Some Exact Solutions in Linearized Magneto-aerodynamics for Arbitrary Magnetic Reynolds Numbers*

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INTRODUCTION

MAXWELL'S equations and Ohm's law combined with the equations governing the motion of a compressible inviscid fluid with scalar electrical conductivity are sufficient to describe many of the phenomena in magnetoaerodynamics. Since this set of equations is formidable, simplifications, where appropriate, are in order. A possible approach is that of treating small perturbations to some initial state and linearizing the equations about that initial state. When the initial state involves uniform flow and magnetic fields, the linear treatment has already been found to be adequate in describing many phenomena in magnetohydrodynamics. This success is not surprising in view of the widespread success of the linear treatment in aerodynamics and in view of the fact that the nonlinear terms that make solution of the general equations difficult occur in the fluid flow equations.

A general treatment of the linearized equations is presented and special solutions for the flow past an insulating sinusoidal wall with various magnetic field geometries are given. Arbitrary values of the conductivity and compressibility are included and some numerical estimates of magnetoaerodynamic effects observable in the laboratory are made.

LINEARIZED EQUATIONS

Throughout this treatment electromagnetic units (emu) are used, i.e., the magnetic flux density $\mathbf{B}$ equals the magnetic field $\mathbf{H}$ for nonferromagnetic media. The equations are linearized about an initial state involving a uniform fluid stream and a uniform magnetic field, as described elsewhere.[1] A feature of this linearization is the specification that the currents are perturbation quantities, thus implying the existence of a uniform electric field to balance the induced electromotive forces. The linearization therefore results in particle isentropic flow. The magnetohydrodynamic interaction term retained in this case is the electric body force in the equation of motion. Suppose the x axis is aligned with the main stream velocity $\mathbf{U}$. If the total fluid velocity vector $\mathbf{q}=\mathbf{U}+\mathbf{v}$ and the total field $\mathbf{H}=\mathbf{H}_0+\mathbf{h}$, where $|\mathbf{v}|\ll U$ and $|\mathbf{h}|\ll H_0$, then the linearized versions of the usual equations of magnetoaerodynamics are

Continuity equation:

$$\partial\rho/\partial t+U\partial\rho/\partial x+\rho_\infty\nabla\cdot\mathbf{v}=0. \tag{1}$$

Equation of motion:

$$\frac{\partial\mathbf{v}}{\partial t}+U\frac{\partial\mathbf{v}}{\partial x}+\frac{1}{\rho_\infty}\nabla p=\frac{J\times\mathbf{H}}{\rho_\infty}=\frac{\boldsymbol{\xi}\times\mathbf{H}}{4\pi\rho_\infty}$$
$$=\frac{1}{4\pi\rho_\infty}\{\mathbf{H}_0\cdot\nabla\mathbf{h}-\tfrac{1}{2}\nabla H^2\}, \tag{2}$$

where Ampère's law $\boldsymbol{\xi}=\nabla\times\mathbf{H}=4\pi J$ has been used. Here ρ is the mass density of the fluid, p the pressure, and J the current density.

Equation for magnetic flux:

$$\partial\mathbf{H}/\partial t=\mathbf{H}_0\cdot\nabla\mathbf{v}-U(\partial/\partial x)\mathbf{H}-\mathbf{H}_0\nabla\cdot\mathbf{v}+(1/4\pi\sigma)\nabla^2\mathbf{H}, \tag{3}$$

obtained by eliminating the electric-field vector $\mathbf{E}$ between Faraday's law $\nabla\times\mathbf{E}=-\partial\mathbf{H}/\partial t$ and Ohm's law $J=\sigma(\mathbf{E}+\mathbf{q}\times\mathbf{H})$ and using Ampère's law and $\nabla\cdot\mathbf{H}=0$; σ denotes the electrical conductivity, which is assumed to be constant.

By taking the divergence of (2) and using the fact that the flow is isentropic so $(dp/d\rho)^{\frac{1}{2}}=a_\infty$ (the ordinary sound speed), and defining the operator D/Dt as $\partial/\partial t+U\partial/\partial x$, one obtains

$$\nabla^2 p-(1/a_\infty^2)(D/Dt)^2p=\mathbf{H}_0/4\pi\cdot\nabla\times\boldsymbol{\xi}. \tag{4}$$

On using the curl operator on (2) and letting $\boldsymbol{\Omega}=\nabla\times\mathbf{v}$, one arrives at

$$D\boldsymbol{\Omega}/Dt=\mathbf{H}_0/4\pi\rho_\infty\cdot\nabla\boldsymbol{\xi}. \tag{5}$$

Upon taking curl of (3), using (1), and again the isentropic relation between p and ρ, it follows that

$$D\boldsymbol{\xi}/Dt=\mathbf{H}_0\cdot\nabla\boldsymbol{\Omega}-(1/\rho_\infty)(\mathbf{H}_0\times\nabla)(Dp/Dt)$$
$$+(1/4\pi\sigma)\nabla^2\boldsymbol{\xi}. \tag{6}$$

By eliminating $\boldsymbol{\Omega}$ between (5) and (6) and subsequently using (4), one obtains an equation for $\boldsymbol{\xi}$ alone:

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[1] E. L. Resler, Jr., and J. E. McCune, "Electromagnetic interaction with aerodynamic flows," in *The Magnetodynamics of Conducting Fluids,* D. Bershader, Editor (Stanford University Press, Stanford, California, 1959), p. 120.

$$\left[\nabla^2-\frac{1}{a_\infty^2}\left(\frac{D}{Dt}\right)^2\right]\left\{\frac{1}{4\pi\sigma}\nabla^2\frac{D\xi}{Dt}-\frac{D^2\xi}{Dt^2}+\frac{1}{4\pi\rho_\infty}(\mathbf{H}_0\cdot\nabla)^2\xi\right\}$$
$$=\frac{1}{\rho_\infty a_\infty^2}(\mathbf{H}_0\times\nabla)\left[\frac{D^2}{Dt^2}\left\{\frac{\mathbf{H}_0}{4\pi}\cdot\nabla\times\xi\right\}\right]. \quad (7)$$

Thus, for small disturbance situations where the unperturbed velocity and magnetic fields are uniform, one need solve only the differential equation (7) for ξ.

LINEARIZED STEADY PLANE FLOW

While Eq. (7) covers many phenomena including stability criteria in appropriate cases, our purpose here is to consider only steady plane flows. For this case it is possible to obtain certain exact solutions to the linearized equations for arbitrary σ while including compressibility effects, and an examination of these solutions yields information concerning magnetoaerodynamic phenomena. For steady flows Eq. (7) becomes

$$\left[\nabla^2-\frac{U^2}{a_\infty^2}\frac{\partial^2}{\partial x^2}\right]\left\{\frac{U}{4\pi\sigma}\nabla^2\frac{\partial\xi}{\partial x}-U^2\frac{\partial^2\xi}{\partial x^2}+\frac{1}{4\pi\rho_\infty}(\mathbf{H}\cdot\nabla)^2\xi\right\}$$
$$=\frac{1}{\rho_\infty a_\infty^2}(\mathbf{H}_0\times\nabla)\left[U^2\frac{\partial^2}{\partial x^2}\left\{\frac{\mathbf{H}_0}{4\pi}\cdot\nabla\times\xi\right\}\right]. \quad (8)$$

For plane flows $\boldsymbol{\xi}=(0,0,\xi)$, and (8) becomes an equation for a scalar rather than a vector quantity. If $\mathbf{H}_0=(H_{0x},H_{0y},0)$ and $\alpha_x^2=H_{0x}^2/4\pi\rho_\infty$, $\alpha_y^2=H_{0y}^2/4\pi\rho_\infty$, $M=U/a_\infty$, then (8) takes the form

$$\left(\nabla^2-M^2\frac{\partial^2}{\partial x^2}\right)\left\{\frac{U}{4\pi\sigma}\nabla^2\frac{\partial\xi}{\partial x}+(\alpha_x^2-U^2)\frac{\partial^2\xi}{\partial x^2}+\alpha_y^2\frac{\partial^2\xi}{\partial y^2}\right\}$$
$$=\left(M^2\alpha_x^2\frac{\partial^2}{\partial x\partial y}-2\alpha_x\alpha_y\nabla^2\right)\frac{\partial^2\xi}{\partial x\partial y}+M^2\alpha_y^2\frac{\partial^4\xi}{\partial x^4}. \quad (9)$$

If the disturbances considered extend to plus and minus infinity in the x direction periodically, e.g., as in the case of flow over a sinusoidal wall or over a system of currents resulting in a periodic magnetic field at the boundary, then the quantity ξ can be synthesized from components of the form $\xi_\lambda=Ke^{i\lambda x}e^{-\beta y}$. In the present discussion we consider only a purely sinusoidal disturbance with wavelength $l=2\pi/\lambda$. Then, substituting in (9) for ξ, we obtain an algebraic equation for $\beta(\lambda)$. If we define the magnetic Reynolds number R_m as σUl, then

$$(\beta^2-\lambda^2)\left[\beta^2\left(\alpha_y^2+\frac{iU^2}{2R_m}\right)-\beta 2i\lambda\alpha_x\alpha_y+\lambda^2\left(U^2-\alpha_x^2-\frac{iU^2}{2R_m}\right)\right]$$
$$=-M^2\left[\beta^2\lambda^2\left(\alpha_y^2+\alpha_x^2+\frac{iU^2}{2R_m}\right)+\lambda^4\left(U^2-\alpha_x^2-\alpha_y^2-\frac{iU}{2R_m}\right)\right]. \quad (10)$$

In general we have a quartic equation for β with complex coefficients, which becomes, if we collect the terms, and let $\alpha^2=\alpha_x^2+\alpha_y^2$,

$$\beta^4\left[\alpha_y^2+\frac{iU^2}{2R_m}\right]+\beta^3[-2i\lambda\alpha_x\alpha_y]+\beta^2\lambda^2\left[U^2+\alpha^2(M^2-1)+\frac{iU^2}{2R_m}(M^2-2)\right]+2\beta i\lambda^3\alpha_x\alpha_y-\lambda^4\left[(U^2-\alpha_x^2)+M^2(\alpha^2-U^2)+\frac{iU^2}{2R_m}(M^2-1)\right]=0. \quad (11)$$

Since β is in general complex, assume $\beta=A+iB$; then equating the real and imaginary parts of (11) separately to zero gives two equations for A and B. On equating the real parts to zero one obtains

$$A^4\alpha_y^4+A^3B\left(-\frac{2U^2}{R_m}\right)+A^2[-6\alpha_y^2B^2+\lambda^2U^2+\lambda^2\alpha^2(M^2-1)+6B\lambda\alpha_x\alpha_y]+A\left[B\lambda^2(2-M^2)\frac{U^2}{R_m}+2B^3\frac{U^2}{R_m}\right]-2\alpha_x\alpha_yB^3\lambda+B^4\alpha_y^2-B^2\lambda^2[U^2+(M^2-1)\alpha^2]$$
$$-2\alpha_x\alpha_yB\lambda^3+\lambda^4[(M^2-1)U^2+\alpha_x^2-M^2\alpha^2]=0, \quad (12)$$

and equating the imaginary part to zero gives

$$A^4\frac{U^2}{2R_m}+A^3(4B\alpha_y^2-2\lambda\alpha_x\alpha_y)+A^2\left(-3B^2\frac{U^2}{R_m}-\lambda^2\frac{U^2}{R_m}+M^2\lambda^2\frac{U^2}{2R_m}\right)+A(-4B^3\alpha_y^2+6B^2\lambda\alpha_x\alpha_y+2B\lambda^2U^2-2B\lambda^2\alpha^2+2BM^2\lambda^2\alpha^2+2\lambda^3\alpha_x\alpha_y)+B^4U^2\frac{1}{2R_m}$$
$$+B^2\lambda^2\frac{U^2}{R_m}-M^2B^2\lambda^2\frac{U^2}{2R_m}+\frac{U^2}{2R_m}\lambda^4(1-M^2)=0. \quad (13)$$

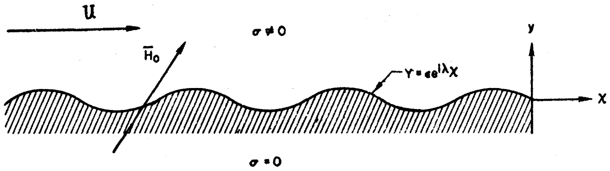


FIG. 1. Flow past a wavy wall where the applied magnetic field is uniform and oriented at an arbitrary angle with respect to the undisturbed free-stream velocity U.

INCOMPRESSIBLE FLOW

As a simple example of the method of solution of a given boundary value problem, consider the flow of a conducting fluid past an insulated "wavy wall" as depicted in Fig. 1. For this case Eq. (10) becomes, after discarding the root $\beta^2=\lambda^2$ as not appropriate,

$$\frac{\beta^2}{\lambda^2}\left[2R_m\frac{\alpha_y^2}{U^2}+i\right]-4R_m i\frac{\alpha_x\alpha_y}{U^2}\frac{\beta}{\lambda}+2R_m\left(1-\frac{\alpha_x^2}{U^2}\right)-i=0 \tag{14}$$

or

$$\frac{\beta}{\lambda}=\frac{4\frac{\alpha_x\alpha_y}{U^2}R_m i\pm\left\{\left(4\frac{\alpha_x\alpha_y}{U^2}R_m i\right)^2-4\left[2R_m\frac{\alpha_y^2}{U^2}+i\right]\left[2R_m\left(1-\frac{\alpha_x^2}{U^2}\right)-i\right]\right\}^{\frac{1}{2}}}{2\left[2R_m\frac{\alpha_y^2}{U^2}+i\right]}. \tag{15}$$

If we let R_m approach infinity in Eq. (15), then β/λ approaches $(iU/\alpha_y)[(\alpha_x/U)\pm1]$. Since β is pure imaginary in this limit, there is no decay with y of the disturbance and ξ remains constant along lines of slope $dy/dx=\alpha_y/(\alpha_x\pm U)$. Thus if $U=0$, the disturbance moves along the magnetic field lines and can be identified with the Alfvén wave mechanism. With a free-stream velocity U, however, the currents also drift with the stream. From a fixed point in the field currents move along the y axis at the speed $\pm\alpha_y$ and along the x axis at the speed $U\pm\alpha_x$; thus if disturbances originate at the wall under consideration, we must retain only the root $\beta/\lambda=(iU/\alpha_y)[(\alpha_x/U)+1]$. If we call the appropriate root β_1, then

$$\xi=K_1e^{i\lambda x-\beta_1 y}. \tag{16}$$

By using Eq. (5) to find Ω and noting that the arbitrary function of y introduced must be zero as it is not periodic in x, it follows that

$$\Omega=\frac{H_x-(\beta_1/i\lambda)H_y}{4\pi\rho U}K_1e^{i\lambda x-\beta_1 y}, \tag{17}$$

where K_1 is a constant. Since $\nabla\cdot\mathbf{v}=0$, it is convenient to define a stream function ψ so that $u=-\psi_y$ and $v=\psi_x$. Then

$$\nabla^2\psi=\Omega, \tag{18}$$

or

$$\psi=K_2e^{i\lambda x-\beta_1 y}+K_3e^{i\lambda(x+iy)}, \tag{19}$$

where

$$K_2=K_1[H_x+(i\beta_1/\lambda)H_y]/4\pi\rho_\infty U(\beta_1^2-\lambda^2).$$

One can now easily find u and v, and to satisfy the boundary condition of the wall being a streamline to first order $[v(y=0)=iU\epsilon\lambda e^{i\lambda x}]$, it follows that

$$K_2+K_3=U\epsilon. \tag{20}$$

If we also define a stream function for the magnetic field $\mathbf{h}$ in the upper half-plane so $h_x=-S_y$ and $h_y=S_x$, then $\nabla^2S=\xi$ and

$$S=K_4e^{i\lambda x-\beta_1 y}+K_5e^{i\lambda(x+iy)}, \tag{21}$$

where $K_4=K_1/(\beta_1^2-\lambda^2)$.

On recalling that $\nabla\times\mathbf{E}=0$ and $\xi=0$ when the perturbations vanish, we find $E_z=\text{constant}=-UH_y$. It follows from Ohm's law that

$$\xi=4\pi\sigma[uH_y+Uh_y-vH_x]. \tag{22}$$

On substituting for the perturbation quantities u, h_y, and v, Eq. (22) imposes a condition on the harmonic parts of the solution which were introduced, namely,

$$K_3(H_x+iH_y)=UK_5. \tag{23}$$

At $y=0$, h_x and h_y are continuous across the boundary since no surface currents can flow for finite conductivity and steady flow. Moreover, since there are no currents flowing within the wall (i.e., in the lower half-plane) as it is an insulator, it follows that the stream function for $\mathbf{h}$ in that region satisfies Laplace's equation. It is then easily shown that necessarily $h_x=ih_y$ at $y=0$, so

$$K_4=K_5[-2\lambda/(\beta_1+\lambda)]. \tag{24}$$

We now have five relations between five complex coefficients K_1 through K_5. In many cases one is also interested in the pressure, which can be found using Eq. (4) and a component of the equation of motion (2) to determine the coefficient of the solution to the homogeneous equation. As $y\rightarrow\infty$, $p\rightarrow p_\infty$, so the pressure is given by

$$p=K_6e^{i\lambda x-\beta_1 y}+K_7e^{i\lambda(x+iy)}+p_\infty, \tag{25}$$

where

$$K_6=(K_1/4\pi)[(H_x\beta_1+i\lambda H_y)/(\lambda^2-\beta_1^2)],$$

$$K_7=-\rho_\infty U\lambda K_3.$$

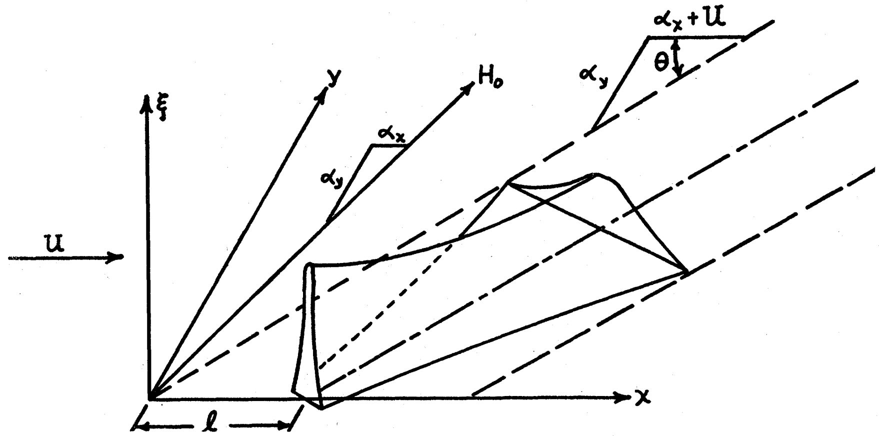
The drag per wavelength of wall can be found:

$$D=\int_0^{2\pi/\lambda}p\,dY=\int_0^{2\pi/\lambda}p\left(\frac{dY}{dx}\right)dx, \tag{26}$$

and if $K_6=F_6e^{i\theta_6}$ and $K_7=F_7e^{i\theta_7}$, then

$$D=\epsilon\pi\{F_6\sin\theta_6+F_7\sin\theta_7\}. \tag{27}$$

FIG. 2. Disturbances produced in an incompressible fluid in a uniform magnetic field propagate along the magnetic field lines as Alfvén waves relative to the fluid.



For convenience the K's are tabulated:

$$K_1 = \frac{2\lambda\epsilon(\beta_1^2-\lambda^2)(H_x+iH_y)}{2\lambda(\alpha_x^2/U^2)-2\beta_1(\alpha_y^2/U^2)+2i(\alpha_x\alpha_y/U^2)(\beta_1+\lambda)-(\beta_1+\lambda)}$$

$$K_2 = \{[H_x+(i\beta_1/\lambda)H_y]/4\pi\rho U(\beta_1^2-\lambda^2)\}K_1$$

$$K_3 = [U/2\lambda(\lambda-\beta_1)(H_x+iH_y)]K_1 \qquad (28)$$

$$K_4 = K_1/(\beta_1^2-\lambda^2)$$

$$K_5 = K_1/2\lambda(\lambda-\beta_1)$$

$$K_6 = (K_1/4\pi)[(H_x\beta_1+i\lambda H_y)/(\lambda^2-\beta_1^2)] = F_6 e^{i\theta_6}$$

$$K_7 = \tfrac{1}{2}(\rho_\infty U^2)[K_1/(\beta_1-\lambda)(H_x+iH_y)] = F_7 e^{i\theta_7}.$$

For the "aligned fields" case, that is, $H_y=0$, so that the velocity and magnetic fields are aligned, one finds

$$\xi = \frac{4i\epsilon\lambda^2 H_x[1-(\alpha_x^2/U^2)]R_m}{2\alpha_x^2-1-\{1+i2R_m[1-(\alpha_x^2/U^2)]\}^{\frac{1}{2}}} \times \exp\left(i\lambda x-\lambda\left\{\left[1+i2R_m\left(1-\frac{\alpha_x^2}{U^2}\right)\right]^{\frac{1}{2}}\right\}y\right), \qquad (29)$$

Note that ξ becomes $1/e$ of its value at $y=0$ at a distance

$$y=\delta=\mathrm{Re}(\lambda^{-1}\{1+i2R_m[1-(\alpha_x^2/U^2)]\}^{-\frac{1}{2}}),$$

where Re denotes the real part. For large R_m the currents are confined in a layer δ whose thickness is proportional to $l/\{R_m[1-(\alpha_x^2/U^2)]\}^{\frac{1}{2}}$, and in the limit as $R_m \rightarrow \infty$ the coefficient of the exponential goes as $(R_m)^{\frac{1}{2}}$ so the current layer approaches a current sheet. When the current layer is thin, the magnetohydrodynamic effects are limited to what can be called a "boundary layer" analogous in many ways to the viscous boundary layer in regular fluid dynamics.

In all the other incompressible cases the results for large R_m can be interpreted as follows. By using Fig. 2, the currents penetrate the fluid at an angle θ, where $\sin\theta=\alpha_y/[(\alpha_x+U)^2+\alpha_y^2]^{\frac{1}{2}}$, i.e., as we have seen for $R_m \rightarrow \infty$, the mechanism of current penetration is Alfvén wave propagation relative to the fluid. As the currents propagate away from the wall they diffuse into one another because of the finite conductivity. If we let a typical diffusion time $\tau=\sigma l^2\sin^2\theta$ and let $\tau=y_d/\alpha_y$, where y_d is a distance typical of the penetration depth before diffusion smears the waves,

$$y_d=\sigma l U\frac{(\alpha_y/U)^3}{[(\alpha_x/U)+1]^2+(\alpha_y/U)^2}. \qquad (30)$$

It can readily be checked that this result is in perfect agreement with the penetration distance obtained from expansion of Eq. (15) in inverse powers of R_m.

Figures 3 and 4 show drag coefficients per wavelength for the wavy wall for the cases $H_y=0$, and $H_x=0$, respectively. In the aligned fields case ($H_y=0$, Fig. 3), the drag is zero at $R_m=0$ and $R_m=\infty$ and reaches a maximum in between. In this case a large R_m approximation is good at about $R_m=10^4$. This is to be contrasted with the "crossed fields" case ($H_x=0$, Fig. 4), where the drag is zero at $R_m=0$ but approaches a finite value as $R_m \rightarrow \infty$. This is because the Alfvén waves transport momentum away from the wall and although the flow is incompressible there is a "wave drag." In this case a large R_m theory is valid at about $R_m=10$.

COMPRESSIBLE FLOWS

While compressible flows for arbitrary angle between the velocity and magnetic field can in principle be treated with the above relations, in practice the computation tends to be long and tedious. To illustrate the effect the electromagnetic interaction has on a compressible flow field, suppose Eq. (10) is treated for the aligned fields case, i.e., $\alpha_y=0$, and also for the moment in the limit $R_m \rightarrow \infty$. (The aligned fields case is of special interest as it is easily attained experimentally.) Then it follows that

$$\beta^2/\lambda^2=\{[(U^2/\alpha_x^2)-1](1-M^2)/[(U^2/\alpha_x^2)+M^2-1]\}. \qquad (31)$$

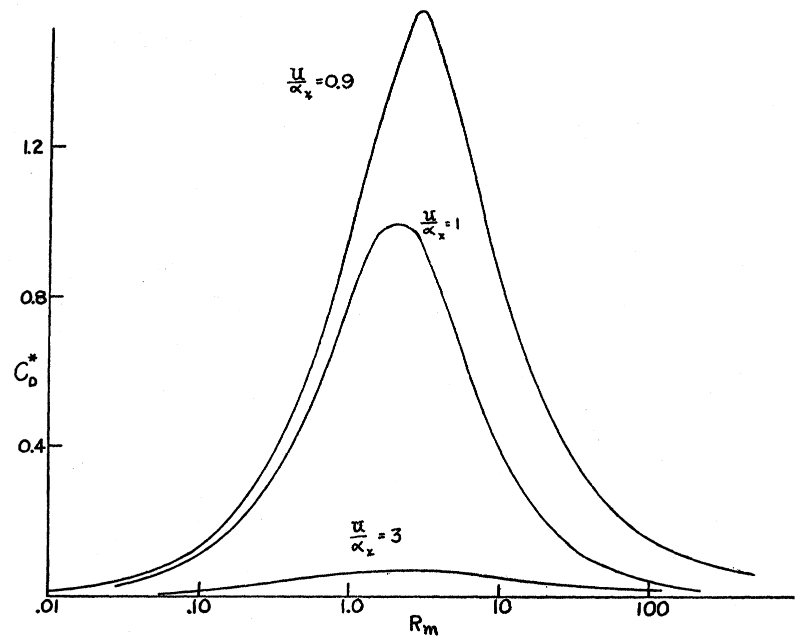


FIG. 3. Drag coefficient $C_D{}^*$ plotted vs $R_m=\sigma Ul$ for various values of the ratio U/α_x for the case where U is aligned with $\mathbf{H}_0$.

$$C_D{}^*=\frac{\text{Drag per wavelength}}{\frac{1}{2}\rho_\infty U^2 l(\lambda\epsilon)^2}.$$

Note that our solution consists of standing waves in the flow if β is imaginary or $\beta^2/\lambda^2<0$, and that otherwise the flow has an elliptic nature. This is consistent with previous work[2] in which other field orientations were also treated in the limit $R_m\to\infty$. In the present case the different regions are conveniently represented on the U/α_x vs M diagram shown in Fig. 5.

Probably the most interesting region in this limiting case is the range $U/\alpha_x<1$ and $M<1$. In the hyperbolic part of this region the standing waves are forward facing. The appearance of these forward-facing waves has recently been interpreted by the authors[1] in terms of the basic "pulse" solutions of magnetoacoustics; they have also been noted by Kogan and by Taniuti.[3]

Forward facing waves are not the only example of disturbances moving counter to the fluid stream in magnetohydrodynamics. Forward facing "wakes" have been noted by Hasimoto, by Greenspan and Carrier, and by Lary,[4] for incompressible flow. It is believed that such forward-facing "wakes" also appear for compressible flows, for example, in the elliptic region in the lower left-hand corner of Fig. 5.

Extension of the analysis to finite R_m, because it introduces a diffusion process, enables one to verify that the forward-facing waves are indeed the proper choice in the above hyperbolic region, since along these waves the disturbances introduced by the wall diminish, while along the rearward-facing waves they are amplified. Also, one can determine in this way the magnetic Reynolds number necessary to observe the phenomenon

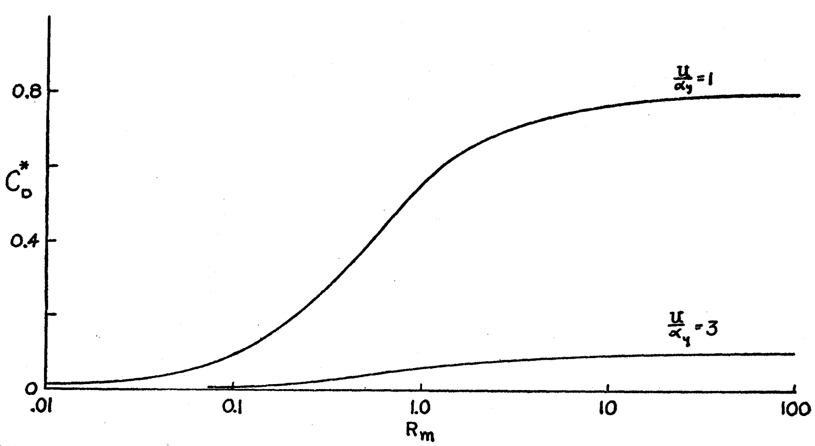


FIG. 4. Drag coefficient $C_D{}^*$ plotted vs $R_m=\sigma Ul$ for various values of the ratio U/α_y for the case, where U is perpendicular to $\mathbf{H}_0$.

$$C_D{}^*=\frac{\text{Drag per wavelength}}{\frac{1}{2}\rho_\infty U^2 l(\lambda\epsilon)^2}.$$

[2] J. E. McCune and E. L. Resler, Jr., J. Aero/Space Sci. **27**, 493 (1960).
[3] T. Taniuti, Progr. Theoret. Phys. (Kyoto) **19**, 749 (1958); M. N. Kogan, Priklad. Mat. Mech. **23**, 70 (1959).
[4] H. Hasimoto, Phys. Fluids **2**, 338 (1959); H. P. Greenspan and G. F. Carrier, J. Fluid Mech. **6**, 77 (1959); E. C. Lary, Ph.D. thesis, Cornell University, 1960.

of forward-facing waves experimentally. For $\alpha_y=0$ Eq. (12) becomes

$$(U^2/R_m)D_2=\lambda^2[U^2+\alpha_x^2(M^2-1)](A^2-B^2)+\lambda^4(M^2-1)(U^2-\alpha_x^2); \quad (32)$$
$$D_2\equiv AB[2(A^2-B^2)+(M^2-2)\lambda^2],$$

while (13) becomes

$$(U^2/R_m)D_1=-AB\lambda^2[2U^2+2\alpha_x^2(M^2-1)];$$
$$D_1=\tfrac{1}{2}A^4+A^2(\tfrac{1}{2}M^2\lambda^2-\lambda^2-3B^2)+\tfrac{1}{2}B^4+B^2\lambda^2-\tfrac{1}{2}M^2B^2\lambda^2+\tfrac{1}{2}\lambda^4(1-M^2). \quad (33)$$

While the last two equations are difficult to solve for A and B, it is rather straightforward to take as our unknowns α_x^2 (or M^2), and R_m. With this point of view, it is convenient to write

$$\frac{\alpha_x^2}{U^2}=\frac{2ABD_2+[A^2-B^2+\lambda^2(M^2-1)]D_1}{-2AB(M^2-1)D_2+(M^2-1)[\lambda^2-A^2+B^2]D_1}, \quad (34)$$

and

$$R_m=\frac{D_2}{\lambda^2[1+(\alpha_x^2/U^2)(M^2-1)](A^2-B^2)+\lambda^4(M^2-1)[1-(\alpha_x^2/U^2)]}. \quad (35)$$

Recall that A is a measure of the damping and B of the slope of the wave. If m is the wave angle, then

$$m=\tan^{-1}(dy/dx)=\tan^{-1}(\lambda/B). \quad (36)$$

Suppose we ask what R_m is necessary so the current generated at the wall is damped to $1/e$ times its wall value a distance l (=the wavelength, $2\pi/\lambda$) along the wave. Then A and B are related as follows:

$$A/\lambda=[(B/\lambda^2)+1]^{\frac{1}{2}}/2\pi. \quad (37)$$

Thus, choosing B/λ determines both the wave slope and A/λ for this condition; and α_x^2/U^2, and subsequently R_m, can be calculated for given M. (Alternatively, M^2 and R_m can be calculated for given α_x^2/U^2.) In Fig. 6 R_m vs B/λ is plotted for $M=\frac{1}{2}$, with α_x/U varying along the vertical dotted line shown in Fig. 5. Also in Fig. 6, R_m is plotted vs B/λ for $\alpha_x/U=2$ with M varying along the horizontal dotted line shown in Fig. 5. In either case R_m in Fig. 6 is the value of the magnetic Reynolds number that must be achieved in order that the forward-facing waves not damp out before propagating into the flow field one wavelength.

For $M=\frac{1}{2}$ there is a minimum R_m of about 16 at $B/\lambda=-0.5$, i.e., for a wave tilted about 26° forward of the normal. This is a difficult R_m to achieve in the laboratory. For a wave swept forward the same amount but along the curve for $\alpha_x/U=2$, one needs an R_m of only 1.6, and the R_m continually decreases as $M=1$ is approached. It seems best to work at a fixed Alfvén number and as close to $M=1$ as possible to facilitate observation of forward facing waves.

However, this calculation has been carried out for disturbances involving only a single harmonic. There

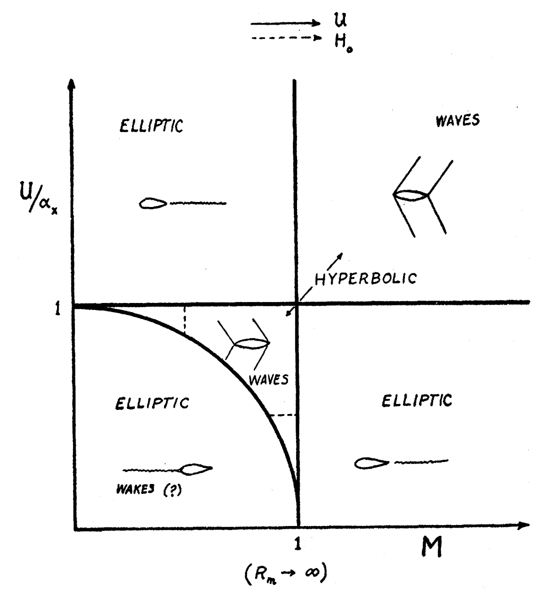


FIG. 5. Various flow regimes in the U/α_x, M plane. The dotted lines refer to the curves plotted in Fig. 6.

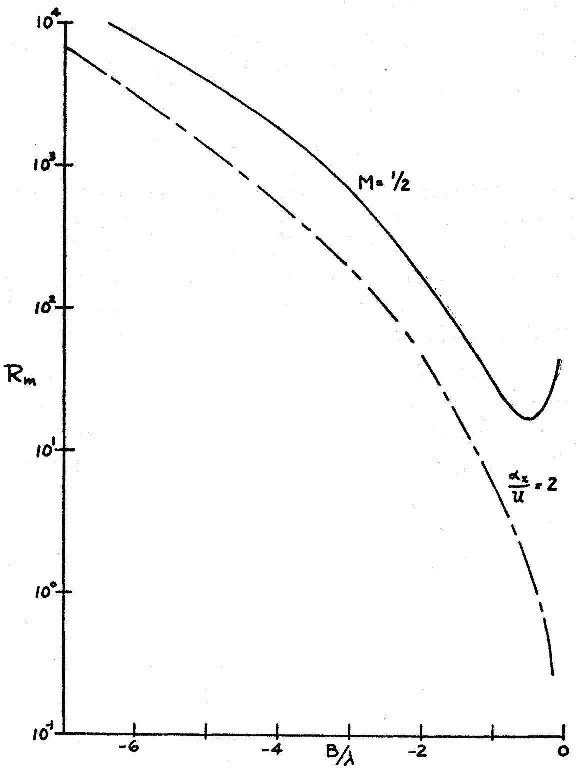


FIG. 6. The magnetic Reynolds number ($R_m=\sigma Ul$) required for currents to damp to $1/e$ of the wall value in a distance l along the waves vs the cotangent of the wave angle B/λ.

is some evidence[5] that sharp disturbances (involving many harmonics) may propagate more deeply into the flow field for a given R_m and thus enhance the possibility of observation.

CONCLUSIONS

The compressible linearized equations of magnetoaerodynamics can be solved in some simple cases. These examples serve to illustrate many of the phenomena common to this field. The addition of the electromagnetic equations to the fluid flows leads to Alfvén waves so that even incompressible flows have a wave character. Finite electrical conductivity introduces current diffusion, while compressibility effects bring in sound waves modified by the currents and Alfvén waves modified by the compressibility. The interaction is complex and depends strongly on the magnetic field strength and geometry.

The drag coefficient per unit wavelength of a "wavy wall" has been plotted for two field geometries and various field strengths. The appropriateness of approximate theories for large electrical conductivity is seen to depend not only on R_m but also on the geometry of the field. The geometry governs whether the flow field is affected by an Alfvén wave mechanism or current diffusion, the wave mechanism being by far the more powerful.

Under certain conditions the magnetic field makes possible the propagation of disturbances counter to the fluid stream. This phenomenon is strikingly different from what is usually observed in conventional fluid mechanics. An estimate of the experimental conditions required to observe these effects has been presented in Fig. 6 and the conditions are found to be attainable in the laboratory.

[5] J. E. McCune, J. Fluid Mech. **7**, 449 (1960).