Magnetohydrodynamic Wakes in a Viscous Conducting Fluid

HIDENORI HASIMOTO*

The Johns Hopkins University, Baltimore, Maryland

1. INTRODUCTION

A TYPICAL feature of magneto-fluid dynamics is the appearance of Alfvén waves[1,2] which propagate along the magnetic field. In the presence of viscosity and finite conductivity of the fluid, however, these waves are so dispersive that it may be difficult to observe them.

The steady two-dimensional flow of an incompressible fluid with finite electric conductivity is the subject of this paper. It is assumed that both the magnetic field and velocity vector are in the plane of the flow and each reaches a constant value at infinity.

A disturbance is placed at the origin and the small-perturbation technique is used to find the disturbance velocity and magnetic field.

Explicit formulas are obtained in three special cases:

(1) The undisturbed velocity and magnetic field are parallel.

(2) The magnetic Prandtl number, $\lambda=\nu/\kappa$, is unity, where ν is the kinematic viscosity, and κ is the magnetic viscosity of the fluid.

(3) The Stokes flow in which the undisturbed velocity U is very small compared to the Alfvén-wave propagation speed a.

The most striking novel feature is the formation of two wakes instead of one, and in general these point in two different directions. These wakes have been introduced by many authors[3–12] implicitly in their studies of boundary-value problems. As far as this author is aware, the general nature of these wakes has been left to further studies.

The well-known integral formulas of Kutta, Joukowski, Filon,[13] and Goldstein[14] for the total force exerted on the disturbance have been extended to these magnetohydrodynamic cases.

Case (1) is considered at some length. It is found that the upstream wake appearing for $U<a$ is dominant in the total mass flux and the total magnetic flux along the wake compared with the downstream wake, except for the case of $a/U\gg\lambda^{\pm\frac{1}{2}}$, in which both wakes have equal strength. As an illustrating example, the drag and the lift for an insulating circular cylinder are determined.

2. FUNDAMENTAL EQUATIONS

We use mks units for the electromagnetic quantities and conventional notation. Then the magnetohydrodynamic equations for a steady incompressible fluid are

$$(\mathbf{V}\cdot\nabla)\mathbf{V}=-\frac{1}{\rho}\nabla P+\nu\nabla^2\mathbf{V}+\frac{\mu}{\rho}(\mathbf{H}\cdot\nabla)\mathbf{H},$$

$$P=p+\frac{\mu}{2}\mathbf{H}^2+\frac{\epsilon}{2}\mathbf{E}^2 \quad (1)$$

$$\nabla\cdot\mathbf{V}=0, \quad (2)$$

$$(\mathbf{V}\cdot\nabla)\mathbf{H}=\kappa\nabla^2\mathbf{H}+(\mathbf{H}\cdot\nabla)\mathbf{V},\quad \kappa=(\mu\sigma)^{-1} \quad (3)$$

$$\nabla\cdot\mathbf{H}=0, \quad (4)$$

$$\mathbf{E}=-\mu\mathbf{V}\times\mathbf{H}+\mathbf{j}/\sigma,\quad \mathbf{j}=\nabla\times\mathbf{H}. \quad (5)$$

We must also note that (1) is derived from the continuity of the total stress tensor $\mathbf{T}$:

$$\nabla\cdot\mathbf{T}=0,\quad \mathbf{T}=-\rho\mathbf{V}\mathbf{V}+\rho\nu\,\mathrm{def}\mathbf{V}-P\mathbf{I}+\mu\mathbf{H}\mathbf{H}+\epsilon\mathbf{E}\mathbf{E}, \quad (6)$$

neglecting free charge. The induction equation (3) is derived from (5) and

$$\nabla\times\mathbf{E}=0. \quad (7)$$

3. BEHAVIOR OF THE FIELDS FAR FROM THE OBSTACLE

We consider a cylindrical obstacle whose generator is perpendicular to the xy plane of the flow and magnetic fields. The electric current $\mathbf{j}$ and the vorticity $\boldsymbol{\omega}=\nabla\times\mathbf{H}$ are both parallel to $\mathbf{e}_z$ (the unit vector parallel to the z axis) (Fig. 1). Then, $\mathbf{E}$ is also parallel to $\mathbf{e}_z$ and is found to be uniform from (7):

$$\mathbf{E}\equiv-\mu UH\mathbf{e}_x\times\mathbf{e}_m=-\mu UH\sin\alpha\,\mathbf{e}_z, \quad (7')$$

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1 H. Alfvén, Arkiv. Mat. Astron. Fysik **B29**, No. 2 (1942).
2 C. Walén, Arkiv. Mat. Astron. Fysik **A30**, No. 15 (1944).
3 K. Stewartson, Proc. Cambridge Phil. Soc. **52**, 301 (1956).
4 W. Chester, J. Fluid Mech. **3**, 304 (1957).
5 E. L. Resler, Jr., and W. R. Sears, J. Fluid Mech. **5**, 257 (1959).
6 H. P. Greenspan and G. F. Carrier, J. Fluid Mech. **6**, 77 (1959).
7 H. Hasimoto, Phys. Fluids **2**, 337 (1959).
8 H. Hasimoto, Phys. Fluids **2**, 575 (1959).
9 H. Yosinobu and T. Kakutani, J. Phys. Soc. Japan **14**, 1433 (1959).
10 H. Yosinobu, J. Phys. Soc. Japan **15**, 175 (1960).
11 K. Gotoh, J. Phys. Soc. Japan **15**, 189 (1960).
12 E. C. Lary, Ph.D. thesis, Cornell University, 1960; also, the report by W. R. Sears at the ARS-Northwestern Gas Dynamics Symposium (August, 1959).
13 L. N. G. Filon, Proc. Roy. Soc. (London) **A113**, 276 (1926).
14 S. Goldstein, Proc. Roy. Soc. (London) **A123**, 216 (1929).

where α denotes the angle of the imposed magnetic field $H\mathbf{e}_m$ with respect to the undisturbed flow $U\mathbf{e}_x$ parallel to the x axis. We may note that $\mathbf{E}\equiv 0$, if we adopt the coordinate system in which the flow is rest at infinity and the obstacle is moving with the velocity $-U\mathbf{e}_x$ [see Eq. (5)].

At a great distance from the obstacle, we may adopt the Oseen-type approximation which becomes better and better as we proceed far away from the obstacle:

$$\mathbf{V}=U\mathbf{e}_x+\mathbf{v}(U\gg v),\quad \mathbf{H}=H\mathbf{e}_m+\mathbf{h}(H\gg h). \tag{8}$$

On introducing (8) into (1) and (3) and neglecting quadratic terms of $\mathbf{v}$ and $\mathbf{h}$, we obtain

$$U(\mathbf{e}_x\cdot\nabla)\mathbf{v}=-(1/\rho)\nabla P+\nu\nabla^2\mathbf{v}+(\mu/\rho)H(\mathbf{e}_m\cdot\nabla)\mathbf{h} \tag{9}$$

$$U(\mathbf{e}_x\cdot\nabla)\mathbf{h}=\kappa\nabla^2\mathbf{h}+H(\mathbf{e}_m\cdot\nabla)\mathbf{v}. \tag{10}$$

The curl operator applied to (9) and (10) yields, for $\boldsymbol{\omega}=\omega\mathbf{e}_z$ and $\mathbf{j}=j\mathbf{e}_z$,

$$\left(\nabla^2-\frac{U}{\nu}\frac{\partial}{\partial x}\right)\boldsymbol{\omega}=-\frac{\mu H}{\rho\nu}(\mathbf{e}_m\cdot\nabla)\mathbf{j} \tag{11}$$

$$\left(\nabla^2-\frac{U}{\kappa}\frac{\partial}{\partial x}\right)\mathbf{j}=-\frac{H}{\kappa}(\mathbf{e}_m\cdot\nabla)\boldsymbol{\omega}. \tag{12}$$

On eliminating $\mathbf{j}$ or $\boldsymbol{\omega}$, we obtain

$$\left[\nabla^4-\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)U\frac{\partial}{\partial x}\nabla^2 +\frac{1}{\nu\kappa}\left\{U^2\frac{\partial^2}{\partial x^2}-a^2(\mathbf{e}_m\cdot\nabla)^2\right\}\right]\boldsymbol{\omega}\mathbf{j}=0. \tag{13}$$

In the several cases mentioned in the Introduction, this fourth-order operator can be factorized into two commutable Oseen-type operators L_+ and L_-, each of which describes one wake:

$$L_+L_-[\boldsymbol{\omega}]=0,\quad L_\pm=\nabla^2-2(\mathbf{k}_\pm\cdot\nabla), \tag{14}$$

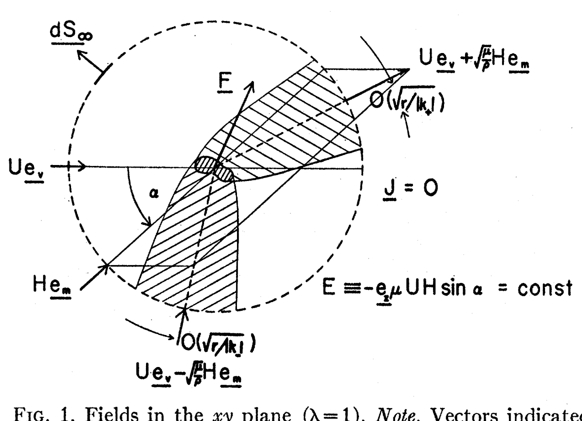

FIG. 1. Fields in the xy plane ($\lambda=1$). *Note.* Vectors indicated in the figure by bars underneath the letters are symbolized by boldface letters in the text.

TABLE I.

	$\mathbf{k}_\pm$	$\gamma_\pm$
(1) $\mathbf{e}_m=\mathbf{e}_x$	$k_\pm\mathbf{e}_x=\frac{1}{4}(U/\nu)(1+\lambda\pm N)\mathbf{e}_x$	$\frac{1}{2}(\rho U/\mu H)(1-\lambda\mp N)$
(2) $\lambda=1$	$(2\nu)^{-1}(U\mathbf{e}_x\pm a\mathbf{e}_m)$	$\mp(\rho/\mu)^{\frac{1}{2}}$
(3) Stokes flow	$\pm\frac{1}{2}(\nu\kappa)^{-\frac{1}{2}}a\mathbf{e}_m$	$\mp(\lambda\rho/\mu)^{\frac{1}{2}}$

where $N=[(1-\lambda)^2+4\beta^2\lambda]^{\frac{1}{2}}$ $\beta=a/U$.

i.e.,

$$\boldsymbol{\omega}=\boldsymbol{\omega}_++\boldsymbol{\omega}_-,\quad L_\pm[\boldsymbol{\omega}_\pm]=0, \tag{15}$$

where $\mathbf{k}_+$ and $\mathbf{k}_-$ are given in Table I, and are parallel to the directions of the two wakes [see Eq. (25)].

After some manipulations of (9)–(15), taking into account two-dimensionality of the fields and the boundary conditions at infinity, we find

$$\mathbf{v}=\mathbf{v}_0+\mathbf{v}_w,\quad \mathbf{h}=\mathbf{h}_0+\mathbf{h}_w, \tag{16}$$

where

$$\mathbf{v}_0=U\nabla(\partial\Phi/\partial x),\quad \mathbf{h}_0=H\nabla(\mathbf{e}_m\cdot\nabla)\Phi, \tag{17}$$

governed by the same harmonic function Φ related to the total pressure P, i.e.,

$$\nabla^2\Phi=0,\quad P=-[\rho U(\mathbf{e}_x\cdot\mathbf{v}_0)-\mu H(\mathbf{e}_m\cdot\mathbf{h}_0)] =-\rho U^2\partial^2\Phi/\partial x^2+\mu H^2(\mathbf{e}_m\cdot\nabla)^2\Phi, \tag{18}$$

and

$$\mathbf{v}_w=\mathbf{v}_++\mathbf{v}_-,\quad \mathbf{h}_w=\mathbf{h}_++\mathbf{h}_-,\quad \mathbf{h}_\pm=\gamma_\pm\mathbf{v}_\pm, \tag{19}$$

(see Table I) governed by

$$L_\pm[\mathbf{v}_\pm]=0,\quad \nabla\cdot\mathbf{v}_\pm=0,\quad \nabla\times\mathbf{v}_\pm=\boldsymbol{\omega}_\pm. \tag{20}$$

$\mathbf{v}_\pm$ and $\mathbf{h}_\pm$ are the rotational fields induced in the $k_\pm$ wake; $\mathbf{v}_0$ and $\mathbf{h}_0$ represent the induced irrotational fields dominant outside of the two wakes.

We can also find a simple formula

$$\boldsymbol{\omega}=\omega\mathbf{e}_z=(U/\nu)[\mathbf{e}_x\times\mathbf{v}_w-(\mu H/\rho U)\mathbf{e}_m\times\mathbf{h}_w] \tag{21}$$

from (11).

If we make use of the complex-variable notation, (17) can be rewritten as

$$\Phi=\tfrac{1}{2}[f(\zeta)+\bar{f}(\bar{\zeta})],\quad \zeta=x+iy,\quad \bar{\zeta}=x-iy \tag{22}$$

and

$$w_0=v_{0x}-iv_{0y}=Uf'',\quad m_0=h_{0x}-ih_{0y}=He^{i\alpha}f''. \tag{23}$$

Equations (16)–(23) constitute the formal solutions of our problem. More concrete expressions for $\mathbf{v}$ are found from the discussions on the Oseen equation[15,16] [see Eq. (58)] or on the asymptotic behaviors of the solutions of the Navier-Stokes equations,[13,17]

$$v_{\pm r}\sim -q_\pm|k_\pm|^{\frac{1}{2}}(2\pi r)^{-\frac{1}{2}}e^{-\eta_\pm}\gg v_{\pm\theta} \tag{24}$$

with

$$\eta_\pm=|k_\pm|r[1-\cos(\theta-\alpha_\pm)]=2|k_\pm|r\sin^2\tfrac{1}{2}(\theta-\alpha_\pm), \tag{25}$$

[15] H. Lamb, *Hydrodynamics* (Cambridge University Press, New York 1932), 6th ed.
[16] I. Imai, Proc. Roy. Soc. (London) **A224**, 141 (1954).
[17] I. Imai, Proc. Roy. Soc. (London) **A208**, 487 (1951).

where (r,θ) are the cylindrical coordinates

$$x=r\cos\theta, \quad y=r\sin\theta, \tag{26}$$

and $\theta=\alpha_\pm$ is the direction of the $k_\pm$ wake. We must note that $\eta_\pm=$constant yields a family of parabolas with an axis parallel to $\mathbf{k}_\pm$, along which $\mathbf{v}_\pm$, $\mathbf{h}_\pm$, $\mathbf{w}_\pm$, and $\mathbf{j}_\pm$ are dominant. We should also note that $q_\pm$ denotes the total inflow along the $k_\pm$ wake:

$$q_\pm=-\lim_{r\to\infty}\int_0^{2\pi} rv_{\pm r}d\theta. \tag{27}$$

Equations (27) and (19) determine the total magnetic influx along the $k_\pm$ wake:

$$q_{m\pm}=\gamma_\pm q_\pm. \tag{28}$$

The irrotational fields dominant outside of the wakes are

$$\begin{aligned} v_{0r}-iv_{0\theta}&=(2\pi r)^{-1}(q+i\Gamma)+o(r^{-1}),\\ h_{0r}-ih_{0\theta}&=(2\pi r)^{-1}(q_m+i\Gamma_m)+o(r^{-1}), \end{aligned} \tag{29}$$

with

$$q_m+i\Gamma_m=(H/U)e^{i\alpha}(q+i\Gamma) \tag{30}$$

from (23), where Γ and Γ_m denote, respectively, the usual and the magnetic circulations. q and q_m are the total outward fluxes of the irrotational field, which should compensate $q_\pm$ and $q_{m\pm}$ according to (2) and (4), i.e.,

$$q=q_++q_-, \quad q_m=q_{m+}+q_{m-}=\gamma_+q_++\gamma_-q_-. \tag{31}$$

4. EXTENSION OF THE KUTTA-JOUKOWSKI-FILON THEOREM

The total force $\mathbf{F}$ acting on the obstacle is given by integrating the total stress $\mathbf{T}\cdot\mathbf{ds}$ over the surface element $\mathbf{ds}$ of an arbitrary closed surface s around the obstacle:

$$\mathbf{F}=\int_s \mathbf{T}\cdot\mathbf{ds}. \tag{32}$$

Let us take as s a large circle on which (24)–(31) are valid, and let its radius be infinitely large. The contributions from the constant terms (e.g., $U^2\mathbf{e}_n\mathbf{e}_x$, $\mathbf{EE}$) and the quadratic terms of $\mathbf{v}$ and $\mathbf{h}$ in (6) are found to be zero (Fig. 1).

Then

$$\begin{aligned}\rho\nu\int_s \text{def}\,\mathbf{v}\cdot\mathbf{ds}&=\rho\nu\int_s \boldsymbol{\omega}\times\mathbf{ds}\\ &=\rho U\int_s [(\mathbf{e}_x\cdot\mathbf{ds})\mathbf{v}_w-(\mathbf{v}_w\cdot\mathbf{ds})\mathbf{e}_x]\\ &\quad-\mu H\int_s [(\mathbf{e}_m\cdot\mathbf{ds})\mathbf{h}_w-(\mathbf{h}_w\cdot\mathbf{ds})\mathbf{e}_m] \end{aligned}\tag{33}$$

from (2) and (21), after some vector calculus. Also,

$$\int_s(-\rho\mathbf{vv}+\mu\mathbf{HH}+\epsilon\mathbf{EE})\cdot\mathbf{ds} =-\rho U\int_s(\mathbf{e}_x\cdot\mathbf{ds})\mathbf{v}+\mu H\int_s(\mathbf{e}_m\cdot\mathbf{ds})\mathbf{h} \tag{34}$$

and

$$-\int_s P\cdot\mathbf{1ds}=\rho U\int_s(\mathbf{e}_x\cdot\mathbf{v}_0)\mathbf{ds}-\mu H\int_s(\mathbf{e}_m\cdot\mathbf{h}_0)\mathbf{ds} \tag{35}$$

from (2), (4), and (18).

On introducing (33)–(35) into (32), we obtain

$$\mathbf{F}=\rho U(q\mathbf{e}_x+\boldsymbol{\Gamma}\times\mathbf{e}_x)-\mu H(q_m\mathbf{e}_m+\boldsymbol{\Gamma}_m\times\mathbf{e}_m), \tag{36}$$

where

$$\boldsymbol{\Gamma}=\int \mathbf{v}_0\times\mathbf{ds}=\Gamma\mathbf{e}_z, \quad \boldsymbol{\Gamma}_m=\int \mathbf{h}_0\times\mathbf{ds}=\Gamma_m\mathbf{e}_z. \tag{37}$$

The first two terms of (36) are the same as the results of the Kutta-Joukowski and Filon theorem[13] expressing the lift by $\rho U\Gamma$ and the drag by ρUq. The second two terms express a characteristic feature of our fields. We may note its negative sign. This is explained by the fact that the magnetic stress $\mu\mathbf{HH}$ is rather tensile in contrast to the inertial stress $-\rho\mathbf{VV}$, and that the force acting on the current $\mathbf{j}$ in the magnetic field $\mathbf{H}$ is $\mu\mathbf{j}\times\mathbf{H}$ in contrast to the force $-\rho\boldsymbol{\omega}\times\mathbf{V}$ acting on the vorticity $\boldsymbol{\omega}$ in the flow field $\mathbf{V}$.

On introducing (30) into (36), we find

$$D+iL=\rho U(q+i\Gamma)(1-\beta^2e^{2i\alpha}) \tag{38}$$

for the drag D and the lift L, for the cylinder.

This is the final formula; the force on the obstacle is expressed by only two quantities q and Γ, i.e., the total inward flow along the wakes and the circulation around the cylinder at infinity.

A corresponding formula, referred to the direction of the imposed magnetic field, is found to be

$$\begin{aligned}D_m+iL_m&=(D+iL)e^{-i\alpha}\\ &=\mu H(\beta^{-2}e^{-2i\alpha}-1)(q_m+i\Gamma_m). \end{aligned}\tag{39}$$

5. CASE OF ALIGNED FIELDS AT INFINITY

We find from Table I that $k_+>0$ and $k_+>|k_-|$, i.e., k_- wake is wider than k_+ wake [see Eq. (25)]. Also, $k_-\gtrless 0$ according as $1\gtrless\beta$; we have k_- wake in front of the body if $a>U$.

Let us consider the strength of two wakes, characterized by $q_\pm$ or $q_{m\pm}$. On letting $\alpha=0$ in (30) and (38), we obtain

$$\begin{aligned} q_m&=(H/U)q,\\ \Gamma_m&=(H/U)\Gamma,\\ D+iL&=(1-\beta^2)\rho U(q+i\Gamma). \end{aligned}\tag{40}$$

Then, (31) can be solved with respect to $q_\pm/q$:

$$q_\pm/q=\tfrac{1}{2}[1\mp(2\beta^2-1+\lambda)/N],$$
$$q_{m\pm}/q_m=\tfrac{1}{2}\beta^{-2}(1-\lambda\mp N)(q_\pm/q). \tag{41}$$

Equations (40) and (41) determine D from the total flux of fluid or the total magnetic flux along only one wake for $\beta>1$.

Equation (40) also shows that

$$q,q_m \gtrless 0 \quad \text{according as} \quad \beta \lessgtr 1, \tag{42}$$

as long as $D>0$. Because $q_-/q>0$ for $\beta>1$ (Table II), (42) implies that the fluid is pushed forward along the upstream wake as if a jet ($q_-<0$) in contrast to the downstream wake along which we have positive inward flow. On the other hand, we have positive outward magnetic flux along both wakes for $\beta>1$, in contrast to the case $\beta<1$; we have outward flux along the inner k_+ wake and the inward flux along the outer k_- wake ($H>0$) (Fig. 2).

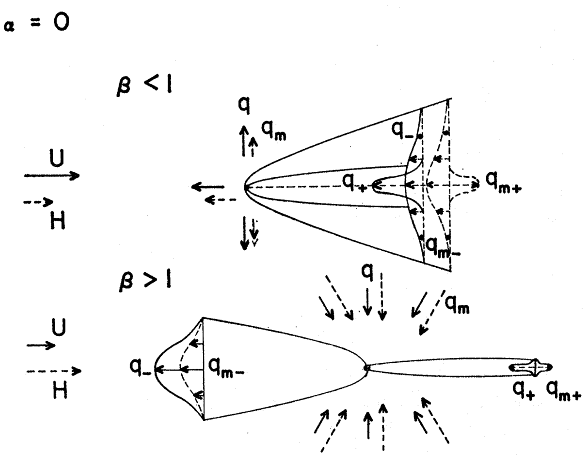

FIG. 2. Fields for $\alpha=0$, $\lambda<1$; for $\lambda>1$ only the relative strength of $\rho q_\pm^2$ with respect to $\mu q_{m\pm}^2$ is changed (see Table II).

The signs of $q_\pm$ and q suggest that the upstream wake is more dominant than the downstream wake, with regard to the total flux. In fact, we have

$$(q_+/q_-)^2 \le 1 \quad \text{for} \quad \infty \ge \beta^2 \ge \tfrac{1}{2}(1-\lambda),$$
$$(q_{m+}/q_{m-})^2 \le 1 \quad \text{for} \quad \infty \ge \beta^2 \ge 0. \tag{43}$$

In a limiting case of the strong magnetic field, we obtain

$$\lim_{\beta\to\infty} q_+/q_- = -1\,(\beta^2 \gg \lambda), \quad \lim_{\beta\to\infty} q_{m+}/q_{m-} = 1\,(\beta^2 \gg \lambda^{\pm 1}). \tag{44}$$

This is related to the case discussed by Stewartson[3] for $\nu=0$. He has shown that the forward and the rear regions bounded by the applied magnetic lines of force through the body are at rest with the body as if solid.

On the other hand, if we fix β and let λ be large enough ($\lambda \gg \beta^2$), we obtain

TABLE II.

	$k_\pm$	$\gamma_\pm$	$q_\pm/q$	$q_{m\pm}/q_m$	$\rho q_\pm^2-\mu q_{m\pm}^2$
$\beta<1$	+	−	+	−	sgn$(1-\lambda)$
	+	+	+	+	sgn$(\lambda-1)$
$\beta>1$	+	−	−	+	sgn$(1-\lambda)$
	−	+	+	+	

$$k_+/k_- \sim \lambda/(1-\beta^2), \quad |v_{+r}| \ll |v_{-r}|, \quad |h_{+r}| \ll |h_{-r}|$$
$$q_+/q_- \sim \beta^2(1-\beta^2)\lambda^{-2} \ll 1, \quad q_{m+}/q_{m-} \sim (1-\beta^2)\lambda^{-1} \ll 1, \tag{45}$$

i.e., the k_+ wake is negligible compared with the k_- wake as regards to the fluxes (Fig. 3).

For the perfectly conducting fluid, it is found that (1)–(5) have a solution derived from

$$\mathbf{H}=(\rho/\mu)^{\frac{1}{2}}\beta_*\mathbf{V}, \quad (\mathbf{V}\cdot\nabla)\beta_*=0, \quad \nabla\cdot\mathbf{V}=0,$$
$$\rho(1-\beta_*^2)(\mathbf{V}\cdot\nabla)\mathbf{V}=-\nabla P+(\rho\nu)\nabla^2\mathbf{V}. \tag{46}$$

β_* is constant along each streamline and is determined to be equal to β in our case, except in the region bounded by the obstacle and the separated streamline

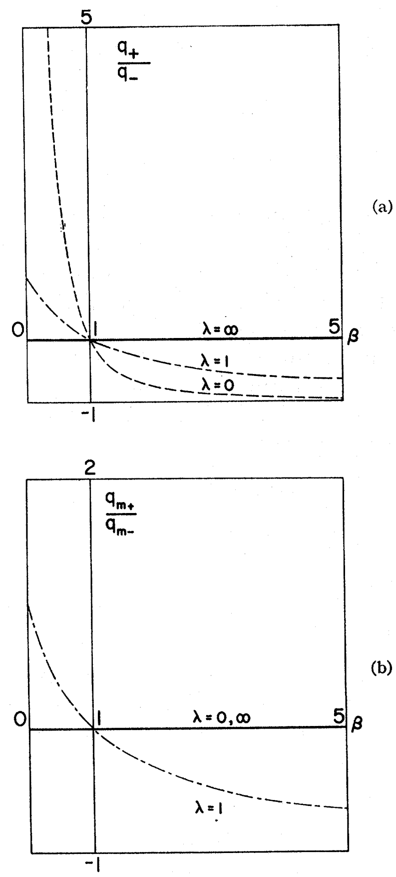

FIG. 3. Relative strengths of the mass and magnetic fluxes along two wakes for $\alpha=0$.

(if it exists) from the surface of the body. Letting $\beta_* = \beta$ everywhere, (this is one possibility and is allowable, at least when one looks at the problem in the large), (46) is reduced to the usual Navier-Stokes equations in which the density of the fluid is $\rho(1-\beta^2)$. For $\beta>1$ we have only to solve the problem for the fluid of density $\rho|1-\beta^2|$ with reversed flow direction at infinity and then reverse the flow and P.[7] In this manner we have an upstream wake for $U<a$. Equation (45) supports this conclusion.

For $\beta>1$, the blocking effect of the obstacle can be propagated upstream as shear waves in which vorticity and current can be shed away. If the inertia of the oncoming flow is negligible, this shedding is made in the equal manner in the upstream and the downstream. Under the influence of the oncoming flow, the vorticity and current pile up in front of the body and form a larger upstream wake, involving larger amounts of magnetic lines of force. This may explain the fact expressed in (43). Especially for $\lambda \gg 1$, the inner k_+ wake ($2k_+ \sim U/\kappa$) is confined near the surface of the body and the disturbance in the outside region is shed mainly into the upstream viscous wake ($2k_- \sim U(1-\beta^2)/\nu$). This feature is enforced for $\beta \gtrsim 1$, as shown in (45).

We may note the other special cases:

$$\lambda=1: \quad q_\pm/q = \tfrac{1}{2}(1\mp\beta), \quad q_{m\pm}/q_m = \mp\tfrac{1}{2}(1\mp\beta)\beta^{-1}, \quad (47)$$

and

$$\lambda \ll 1: \quad q_\pm/q \sim \begin{cases} 1-\beta^2 \\ \beta^2 \end{cases}, \quad q_{m\pm}/q_m \sim \begin{cases} -\lambda(1-\beta^2) \\ 1 \end{cases}. \quad (48)$$

For $\lambda=1$, q_+/q_-, and $-q_{m+}/q_{m-}$ are equal to $k_+/k_- = (U-a)/(U+a)$. Equation (48) is related to Lary's result[12] in the thin-wing theory showing an upstream wake for $\nu=0$, $|k_- t| \ll 1$ [t is the thickness of the body, and $2k_+ \sim U/\nu$, $2k_- \sim U(1-\beta^2)/\kappa$].

Another interesting quantity is the increase of the total radial stress in the wakes compared with that in the outside region, $\mathrm{sgn}(k_\pm)\delta\tau_\pm$. On making use of (6), (40), and (41), we obtain

$$\begin{aligned} \delta\tau_\pm &= 2(\rho U q_\pm - \mu H q_{m\pm}) = [1 \pm \epsilon\, \mathrm{sgn}(1-\lambda)]D, \\ \epsilon &= [1+4\beta^2\lambda(1-\lambda)^{-2}]^{-\frac{1}{2}}, \end{aligned} \quad (49)$$

which yields

$$\delta\tau_+ + \delta\tau_- = 2D, \quad \delta\tau_- \gtrless \delta\tau_+ \quad \text{for} \quad \lambda \gtrless 1, \quad (50)$$

i.e., the magnetic Prandtl number λ plays an important role in the partition of the drag into two wakes. We must note that P contributes $-D\mathbf{e}_x$ in (35).

Table II shows the signs of the main quantities in this section.

6. CASE $\lambda=1$

According to Table I, the main vorticity and current are shed into the direction parallel to $U\bar{e}_x \pm a\bar{e}_m$ rather than into the direction of the uniform flow; if $\alpha \neq 0$. We also find the equipartition of the induced flow energy and the induced magnetic energy in each wake:

$$\mathbf{h}_\pm = \mp(\rho/\mu)^{\frac{1}{2}}\mathbf{v}_\pm, \quad \text{i.e.,} \quad \tfrac{1}{2}\rho v_\pm^2 = \tfrac{1}{2}\mu h_\pm^2. \quad (51)$$

It is interesting to note that $\mathbf{v}_\pm$ and $\mathbf{h}_\pm$ are the exact solutions of (1)–(4) for $\lambda=1$ and $P=$constant.[8] For the negative value of $q_\pm$ they also represent a jet in the uniform magnetic field (we may put $U=0$ in this case).

7. CASE OF STOKES FLOW

In this case $\mathbf{k}_+$ and $\mathbf{k}_-$ are, respectively, parallel and antiparallel to $\mathbf{e}_m$, and both wakes have equal widths.

From (20) and Table I we obtain (taking x axis parallel to $\mathbf{e}_m$)

$$[\nabla^2 \mp (M/L)\partial/\partial x]\mathbf{v}_\pm = 0, \quad M = aL(\nu\kappa)^{-\frac{1}{2}} \quad (52)$$

and

$$\mathbf{h}_\pm = \mp(\rho\nu\sigma)^{\frac{1}{2}}\mathbf{v}_\pm, \quad \text{i.e.,} \quad \tfrac{1}{2}\mu h_\pm^2 = \lambda\tfrac{1}{2}\rho v_\pm^2, \quad (53)$$

where L is the typical length and M denotes the Hartmann number. We may consider $v_\pm$ to be of the same order as (or smaller than) U. Then,

$$(\mathbf{h}_\pm/H)^2 \sim \lambda\beta^{-2} \ll 1 \quad (\text{if } \beta^2 \gg \lambda). \quad (54)$$

Equations (52) and (54) are in accordance with the results of Chester in the study on the flow past a nonmagnetic body at very small Reynolds numbers, with $\alpha=0$.

8. AXISYMMETRIC CASE

Let $\mathbf{e}_m$ be parallel to the axis of symmetry (x axis) of the axisymmetric body, and $\mathbf{e}_z$ be perpendicular to the meridian plane. Then every discussion and formula in the previous section is valid with the following slight modifications, denoted by the prime applied to the corresponding equations. It is obvious that we should put $\alpha=0$, $\Gamma=0$ and replace a circle by a sphere.

$$v_{\pm r} = -(2\pi r)^{-1}|k_\pm| q_\pm e^{-\eta_\pm}, \quad (24')$$

$$-q_\pm = \lim_{r\to\infty} 2\pi \int_0^\pi r^2 \sin\theta v_{\pm r} d\theta \quad (27')$$

$$v_{0r} \sim q(4\pi r^2)^{-1}, \quad h_0 = (H/U)v_0 \quad (29')$$

$$D = \rho U q(1-\beta^2) = \mu H(\beta^{-2}-1)q_m, \quad (38')$$

which is an extension of Goldstein's theorem.[14]

9. DRAG AND LIFT FOR AN INSULATING CIRCULAR CYLINDER AT SMALL REYNOLDS AND HARTMANN NUMBERS

Let us consider an insulating circular cylinder of radius l with its center at the origin. We assume

(i) The magnetic permeability of the cylinder is the same as that of the fluid.

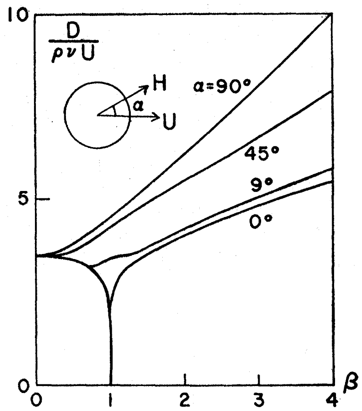

FIG. 4. Drag for an insulating circular cylinder for $R=R_m=0.1$.

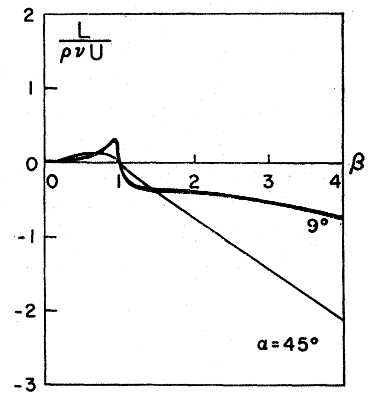

FIG. 5. Lift for an insulating circular cylinder for $R=R_m=0.1$.

(ii) The Reynolds number R and the magnetic Reynolds number R_m are small, i.e.,

$$R=Ul/\nu\ll 1,\quad R_m=Ul/\kappa\ll 1. \tag{55}$$

(iii)

$$M=(RR_m)^{\frac{1}{2}}\beta\ll 1, \tag{56}$$

which yields, along with (ii),

$$|k_\pm|l\ll 1 \tag{57}$$

Then, Eqs. (9)–(23) are considered to be uniformly valid in the whole domain outside of the cylinder, and $v_\pm$ is given by[16]

$$w_\pm=(v_x-iv_y)_\pm=-\epsilon_\pm e^{|k_\pm|r\cos\theta_\pm}\Big[A_{0\pm}K_0(|k_\pm|r)+\sum_{n=1}^{\infty}(A_{n\pm}e^{ni\theta_\pm}+\bar{A}_{n-1,\pm}e^{-ni\theta_\pm})K_n(|k_\pm|r)\Big], \tag{58}$$

$$\epsilon=e^{-i\alpha},\quad \epsilon_\pm=e^{-i\alpha_\pm},\quad \theta_\pm=\theta-\alpha_\pm$$
$$q_\pm=(\pi|k_\pm|)^{-1}\sum_{n=0}^{\infty}(A_n+\bar{A}_n)_\pm, \tag{59}$$

where K_n denotes the modified Bessel function of order n, and f'' of (23) is an analytic function of ζ for $|\zeta|\geq l$. The unknown coefficients A_n and f'' should be determined by the boundary conditions on the surface of the cylinder $\zeta\bar{\zeta}=l^2$:

$$w\equiv w_++w_-+Uf''=-U,\quad f''=\sum_{n=1}^{\infty}b_n\zeta^{-n} \tag{60}$$

$$m\equiv\gamma_+w_++\gamma_-w_-+H\bar{\epsilon}f''=g'\equiv\sum_{n=0}^{\infty}C_n\zeta^n, \tag{61}$$

where g' is an analytic function of ζ for $|\zeta|<l$ and actually expresses the harmonic magnetic field in the insulating cylinder, which is continuous to m from (i).

It is found that

$$A_{n\pm}=0[(k_\pm l)^n],\quad A_{0\pm}=A_\pm+0(k_\pm l). \tag{62}$$

Letting $k_\pm l\rightarrow 0$ and comparing the coefficients of ζ^{-1}, $\zeta^0\cdots$ in (60) and (61) we obtain for $A_\pm$, b_1, and C_0

$$Ub_1=\bar{A}_+/|k_+|+\bar{A}_-/|k_-|, \tag{63}$$

$$U=\epsilon_+[A_+T_++\tfrac{1}{2}\bar{A}_+]+\epsilon_-[A_-T_-+\tfrac{1}{2}\bar{A}_-], \tag{64}$$

$$H\bar{\epsilon}b_1=\gamma_+\bar{A}_+/|k_+|+\gamma_-\bar{A}_-/|k_-|, \tag{65}$$

$$C_0=-\gamma_+\epsilon_+[A_+T_++\tfrac{1}{2}\bar{A}_+]-\gamma_-\epsilon_-[A_-T_-+\tfrac{1}{2}\bar{A}_-], \tag{66}$$

where

$$T_\pm=-\gamma-\log(\tfrac{1}{2}|k_\pm|l),\qquad \gamma=0.5772\cdots\text{(Euler's constant)}. \tag{67}$$

On solving for b_1, we obtain

$$\frac{D+iL}{8\pi\rho\nu U}=\frac{(1-\beta^2\bar{\epsilon}^2)Ub_1}{4\nu}=\begin{cases}N[(N+1-\lambda)(T_++\tfrac{1}{2})+(N-1+\lambda)(T_-+\tfrac{1}{2})]^{-1}, & :\alpha=0 \quad (68)\\[1ex] \dfrac{[T\Theta-(1-\beta^2)(1-\beta^2\cos 2\alpha)]+i\beta^2(1-\beta^2)\sin 2\alpha}{T^2\Theta-(1-\beta^2)^2} & :\lambda=1 \quad (69)\\[1ex] \dfrac{[2T_M-\cos 2\alpha]-i\sin 2\alpha}{4T_M{}^2-1} & :\beta^2\gg 1, \quad (70)\end{cases}$$

where

$$T=-2\gamma-2\log(R/4)-\tfrac{1}{2}\log\Theta,$$
$$\Theta=1-2\beta^2\cos 2\alpha+\beta^4, \tag{71}$$
$$T_M=-\gamma-\log(M/4).$$

Equation (68) is in accordance with the result obtained by Yosinobu[10] independently. Equation (70) is found to be derived from (69) by letting $\beta\rightarrow\infty$ and is in accordance with Yosinobu and Kakutani's results[9] for $\alpha=0$ and $\alpha=\pi/2$, to the order so far obtained.

We may remark that:

(1) $D\sim 0$ for $\beta=1$, $\alpha=0$, since $T_{-}\sim\log|1-\beta^2|$. This is also related to the vanishing of k_{-}, and is also found in the other examples[6,7] of two-dimensional flow.

(2) $L=0$ for $\beta=0$, $\beta=1(\lambda=1)$, i.e., the inversion of the lift at $\beta=1$ from its positive value $(0<\alpha<\pi/2)$ for $\beta<1$ to the negative value for $\beta>1$.

Figures 4 and 5 show the value of $D/(\rho\nu W)$ and $L/(\rho\nu W)$ for $R=R_m=0.1$ and $\alpha=0°, 9°, 45°, 90°$. We may remark a peculiar behavior of D for small α, near $\beta\sim 1$. The increase of drag with α is expected.

On the other hand, the maximum lift angle is attained by an extremely small α if $\beta\sim 1$, in contrast to $\sim 45°$, which is the case for β far from 1.

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Some Remarks on Hydromagnetic Waves for Finite Conductivity

E. L. RESLER, JR.

Graduate School of Aeronautical Engineering, Cornell University, Ithaca, New York

IT is of some interest to discuss the form a "Friedrichs diagram"[1] might take for hydromagnetic waves when the conductivity of the fluid is finite. Plane waves in an otherwise undisturbed, electrically conducting fluid containing a uniform magnetic field $\mathbf{H}_0$ are governed by the equation[2]

$$\left[\nabla^2-\frac{1}{a_\infty^2}\frac{\partial^2}{\partial t^2}\right]\left[\frac{1}{4\pi\sigma}\nabla^2\frac{\partial\xi}{\partial t}-\frac{\partial^2\xi}{\partial t^2}+\frac{1}{4\pi\rho_\infty}(\mathbf{H}_0\cdot\nabla)^2\xi\right]=\frac{1}{\rho_\infty a_\infty^2}\mathbf{H}_0\times\nabla\left[\frac{\partial^2}{\partial t^2}\left\{\frac{\mathbf{H}_0}{4\pi}\cdot\nabla\times\xi\right\}\right]. \quad (1)$$

In Eq. (1) a_∞ is the ordinary sound speed, σ the conductivity, $\mathbf{H}_0$ the applied magnetic field, and t the time. $\xi=\nabla\times\mathbf{H}$, where $\mathbf{H}$ is the total magnetic vector made up of the applied field $\mathbf{H}_0$ and the wave-induced field $\mathbf{h}$. In the derivation of Eq. (1) it was assumed that $|\mathbf{h}|\ll|\mathbf{H}_0|$. Suppose we restrict attention to waves where the currents associated with the wave are perpendicular to the plane of the magnetic field and the wave normal. Consider Fig. 1 where the angle φ between the wave normal and the magnetic field vector is defined and treat waves propagating in the direction as shown.

Assume that at $y=0$, ξ varies as $e^{i\omega t}$, where ω is the frequency in radians per second. For a wave of the type under consideration ξ is proportional to $e^{i\omega t-(i\omega/c)y}$ $\cdot\partial/\partial x=\partial/\partial z=0$ and if $\mathbf{H}_0=(H_x,H_y)$, Eq. (1) leads to

$$\left[-\frac{\omega^2}{c^2}+\frac{\omega^2}{a_\infty^2}\right]\left[\frac{-i\omega^3}{4\pi\sigma c^2}+\omega^2-\frac{H_y^2\omega^2}{4\pi\rho_\infty c^2}\right]=\frac{\omega^4}{c^2}\frac{H_x^2}{4\pi\rho_\infty a_\infty^2}. \quad (2)$$

Denote the Alfvén speed by α so $\alpha^2=H_0^2/4\pi\rho_\infty$; then for $f=\omega/2\pi$ (2) becomes

$$c^4-c^2[a_\infty^2+\alpha^2+(if/2\sigma)]+a_\infty^2\alpha^2\cos^2\varphi+(if/2\sigma)a_\infty^2=0. \quad (3)$$

In the various limits (3) reduces to well-known expressions for c. As $\sigma\rightarrow\infty$ Friedrichs' expression follows for c; as $\sigma\rightarrow 0$ c equals the ordinary sound speed; as $\alpha\rightarrow 0$ we get ordinary sound waves and current diffusion (skin-depth phenomena); and as $a_\infty\rightarrow\infty$, incompressible Alfvén waves and again current diffusion. In all the limiting cases c is real and is the phase velocity.

If all the terms are retained in (3), then c is complex and its value depends upon the frequency f. Let

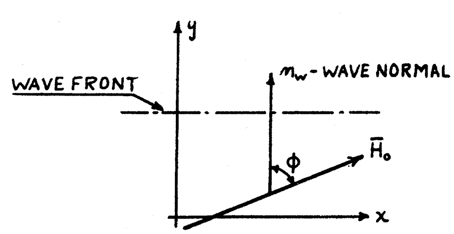

FIG. 1. Wave front traveling in the y direction whose normal n_w makes an angle φ with the magnetic field $\mathbf{H}_0$.

[1] K. O. Friedrichs and H. Kranzer, New York University Rept. No. 6486 (1954), reissued 1958.

[2] See Eq. (7) of E. L. Resler, Jr., and J. E. McCune, Revs. Modern Phys. 32, 848 (1960), this issue.