} = \left(2p_0 + \frac{B_0^2}{4\pi}\right)\frac{U_0^2}{u^3}\frac{du}{dx}, \tag{31}$$

$$\rho_0 U_0 \frac{dv_i}{dx} + \frac{d(p_i)_{12}}{dx} = 0, \tag{32}$$

$$u\frac{d(p_i)_{11}}{dx} + 3(p_i)_{11}\frac{du}{dx} = 2\omega_i (p_i)_{12}, \tag{33}$$

$$u\frac{d(p_i)_{22}}{dx} + (p_i)_{22}\frac{du}{dx} + 2(p_i)_{12}\frac{dv_i}{dx} = -2\omega_i (p_i)_{12}, \tag{34}$$

$$u\frac{d(p_i)_{33}}{dx} + (p_i)_{33}\frac{du}{dx} = 0, \tag{35}$$

$$u\frac{d(p_i)_{12}}{dx} + 2(p_i)_{12}\frac{du}{dx} + (p_i)_{11}\frac{dv_i}{dx} = -\omega_i[(p_i)_{11} - (p_i)_{22}]. \tag{36}$$

To simplify the writing, we omit the subscript i when it is evident that the equations refer to ions.

From this set, an equation of energy can be deduced by adding together Eq. (31), multiplied by u; Eq. (32), multiplied by v; and one-half of the sum of Eqs. (33)–(35). The result is

$$\rho_0 U_0 \frac{d}{dx}\left(\frac{u^2+v^2}{2}\right) + \frac{d}{dx}\left\{\frac{u}{2}(p_{11}+p_{22}+p_{33})\right\} + \frac{d}{dx}(up_{11}+vp_{12}) - \left(2p_0 + \frac{B_0^2}{4\pi}\right)\frac{U_0^2}{u^2}\frac{du}{dx} = 0. \tag{37}$$

Equations (31), (32), (35), and (37) can be integrated immediately and give

$$\rho_0 U_0 u + p_{11} + \left(2p_0 + \frac{B_0^2}{4\pi}\right)\frac{U_0^2}{2u^2} = \text{const} = \rho_0 U_0^2 + 2p_0 + \frac{B_0^2}{8\pi}, \tag{31a}$$

$$\rho_0 U_0 v + p_{12} = \text{const} = 0, \tag{32a}$$

$$up_{33} = U_0 p_0 \text{ [already given earlier as (24)]}, \tag{35a}$$

$$\rho_0 U_0 \frac{u^2+v^2}{2} + \frac{u}{2}(p_{11}+p_{22}+p_{33}) + up_{11} + vp_{12} + \left(2p_0 + \frac{B_0^2}{4\pi}\right)\frac{U_0^2}{u} = \text{const} = \frac{1}{2}\rho_0 U_0^2 + \frac{9}{2}U_0 p_0 + \frac{U_0 B_0^2}{4\pi}. \tag{37a}$$

Our system of equations now contains the following variables: u, v, p_{11}, p_{22}, p_{33}, p_{12} (all referring to the ions). For these six variables we have four integrated equations: (31a), (32a), (35a), (37a); and two differential equations: (33), (36). Equation (34) has become redundant. There are further Eqs. (23a) for n and Eq. (28) for B.

It is evident from Eq. (31a) that the calculations are applicable only when the original value of p_{11} in the undisturbed flow (i.e., p_0) is high enough to prevent p_{11} from becoming negative. The equations are not valid for a gas of zero temperature.

7. SEARCH FOR A SECOND UNIFORM STATE OF FLOW

We started from a uniform state of flow in the domain $x<0$. All equations are satisfied by this state, in which the derivatives with respect to x vanish. The question presents itself as to whether the equations can also be satisfied by another uniform state. To abbreviate writing it is convenient to introduce nondimensional parameters and variables. We write

$$\frac{2p_0}{\rho_0 U_0^2} = \bar{\omega}; \quad \frac{2p_0 + B_0^2/4\pi}{\rho_0 U_0^2} = \gamma, \text{ with } 1 + \tfrac{1}{2}\bar{\omega} + \tfrac{1}{2}\gamma = \beta; \tag{38a}$$

$$\frac{u}{U_0} = z; \quad \frac{v}{U_0} = z^*; \quad \frac{p_{11}}{\rho_0 U_0^2} = y; \quad \frac{p_{22}}{\rho_0 U_0^2} = y^*; \quad \frac{p_{12}}{\rho_0 U_0^2} = \eta \tag{38b}$$

(as before, v, p_{11}, p_{22}, p_{12} refer to the ions). The quantities $\bar{\omega}$, γ, and β defined by (38a) are constants.

It follows from Eqs. (33) and (34) that $p_{12}=0$ in a uniform state; further, from (32a), $v=0$; from (36), $p_{11}=p_{22}$.

We substitute the notations (38a) and (38b) into the equation of momentum (31a) and obtain

$$z + y + (\gamma/2z^2) = 1 + \tfrac{1}{2}\bar{\omega} + \tfrac{1}{2}\gamma = \beta. \tag{39}$$

This equation holds everywhere in the field, whether the field is uniform or not.

We also make the substitution in the equation of energy (37a), applying it to a uniform state; we then arrive at

$$\tfrac{1}{2}z^2 + 2zy + (\gamma/z) = \tfrac{1}{2} + \bar{\omega} + \gamma = 2\beta - \tfrac{3}{2} \tag{40}$$

(not valid for a nonuniform state).

In a certain sense Eqs. (39) and (40) can be considered as nondimensional Hugoniot conditions for a shock wave in a magnetic field, when it is observed that use has already been made of Eq. (23a) for the density and of Eq. (28) for the magnetic field. While Eq. (39) is of the form used by most authors, with y standing for the dimensionless pressure component $(p_i)_{11}$ [or $(p_i)_\perp$ in a notation used sometimes], the occurrence of the term $2zy$ in Eq. (40) is peculiar to the theory developed here for the behavior of the electrons and the ions, as expressed by Eqs. (24) and (26), together with the result $(p_i)_{11}=(p_i)_{22}$ in a uniform state of motion. When other assumptions are introduced, for instance, that the pressure would be isotropic, the form of this term changes; it also changes when there would be exchange of energy between ions and electrons through collisions.

The value 2 for the numerical coefficient of this term brings the consequence that elimination of y between Eqs. (39) and (40) leads to a quadratic equation for z. Since this equation must be satisfied by the original state for which $z=1$, it can be divided by $(z-1)$, leaving a linear equation for the determination of the remaining root. The latter is found to be

$$z_0=\tfrac{4}{3}\beta-1=\tfrac{1}{3}(1+2\bar{\omega}+2\gamma). \tag{41}$$

The value z_0 is smaller than one when

$$\bar{\omega}+\gamma<1. \tag{41a}$$

8. CAN THE SYSTEM OF EQUATIONS (31a), (32a), (35a), (37a), (33), (36) DESCRIBE A CONTINUOUS TRANSITION FROM THE STATE CORRESPONDING TO $z=1$ TOWARDS THE STATE $z=z_0$?

We obtain from (39)

$$y=\beta-z-\gamma/2z^2,$$

and from (32a)

$$z^*=-\eta.$$

Next from (33) (using a prime to denote derivatives with respect to x),

$$\eta=\tfrac{1}{2}L(z^2y'+3yzz')=-\tfrac{1}{2}L[4z^2-3\beta z+(\gamma/2z)]z'.$$

From Eq. (36) we then find

$$y^*=y+L(z^2\eta'-yz\eta'+2\eta zz').$$

The results are substituted into the equation of energy (37a); after a number of intermediate calculations we arrive at the following differential equation for z:

$$\varphi_1(d^2z/dx^2)+\varphi_2(dz/dx)^2+[(z-1)\varphi_0/L^2]=0, \tag{42}$$

where

$$\varphi_1=z^3[2z-\beta+(\gamma/2z^2)][2z-\tfrac{3}{2}\beta+(\gamma/4z^2)],$$
$$\varphi_2=16z^4-16\beta z^3+(15/4)\beta^2z^2+3\gamma z-(5/4)\beta\gamma-(\gamma^2/16z^2), \tag{43}$$
$$\varphi_0=3(z-\tfrac{4}{3}\beta+1).$$

A formal solution of the differential equation can be constructed when z is used as independent variable and dz/dx as the dependent variable. Before considering this solution, it is useful to give attention to a linearized form of Eq. (42), which is obtained by setting $z=1+\zeta$, neglecting $(d\zeta/dx)^2$ and taking $\zeta=0$ in the expressions for φ_1, φ_2, φ_0. At the same time we adjust the scale for x in such a way that L can be replaced by unity. We then obtain

$$\varphi_1(d^2\zeta/dx^2)+\varphi_0\zeta=0, \tag{44}$$

with

$$\varphi_1=\tfrac{1}{2}(1-\tfrac{1}{2}\bar{\omega})(1-\tfrac{3}{2}\bar{\omega}-\gamma);$$
$$\varphi_0=2(1-\bar{\omega}-\gamma).$$

We observe that the same equation can be obtained by making use of linearized forms of Eqs. (31)–(36).

When we replace (41a) by the stronger condition

$$\tfrac{3}{2}\bar{\omega}+\gamma<1, \tag{45}$$

the coefficients φ_1 and φ_0 in (44) are positive and the equation admits oscillating solutions for ζ and for the quantities depending on it. The wavelength of the oscillations, expressed with L as unit of length, depends upon the ratio of φ_1 to φ_0.

It may be added that condition (45) entails $\bar{\omega}+\gamma<1$, so that $\beta<\tfrac{3}{2}$ in consequence of (38a). When expressed in terms of velocities and pressures, condition (45) becomes

$$U_0>[(5p_0/\rho_0)+(B_0^2/4\pi\rho_0)]^{\frac{1}{2}}, \tag{45a}$$

which can be considered as a condition defining "supersonic" flow. It should be kept in mind that the original pressure of the gas is $2p_0$, since both ions and electrons contribute the value p_0.

We return to Eq. (42) and write it in the form

$$\varphi_1(d^2\zeta/dx^2)+\varphi_2(d\zeta/dx)^2+\varphi_0\zeta=0, \tag{42a}$$

with φ_1, φ_2, φ_0 as given in (43), z being expressed with the aid of ζ. We then write

$$d\zeta/dx=S^{\frac{1}{2}}. \tag{46}$$

Equation (42a) changes into

$$\varphi_1(dS/d\zeta)+2\varphi_2S=-2\varphi_0\zeta.$$

We introduce an auxiliary variable ψ defined by

$$\psi=\int_0^\zeta d\zeta\frac{2\varphi_2}{\varphi_1}.$$

The solution of the equation for S can be written

$$S=e^{-\psi}\left[C-2\int_0^\zeta d\zeta\frac{\varphi_0\zeta}{\varphi_1}e^{\psi}\right], \tag{47}$$

where C is an integration constant which must be positive. The integral between the square brackets [ ] assumes positive values both for $\zeta>0$ and for $\zeta<0$. In the domain $\zeta>0$, both φ_0 and φ_1 are positive; the integral increases with ζ, and we can always find a value ζ_{II} for which it reaches the value of the constant C, so that $S=0$ for $\zeta=\zeta_{II}$. When we go to the domain $\zeta<0$, we find that φ_0 becomes zero if $\zeta=z_0-1=\frac{4}{3}\beta-2$ (which is <0, since $\beta<\frac{3}{2}$). It can be proved that $\varphi_1<0$ for this value of ζ, so that φ_1 becomes zero earlier than φ_0. When φ_1 approaches zero, the integral diverges logarithmically and consequently reaches the value of C for a value $\zeta_I>\zeta_0$.

The domain of admissible values of ζ is defined by $\zeta_I<\zeta<\zeta_{II}$, with $S=0$ at both end points.

The integration of Eq. (46a), which can be written

$$x=\int\frac{d\zeta}{S^{\frac{1}{2}}},\tag{48}$$

now leads to an oscillatory dependence of ζ on x (a "libration") of periodic nature, but not of simple harmonic type. There is no progressive advance either in the positive or in the negative direction for ζ; the libration is always around $\zeta=0$, and it never reaches the point $\zeta=\zeta_0$.

Thus, the equations do not describe a continuous transition from the state $z=1$ to a state $z=z_0$.

9. SUMMARY OF RESULTS

In the following lines we collect a number of results obtained from the nonlinearized equations. We use the nondimensional variables defined by Eqs. (38a) and (38b) and take $L(=L_i)$ as unit of length, so that it can be omitted as a factor in the formulas.

Results for the Positive Ions

$$p_{11}/\rho_0U_0^2=y=\beta-z-\gamma/2z^2\quad\text{(from 39)};\tag{49a}$$

$$\frac{p_{22}}{\rho_0U_0^2}=y^*=\left(4z^3-6\beta z^2+\frac{9}{4}\beta^2z+\gamma-\frac{3\beta\gamma}{4z}+\frac{\gamma^2}{16z^3}\right)(z')^2+2z-3\beta+\frac{4\beta-3}{z}-\frac{\gamma}{2z^2}\quad\text{(from 36)};\tag{49b}$$

$$p_{33}/\rho_0U_0^2=\bar{\omega}/2z\quad\text{(from 35a)}\tag{49c}$$

(expressions for $p/\rho_0U_0^2$ and for $p/\rho U_0^2$ can be deduced from these formulas);

$$\frac{p_{12}}{\rho_0U_0^2}=\eta=-\left(2z^2-\tfrac{3}{2}\beta z+\frac{\gamma}{4z}\right)z'\quad\text{(from 33)};\tag{49d}$$

$$\frac{v}{U_0}=-\eta=+\left(2z^2-\tfrac{3}{2}\beta z+\frac{\gamma}{4z}\right)z'\quad\text{(from 32a)}.\tag{49e}$$

Results for the Electrons

$$(p_e)_{11}/\rho_0U_0^2=\bar{\omega}/2z^2,\tag{50a}$$

$$(p_e)_{22}/\rho_0U_0^2=\bar{\omega}/2z^2,\tag{50b}$$

[from (26), with neglect of terms of the relative order m_e/m_i; compare with Eq. (51), for the difference between the two components];

$$(p_e)_{33}/\rho_0U_0^2=\bar{\omega}/2z\quad\text{(from 24)};\tag{50c}$$

$$(p_e)_{12}/\rho_0U_0^2=-(m_e/m_i)(\bar{\omega}/4z)z'\quad\text{(from 26a)};\tag{50d}$$

$$\frac{v_e}{U_0}=\left(2z^2-\frac{3}{2}\beta z+\frac{4\bar{\omega}-3\gamma}{4z}\right)z'\quad\text{(from 29)}.\tag{50e}$$

We can make use of (22), applied to the electrons, in order to obtain an expression for $(p_e)_{22}-(p_e)_{11}$. The only term of importance on the left-hand side is $(p_e)_{11}(dv_e/dx)$, since the other terms are of order m_e/m_i in comparison with it. Thus we find

$$\frac{(p_e)_{22}-(p_e)_{11}}{\rho_0U_0^2}=-\frac{m_e}{m_i}\frac{\bar{\omega}}{2z}\frac{d}{dx}\left\{\left(2z^2-\frac{3}{2}\beta z+\frac{4\bar{\omega}-3\gamma}{4z}\right)z'\right\}+\text{terms of order }(m_e/m_i)^2.\tag{51}$$

Equation (18), applied to the electrons, can be used to find how far the product uB deviates from the value U_0B_0 given in (28). After some calculations the following result is obtained:

$$\frac{uB}{U_0B_0}=1+\frac{m_e}{m_i}z\frac{d}{dx}\left\{\left(2z^2-\frac{3}{2}\beta z+\frac{3\bar{\omega}-3\gamma}{4z}\right)z'\right\}.\tag{52}$$

Hence the deviations are of the order m_e/m_i.

Finally we can use Eq. (30) to obtain the value of E_1. The result is

$$E_1=-\frac{\rho_0U_0^2}{N_0e}\left(2z-\frac{3}{2}\beta-\frac{3\gamma}{4z^2}\right)z'.\tag{53}$$

Since we have made $L=m_iU_0c/eB_0=1$, the factor before the parentheses ( ) expression can also be written U_0B_0/c, so that E_1 is of the same order as E_2.

Equation (14) finally gives

$$\frac{n_i-n_e}{N_0}=\frac{1}{4\pi N_0e}\frac{dE_1}{dx}=-\frac{U_0B_0}{4\pi N_0ec}\frac{d}{dx}\left\{\left(2z-\frac{3}{2}\beta-\frac{3\gamma}{4z^2}\right)z'\right\}.\tag{54}$$

Again by making use of the relation $m_iU_0c/eB_0=1$ in order to eliminate e, the factor before d/dx becomes

$$B_0^2/(4\pi N_0m_ic^2),$$

which is of the order U_0^2/c^2 and hence small compared with unity, as had been stated in Eq. (27).

10. SECOND APPROXIMATION

The negative result obtained with the first approximation makes it desirable to investigate what can be obtained when the third-order moments $(p_a)_{hki}$ are not neglected. It is necessary to make use of Eq. (9), but we must introduce a hypothesis about the values to be given to the fourth-order moments $(p_a)_{hkij}$. The simplest assumption seems to be that they have the values which would hold for a Maxwellian velocity distribution, as given by

$$(p_a)_{hkij}=(\delta_{hk}\delta_{ij}+\delta_{hi}\delta_{kj}+\delta_{hj}\delta_{ki})(n_ak^2T_a{}^2/m_a), \quad (55)$$

where the temperature T_a is determined by (4a).

At the same time we introduce a further approximation by replacing

$$(p_a)_{hk} \quad \text{by} \quad \delta_{hk}p_a$$

in the terms between the square brackets [ ] of Eq. (9), which terms are of the second degree in the pressure components. These terms then reduce to

$$-(\delta_{hk}\delta_{ij}+\delta_{hi}\delta_{kj}+\delta_{hj}\delta_{ki})(p_a/\rho_a)(\partial p_a/\partial x_j),$$

and the combination of the derivative of $(p_a)_{hkij}$ and the [ ] terms gives

$$(\delta_{hk}\delta_{ij}+\delta_{hi}\delta_{kj}+\delta_{hj}\delta_{ki})p_a(\partial/\partial x_j)(p_a/\rho_a). \quad (55\text{a})$$

The next problem is how many of the third-order moments are needed. As before, we assume that the velocity in the x_3 direction is zero, and also that $(p_a)_{13}$ and $(p_a)_{23}$ are zero. From the basic equations (8) it follows that Eqs. (19)–(22) for $(p_a)_{11}$, $(p_a)_{22}$, $(p_a)_{33}$, $(p_a)_{12}$, respectively, must be completed on the left-hand side with terms

$$d(p_a)_{111}/dx, \quad d(p_a)_{221}/dx, \quad d(p_a)_{331}/dx, \quad d(p_a)_{121}/dx.$$

We note that $(p_a)_{121}=(p_a)_{112}$. It appears that we also need $(p_a)_{222}$ and $(p_a)_{332}$. The equations then form a closed system.

We must investigate which part of the conclusions of Secs. 4 and 5 can be retained. We keep to the assumption that E_1 is of the same order of magnitude as E_2, and that the scale of the field is determined by $L=L_i$, as given in the first formula of (25). We can then retain Eq. (27), so that we may take $n_i=n_e=n$ and $u_i=u_e=u$ in all those cases where the difference n_i-n_e or u_i-u_e does not occur explicitly.

Looking at Eq. (9), applied to the *electrons*, it should be noted that the components of the tensor $(M_e)_{hki}$ contain terms of the type $\omega_e(p_e)_{hki}$, which have the electron mass in the denominator. The same holds for the terms derived from the expression (55a). The other terms have the order of magnitude

$$(U_0/L)(p_e)_{hki}=\omega_i(p_e)_{hki},$$

and thus are much smaller. Consequently Eq. (9) for the six components of $(p_e)_{hki}$ mentioned earlier, with the $(M_e)_{hki}$ components written explicitly, reduces to

$$\begin{aligned}
&3p_e(d/dx)(p_e/\rho_e)=3\omega_e(p_e)_{112},\\
&p_e(d/dx)(p_e/\rho_e)=\omega_e[(p_e)_{222}-2(p_e)_{112}],\\
&p_e(d/dx)(p_e/\rho_e)=\omega_e(p_e)_{332},\\
&0=\omega_e[-(p_e)_{111}+2(p_e)_{221}],\\
&0=-3\omega_e(p_e)_{221},\\
&0=-\omega_e(p_e)_{331}.
\end{aligned} \quad (56)$$

Thus we find

$$\begin{aligned}
&(p_e)_{111}=(p_e)_{221}=(p_e)_{331}=0,\\
&(p_e)_{112}=(p_e)_{332}=(p_e/\omega_e)(d/dx)(p_e/\rho_e),\\
&(p_e)_{222}=(3p_e/\omega_e)(d/dx)(p_e/\rho_e).
\end{aligned} \quad (57)$$

Since the order of magnitude of p_e is that of p_0, that is, the order of $\rho_0U_0{}^2$, we easily find—in connection with (25)—that the order of magnitude of the nonzero components of the tensor $(p_e)_{hki}$ is that of $\rho_0U_0{}^3$.

The derivatives of these third-order moments occur in Eq. (8) for the electrons, along with derivatives of terms which are of the order of magnitude of p_eu, that is, again of the order $\rho_0U_0{}^3$. Hence the introduction of the third-order moments does not upset the order of magnitude of the left-hand sides of these equations. Consequently, when Eqs. (19)–(22) for the electrons are completed with the third-order terms, we still can conclude that

$$(p_e)_{12} \quad \text{and} \quad (p_e)_{11}-(p_e)_{22}$$

are small compared with $(p_e)_{11}$ or $(p_e)_{22}$.

Equation (19) for the electrons must be completed on the right-hand side with the term $d(p_e)_{111}/dx$; however, as we have seen, this term is zero. Similarly, there is no change in Eq. (20) or in Eq. (21) for the electrons. Hence Eqs. (26) and (24) can be retained. We can also retain Eq. (26a).

Equation (22) for the electrons must be completed on the left-hand side with the term $d(p_e)_{112}/dx$, which is not zero, according to (57). However, this equation is of interest only when we wish to calculate the difference $(p_e)_{11}-(p_e)_{22}$, which is not of great importance.

Thus, apart from the last point, the conclusions of Secs. 4 and 5 remain valid, so that again we can make use of (28), of (29), and of (30) [which was based upon the result (26) for $(p_e)_{11}$].

Consequently, also in the present case the unknowns of the problem refer to the ion gas, viz., u, v_i, $(p_i)_{11}$, $(p_i)_{22}$, $(p_i)_{33}$, $(p_i)_{12}$, $(p_i)_{111}$, $(p_i)_{221}$, $(p_i)_{331}$, $(p_i)_{112}$, $(p_i)_{222}$, $(p_i)_{332}$. As before, we omit the subscript i when no ambiguity is to be expected.

11. DISCUSSION OF THE SYSTEM OF EQUATIONS

For the 12 unknowns mentioned, we have the following set of 12 equations:

(a) The integrated equations (31a) and (32a), which do not change, in consequence of the results obtained for $(p_e)_{11}$ and $(p_e)_{12}$.

(b) The following system of equations for the second-order moments, replacing (33)–(36), respectively:

$$u\frac{dp_{11}}{dx}+3p_{11}\frac{du}{dx}+\frac{dp_{111}}{dx}=2\omega_i p_{12}; \tag{58}$$

$$u\frac{dp_{22}}{dx}+p_{22}\frac{du}{dx}+2p_{12}\frac{dv}{dx}+\frac{dp_{221}}{dx}=-2\omega_i p_{12}; \tag{59}$$

$$u\frac{dp_{33}}{dx}+p_{33}\frac{du}{dx}+\frac{dp_{331}}{dx}=0; \tag{60}$$

$$u\frac{dp_{12}}{dx}+2p_{12}\frac{du}{dx}+p_{11}\frac{dv}{dx}+\frac{dp_{112}}{dx}=-\omega_i[p_{11}-p_{22}]. \tag{61}$$

As before, Eq. (60) can be integrated and gives

$$up_{33}+p_{331}=U_0p_0. \tag{60a}$$

Moreover, by taking one-half the sum of (58)–(60), to which we add the sum of Eq. (31), multiplied by u, and Eq. (32), multiplied by v, we obtain the new equation of energy

$$\rho_0U_0\frac{d}{dx}\left(\frac{u^2+v^2}{2}\right)+\frac{d}{dx}\left\{\frac{u}{2}(p_{11}+p_{22}+p_{33})\right\}+\frac{d}{dx}(up_{11}+vp_{12})+\frac{d}{dx}\left(\frac{p_{111}+p_{221}+p_{331}}{2}\right)-\left(2p_0+\frac{B_0^2}{4\pi}\right)\frac{U_0^2}{u^2}\frac{du}{dx}=0, \tag{62}$$

which can be integrated and then takes the form

$$\rho_0U_0\frac{u^2+v^2}{2}+\frac{u}{2}(p_{11}+p_{22}+p_{33})+up_{11}+vp_{12}+\tfrac{1}{2}(p_{111}+p_{221}+p_{331})+\left(2p_0+\frac{B_0^2}{4\pi}\right)\frac{U_0^2}{u}=\tfrac{1}{2}\rho_0U_0^3+\frac{9}{2}U_0p_0+\frac{U_0B_0^2}{4\pi}. \tag{62a}$$

Hence this group contains two integrated equations (60a) and (62a), and two remaining differential equations (58) and (61).

(c) A set of six equations for the third-order moments:

$$u\frac{dp_{111}}{dx}+4p_{111}\frac{du}{dx}+3p\frac{d}{dx}\left(\frac{p}{\rho}\right)=3\omega_i p_{112}; \tag{63}$$

$$u\frac{dp_{221}}{dx}+2p_{221}\frac{du}{dx}+2p_{112}\frac{dv}{dx}+p\frac{d}{dx}\left(\frac{p}{\rho}\right)=\omega_i[p_{222}-2p_{112}]; \tag{64}$$

$$u\frac{dp_{331}}{dx}+2p_{331}\frac{du}{dx}+p\frac{d}{dx}\left(\frac{p}{\rho}\right)=\omega_i p_{332}; \tag{65}$$

$$u\frac{dp_{112}}{dx}+3p_{112}\frac{du}{dx}+p_{111}\frac{dv}{dx}=\omega_i[-p_{111}+2p_{221}]; \tag{66}$$

$$u\frac{dp_{222}}{dx}+p_{222}\frac{du}{dx}+3p_{221}\frac{dv}{dx}=-3\omega_i p_{221}, \tag{67}$$

$$u\frac{dp_{332}}{dx}+p_{332}\frac{du}{dx}+p_{331}\frac{dv}{dx}=-\omega_i p_{331}. \tag{68}$$

The complete system contains eight differential equations and four integrated equations.

12. INTRODUCTION OF THE HEAT FLOW COMPONENTS

The system of equations obtained in the preceding section appears to be considerably more difficult than that which was used in the first approximation, where we had only two differential equations. It may therefore be attempted to develop an "intermediate approximation," in which not all six third-order moments are treated as unknowns, but where we admit only two unknowns, viz., the components of the heat flow defined by

$$q_1=\tfrac{1}{2}(p_{111}+p_{221}+p_{331});\quad q_2=\tfrac{1}{2}(p_{112}+p_{222}+p_{332}). \tag{69}$$

We then assume the following relations:

$$p_{111}=(6/5)q_1;\quad p_{221}=p_{331}=(2/5)q_1;\quad p_{222}=(6/5)q_2;\quad p_{112}=p_{332}=(2/5)q_2. \tag{70}$$

These values are substituted into Eqs. (63)–(68); we add together Eqs. (63)–(65), and also Eqs. (66)–(68), and use the resulting equations only, which have the forms

$$u\frac{dq_1}{dx}+\frac{16}{5}q_1\frac{du}{dx}+\frac{2}{5}q_2\frac{dv}{dx}+\frac{5}{2}p\frac{d}{dx}\left(\frac{p}{\rho}\right)=\omega_i q_2; \tag{71}$$

$$u\frac{dq_2}{dx}+\frac{7}{5}q_2\frac{du}{dx}+\frac{7}{5}q_1\frac{dv}{dx}=-\omega_i q_1. \tag{72}$$

At the same time we replace (60a) by

$$up_{33}+\tfrac{2}{5}q_1=U_0p_0, \tag{60b}$$

while in (62a) the term

$\tfrac{1}{2}(p_{111}+p_{221}+p_{331})$ is replaced by q_1.

[It should be noted that for the electrons $(p_e)_{111}=(p_e)_{221}=(p_e)_{331}=0$, so that also $(q_e)_1=0$. This makes Eq.

(24) still hold for the electrons.] We now have a system with eight unknowns:

$$u,\ v,\ p_{11},\ p_{22},\ p_{33},\ p_{12},\ q_1,\ q_2,$$

for which we have four integrated equations: (31a), (32a), (60b), (62a); and four differential equations: (58), (61), (71), and (72). We can eliminate p_{33} from (62a) and omit Eq. (60b); we can also eliminate v and express it through p_{12} by means of (32a). On the other hand, it may be convenient to introduce p as a separate variable given by the relation

$$p=\tfrac{1}{3}[p_{11}+p_{22}+(U_0p_0/u)-(2q_1/5u)]. \qquad (73)$$

We remember that

$$\rho=\rho_0U_0/u;\quad \omega_i=U_0^2/L_iu.$$

13. LINEAR APPROXIMATION

In view of the complicated nature of the system, we investigate a linear approximation. For this purpose we need not use Eqs. (69) and (70), but can start directly from the set (63)–(68).

We use dimensionless variables, so that p_{hki} stands for $p_{hki}/\rho_0U_0^3$; we take $L_i=L=1$; write ν for the differential operator d/dx, and

$$\alpha=\frac{p_0}{\rho_0U_0^4}\frac{d}{dx}\left(\frac{p}{\rho}\right)=\frac{p_0}{\rho_0^2U_0^5}\frac{d}{dx}(pu). \qquad (74)$$

Equations (63)–(68) reduce to

$$\begin{aligned}\nu p_{111}+3\alpha&=3p_{112},\\ \nu p_{221}+\alpha&=p_{222}-2p_{112},\\ \nu p_{131}+\alpha&=p_{232},\\ \nu p_{112}&=-p_{111}+2p_{221},\\ \nu p_{222}&=-3p_{221},\\ \nu p_{332}&=-p_{331}.\end{aligned} \qquad (75)$$

The solutions of these equations are

$$\begin{aligned}p_{111}&=-[3\nu/(\nu^2+1)]\alpha,\\ p_{221}=p_{331}&=-[\nu/(\nu^2+1)]\alpha,\\ p_{112}=p_{332}&=[1/(\nu^2+1)]\alpha,\\ p_{222}&=[3/(\nu^2+1)]\alpha,\end{aligned} \qquad (76)$$

which results automatically satisfy Eqs. (69) and (70). When they are substituted into Eqs. (31a), (58), and (61), neglecting all terms of higher degree, we obtain

$$\delta y=-(1-\gamma)\zeta \text{ (as before)}, \qquad (77a)$$

$$\eta=-\tfrac{1}{2}(1-\tfrac{3}{2}\bar{\omega}-\gamma)\nu\zeta-[3\nu^2/2(\nu^2+1)]\alpha; \qquad (77b)$$

$$\delta y^*=\delta y-\tfrac{1}{2}(1-\tfrac{1}{2}\bar{\omega})(1-\tfrac{3}{2}\bar{\omega}-\gamma)\nu^2\zeta +\frac{\nu}{\nu^2+1}\{1-\tfrac{3}{2}(1-\tfrac{1}{2}\bar{\omega})\nu^2\}\alpha. \qquad (77c)$$

When account is taken of Eq. (60a), the energy equation (62a) appears to reduce to

$$\zeta+\tfrac{3}{2}\delta y+\tfrac{1}{2}\delta y^*+\bar{\omega}\zeta-[2\nu/(\nu^2+1)]\alpha-\gamma\zeta=0. \qquad (78)$$

After substitution for δy, δy^* this gives

$$-(1-\bar{\omega}-\gamma)\zeta-\tfrac{1}{4}(1-\tfrac{1}{2}\bar{\omega})(1-\tfrac{3}{2}\bar{\omega}-\gamma)\nu^2\zeta -\frac{3\nu}{2(\nu^2+1)}\{1+\tfrac{1}{2}(1-\tfrac{1}{2}\bar{\omega})\nu^2\}\alpha=0. \qquad (79)$$

It remains to find the value of α. For this purpose we calculate the value of δp with the aid of Eqs. (58), (59), and (61), which can be summed to give dp/dx. We omit the details of the calculation and give only the result

$$\alpha=-[\nu(\nu^2+1)\bar{\omega}^2/6\{(1-5\bar{\omega}/6)\nu^2+1\}]\zeta. \qquad (80)$$

When this is introduced into (79) the final equation (after some rearrangement) becomes

$$(P_1\nu^4+P_2\nu^2+P_3)\zeta=0, \qquad (81)$$

with

$$\begin{aligned}P_1&=(1-\tfrac{1}{2}\bar{\omega})(1-\tfrac{5}{6}\bar{\omega})(1-\tfrac{3}{2}\bar{\omega}-\gamma)-\tfrac{1}{2}(1-\tfrac{1}{2}\bar{\omega})\bar{\omega}^2,\\ P_2&=(1-\tfrac{1}{2}\bar{\omega})(1-\tfrac{3}{2}\bar{\omega}-\gamma)\\ &\quad+4(1-\tfrac{5}{6}\bar{\omega})(1-\bar{\omega}-\gamma)-\bar{\omega}^2,\\ P_3&=4(1-\bar{\omega}-\gamma).\end{aligned} \qquad (81a)$$

When condition (45) is satisfied, P_1, P_2, and P_3 are usually positive if $\bar{\omega}$ is not too large. A special case is that where $\bar{\omega}$ is almost zero; Eq. (81) then reduces to [after division by $(1-\gamma)$]

$$(\nu^4+5\nu^2+4)=(\nu^2+1)(\nu^2+4)=0,$$

with the roots

$$\nu=\pm i;\quad \nu=\pm 2i.$$

The first root, however, is spurious, since Eq. (80) shows that $\alpha=0$ in this case, which reduces Eq. (79) to

$$-(1-\gamma)\{\zeta+\tfrac{1}{4}\nu^2\zeta\}=0.$$

After division by the factor ζ (which would give the root $\zeta=0$, corresponding to the original state), this equation is the same as the former Eq. (44), when here too, we make the substitution $\bar{\omega}=0$.

Although this result is not conclusive, it does not give much ground for the expectation that the nonlinearized equations present a solution describing a transition from the original state to the second uniform state.

14. REDUCTION OF THE SYSTEM OF EQUATIONS CONSIDERED IN SEC. 12

We have not succeeded in obtaining a solution of the system of nonlinear equations which was obtained in Sec. 12. In the following lines a few steps are presented which can be used in its reduction to a more compact form.

Again we use the dimensionless variables defined in (38a) and (38b), to which we add

$$p_{33}/\rho_0 U_0^2 = y^{**};\quad q_1/\rho_0 U_0^3 = \sigma_1;\quad q_2/\rho_0 U_0^3 = \sigma_2.$$

As before, we have [from (31a) and (32a)]

$$y = \beta - z - \gamma/2z^2;\quad z^* = -\eta.$$

When L_i again is taken as unit of length, Eq. (58) leads to

$$\eta = -z[2z - \tfrac{3}{2}\beta + (\gamma/4z^2)]z' + \tfrac{3}{5}z\sigma_1'. \qquad (82)$$

Introduction of this result into (61) gives

$$y^* = y - \frac{\varphi_1}{z}z'' - \left(12z^3 - 10\beta z^2 + \tfrac{3}{2}\beta^2 z + 2\gamma - \frac{\beta\gamma}{2z} - \frac{\gamma^2}{8z^3}\right)(z')^2 + \tfrac{3}{5}z^2\left(2z - \beta + \frac{\gamma}{2z^2}\right)\sigma_1'' + \tfrac{3}{5}z\left(4z - \beta + \frac{\gamma}{2z^2}\right)z'\sigma_1' + \tfrac{3}{5}z\sigma_2'. \qquad (83)$$

From (60b) we derive

$$zy^{**} = \tfrac{1}{2}\bar{\omega} - \tfrac{2}{5}\sigma_1.$$

The results for y, y^*, y^{**}, z^* must be substituted into the equation of energy (62a), which leads to the following result:

$$\varphi_1 z'' + \varphi_2(z')^2 + \varphi_0(z-1) - \tfrac{3}{5}z^3\left(2z - \beta + \frac{\gamma}{2z^2}\right)\sigma_1'' - \frac{12}{5}z^2\left(2z - \beta + \frac{\gamma}{4z^2}\right)z'\sigma_1' + (9/25)z^2(\sigma_1')^2 - (8/5)\sigma_1 - \tfrac{2}{5}z^2\sigma_2' = 0. \qquad (84)$$

This equation takes the place of our former Eq. (42). It must be supplemented by equations for σ_1 and σ_2 to be deduced from (71) and (72). Writing

$$a = (L_i/\rho_0^2 U_0^6)pu(d/dx)(pu) = z(p/p_0)\alpha \qquad (85)$$

(with $L_i = 1$), we arrive at

$$z^2\sigma_1' + (16/5)zz'\sigma_1 - (2/5)z\eta'\sigma_2 + (5/2)a = \sigma_2, \qquad (86a)$$

$$z^2\sigma_2' + (7/5)zz'\sigma_2 - (7/5)z\eta'\sigma_1 = -\sigma_1. \qquad (86b)$$

We still need an expression for p, which occurs in Eq. (85) for α. The following result is obtained [use has been made of (84) in order to simplify the final expression]:

$$\frac{pu}{\rho_0 U_0^3} = \frac{\bar{\omega}}{2} - \frac{\bar{\omega}(z-1)}{3} - \frac{2}{3}\sigma_1 + \frac{(z-\gamma)(z-1)^2}{3z} + \frac{1}{3}\left(4z^4 - 6\beta z^3 + \frac{9}{4}\beta^2 z^2 + \gamma z - \frac{3\beta\gamma}{4} + \frac{\gamma^2}{16z^2}\right)(z')^2 - \frac{z^2}{5}\left(4z - 3\beta + \frac{\gamma}{2z^2}\right)z'\sigma_1' + \frac{3}{25}z^2(\sigma_1')^2. \qquad (87)$$

The problem now is condensed in Eq. (84), which is an extension of our former Eq. (42). The additional terms, depending upon σ_1 and σ_2 must be obtained from Eqs. (86a) and (86b), with a expressed in terms of the other variables by means of (85) and (87).

A slight simplification can be obtained by assuming $\bar{\omega} = 0$. In that case it is found that $(pu)/(\rho_0 U_0^3)$ is of the second order with respect to dz/dx and ζ, while the quantities a, σ_1, and σ_2 become of the fourth order with respect to dz/dx and ζ. In principle this does not mean very much, since in our system of units dz/dx and ζ are of order unity. Nevertheless, it may perhaps give some hint for a treatment of the system of the equations in which one would start with small deviations from the original state with $z = 1$.

DISCUSSION

Session Reporter: P. S. Lykoudis

R. Lüst, *Max-Planck-Institut für Physik und Astrophysik, Munich, Germany:* We attacked a problem quite similar to the one treated by Professor Burgers. We tried to solve all the state equations, but found it highly difficult to solve the equations including those for the electrons. Professor Burgers assumed that the mass ratio of the electrons and the ions is very small, but in the expression for the current one cannot directly neglect the coupling between the electrons and the ions. I would have some doubt concerning the validity of the approximations made in Burgers' treatment.

J. M. Burgers (*reply added after the conclusion of the Symposium*)*:* There is actually no contradiction between the method applied in my paper and the method followed by Dr. Lüst, but they refer to two different types of solution. When Dr. Lüst presented the paper on "Hydromagnetic waves of finite amplitude in a plasma with isotropic and nonisotropic pressure perpendicular to a magnetic field," during this Symposium,[a] it appeared that in the treatment with nonisotropic pressure he uses a scale of length based upon the *geometric mean* of the Larmor radius for the ions and that for the electrons. Looking into the matter, I have found that this assumpton can also be introduced into Eqs. (16)–(22) of my paper, as an alternative to the assumption of a length scale based upon the ion Larmor radius. When the new assumption is applied, a factor $(m_e/m_i)^{\frac{1}{2}}$ appears at various places and since this still is a small quantity, some terms carrying this factor can be neglected. This leads to an approximation different from that which was obtained in my paper. The interesting point is that *the new approximation leads to the equations given by Dr. Lüst.*

[a] K. Hain, R. Lüst, and A. Schlüter, Revs. Modern Phys. **32**, 967 (1960), this issue.

It may be of interest to present the relevant calculations. We start from Eqs. (16)–(22), applied both to the ions and to the electrons, together with Eqs. (14) and (15), and Eq. (13) determining the value of E_2. We again make use of the results $n_i-n_e\ll n_i$, and $u_i-u_e\ll u_i$ (obtained in Sec. 4), so that we can write n and u, respectively, for these quantities whenever their difference is not of importance. As before, Eq. (23) holds: $nu=N_0U_0$. We make use of dimensionless variables as defined in Eq. (38b), adding subscripts or superscripts i and e to distinguish between quantities referring to ions or to electrons where this is necessary. We also make use of the quantities defined in (38a) and observe that $B_0^2/(4\pi\rho_0U_0^2)=\gamma-\varpi$. It is convenient to write $b=B/B_0$. Finally we introduce a coordinate ξ defined by

$$x=[(m_im_e)^{\frac{1}{2}}U_0c/eB_0]\xi,$$

and we write $\lambda=(m_e/m_i)^{\frac{1}{2}}$.

Some reductions are possible before we introduce the new variables. Equation (17), when divided by m_a and applied to ions and well as to electrons, yields two equations which can be added. On the right-hand side the variable E_1 disappears, and the difference v_i-v_e can be expressed in terms of dB/dx with the aid of (15). The resulting equation can be integrated and gives the total momentum equation for the flow. Equation (18) when divided by m_a and applied to ions as well as to electrons, yields two equations of which the second one can be subtracted from the first one. We obtain an equation containing $d(v_i-v_e)/dx$, which can be transformed into a second derivative of B.

After introduction of the new variables the equations can be written in such a form that the derivative with respect to ξ always is combined with a factor z. We therefore write

$$z(d/d\xi)=d/ds.$$

Since s is a dimensionless time, derivatives with respect to s correspond to the time derivatives in Lüst's equations. By omitting further details the results can be given as follows:

momentum equation for the total flow:

$$z+y_i+y_e+\tfrac{1}{2}(\gamma-\varpi)b^2=1+\tfrac{1}{2}(\gamma+\varpi)=\beta; \tag{91}$$

equation for the magnetic field:

$$(\gamma-\varpi)\frac{d^2b}{ds^2}-\lambda\frac{d\eta_i}{ds}+\frac{1}{\lambda}\frac{d\eta_e}{ds}=zb-1; \tag{92}$$

equations derived from (18):

$$\frac{1}{\lambda}\frac{dz_i^*}{dz}+\frac{1}{\lambda}\frac{d\eta_i}{ds}=-(zb-1); \tag{93a}$$

$$\lambda\frac{dz_e^*}{ds}+\frac{1}{\lambda}\frac{d\eta_e}{ds}=(zb-1); \tag{93b}$$

equations derived from (19):

$$\frac{dy_i}{ds}+3\frac{y_i}{z}\frac{dz}{ds}=2\lambda b\eta_i; \tag{94a}$$

$$\frac{dy_e}{ds}+3\frac{y_e}{z}\frac{dz}{ds}=-\frac{2b\eta_e}{\lambda}; \tag{94b}$$

equations derived from (20):

$$\frac{dy_i^*}{ds}+\frac{y_i^*}{z}\frac{dz}{ds}+\frac{2\eta_i}{z}\frac{dz_i^*}{ds}=-2\lambda b\eta_i; \tag{95a}$$

$$\frac{dy_e^*}{ds}+\frac{y_e^*}{z}\frac{dz}{ds}+\frac{2\eta_e}{z}\frac{dz_e^*}{ds}=\frac{2b\eta_e}{\lambda}; \tag{95b}$$

equations derived from (22):

$$\frac{d\eta_i}{ds}+\frac{2\eta_i}{z}\frac{dz}{ds}+\frac{y_i}{z}\frac{dz_i^*}{ds}=-2\lambda b(y_i-y_i^*); \tag{96a}$$

$$\frac{d\eta_e}{ds}+\frac{2\eta_e}{z}\frac{dz}{ds}+\frac{y_e}{z}\frac{dz_e^*}{ds}=\frac{2b(y_e-y_e^*)}{\lambda}; \tag{96b}$$

equations derived from (21):

$$z(p_i)_{33}=z(p_e)_{33}=p_0. \tag{97}$$

The only approximation introduced so far is that the electron mass has been neglected where it is added to the ion mass, and the electron density where it is added to the ion density.

With some differences in notation the system of equations (91), (92), (94a), (94b), (95a), (95b), (96a), and (96b) has the same structure as the system given by Lüst.

Equations (97) do not play an important part.

Equations (93a) and (93b) can be added; the result can be integrated and gives (after multiplication by λ)

$$z_i^*+\lambda^2z_e^*+\eta_i+\eta_e=\text{const}=0. \tag{98a}$$

We can make use of the fact that $1/\lambda$ is a relatively large number in comparison with unity. It is evident from (91), in which no quantity can be negative, that z, y_i, y_e, and b must remain of the order unity. We then see from (93a) that $d(z_i^*+\eta_i)/ds=$ order λ, so that

$$z_i^*+\eta_i=0, \text{ neglecting amounts of order } \lambda. \tag{98b}$$

We can assume that z_i^* and η_i are at most of order unity, since otherwise the transverse velocity of the ions would become much larger than the velocity in the x direction, which is highly improbable. From (94b) we see that η_e is of order λ, so that there is no contradiction between (98a) and (98b).

In Eqs. (94a), (95a), and (96a) the right-hand sides are of order λ. By neglecting these right-hand sides, we obtain, from (94a),

$$y_iz^3=\text{const}=\varpi/2;$$

from (95a), after substituting $z_i^*=-\eta_i$,

$$zy_i^*-\eta_i^2=\text{const}=\varpi/2;$$

from (96a),

$$\frac{d\eta_i}{ds}\left(1-\frac{y_i}{z}\right)+\frac{2\eta_i}{z}\frac{dz}{ds}=0.$$

After substitution of the value of y_i just obtained, this equation can be integrated and gives

$$\eta_i(2z^4-\varpi)^{\frac{1}{2}}=\text{const}.$$

Since $\eta_i=0$ for the uniform state, the constant must be zero and η_i must be zero in the whole field. This entails $z_i^*=0$ and $zy_i^*=\varpi/2$.

In this way we have disposed of four of the unknowns (y_i; y_i^*; η_i; z_i^*). There remain z, b; y_e, y_e^*; η_e, and z_e^*.

We have already seen [from (94b)] that η_e is of order λ. From (95b) we then see that z_e^* can be of order $1/\lambda$. This makes the speed v_e of the electron transverse velocity large compared with u and large compared with v_i. Hence the electric current is almost completely due to the electrons. The kinetic energy per unit volume associated with the transverse motion of the electrons is of the order $\rho_0U_0^2$, which is not excessive. It is convenient to write

$$z_e^*=\sigma/\lambda;\quad \eta_e=\tau\lambda.$$

We then obtain the following equations: from (93b),

$$(d\sigma/ds)+(d\tau/ds)=zb-1; \tag{99}$$

from (94b),

$$\frac{dy_e}{ds}+\frac{3y_e}{z}\frac{dz}{ds}=-2b\tau; \tag{100}$$

from (95b),

$$\frac{dy_e^*}{ds}+\frac{y_e^*}{z}\frac{dz}{ds}+\frac{2\tau}{z}\frac{d\sigma}{ds}=2b\tau; \tag{101}$$

from (96b), in which the first two terms on the left-hand side can be neglected,

$$(y_e/z)(d\sigma/ds)=2b(y_e-y_e^*). \tag{102}$$

In Eq. (92) we can omit the term with $d\eta_i/ds$, which is zero; by expressing η_e with the aid of τ and making use of (99) we can transform this equation into an expression which can be integrated once, with the result

$$\sigma=(\gamma-\varpi)db/ds. \tag{103}$$

Finally we have the following integrated equation which follows from (91):

$$z+\varpi/(2z^3)+y_e+\tfrac{1}{2}(\gamma-\varpi)b^2=1+\tfrac{1}{2}(\gamma+\varpi). \tag{104}$$

Thus we have six equations [(99)–(104)] for the six unknowns. It is possible, however, to derive one integrable expression from these equations; this is obtained by adding together

$\tfrac{1}{2}z$ times Eq. (100),

$\tfrac{1}{2}z$ times Eq. (101),

σ times Eq. (99),

and z times the derivative of (104).

After integration we arrive at

$$\tfrac{1}{2}(z^2+\sigma^2)+(3\varpi/4z^2)+\tfrac{3}{2}zy_i+\tfrac{1}{2}zy_i^*+\sigma\tau+(\gamma-\varpi)b =\text{const}=\tfrac{1}{2}+\gamma+\tfrac{3}{4}\varpi. \tag{105}$$

This makes it possible to leave aside one of the differential equations, so that the final system is of the fourth order. It can be reduced to a system of the third order by eliminating ds, using z as the independent variable. Apparently the system is too complicated to permit direct integration, as could be done in the case where the length scale was determined by the ion Larmor radius; hence it is understandable that Dr. Lüst and his co-workers have used machine integration.

The interesting result of this discussion is that the system of equations admits two special solutions, one with the ion Larmor radius as length scale, the other one with the geometric mean Larmor radius as its scale, while no acceptable solution can be constructed with the electron Larmor radius as scale factor.

M. Łunc, *Polska Akademia Nauk, Warsaw, Poland:* In your work you considered the ratio of the excess charge divided by the number of particles; for high number of particles this ratio is very small. Nevertheless, the excess charge could be by itself rather high and in this case one cannot neglect the electrostatic force when compared with the purely ponderomotive force.

J. M. Burgers: This is true, and the charge separation in the direction of motion creates an electrostatic force which is not negligible compared to the $\mathbf{J}\times\mathbf{B}$ force. I did not calculate the charge separation directly but obtained it from E. Then I used this result to show that the difference between u_i and u_e is negligible.

A. R. Kantrowitz, *Avco-Everett Research Laboratory, Everett, Massachusetts:* It would be surprising if you had transition from one state to a different final state without any irreversible mechanism being present in the physical picture. This kind of picture must give you no transition.

H. Grad, *Institute of Mathematical Sciences, New York University, New York, New York:* It is not correct to say that there is no irreversible mechanism. Collisions are not the only irreversible mechanism that exists. The fact that no shock was found is not surprising, but simply means that the irreversible mechanism, which is very subtle, has been omitted.[b]

L. J. F. Broer, *Laboratorium voor Aero- en Hydrodynamica, Technische Hogeschool, Delft, Netherlands:* You started from the Boltzmann equation without the collision term. In that case there is no dissipation and therefore there is no shock solution. Now in the further treatment a certain randomization was introduced by means of assumptions about the moments. Has it been ascertained that these assumptions mean a true randomization? The question is, do they really entail an increase of entropy for all distribution functions involved? If there would exist distributions with negative dissipation the oscillatory behavior of the solutions would perhaps be more clear.

J. M. Burgers: I introduced my assumptions concerning the third- or fourth-order moments as a simplifying mathematical assumption. I do not introduce a definite form for the distribution function and consequently there is no definite value of the entropy. It would be necessary to make additional assumptions (not needed for the mathematical treatment given in my paper) before anything could be said about the entropy.

[b] For a discussion of this point see C. S. Gardner, H. Goertzel, H. Grad, C. Morawetz, M. H. Rose, and H. Rubin, "Hydromagnetic shock waves in high-temperature plasmas" in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, New York, 1958), Vol. 31, p. 230.