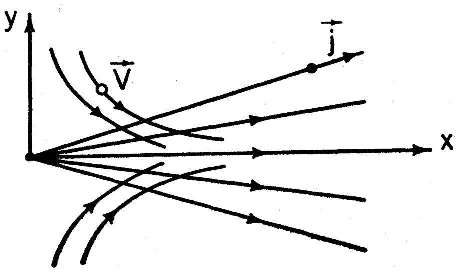

FIG. 7. Ejection of a gas by a discharge.

basis of the equation of conservation of impulse is always accompanied by motion of the medium, even if the latter is at rest at infinity.[6]

The main parameter that is an analogy of the magnetic Reynolds number for this type of discharge is the value $R=I/\nu_m c\rho^{\frac{1}{2}}$ (I denotes the total current in a discharge).

In case of large values R ($R\gg1$), the discharge has the character of a boundary layer with the thickness L/R, and the y component of the momentum equation is

$$\partial p_m/\partial y=-(H^2/4\pi y),$$

hence

$$p_m=p_{m\infty}+\int_y^\infty \frac{H^2dy}{4\pi y}$$

[6] V. N. Zhigulev, Doklady Akad. Nauk. S. S. S. R. **130**, 280 (1960).

taking into consideration that $\partial|H|/\partial x<0$ for the discharge in Fig. 7, we obtain for a case of an inviscid gas

$$du/dt>0,$$

where u is the horizontal component of the velocity vector. The pattern of the flow is as shown in Fig. 7.

Thus, a diverging axially symmetric discharge ejects the gas.

The equations of ejection have been given by the author.[6] In case of incompressible fluid they admit a class of similar solutions of the kind

$$H=x^\gamma h(\zeta);\quad \psi=xf(\zeta);\quad p_m=x^\delta g(\zeta);\quad \zeta=x^\alpha y;$$
$$T=x^q t(\zeta);\quad (\delta=2+4\alpha;\quad \gamma=-\alpha^{-1}-2;\quad q=-2\alpha\gamma).$$

The most interesting solution of this class is the case (Fig. 7) in which the total current in a discharge is independent on x; then $\gamma=\alpha=-1$; $\delta=q=-2$; $\zeta=y/x$.

Note especially that in the last case the kind of solution remains the same for a compressible viscous heat-conducting fluid. Thus for the Navier-Stokes equations, taking into consideration magnetic terms, we have the following axially symmetric (and plane) solution:

$$\psi=xf(\zeta);\quad H=h(\zeta)/y;\quad p_m=g(\zeta)/x^2;\quad T=t(\zeta)/x^2;$$
$$\rho=l(\zeta);\quad s=m(\zeta);\quad \zeta=y/x.$$

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Reducible Problems in Magneto-Fluid Dynamic Steady Flows*

HAROLD GRAD

Institute of Mathematical Sciences, New York University, New York, New York

1. INTRODUCTION

THE magnitude and degree of complexity of the phenomena encompassed in the subject of nondissipative magneto-fluid dynamics is perhaps best illustrated by the presence of three distinct and strongly anisotropic modes of signal propagation. The linearized problem of the propagation of small disturbances in an unbounded medium is fairly well understood (but is by no means complete).[1] Boundary value problems, even when linearized, are considerably more abstruse. One reason is that, although the various modes of propagation are inherently coupled even in an unbounded domain, they may be decoupled (somewhat artificially, to be sure) by introducing Fourier components. In a boundary-value problem, a higher-order system is, generally speaking, solvable in useful terms only when the boundary conditions as well as the differential equations separate into subsystems.

An alternative technique which is very useful in the early development of a new subject is the discovery of special classes of flows which yield conventional fluid-dynamical or classical second-order mathematical structures. This paper lists and to some extent develops a number of such reducible fluid-magnetic boundary-

* This work was supported by the U. S. Atomic Energy Commission under contract.

[1] For example, see (a) A. Baños, Phys. Rev. **97**, 1435 (1955); (b) A. Baños, Proc. Roy. Soc. (London) **A233**, 350 (1955); (c) J. Bazer and O. Fleischman, Phys. Fluids **2**, 366 (1959); (d) H. Grad, in *The Magnetodynamics of Conducting Fluids*, D. Bershader, Editor (Stanford University Press, Stanford, California, 1959).

value problems in order to map out solvable mathematical structures and analyze the similarities and contrasts with ordinary fluid dynamics. This analysis of reducible flows serves to separate large classes of fluid-magnetic problems which are relatively accessible using fluid-dynamical techniques from the remainder which may require the development of significantly new techniques in order to handle boundary-value problems. Several of these unusual problems which one hopes will lead to deeper understanding of the equations are also pointed out.

The term "reducible" is intended to be quite flexible. One meaning is the reduction to a set of equations which is mathematically identical to some system common to ordinary fluid dynamics. While such an analog does not necessarily yield explicitly solvable problems, it may permit the application of a large variety of tested approximation techniques. In Sec. 4 we present such an analog which allows the application of a large part of the entire literature of inviscid fluid dynamics to a certain part of magneto-fluid dynamics. Specifically, any special solution or general technique for a gas-dynamical flow which is two-dimensional nonsteady (or more special), compressible or incompressible, linear or nonlinear, can be applied. For example, one can transfer conventional results connected with simple waves, shock waves, Riemann invariants, flows around cylinders, airfoils, and bends. One must be cautious with the interpretation of boundary layers (which involve dissipation) and most stability and turbulence analyses (which are usually three-dimensional). A significant point is that only a very small part of the fluid-magnetic universe is covered by this analog.

Another reduction to a system with essentially one characteristic cone (i.e., one signal speed) is given by the parallel flows (fluid velocity and magnetic field locally parallel) of Sec. 5. This constraint is compatible in the sense that it does not restrict the solution to a few isolated flows but allows the imposition of a class of arbitrary boundary conditions. In the incompressible case, the parallel flow is an exact mathematical analog of a fluid-dynamical problem. Linearized compressible parallel-flow problems reduce to consideration of the potential or wave equations which, except for certain subtleties involving the domain of dependence, are solvable by conventional aerodynamic techniques.[2] The nonlinear system, although essentially second order, exhibits unusual transonic features which are only touched on here. An example is given of a nozzle flow which is identical to the ordinary gas-dynamical solution when solved in the hydraulic approximation, but which has three distinct sonic transitions, only one of which is at the throat. A combination of the transverse and parallel flows results in a system which is in a formal sense very similar to the simple parallel flow but which has certain other very strange properties.

A linear system with constant coefficients in two independent variables can always be separated. Specifically, one can compute a Riemann invariant for each real characteristic line and obtain a pair of equations affinely equivalent to the Cauchy-Riemann equations for each pair of complex roots. This reduction has been carried out previously for the case of one-dimensional nonsteady flow.[1d] In Sec. 6, we obtain this reduction for two-dimensional linearized steady flows. For parameter values which yield a totally hyperbolic system (all characteristic roots are real), boundary-value problems (such as flow around a thin airfoil) can be solved explicitly, although possibly tediously. When the problem is partly elliptic and partly hyperbolic, a solution by elementary means is only possible when the boundary conditions do not mix the elliptic and hyperbolic variables. This splitting occurs in some special and limiting cases, but in general, the appropriate physical boundary conditions do not have this simple structure. It is interesting to note that fluid-magnetic flows do *not*, in general, reduce to conventional fluid flows in the limit of vanishing magnetic field although this has been observed in special cases.[3]

The three-dimensional limiting case of very large magnetic field (more precisely, large Alfvén speed compared with the gas sound speed) is considered in Sec. 7. For time-dependent problems, it has been previously found that in this limit the full three-dimensional linearized system splits (in a certain sense) into three uncoupled systems, one for each characteristic cone.[4] The steady-flow problem is more complex, first because of the appearance of an additional parameter, the speed at infinity as compared to either the Alfvén or the gas speed, and second because the variables in which the differential equations split are not necessarily those on which boundary conditions are imposed. The results in this limit of large magnetic field are particularly interesting. It is found that the magnetic field is not perturbed to lowest order. The disturbance is entirely in the fluid variables. However, this is a very unconventional fluid disturbance. Only the velocity component parallel to the undisturbed magnetic field is perturbed, and the domain of dependence is two-dimensional, lying within the plane of the undisturbed velocity and magnetic field vectors.

Sections 2 and 3 contain supplementary material on the characteristics and appropriate boundary conditions of the systems being studied.

[2] (a) W. R. Sears, J. Am. Rocket Soc. **29**, 397 (1959); (b) E. L. Resler, Jr., and J. E. McCune, in *The Magnetodynamics of Conducting Fluids*, D. Bershader, Editor (Stanford University Press, Stanford, California, 1959); (c) W. R. Sears and E. L. Resler, Jr., J. Fluid Mech. **5**, 257 (1959); (d) J. E. McCune and E. L. Resler, Jr., J. Aero/Space Sci. **27**, 493 (1960).

[3] Reference 2d. The reason why this example is solvable and the solution approaches the gas-dynamical limit as the field goes to zero is seen later.

[4] See reference 1d; a partial splitting is given in 1a.

2. CHARACTERISTICS

For a self-contained exposition, it is necessary to give a brief description of the characteristic surfaces of the system under study. The complete nonlinear nonsteady system is taken as

$$\begin{aligned} &\partial\rho/\partial t+\operatorname{div}(\rho\mathbf{u})=0,\\ &\rho\partial\mathbf{u}/\partial t+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=(1/\mu_0)\operatorname{curl}\mathbf{B}\times\mathbf{B},\\ &\partial\mathbf{B}/\partial t+\operatorname{curl}(\mathbf{B}\times\mathbf{u})=0,\\ &\rho\partial\eta/\partial t+(\mathbf{u}\cdot\nabla)\eta=0,\quad p=f(\rho,\eta). \end{aligned} \tag{1}$$

We have taken a classical compressible fluid with density ρ, pressure p, and entropy η, and have eliminated the current and electric field from Maxwell's equations (without displacement current) and Ohm's law for a perfect conductor, $\mathbf{E}+\mathbf{u}\times\mathbf{B}=0$.

The possible characteristic manifolds $\phi(x,y,z,t)=$ constant are given[5] by solutions of the first-order partial differential equation for ϕ:

$$\begin{aligned} \phi_t\phi'[(\phi')^2-(\mathbf{A}\cdot\nabla\phi)^2][(\phi')^4-(a^2+A^2)(\phi')^2(\nabla\phi)^2 \\ +a^2(\mathbf{A}\cdot\nabla\phi)^2(\nabla\phi)^2]=0, \end{aligned} \tag{2}$$

$$\phi'\equiv\phi_t+\mathbf{u}\cdot\nabla\phi.$$

Here a^2 is written for the ordinary gas sound speed and $\mathbf{A}$ for the (vector) Alfvén speed:

$$a^2=\partial p/\partial\rho,\quad \mathbf{A}=\mathbf{B}/(\mu_0\rho)^{\frac{1}{2}}. \tag{3}$$

The characteristic equation (2) is of eighth degree, consistent with the eight scalar variables ρ, $\mathbf{u}$, $\mathbf{B}$, η. As it stands, the system (1) allows solutions in which div$\mathbf{B}$ is not zero; if div$\mathbf{B}=0$ is taken as a constraint, the root $\phi_t=0$ is lost in (2) leaving (1) as an essentially seventh-order system. If the flow is assumed to be isentropic, $\eta=$ constant, the root $\phi'=0$ is lost, and we are left with a sixth-order system. The first bracket in (2) refers to the transverse characteristic cone and the second bracket to the compressive or slow and fast cone.

These results can be visualized graphically by plotting the characteristic locus, which is the intersection in physical space of the characteristic cone at a fixed time (Fig. 1). This is a three-dimensional locus with rotational symmetry about the axis of $\mathbf{B}$. One interpretation of this figure is as a snapshot of the wave fronts resulting from a point explosion at an earlier instant. The outside wave front is that of the fast wave, the two cusped regions arise from the slow wave, and the two encircled points are the transverse wave front. Another interpretation of this figure is as the appropriate "wavelet" to be used (instead of a sphere) in applying Huygens' principle. This figure contains the essential content of the general fluid magnetic theory of the propagation of discontinuous wave fronts (i.e., ray optics).[6]

In two dimensions (Fig. 1 is plane, exactly as represented), the fundamental solution for the linearization about a uniform field has been computed explicitly[7] giving the entire disturbance within the domain interior to the fast locus in addition to the location and amplitude of the wave fronts. It is an interesting fact that the disturbance is identically zero within the two cusped regions in this two-dimensional case.

An important point is that, although wave fronts separate into three independently propagating signals, the disturbance which is left behind after passage of the fast front generally exhibits a complicated interaction among the modes even in a linearized problem.[1d] In a steady flow, this interaction is crucial and causes most of the difficulty.

The real characteristic cones for a steady flow can be obtained from Fig. 1 by a simple geometric construction. On the characteristic locus, drawn at time $t=1$, we place the flow velocity vector reversed in sign, $-\mathbf{u}$. A real (hyperbolic) characteristic cone is ruled by the tangent lines from the terminus of $-\mathbf{u}$ to the locus, see Fig. 2 (the transverse cone has been omitted in order to simplify the drawings). In two dimensions, the characteristic "cones" are lines, exactly as shown. It is interesting to note that the distinction between the slow and fast cones is lost in a steady flow. These terms merely distinguish two real traces of a single analytic complex manifold.

In principle, a characteristic cone of the differential equations governing steady flow is complete with both

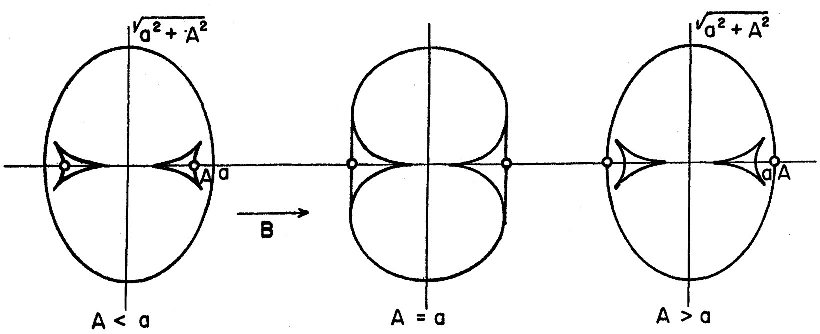

FIG. 1. Characteristic loci.

[5] K. O. Friedrichs, Los Alamos Sci. Lab. Rept. LAMS-2105 (1954); K. O. Friedrichs and H. Kranzer, New York University Rept. NYO-6486-VIII (1958); also see reference 1d.

[6] A complete account is given in reference 1c.

[7] H. Weitzner, Bull. Am. Phys. Soc. Ser. II, 5, 321 (1960).

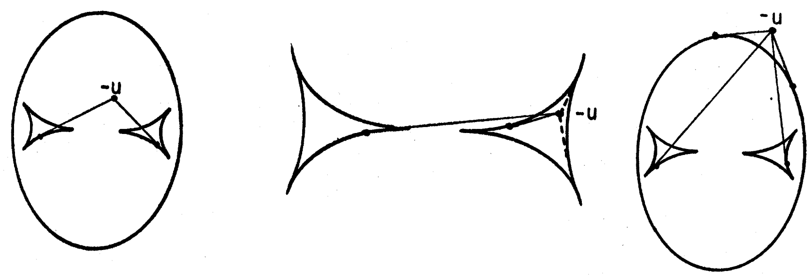

FIG. 2. Steady-flow characteristics.

nappes present. However, it is well known that steady-flow problems are frequently improperly posed mathematically unless one adds some sort of extraneous "causality" restriction derived from study of the time-dependent problem. This can take the form of a radiation condition in an elliptic problem or a statement that the upstream flow is unperturbed in a hyperbolic problem. More precisely, a simple investigation of the time-dependent propagation of disturbances shows that the correct domain of dependence in the hyperbolic case is given by the single-napped cone (or ray in two dimensions) which is explicitly given by the above construction. On occasion, this yields the upstream (or forward) rather than the downstream (or backward) characteristic,[2a] e.g., as indicated by the tangents shown dashed in Fig. 2. This is no violation of any intuition about supersonic flow since the "wrong" direction for a cone is found only when there exists a signal speed with respect to which the flow is subsonic. In other words, signals can reach everywhere in space, and the real cone represents a locus of possible discontinuities rather than a strict boundary to the domain of dependence. What is possibly confusing is that, in some special cases, the forward characteristic does accidentally define the domain of dependence, e.g., in two dimensions.

The characteristics for the special problems considered in this paper are easily obtained by specialization and limiting processes. For a parallel flow (Sec. 5), **u** is taken on the axis in Fig. 2. In addition to the streamline (or magnetic line) multiply counted, there is a single nondegenerate real cone when **u** falls inside the cusped region [between $aA/(a^2+A^2)^{\frac{1}{2}}$ and the smaller of a and A] or outside the fast locus (greater than both a and A). The characteristic direction is parallel to the flow at one transonic point (at the cusp) and is normal to the flow at the other two transonic points; the latter is conventional in gas dynamics.

For the transverse flow (Sec. 4), one must integrate out the ignorable coordinate parallel to **B**. This gives a single cone corresponding to the signal speed $(a^2+A^2)^{\frac{1}{2}}$ as well as the particle path (zero speed) counted several times. In the two-dimensional problem (Sec. 6), one must distinguish whether the ignorable coordinate is or is not in the direction of the magnetic field. In the former case, real characteristics are given by the appropriate tangents in the plane represented in Fig. 2. In the latter case, the three-dimensional characteristic locus must first be integrated in the (skew) ignorable direction before drawing tangents. The result is hard to visualize geometrically, but one interesting conclusion is that the transverse disturbance no longer falls on top of the slow-fast front.[8]

3. BOUNDARY CONDITIONS

From the fact that there exist three nontrivial characteristic cones, one would usually expect to be able to impose three boundary conditions at a rigid boundary across which there is a nonvanishing normal component of **B**. The characteristics carried by the particle path are not expected to contribute boundary data at an impermeable wall. If $B_n=0$ on the boundary, the boundary surface is characteristic with regard to both the transverse and the slow cones (as well as the particle paths), and one can only expect to impose a single boundary condition. These expectations are found to be fulfilled in the special problems which have been solved.

At a rigid wall, one always imposes the condition $u_n=0$. This is sufficient to determine a unique solution in the case where the magnetic field does not cross the boundary, $B_n=0$, cf. Secs. 4 and 5. If B_n does not vanish, two more boundary conditions must be found. We consider separately the cases of perfectly conducting and nonconducting boundaries.

The perfectly conducting boundary is described by the boundary condition that there is no tangential electric field, $\mathbf{E}_t=0$. If $B_n\neq 0$, using $(\mathbf{E}+\mathbf{u}\times\mathbf{B})_t=0$, we conclude that $\mathbf{u}_t=0$. Together with $u_n=0$, we state simply that the fluid sticks; the vector velocity is zero. We conjecture that the steady-flow problem is, in this case, trivial; i.e., the only steady flow of a perfectly conducting fluid around a perfectly conducting object across which $B_n\neq 0$ is $\mathbf{u}\equiv 0$. This is proved for two-dimensional flow in Appendix 1.

For a nonconducting object, the situation is more complex. It is necessary to solve the electromagnetic (interior) problem in the nonconductor as well as the fluid magnetic (exterior) problem, joining them with appropriate jump conditions. Only in special cases can the interior problem be neglected and replaced by a boundary condition which is sufficient to make the exterior problem well posed. We claim that the appropriate matching conditions (when $B_n\neq 0$) are that the

[8] This fact was pointed out to the author by H. Weitzner.

vector $\mathbf{B}$ be continuous across the boundary (there is no surface current). This claim is supported by heuristic arguments (essentially counting) and later by direct verification that the resulting problem is well posed in special cases. Let us suppose that a solution to the exterior problem satisfying $u_n=0$ has been found. The conventional electromagnetic boundary conditions that B_n and $\mathbf{E}_t$ be continuous then allow us to solve a pure electromagnetic problem for $\mathbf{B}$ and $\mathbf{E}$ in the interior. We assume that the fluid is inviscid and can withstand no shear stress. As a consequence, the tangential component of the Maxwell stress must be continuous across the interface. The stress tensor is

$$(1/\mu_0)(B_iB_j-\tfrac{1}{2}B^2\delta_{ij}).$$

The force per unit area on an element of the interface is

$$(1/\mu_0)(B_iB_n-\tfrac{1}{2}B^2n_i).$$

The requirement that the tangential component of this force be continuous is

$$[B_iB_n]_t=B_n[\mathbf{B}_t]=0,$$

where $[Q]$ denotes the jump in the quantity Q. If $B_n=0$, this condition is automatically satisfied and we obtain no additional matching conditions; i.e., the exterior and interior problems are essentially decoupled. If $B_n\neq 0$, we have $[\mathbf{B}_t]=0$ which, combined with $B_n=0$ is exactly $[\mathbf{B}]=0$. For purposes of counting, on postulating an exterior solution with $u_n=0$, we have found a unique interior solution and, in addition, the two scalar matching conditions $[\mathbf{B}_t]=0$. The requirement that the fluid support no shear stress has therefore led us to impose on the exterior problem the equivalent of two boundary conditions in addition to $u_n=0$.

The possibility of a normal component of current J_n is of interest. It is clear that J_n must vanish at the surface of a nonconductor. This, we claim, is a physical fact, but not necessarily a boundary condition. The requirement $[\mathbf{B}_t]=0$ (when $B_n\neq 0$) implies that $[J_n]=0$. Moreover, since $\mathbf{J}=0$ in the nonconductor, we conclude that $J_n=0$ follows from the already adopted boundary conditions in the case $B_n\neq 0$. For flows with $B_n=0$, however, we find solutions to the fluid magnetic problem in which $J_n\neq 0$ (Sec. 4). It is indicated in Appendix 2 that the relation $J_n=0$ is a legitimate boundary condition for a *finitely* conducting fluid, but this boundary condition is lost in the limit as the conductivity approaches infinity. It *must not* be imposed in the perfectly conducting problems which we treat. The situation is entirely analogous to ordinary viscous vs nonviscous flow. The physical fact that the fluid sticks to a wall must be ignored when nonviscous equations are used.

Quite generally, this question is the universal one of relating a physical problem to a mathematical model. Once the differential equations are chosen, the question of how many and, to some extent, what kind of boundary conditions to impose is a purely mathematical one governed by the theory of the differential equations. One must judiciously select from what is, in principle, an infinite list of physical facts (or approximate facts) the correct number and type of boundary conditions required by the theory.

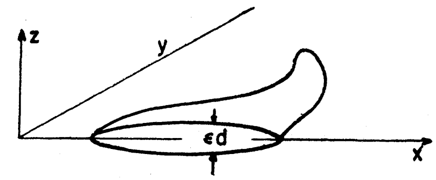

FIG. 3. Three-dimensional airfoil.

In certain limiting cases, it is possible to eliminate the necessity for the joint solution of interior and exterior problems with matching conditions and formulate a self-contained exterior problem subject to boundary conditions even when $B_n\neq 0$. Specifically, we consider the flow about a thin airfoil, linearized about a uniform magnetic field and a uniform flow (cf. Secs. 6 and 7). The unperturbed velocity must lie in the plane of the airfoil but the magnetic field orientation is arbitrary. The airfoil is a thickened lamina of thickness $\epsilon d(x,y)$, see Fig. 3. We suppose that the exact solution to the complete problem yields a perturbed magnetic field $B_n^+(x,y)$ on the top of the lamina and $B_n^-(x,y)$ below. The magnetic field inside is obtained as the harmonic vector (curl$\mathbf{B}=0$, div$\mathbf{B}=0$) which takes these boundary values. We write $\mathbf{B}'$ for the two-dimensional vector projection of the perturbation field $\mathbf{B}$ into the (x,y) plane and verify that, to low order in ϵ,

$$B_z=\tfrac{1}{2}(B_n^++B_n^-), \tag{4}$$

and $\mathbf{B}'$ is given as the solution of the two-dimensional Poisson equation

$$\mathbf{B}'=\nabla\phi,\quad \operatorname{div}(\epsilon d\mathbf{B}')=B_n^+-B_n^-. \tag{5}$$

In the limit as ϵ approaches zero, we obtain the jump condition $[B_n]=0$ across the lamina. Also, since $\mathbf{B}_t$ is continuous on top and on bottom and $\mathbf{B}'$ is, to lowest order, independent of z, we have $[\mathbf{B}_t]=0$. Finally, we must require that $\mathbf{B}_t$ be a surface gradient, $\mathbf{B}_t=\nabla\phi$. We summarize

$$[\mathbf{B}]=0,\quad \mathbf{B}_t=\nabla\phi. \tag{6}$$

This yields two boundary conditions which should be sufficient to complete the exterior problem. Since a boundary condition must be specified on both the top and bottom of the lamina, a jump condition on a scalar quantity amounts to one-half a boundary condition. Therefore, $[\mathbf{B}]=0$ counts as three-halves, and $\mathbf{B}_t=\nabla\phi$ (which need only be applied on one side) as the remaining half. Or, $\mathbf{B}_t=\nabla\phi$ can be counted as a single boundary condition on each side and $[\phi]=[B_n]=0$ as the other.

In a two-dimensional problem (the lamina is a thickened slit) one must distinguish between a *restricted* two-dimensional problem in which all vectors have two components and a *general* two-dimensional problem in which vector variables may also have components in the ignorable direction. For the restricted problem, B_t has only a single component and the statement $B_t = \nabla\phi$ is redundant; we impose only $[\mathbf{B}]=0$.[9] For the general problem, the requirement $\mathbf{B}_t = \nabla\phi$ implies that the third component (in the ignorable direction) has a constant magnitude on the slit.

The linearized airfoil flow in which the unperturbed flow velocity is parallel to the unperturbed magnetic field must be treated as a special case since the airfoil becomes characteristic and boundary conditions are lost; see Sec. 6.

4. TRANSVERSE FLOWS[10]

By a transverse flow we mean one in which the magnetic field is unidirectional, the fluid-flow vector lies in the plane perpendicular to **B**, and the coordinate in the magnetic field direction is ignorable. We take **B** to point in the z direction while **u** has no z component; all quantities are functions of x, y, and t. It is easy to verify that these constraints are compatible; if they are assumed to hold initially, they perpetuate. On using two-dimensional vector notation, we find for the scalar B the equation

$$\partial B/\partial t + \operatorname{div}(B\mathbf{u}) = 0. \tag{7}$$

Combined with the continuity equation, this can be put in the form

$$(d/dt)(B/\rho) = 0, \tag{8}$$

where $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$ is the Lagrangian time derivative following the fluid. We introduce the notation

$$\begin{aligned} p_* &= p + B^2/2\mu_0 \\ e_* &= e + B^2/2\mu_0\rho \\ h_* &= e_* + p_*/\rho = e + p/\rho + B^2/\mu_0\rho \\ \eta_* &= B/\rho. \end{aligned} \tag{9}$$

The thermodynamic variables e and h are the internal energy and enthalpy per unit mass. Just as the conventional energy equation takes one of the two forms

$$de/dt + p \operatorname{div}\mathbf{u} = 0, \quad d\eta/dt = 0, \tag{10}$$

the magnetic field equation can be combined with the others to give the alternative forms

$$de_*/dt + p_* \operatorname{div}\mathbf{u} = 0, \quad d\eta_*/dt = 0. \tag{11}$$

The momentum equation is

$$\rho d\mathbf{u}/dt + \nabla p_* = 0. \tag{12}$$

The complete system consists of the continuity and momentum equations and one version each of (10) and (11). This system is very similar to that of ordinary fluid dynamics when expressed in terms of the starred variables. The two are precisely identical except that there is an extra variable, say B, and an extra equation. This equation can be made to look like a continuity equation (7), or an energy or entropy equation (11).

Several suggestive pseudothermodynamic relations are easily derived:

$$Td\eta + (B/\mu_0)d\eta_* = de_* + p_*d(1/\rho). \tag{13}$$

The left-hand side can also be written

$$Td\eta + \rho d[(1/2\mu_0)\eta_*^2] \quad \text{or} \quad Td\eta + \rho d(\tfrac{1}{2}A^2/\rho), \tag{14}$$

and the right-hand side,

$$dh_* - \rho^{-1}dp_*. \tag{15}$$

Bernoulli's laws are easily derived under essentially the same restrictions as classically. For a steady flow (no further restrictions), we have

$$\tfrac{1}{2}u^2 + h_* = \text{constant on a streamline}. \tag{16}$$

For an irrotational ($\mathbf{u} = \nabla\phi$) strongly isentropic flow (η and $\eta_* = B/\rho$ constant throughout), we have

$$\partial\phi/\partial t + \tfrac{1}{2}u^2 + h_* = \text{constant throughout}. \tag{17}$$

There is an additional slight specialization which reduces the present system *identically* to that of ordinary compressible fluid dynamics. This is to assume that η_* is a function of η,

$$\eta_* = f(\eta). \tag{18}$$

By this we mean a very general type of functional relation, namely, that the two families of surfaces $\eta_* =$ constant and $\eta =$ constant are the same. If the initial state is such that $\eta_* = f(\eta)$, in virtue of $d\eta/dt = d\eta_*/dt = 0$, this functional relation perpetuates. In this case, the extra differential equation for the variable B can be integrated and the result written in the form of an equation of state for (p_*, ρ, η):

$$p_*(\rho,\eta) = p(\rho,\eta) + (1/2\mu_0)\rho^2 f^2(\eta). \tag{19}$$

It is easy to verify that this equation of state is compatible with the conventional thermodynamic convexity conditions. The adiabatic sound speed is given by

$$a_*^2 = \partial p_*/\partial\rho = \partial p/\partial\rho + (1/\mu_0)\rho\eta_*^2 = a^2 + A^2 \tag{20}$$

as would be expected from the general theory (Sec. 2).

We now list the most interesting special cases in which $\eta_* = f(\eta)$ is automatically realized.

(*a*) *One-dimensional nonsteady flow.* With arbitrary initial values, we have $\eta(x)$ and $\eta_*(x)$, from which we obtain $\eta_* = f(\eta)$ (in the extended sense described before) by eliminating x.

(*b*) *Two-dimensional steady flow.* Both η and η_* are constant on streamlines, say $\eta(\psi)$ and $\eta_*(\psi)$, where ψ is the stream function. The required relation follows by elimination of ψ.

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[9] This boundary condition is given in reference 2d.

[10] This material was presented at a classified Sherwood meeting at Princeton in 1954, was issued as part of a set of lecture notes at New York University in 1954, and was reissued in 1958 as Institute of Mathematical Sciences Rept. NYO-6486-VII (unpublished) by A. A. Blank and H. Grad. Some of this material is covered in a paper by M. Mitchener in *The Magnetodynamics of Conducting Fluids*, D. Bershader, Editor (Stanford University Press, Stanford, California, 1959).

(*c*) $\eta_* = B/\rho$ *is constant initially*. From this we deduce that η_* is always constant and have an explicit equation of state for $p_*(\rho,\eta)$.

(*d*) *Isentropic flow*; $\eta =$ *constant initially*. It is necessary to reinterpret some of the expressions in which η_* was assumed to be a solved function of η. The simplest procedure is to take η_* as the quantity which is to take the place of entropy in the fluid analog. The analog of fluid magnetic *isentropic* flow is then fluid *adiabatic* flow. We have the equation of state

$$p_*(\rho,\eta_*) = p(\rho) + (1/2\mu)\rho^2\eta_*^2 \tag{21}$$

and we may, if we wish, adopt the identification [cf. Eq. (13)]

$$T_* = B/\mu_0 = \rho\eta_*/\mu_0. \tag{22}$$

In summary, we have shown that general transverse flow is similar to ordinary compressible fluid flow, and with a slight specialization [viz., $\eta_* = f(\eta)$ initially], it becomes identical to two-dimensional nonsteady compressible adiabatic flow. This allows the transfer of almost the entire literature of linear and nonlinear compressible flows including the theory of Riemann invariants, simple waves, wave interactions, discontinuous shock theory, free boundary flows, provided that the results are not special with regard to the equation of state. On the other hand, the part of fluid-magnetic theory which is covered by this analog is quite small! The basic reason for the simplicity of this special geometry is that the Maxwell stress tensor, which can never be isotropic in three dimensions, is in this case equivalent to a two-dimensional scalar pressure.

5. PARALLEL FLOWS

We look for flows in which **u** is parallel to **B**,

$$\mathbf{B} = \lambda\mathbf{u}. \tag{23}$$

Specifically, we look for circumstances under which the constraint (23) yields not just occasional special flows but is compatible with the imposition of a general class of arbitrary boundary conditions. Presumably, the imposition of certain boundary conditions in the original problem guarantees that the flow is parallel, with the remaining boundary conditions left open. We note first that $\partial\mathbf{B}/\partial t = \mathrm{curl}(\mathbf{u}\times\mathbf{B}) = 0$ and confine our attention henceforth to steady flows.

First we examine the relatively simple case of incompressible flow,[11]

$$\mathrm{div}\mathbf{u} = 0, \quad (\mathbf{u}\cdot\nabla)\rho = 0. \tag{24}$$

The flow is called homogeneous if ρ is constant everywhere instead of constant on each streamline. From $\mathrm{div}\mathbf{B} = 0$ and $\mathrm{div}\mathbf{u} = 0$, we deduce

$$(\mathbf{u}\cdot\nabla)\lambda = 0; \tag{25}$$

i.e., λ is constant on a streamline. The equation of motion is

$$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = (1/\mu_0)(\mathbf{B}\cdot\nabla)\mathbf{B} - (1/\mu_0)\nabla(\tfrac{1}{2}B^2), \tag{26}$$

which can be put in the form

$$\rho_*(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p_* = 0 \tag{27}$$

where

$$p_* = p + B^2/2\mu_0, \quad \rho_* = \rho - \lambda^2/\mu_0. \tag{28}$$

Equation (27) together with

$$\mathrm{div}\mathbf{u} = 0, \quad (\mathbf{u}\cdot\nabla)\rho_* = 0 \tag{29}$$

completely describes a classical incompressible flow in the variables ρ_*, p_*, **u**. No equation of state is required. From any classical incompressible steady flow (in general, inhomogeneous), calling the fluid flow variables ρ_* and p_*, we obtain a fluid-magnetic incompressible flow.[12] We are at liberty to prescribe λ or ρ arbitrarily as a constant on each streamline, after which we can compute the remaining fluid magnetic variables. A homogeneous incompressible fluid flow can give rise to either homogeneous or inhomogeneous fluid-magnetic flows. A potential flow ($\mathrm{curl}\mathbf{u} = 0$, necessarily homogeneous in ρ_*) can be nontrivial magnetically ($\mathrm{curl}\mathbf{B} \neq 0$) if λ is taken to be inhomogeneous.

Bernoulli's law takes the form

$$p_* + \tfrac{1}{2}\rho_* u^2 = p + \tfrac{1}{2}\rho u^2 = \text{constant on a streamline}; \tag{30}$$

it is satisfied in either the original or starred variables.

Now we turn to the case of compressible flow.[13] From $\mathrm{div}(\rho\mathbf{u}) = 0$ and $\mathrm{div}\mathbf{B} = 0$, we conclude that

$$\lambda = \alpha\rho, \quad (\mathbf{u}\cdot\nabla)\alpha = 0. \tag{31}$$

Although λ is not constant on a streamline, λ/ρ is. It is convenient to eliminate the vector **B** from the differential equations, expressing it in terms of **u** and the scalar α,

$$\mathbf{B} = \alpha\rho\mathbf{u}. \tag{32}$$

We obtain the system

$$\begin{aligned} &\mathrm{div}(\rho\mathbf{u}) = 0 \\ &(\mathbf{u}\cdot\nabla)\mathbf{u} + \rho^{-1}\nabla p = (\alpha^2/\mu_0)\,\mathrm{curl}(\rho\mathbf{u})\times\mathbf{u} \\ &(\mathbf{u}\cdot\nabla)\alpha = 0 \\ &(\mathbf{u}\cdot\nabla)\eta = 0, \quad p = f(\rho,\eta). \end{aligned} \tag{33}$$

If we take the scalar product of the momentum equation with the velocity **u**, we find that the magnetic terms on the right disappear. Consequently, we obtain the con-

[11] See NYO-6486-VII, footnote 10. A special case of this incompressible parallel flow is analyzed for stability by S. Chandrasekhar, Proc. Natl. Acad. Sci. U. S. **42**, 273 (1956).

[12] There exists another analog of incompressible fluid flow with an entirely different fluid-magnetic problem; see H. Grad and H. Rubin, "Hydromagnetic equilibria and force-free fields," in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, New York, 1959), Vol. 31, p. 190.

[13] In this connection see M. N. Kogan, Priklad. Mat. Mech. **23**, 70 (1959). During this Symposium an exact fluid analog for a parallel flow which is two-dimensional, isentropic, and irrotational was presented by I. Imai [Revs. Modern Phys. **32**, 992 (1960), this issue]. This analog is discussed and extended in Appendix 3.

ventional gas-dynamical Bernoulli's law

$$\tfrac{1}{2}u^2+h=\text{constant on a streamline.} \tag{34}$$

This should be compared with the transverse flows of the last section where Bernoulli's law contained modified variables, and the incompressible parallel flow (30) where two forms of Bernoulli's law coalesce.

The characteristics of the system (33) are easily computed. If we denote by ϕ the angle between the vector $\mathbf{u}$ (or $\mathbf{B}$) and the normal to the characteristic surface element, we find the single cone given by[14]

$$\begin{aligned}\cos^2\phi &= a^2/u^2+(\alpha^2\rho/\mu_0)(1-a^2/u^2)\\ &= a^2/u^2+(A^2/u^2)(1-a^2/u^2).\end{aligned} \tag{35}$$

A positive value less than one for $\cos^2\phi$ yields a real hyperbolic cone; otherwise, this root is elliptic. In addition to this cone we find the streamline counted four times. Three of these roots correspond to α, η, and the Bernoulli constant.

The problem of a linear perturbation about a uniform flow $\mathbf{u}_0$ and constant state at infinity has been solved and is almost but not quite conventional aerodynamics.[2c] The streamline characteristics integrate out explicitly (almost unconsciously!) and one is left with either a wave equation or a potential equation stretched in the direction of $\mathbf{u}_0$. Any classical elliptic solution (flow over a wavy wall, past an airfoil, etc.) is immediately transferable with only minor modifications. A classical hyperbolic solution is directly transferrable only in the hyperbolic range where u_0 is larger than the fast sound speed (u_0 larger than both a_0 and A_0). In the intermediate hyperbolic range (u_0 inside the cusped region, see Sec. 2 and Fig. 2), the wave equation is solved with the *forward* characteristics determining the domain of dependence.

To observe the full implications of the three distinct sonic transitions, one must attack a nonlinear problem. We can solve one such problem explicitly after making appropriate approximations: the flow in a channel of slowly varying cross section. We make the same assumptions as in the classic hydraulic approximation, i.e., all quantities are approximately constant over the cross section and depend only on the axial distance along the channel. To this approximation, the solution is obtain in terms of the cross-sectional area $\mathcal{A}$ as a parameter using only the conservation of mass, the entropy constant, and the Bernoulli constant,

$$\begin{aligned}&\rho u\mathcal{A}=\text{const}\\ &\eta=\text{const}\\ &h+\tfrac{1}{2}u^2=\text{const},\quad h=f(\rho,\eta).\end{aligned} \tag{36}$$

These equations hold in both the fluid and the fluid-magnetic problems even though the complete flow equations are different in the two cases. Consequently, the classical compressible channel flow carries over without change for the fluid variables u, ρ, p, h, etc. The magnetic field is then determined once we choose a value for the constant α. Specifically, let us examine the transonic nozzle flow in which we have $u<a$ upstream, $u>a$ downstream, and $u=a$ at the throat. It so happens that $u=a$ is also a characteristic speed of the fluid-magnetic flow; there is a sonic transition at the throat. However, there are two other critical points (which are attained in the flow if the area $\mathcal{A}$ varies through a sufficient range), viz., at $u=A$ and $u=aA/(a^2+A^2)^{\frac{1}{2}}$. The latter is always upstream of the throat while the former may be upstream or downstream depending on the relative magnitudes of a and A.

In the classical nozzle flow, the reverse transition from supersonic to subsonic flow is known to be unstable without the occurrence of shocks. This raises the question of the legitimacy of the fluid-magnetic solution, since it involves transitions in both directions between elliptic and hyperbolic regimes. However, it is easy to verify that the heuristic argument which indicates shock formation in the classical case does not apply in the present problem. The classical argument, roughly given, is that a small disturbance upstream propagates only along the Mach line in the downstream direction. This Mach line becomes approximately normal to the channel axis as it approaches the throat; consequently, disturbances tend to collect at the upstream side of the throat. In the corresponding fluid-magnetic case in which the characteristic direction becomes approximately normal to the channel axis, the argument quoted in Sec. 2 requires that it is the *upstream* characteristic along which a discontinuity propagates; this is carried safely out of the hyperbolic region.

This argument does not settle the question of the existence or stability of the fluid-magnetic channel flow. What is necessary is to establish the existence and stability (the latter either in the sense of allowing continuous deformation of the steady-flow boundary data or in the transient sense) of the full two- or three-dimensional transonic flow. Presumably, a unique solution exists which takes on boundary data on an upstream cross section in the initial elliptic regime (e.g., u_n given), with only regularity imposed at the far-downstream end. If such a solution can be shown to exist, the hydraulic approximation is probably quite accurate. But it is also possible that the hydraulic solution is an "approximation" to a nonexistent flow.

It is possible to combine the results of this and the last section in a "parallel-transverse" flow. Consider a two-dimensional parallel flow with superposed third components of velocity and magnetic field. The surprising result (cf. Sec. 6) is that the combination is a system with only one nondegenerate characteristic cone; the transverse characteristic does not appear.

We use the earlier notation of this section, $\mathbf{B}=\alpha\rho\mathbf{u}$, to represent the (xy)-plane projections of the magnetic field and velocity vectors and add the components B_z and u_z in the ignorable z direction. Instead of (33), we

[14] This coincides with the "Mach" lines computed in the linearized problem, reference 2c.

have

$$\begin{aligned}&\operatorname{div}(\rho\mathbf{u})=0\\&\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p+(\alpha^2/\mu_0)\rho\mathbf{u}\times\operatorname{curl}(\rho\mathbf{u})=-\nabla(B_z^2/2\mu_0)\\&(\mathbf{u}\cdot\nabla)\alpha=0\\&(\mathbf{u}\cdot\nabla)\eta=0\end{aligned}\tag{37}$$

supplemented by

$$\begin{aligned}&(\mathbf{u}\cdot\nabla)u_z-(\alpha/\mu_0)(\mathbf{u}\cdot\nabla)B_z=0\\&(\mathbf{u}\cdot\nabla)B_z-\alpha\rho(\mathbf{u}\cdot\nabla)u_z+B_z\operatorname{div}\mathbf{u}=0.\end{aligned}\tag{38}$$

From the first line of (38),

$$(\mathbf{u}\cdot\nabla)(u_z-\alpha B_z/\mu_0)=0,$$

we conclude that

$$u_z-\alpha B_z/\mu_0=\beta\tag{39}$$

where β is constant on a streamline,

$$(\mathbf{u}\cdot\nabla)\beta=0.\tag{40}$$

By using the continuity equation and the second line of (38), we compute

$$\begin{aligned}\rho(\mathbf{u}\cdot\nabla)(B_z/\rho)&=(\mathbf{u}\cdot\nabla)B_z-(B_z/\rho)(\mathbf{u}\cdot\nabla)\rho\\&=(\mathbf{u}\cdot\nabla)B_z+B_z\operatorname{div}\mathbf{u}\\&=\alpha\rho(\mathbf{u}\cdot\nabla)u_z,\end{aligned}$$

or

$$(\mathbf{u}\cdot\nabla)[(B_z/\rho)-\alpha u_z]=0,$$

which can be written

$$B_z=\rho(\alpha u_z+\gamma),\tag{41}$$

where

$$(\mathbf{u}\cdot\nabla)\gamma=0.\tag{42}$$

The system (38) has been integrated completely:

$$\begin{aligned}B_z&=(\alpha\beta+\gamma)\mu_0\rho/(\mu_0-\alpha^2\rho)\\u_z&=(\mu_0\beta+\alpha\gamma\rho)/(\mu_0-\alpha^2\rho).\end{aligned}\tag{43}$$

The net effect in the two-dimensional parallel flow (33) is to replace p by p_*, where

$$p_*=p+\frac{B_z^2}{2\mu_0}=p(\rho,\eta)+\frac{(\alpha\beta+\gamma)^2\mu_0\rho^2}{2(\mu_0-\alpha^2\rho)^2}=p+\frac{\delta^2\rho M^2}{2(M^2-1)^2}\tag{44}$$

and δ and M (the Alfvén Mach number) are given by

$$\begin{aligned}&\delta=\beta+\gamma/\alpha\\&M^2=u^2/A^2=\mu_0\rho u^2/B^2=\mu_0/\alpha^2\rho.\end{aligned}\tag{45}$$

The equation of state (44) expresses p in terms of ρ and the quantities η, α, δ, all of which are constant on a streamline. In addition to the streamline counted many times, we have a single characteristic cone which is given by replacing a^2 in (35) by the expression

$$a_*^2=\partial p_*/\partial\rho=a^2+\delta^2M^4/(M^2-1)^3.\tag{46}$$

The value of a_*^2 may be positive or negative and can apparently switch from plus to minus infinity at $M=1$, i.e., as ρ passes through the value μ_0/α^2. However, it is easily verified that this transition cannot occur on a streamline. The modified Bernoulli's law takes the form

$$h_*+\tfrac{1}{2}u^2=\text{constant on a streamline,}\tag{47}$$

where

$$h^*=\int a_*^2\frac{d\rho}{\rho}=h+\frac{\delta^2(2M^2-1)}{2(M^2-1)^2}.\tag{48}$$

The ordinary enthalpy h would have to go to minus infinity at a transition through $M=1$. We have reached the strange conclusion that in a transverse-parallel flow, a given streamline is characterized by either $M>1$ or $M<1$, whereas in the limiting case of a pure parallel flow, there seems to be the possibility of crossing the value $M=1$ on a single streamline. It should be noted that $M=1$ is no longer characteristic for the parallel-transverse flow. There can be at most one critical point on a streamline with $M>1$ and two when $M<1$.

It is possible to pass through the value $M=1$ in a parallel-transverse flow by having an entire streamline on which $M=1$. The parameters β and γ would be zero on this line and one would have $B_z=\alpha\rho u_z$. On this sonic line the flow is strictly parallel in three dimensions.

Steady transverse flow results from setting $\alpha=\beta=0$ and two-dimensional parallel flow from $\beta=\gamma=0$ in the foregoing formulas.

6. TWO-DIMENSIONAL FLOWS

The characteristics of a system of first-order partial differential equations can be defined in a special way when there are only two independent variables.[15] We assume that the system is homogeneous and linear in first derivatives. We take linear combinations of the equations (say n in number) and look for multipliers such that the linear combination is a directional derivative in a single direction for each of the dependent variables. This poses an algebraic problem, viz., the solution of a homogeneous system of linear equations for the n multipliers; the common direction of differentiation (say described by its slope λ) enters as an eigenvalue parameter. The allowable values of λ are the characteristic directions, and if, for example, they are distinct, we can replace the original system of partial differential equations by a characteristic system in which each equation contains derivatives in a single characteristic direction. If such a characteristic equation can be integrated, we obtain a Riemann invariant. This is a function which is constant on each characteristic curve of a given family. If the system of equations is linear with constant coefficients and every characteristic is real, then each characteristic equation can be integrated and we have a complete set of n Riemann invariants. Each one is a real linear combination of the original variables which is constant on a given family of real parallel straight lines. Corresponding to a pair of complex conjugate roots λ, we find that the real and imaginary parts of the corresponding

[15] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves* (Interscience Publishers, Inc., New York, 1948), Chap. II.

complex Riemann invariant satisfy an elliptic system which can be obtained from the Cauchy-Riemann equations by an affine transformation. Alternatively, both the real and imaginary parts of the Riemann invariant satisfy a second-order elliptic equation which is affinely equivalent to the potential equation.

This reduction for the fluid-magnetic equations has been carried out elsewhere in the case of one-dimensional nonsteady flow,[1d] and we now examine this question for two-dimensional steady flow treating only the linear problem of perturbations about a uniform magnetic field $\mathbf{B}_0$, arbitrarily oriented, and a uniform flow $\mathbf{u}_0$. The results are applied to the flow around a thin airfoil.

There is an essential difference between the restricted two-dimensional problem in which the velocity and magnetic-field vectors lie in the relevant plane and a general two-dimensional problem in which they have components in the ignorable direction as well. In the former case, the system of equations is of fourth order with characteristics corresponding to the slow and fast cones of the transient problem. In the latter case, a transverse cone is also present, and the system is of sixth order. Moreover, we distinguish an intermediate case in which $\mathbf{B}_0$ has no third component but $\mathbf{u}_0$ and the perturbations $\mathbf{u}$ and $\mathbf{B}$ are general. In this case, the third components of $\mathbf{u}$ and $\mathbf{B}$ decouple from the remaining variables, and this (transverse) solution is merely superimposed on the restricted two-dimensional solution for the remaining variables. We treat these cases in order of complexity.

For simplicity we consider isentropic flow for which the linearized equations are

$$
\begin{aligned}
&(\mathbf{u}_0\cdot\nabla)\rho+\rho_0\,\mathrm{div}\mathbf{u}=0\\
&\rho_0(\mathbf{u}_0\cdot\nabla)\mathbf{u}+a_0^2\nabla\rho+(1/\mu_0)\nabla(\mathbf{B}_0\cdot\mathbf{B})=(1/\mu_0)(\mathbf{B}_0\cdot\nabla)\mathbf{B}\\
&\mathrm{curl}(\mathbf{u}\times\mathbf{B}_0+\mathbf{u}_0\times\mathbf{B})\\
&\qquad\equiv(\mathbf{B}_0\cdot\nabla)\mathbf{u}-(\mathbf{u}_0\cdot\nabla)\mathbf{B}-\mathbf{B}_0\,\mathrm{div}\mathbf{u}=0\\
&\mathrm{div}\mathbf{B}=0.
\end{aligned}
\tag{49}
$$

The subscripts (0) refer to unperturbed quantities, all others being perturbations. The pressure has been eliminated using $\nabla p=a_0^2\nabla\rho$. The relation $\mathrm{div}\mathbf{B}=0$ is needed since from the preceding line (flux equation) there follows only $(\mathbf{u}_0\cdot\nabla)\,\mathrm{div}\mathbf{B}=0$.

We introduce x and y as the relevant coordinates with z ignorable. It is convenient to introduce dimensionless variables in terms of the magnitudes u_0 and B_0 of the two-dimensional projections of the vectors $\mathbf{u}_0$ and $\mathbf{B}_0$. We write

$$
\begin{aligned}
\rho&=\rho_0\sigma\\
\mathbf{u}_0&=u_0(1,0,\mu)\\
\mathbf{B}_0&=B_0(\cos\theta,\sin\theta,b)\\
\mathbf{u}&=u_0(u,v,w)\\
\mathbf{B}&=B_0(\xi,\eta,\zeta)\\
A_0^2&=B_0^2/\mu_0\rho_0.
\end{aligned}
\tag{50}
$$

Note that the Alfvén speed A_0 is also defined in terms of the plane component B_0. We normalize the angle θ between $\mathbf{u}_0$ and $\mathbf{B}_0$, taking $0<\theta<\pi/2$.

In components, the equations are

$$
\begin{aligned}
&\sigma_x+u_x+v_y=0\\
&u_0^2u_x+a_0^2\sigma_x+A_0^2(\sin\theta)(\eta_x-\xi_y)+bA_0^2\zeta_x=0\\
&u_0^2v_x+a_0^2\sigma_y-A_0^2(\cos\theta)(\eta_x-\xi_y)+bA_0^2\zeta_y=0\\
&\xi_x+\eta_y=0\\
&\eta+u\sin\theta-v\cos\theta=0
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
&\cos\theta\,w_x+\sin\theta\,w_y-\zeta_x-b(u_x+v_y)=0\\
&u_0^2w_x-A_0^2(\cos\theta\,\zeta_x+\sin\theta\,\zeta_y)=0,
\end{aligned}
\tag{52}
$$

where we have separated out the z-component equations. The last line of (51) is an integrated relation which states that the plane component of $\mathbf{B}$ perpendicular to $\mathbf{u}_0$ is equal to the plane component of $\mathbf{u}$ perpendicular to $\mathbf{B}_0$. It can be derived most easily by noting that $\mathrm{curl}\,(\mathbf{u}\times\mathbf{B}_0+\mathbf{u}_0\times\mathbf{B})=0$ implies that the third component of $\mathbf{u}\times\mathbf{B}_0+\mathbf{u}_0\times\mathbf{B}$ is a constant. If the perturbed flow approaches zero as we approach infinity in any direction, this constant must be taken to be zero.

There are six differential equations and one finite relation in the seven variables. We find it convenient to eliminate u in terms of v and η since the boundary conditions will be imposed on v and on the magnetic-field components. This elimination is invalid if $\theta=0$. However, the case $\theta=0$ is degenerate and must be considered separately in any event. After a little manipulation, we obtain

$$
\begin{aligned}
&\sin\theta\,\sigma_x+\cos\theta\,v_x+\sin\theta\,v_y-\eta_x=0\\
&a_0^2\sin\theta\,\sigma_x+u_0^2\cos\theta\,v_x+(A_0^2\sin^2\theta-u_0^2)\eta_x\\
&\qquad-A_0^2\sin^2\theta\,\xi_y=-bA_0^2\sin\theta\,\zeta_x\\
&a_0^2\sin\theta(\cos\theta\,\sigma_x+\sin\theta\,\sigma_y)+u_0^2v_x-u_0^2\cos\theta\,\eta_x\\
&\qquad=-bA_0^2\sin\theta(\cos\theta\,\zeta_x+\sin\theta\,\zeta_y)\\
&\xi_x+\eta_y=0,
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
&u_0^2w_x-A_0^2(\cos\theta\,\zeta_x+\sin\theta\,\zeta_y)=0\\
&\sin\theta(\cos\theta\,w_x+\sin\theta\,w_y)-\sin\theta\,\zeta_x\\
&\qquad-b(\cos\theta\,v_x+\sin\theta\,v_y-\eta_x)=0.
\end{aligned}
\tag{54}
$$

For the restricted two-dimensional problem we take the system (53) and set $b=0$. We introduce $\lambda=\tan\phi$ as the slope of the normal to the characteristic direction; in other words, we introduce the directional derivative $-\lambda\partial/\partial x+\partial/\partial y$. Multiplying Eqs. (53) by p, q, r, s in order and adding, then setting the ratio of the coefficient of $\partial/\partial x$ to $\partial/\partial y$ for each variable equal to $-\lambda$, we obtain the system of linear equations

$$
\begin{aligned}
(\lambda\sin\theta+\cos\theta)p+u_0^2\cos\theta q+u_0^2r&=0\\
-\lambda A_0^2\sin^2\theta q+s&=0\\
p+a_0^2q+a_0^2(\lambda\sin\theta+\cos\theta)r&=0\\
p+(u_0^2-A_0^2\sin^2\theta)q+u_0^2(\cos\theta)r-\lambda s&=0.
\end{aligned}
\tag{55}
$$

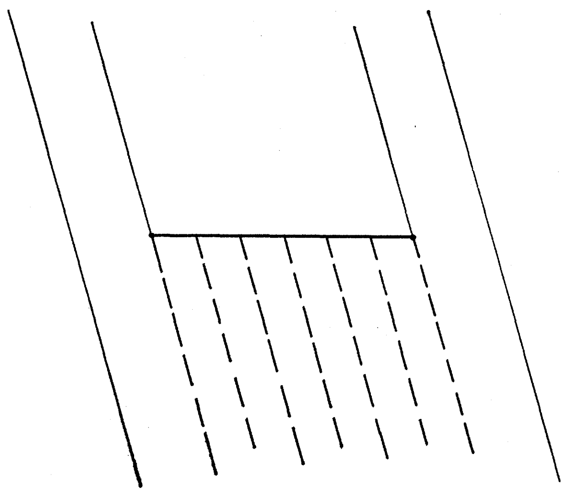

FIG. 4. Characteristic shadow region.

The condition that a nontrivial solution exists is found to be

$$(1+\lambda^2)(\cos\theta+\lambda\sin\theta)^2-(1+\lambda^2)(m_0^2+M_0^2)+m_0^2M_0^2=0, \quad (56)$$

where m_0 and M_0 are the Mach numbers

$$m_0=u_0/a_0, \quad M_0=u_0/A_0. \quad (57)$$

An alternative form in terms of the angle ϕ and also

$$\delta=\theta-\phi \quad (58)$$

is

$$\cos^2\delta-(m_0^2+M_0^2)\cos^2\phi+m_0^2M_0^2\cos^4\phi=0. \quad (59)$$

The solution for p, q, r, s can be written

$$p:q:r:s=u_0^2\sin\theta\sin\delta : M_0^2\cos\phi(1-m_0^2\cos^2\phi) : (m_0^2\cos\theta\cos\phi-\cos\delta) : \lambda u_0^2\sin^2\theta\cos\phi(m_0^2\cos^2\phi-1), \quad (60)$$

and the Riemann invariant can be put in the form

$$(\cos\theta+\lambda\sin\theta-m_0^2\cos\theta)\sigma-m_0^2(\sin\theta-\lambda\cos\theta)v+m_0^2[m_0^2/(1+\lambda^2)-1](\lambda\eta-\xi)=\text{const} \quad (61)$$

or, in terms of the angles ϕ and δ,

$$(\cos\delta-m_0^2\cos\theta\cos\phi)\sigma-m_0^2(\sin\delta)v+m_0^2(m_0^2\cos^2\phi-1)(\eta\sin\phi-\xi\cos\phi)=\text{const.} \quad (62)$$

By using (59) as an identity, one can obtain many equivalent forms for the Riemann invariant; for later use we list the form

$$(M_0^2\sin\phi\cos^2\phi-\sin\theta\cos\delta)\sigma+m_0^2\cos\phi(1-M_0^2\cos^2\phi)v+m_0^2\cos^2\phi\sin\delta(\lambda\eta-\xi). \quad (63)$$

There is a very large amount of information hidden in these relations, but we illustrate their significance with only a few examples.

In a purely hyperbolic problem with four real roots, we can solve the airfoil problem explicitly. From domain of dependence considerations (see Sec. 2), we can assign to each characteristic an orientation such that all quantities approach zero at infinity in this direction. The corresponding Riemann invariant is then zero except in a shadow zone intercepted by the airfoil (see Fig. 4). The solution of the problem is given by the linear superposition of four such shadows; we must compute only the values of the four Riemann invariants in the shadow zones. Consider the boundary values taken by σ, v, ξ, η on the airfoil and count the distinct values above and below as eight functions defined on the slit which represents the airfoil. In general terms, the boundary conditions are four linear relations in these eight variables. Combined with the four Riemann invariants (also linear combinations of these variables) which vanish on the airfoil, two above and two below, we are able to solve for all eight boundary values and thus evaluate the four remaining Riemann invariants. For the specific problem at hand, we have the two boundary conditions that ξ and η are continuous and v is given above (v^+) and below (v^-). This leaves four unknowns σ^+, σ^-, ξ, η to be determined from the four vanishing Riemann invariants; these four equations are inhomogeneous because they contain the given values of v^+ and v^-.

We illustrate this procedure by taking the hypersonic limiting case of large m_0 and M_0, the ratio m_0/M_0 being considered fixed. We find λ to be large on the order of m_0. To lowest order in an expansion in powers of $1/m_0$, we find

$$\frac{1}{\lambda^2}=\frac{m_0^2+M_0^2}{2m_0^2M_0^2}\left\{1\pm\left[1-\frac{4\sin^2\theta m_0^2M_0^2}{(m_0^2+M_0^2)^2}\right]^{\frac{1}{2}}\right\}, \quad (64)$$

and for the Riemann invariant,

$$\cos\theta(\sigma-\lambda v)+(m_0^2\lambda^{-2}-1)(\xi-\lambda\eta). \quad (65)$$

The four values of λ are given by an independent choice of the sign $\pm\lambda$ for each choice of sign in the expression for λ^2. The four vanishing Riemann invariants correspond to $\lambda>0$ above the airfoil and $\lambda<0$ below, viz.,

$$\begin{aligned}\cos\theta(\sigma^+-|\lambda|v^+)+(m_0^2\lambda^{-2}-1)(\xi-|\lambda|\eta)=0,\\ \cos\theta(\sigma^-+|\lambda|v^-)+(m_0^2\lambda^{-2}-1)(\xi+|\lambda|\eta)=0;\end{aligned} \quad (66)$$

each of these equations is counted twice by a choice of sign in (64). The solution is effected most easily in terms of the variables

$$\begin{aligned}\bar\sigma=\tfrac{1}{2}(\sigma^++\sigma^-), \quad \bar v=\tfrac{1}{2}(v^++v^-)\\ \sigma'=\tfrac{1}{2}(\sigma^+-\sigma^-), \quad v'=\tfrac{1}{2}(v^+-v^-)\end{aligned} \quad (67)$$

and we compute

$$\begin{aligned}\xi&=-(M_0^2/Q_0)\cot\theta\, v'\\ \eta&=M_0\cos\theta\,\bar v/(m_0\sin\theta+M_0)\\ \bar\sigma&=m_0(m_0+M_0\sin\theta)v'/Q_0\sin\theta\\ \sigma'&=m_0Q_0\bar v/(m_0\sin\theta+M_0),\end{aligned} \quad (68)$$

where

$$Q_0^2=m_0^2+M_0^2+2m_0M_0\sin\theta. \qquad (69)$$

It should be recalled that $\sin\theta \geq 0$. The values of σ^- and σ^+ and of the nonvanishing Riemann invariants are easily calculated, and the solution away from the airfoil is then given by certain lengthy expressions.

Other limiting cases in which the ratio m_0/M_0 becomes large or small are of particular interest. This limit is equivalent to A_0/a_0 large or small; i.e., the magnetic field either dominates or becomes negligible. An interesting mathematical feature of some of these examples is that they are partly elliptic and partly hyperbolic. Only in special cases can a problem which is partly elliptic and partly hyperbolic can be solved by simple means. In the general case, the elliptic variables (i.e., the real and imaginary parts of the complex invariant) do not satisfy one of the boundary conditions. In applying a boundary condition to a linear combination of elliptic and hyperbolic variables, we are led to a pair of integral equations over the slit for assumed values of either the elliptic variables or of the real Riemann invariants. The integral equation is derived from the Green's function of the slit domain, or, if the continuity of ξ and η are appropriately used, from the simpler Green's function of the entire plane.

Consider first the case $A_0 \gg a_0$ or $M_0 \ll m_0$. The two sets of characteristics are given by

$$\begin{aligned} \lambda &= \pm(M_0^2-1)^{\frac{1}{2}} \quad \text{(fast)} \\ \lambda &= (\pm m_0-\cos\theta)/\sin\theta \quad \text{(slow)}. \end{aligned} \qquad (70)$$

The fast wave is elliptic or hyperbolic depending on whether M_0 is smaller or larger than unity; the slow wave is hyperbolic. To be strictly consistent with our hypothesis, we should take $\lambda=\pm i$ for the fast wave if m_0 is not large and $\lambda=\pm m_0/\sin\theta$ for the slow wave if m_0 is large. For the fast wave, we have $m_0^2/(1+\lambda^2) = m_0^2/M_0^2$ which is large. Inspection of the Riemann invariant (61) shows that the dominant term is the one in ξ and η. We conclude that ξ and η by themselves satisfy an elliptic or hyperbolic second-order system. The boundary conditions that ξ and η are continuous imply that ξ and η vanish identically in both the elliptic and hyperbolic cases. It remains to find σ and v from the slow Riemann invariant which takes the form

$$\sigma-\lambda v=\text{const}, \quad \lambda=\pm m_0/\sin\theta. \qquad (71)$$

From this we easily compute σ and v in the whole plane, viz.,

$$\begin{aligned} v&=v^+, \quad \sigma=m_0v^+/\sin\theta \quad \text{(above)} \\ v&=v^-, \quad \sigma=-m_0v^-/\sin\theta \quad \text{(below)}. \end{aligned} \qquad (72)$$

Here v^+ and v^- signify the boundary values which are taken where the appropriate characteristic through the given point intersects the airfoil. In other words, the values of v and $\sigma=\pm m_0 v/\sin\theta$ are carried as constants on the relevant characteristics in the appropriate shadow zone.

The most interesting conclusion is that the vector velocity perturbation is in the direction of $\mathbf{B}_0$. From the last line of (51), we see that $\eta=0$ implies the vanishing of the component of velocity perpendicular to $\mathbf{B}_0$. We can summarize by stating that in the limit of very large magnetic field, $A_0 \gg a_0$, the airfoil produces a purely gas-dynamical disturbance with no influence on the magnetic field, but this gas disturbance is quite unconventional and is very strongly affected by the presence of the field.

The opposite limiting case $M_0 \gg m_0$ can also be solved in cases of combined elliptic-hyperbolic characteristics. With m_0 finite and $M_0 \gg 1$, we have

$$\begin{aligned} \lambda &= \pm(m_0^2-1)^{\frac{1}{2}} \quad \text{(fast)} \\ \lambda &= \pm M_0/\sin\theta \quad \text{(slow)}. \end{aligned} \qquad (73)$$

For the fast wave, λ is finite and $[m_0^2/(1+\lambda^2)-1]$ is small; consequently ξ and η can be dropped from the Riemann invariant. We are left with

$$\lambda(\sin\theta-\lambda\cos\theta)\sigma-m^2(\sin\theta-\lambda\cos\theta)v=\text{const}.$$

Provided that $\lambda \neq \tan\theta$ (which case requires special treatment), we have

$$\lambda\sigma-m^2v=\text{const}, \quad \lambda=\pm(m_0^2-1)^{\frac{1}{2}}. \qquad (74)$$

This in an elliptic or hyperbolic problem depending on whether $m_0^2<1$ or $m_0^2>1$, and in either case it can be solved for σ and v using the given boundary condition on v. The solution is exactly the conventional gas-dynamical solution for this boundary-value problem. Assuming that σ and v have been evaluated, we now turn to the slow Riemann invariants to find the magnetic field components. In this case λ is large, and to lowest order, only the variables σ, v, and η appear:

$$(\sin\theta)\sigma+m_0^2(\cos\theta)v-m_0^2\eta=\text{const}, \quad \lambda=\pm M_0/\sin\theta. \qquad (75)$$

To this order, the invariant has the same form for each value of λ; this implies that the invariant is identically zero, or

$$\eta=(1/m_0^2)(\sin\theta)\sigma+(\cos\theta)v. \qquad (76)$$

This result is more strking in terms of the velocity component u [cf. last line of (51)],

$$u=-\sigma/m_0^2. \qquad (77)$$

This result for u is also the classical gas-dynamical solution for the airfoil problem. The foregoing analysis is incorrect, however, since the value found for η is not necessarily continuous across the slit. The correct boundary condition can be imposed only if we remove the degeneracy in the invariant (75) and keep the next higher order term in ξ,

$$(\sin\theta)\sigma/m_0^2+(\cos\theta)v-\eta+\xi/\lambda, \quad \lambda=\pm M_0/\sin\theta. \qquad (78)$$

The solution to this problem under the correct boundary conditions yields a large value of ξ on the order of M_0. In other words, if we look for solutions in which ξ is finite (e.g., without specifying any boundary conditions), we obtain the conventional fluid solution in the limit of vanishing magnetic field. But, the boundary condition that η be continuous across the slit implies that a certain velocity component, $\eta = v\cos\theta - u\sin\theta$, be continuous, and this condition is violated in all but certain very special fluid flows. In order to satisfy this boundary condition, the magnetic field component ξ must be large and the x component of velocity is finitely affected by even a small ambient magnetic field.

The solution for ξ and η (also u) is easily obtained from the invariant (78) and is most easily expressed in terms of the solution of the equivalent gas-dynamical problem. In particular, we introduce V as the projection of the flow velocity in the fluid problem in the direction perpendicular to $\mathbf{B}_0$ (which direction is extraneous to the fluid solution),

$$V = \cos\theta\, v + \sin\theta\, \sigma/m_0^2; \tag{79}$$

σ and v are the same in the fluid and in the magnetic problems. In the shadow region above the airfoil, we find

$$\begin{aligned} \xi &= -M_0V'/\sin\theta \\ \eta &= V - V' \\ u &= -\sigma/m_0^2 + V'/\sin\theta, \end{aligned} \tag{80}$$

and in the shadow below,

$$\begin{aligned} \xi &= -M_0V'/\sin\theta \\ \eta &= V + V' \\ u &= -\sigma/m_0^2 - V'/\sin\theta. \end{aligned} \tag{81}$$

Outside the shadow regions, we have

$$\xi = 0, \quad \eta = V, \quad u = -\sigma/m_0^2, \tag{82}$$

which is identical to the fluid solution. The symbol V' represents the difference, $\frac{1}{2}(V^+ - V^-)$, of the boundary values taken by the component V of the fluid solution at the point on the slit which is intercepted by the appropriate characteristic through the given point.

If a_0 and u_0 are taken to be finite and A_0 approaches zero, the shadow regions are very narrow; we can say that the solution converges to the fluid solution except in a boundary layer and wake. However, if only a_0 is finite and both u_0 and A_0 approach zero, the magnetic flow is entirely different.

There is an interesting special case treated by McCune and Resler[2d] in which the solution for small magnetic field *does* reduce to the fluid flow in the entire plane. He takes $\theta = \pi/2$ (magnetic field perpendicular to the flow) and considers a symmetric body for which $v^+ = -v^-$. In this problem, we have $\eta = -u$. The condition that η be continuous is that u be continuous; but this is clearly satisfied in virtue of the symmetry, and the boundary condition is redundant. In the formulas (80) and (81), we see this explicitly when we set $V' = -u' = 0$.

We now turn to the case where $\mathbf{u}$ and $\mathbf{B}$ have components in the z direction, but first look at the special case where $\mathbf{B}_0$ is two-dimensional, $b=0$, in (53) and (54). The two sets of equations are decoupled. Furthermore, the two-dimensional system (53) is exactly the same as the one previously analyzed. We need solve only the system (54) for the z components of $\mathbf{u}$ and $\mathbf{B}$. The characteristic form of these equations is easily computed:

$$[u_0\partial/\partial x \pm A_0(\cos\theta\partial/\partial x + \sin\theta\partial/\partial y)] \times (u_0 w \mp A_0\zeta) = 0. \tag{83}$$

The differential operator is, in vector notation, $\mathbf{u}_0\cdot\nabla \pm \mathbf{A}_0\cdot\nabla$. With the boundary condition that ζ is constant on the airfoil (cf. Sec. 3), we obtain the simple solution that this constant value of ζ is propagated in both shadow regions defined by the backward characteristics, while w takes the constant value $+A_0\zeta/u_0$ in the shadow above the airfoil and $-A_0\zeta/u_0$ below. This is a pure transverse or Alfvén wave.

Finally, we consider the general case with $b \neq 0$. The system (53) and (54) is of sixth order. After a certain amount of manipulation, we find the Riemann invariant

$$\begin{aligned} &(M_0^2\sin\phi\cos^2\phi - \sin\theta\cos\delta)\sigma \\ &\quad + m_0^2\cos\phi(1 - M_0^2\cos^2\phi)v \\ &\quad + m_0^2\cos^2\phi\sin\delta(\lambda\eta - \xi) \\ &\qquad + bm_0^2\cos\phi(\sin\theta\, w + \sin\phi\cos\phi\,\zeta) \end{aligned} \tag{84}$$

corresponding to the four roots

$$\cos^2\delta - \cos^2\phi[M_0^2 + (1+b^2)m_0^2] + m_0^2M_0^2\cos^4\phi = 0, \tag{85}$$

and the Riemann invariant

$$\frac{\sigma}{m_0^2} + \left[\cos\theta \pm M_0\left(\frac{\sin^2\theta}{\cos^2\delta} - 1\right)\right]\left[\frac{w}{b} - \frac{v}{\sin\theta} \pm \frac{\zeta}{\beta M_0}\right] + \frac{b}{M_0^2}\zeta \mp \frac{1}{M_0}(\lambda\eta - \xi) \tag{86}$$

corresponding to the two roots

$$\lambda = (\pm M_0 - \cos\theta)/\sin\theta, \quad \cos\delta = \pm M_0\cos\phi. \tag{87}$$

We do not go into the properties of these solutions except to show that the special case of a symmetric airfoil, $v^+ = -v^-$, with $\theta = \pi/2$ is solvable in a combined elliptic and hyperbolic regime just as it is for $b=0$. The general argument is quite simple, but the algebra is rather formidable and has not been done. Under the assumptions made, it is easy to verify that the real roots occur in pairs, $\pm\lambda$, and the complex roots occur in pure imaginary pairs, $\lambda = \pm i\nu$. Inspection of the Riemann invariants shows that the variables σ, η, are even and v, ξ, w are odd in y. The boundary condition on ξ implies that $\xi = 0$; the boundary condition on η is redundant. The variables v and ξ are given on the boundary. This leaves the boundary values of σ, η, and

w open. The conjugate elliptic variables are linear combinations of the odd and even variables, respectively. On using the two vanishing real Riemann invariants, we can find a linear combination of the two elliptic variables which has known boundary values. For example, we can solve for σ and η in terms of w using the two vanishing Riemann invariants and then eliminate w between the two elliptic variables. This linear combination of the elliptic variables is itself a solution of the elliptic second-order equation, and it can now be found. For the case $b=0$, the odd elliptic variable is v itself (since ξ vanishes), so no algebraic manipulation is necessary.

A word should be said about the special case $\theta=0$. This problem can be treated directly and solved explicitly.[2c] However, this result is not the same as that obtained by letting θ approach zero. The reason is that the airfoil becomes characteristic and boundary conditions are lost for $\theta=0$; we observe bad behavior in all the explicit solutions for small θ. It is not the equations themselves that become singular; it is only the solution to the specific type of boundary value problem under consideration. However, the elimination of u in terms of v and η becomes singular for $\theta=0$, so the later forms of the differential equations are not appropriate for a study of the case $\theta=0$.

It is an easy matter to reduce the problem with $\mathbf{B}_0$ three-dimensional to the restricted problem $b=0$ in the special case $\theta=0$. This is exactly the parallel-transverse flow that was treated nonlinearly in Sec. 5. The equivalence of the two problems is obtained by replacing a_0 by a_{0*}, where [in our present notation, cf. Eq. (46)]

$$a_{0*}^2=a_0^2+[b^2u_0^2/(M_0^2-1)]. \qquad (88)$$

7. THREE-DIMENSIONAL FLOWS WITH A LARGE MAGNETIC FIELD[16]

We investigate the flow about a three-dimensional plane lamina (cf. Fig. 3) with an arbitrary cross section which is described in terms of the given normal component of velocity as a boundary condition.

We rewrite the linearized isentropic steady-flow equations (49) in the variables

$$\sigma=\rho/\rho_0,\quad \mathbf{A}=\mathbf{B}/(\mu_0\rho_0)^{\frac{1}{2}},\quad \mathbf{A}_0=\mathbf{B}_0/(\mu_0\rho_0)^{\frac{1}{2}} \qquad (89)$$

obtaining

$$\begin{aligned}&(\mathbf{u}_0\cdot\nabla)\sigma+\mathrm{div}\mathbf{u}=0\\&(\mathbf{u}_0\cdot\nabla)\mathbf{u}+a_0^2\nabla\sigma=\mathrm{curl}\mathbf{A}\times\mathbf{A}_0=(\mathbf{A}_0\cdot\nabla)\mathbf{A}-\nabla(\mathbf{A}_0\cdot\mathbf{A})\\&(\mathbf{A}_0\cdot\nabla)\mathbf{u}-(\mathbf{u}_0\cdot\nabla)\mathbf{A}-\mathbf{A}_0\,\mathrm{div}\mathbf{u}=0\\&\mathrm{div}\mathbf{A}=0.\end{aligned} \qquad (90)$$

On taking the curl of the second and third lines, then multiplying by $\mathbf{A}_0$, we find

$$\begin{aligned}&(\mathbf{u}_0\cdot\nabla)\mathbf{A}_0\cdot\mathrm{curl}\mathbf{u}-(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0\cdot\mathrm{curl}\mathbf{A}=0\\&(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0\cdot\mathrm{curl}\mathbf{u}-(\mathbf{u}_0\cdot\nabla)\mathbf{A}_0\cdot\mathrm{curl}\mathbf{A}=0.\end{aligned} \qquad (91)$$

[16] See reference 1d for an analysis of the time-dependent problem.

The pair of variables ($\mathbf{A}_0\cdot\mathrm{curl}\mathbf{u}$, $\mathbf{A}_0\cdot\mathrm{curl}\mathbf{A}$) satisfies a *two-dimensional* hyperbolic system which can be solved separately in each plane parallel to $\mathbf{u}_0$ and $\mathbf{A}_0$, given appropriate boundary conditions. In particular, we obtain the characteristic system

$$(\mathbf{A}_0\cdot\nabla\pm\mathbf{u}_0\cdot\nabla)(\mathbf{A}_0\cdot\mathrm{curl}\mathbf{u}\mp\mathbf{A}_0\cdot\mathrm{curl}\mathbf{A})=0. \qquad (92)$$

In allowing A_0 to become large compared to a_0, we consider separately the two cases $u_0\sim a_0$ and $u_0\sim A_0$. In determining relative orders of magnitude, we consider A to be on the order of A_0 and u on the order of u_0.

First, suppose $A_0\gg a_0$ and $A_0\gg u_0$. From the second line of (90), we conclude

$$\mathrm{curl}\mathbf{A}\times\mathbf{A}_0=0. \qquad (93)$$

From the boundary condition that $\mathbf{A}$ is a surface gradient, $(\mathrm{curl}\mathbf{A})_n=0$, we conclude that the vector $\mathrm{curl}\mathbf{A}$ vanishes at the boundary. In particular, $\mathbf{A}_0\cdot\mathrm{curl}\mathbf{A}=0$ at the boundary. We conclude from (92) that $\mathbf{A}_0\cdot\mathrm{curl}\mathbf{A}=0$ everywhere. Combined with (93), we have $\mathrm{curl}\mathbf{A}=0$ everywhere. On recalling that $\mathrm{div}\mathbf{A}=0$ and all components of $\mathbf{A}$ are continuous across the airfoil, we finally conclude that $\mathbf{A}=0$. The correct interpretation of this result is $A/A_0\ll 1$; more precisely, $A/A_0\sim u_0^2/A_0^2$ or a_0^2/A_0^2.

Next we compute $\mathbf{u}$ and σ. It is convenient to consider the component of $\mathbf{u}$ parallel to $\mathbf{A}_0$, $\mathbf{A}_0\cdot\mathbf{u}$, and the two-component vector perpendicular to $\mathbf{A}_0$ which we denote by $\mathbf{u}'$. From the third line of (90), we obtain

$$(\mathbf{A}_0\cdot\nabla)\mathbf{u}'=0 \qquad (94)$$

$$(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0\cdot\mathbf{u}+A_0^2(\mathbf{u}_0\cdot\nabla)\sigma=0 \qquad (95)$$

after eliminating $\mathrm{div}\mathbf{u}$ by use of the continuity equation. Also, from the second line of (90),

$$(\mathbf{u}_0\cdot\nabla)\mathbf{A}_0\cdot\mathbf{u}+a_0^2(\mathbf{A}_0\cdot\nabla)\sigma=0. \qquad (96)$$

The vector $\mathbf{u}'$ is a constant on each magnetic line $\mathbf{A}_0$. On using the conventional regularity condition for three-dimensional problems that all quantities approach zero at infinity whether elliptic or hyperbolic, one would conclude that $\mathbf{u}'=0$. This is too hasty because the hyperbolic characteristics of this problem behave two-dimensionally. A more precise argument is as follows. The value of $\mathbf{u}'$ cannot be carried along the line $\mathbf{A}_0$, since $\mathbf{A}_0$ is not a characteristic. However, it does approximate the two characteristic directions $(\mathbf{A}_0\cdot\nabla)\pm(\mathbf{u}_0\cdot\nabla)$. By reinserting the vector $\mathbf{A}$, we can derive the following exact relations:

$$(\mathbf{A}_0\cdot\nabla\pm\mathbf{u}_0\cdot\nabla)(\mathbf{u}'\mp\mathbf{A}')=\pm\nabla'(\mathbf{A}_0\cdot\mathbf{A}+a_0^2\sigma). \qquad (97)$$

The symbol ∇' represents a two-component gradient. According to our order of magnitude estimates, we can drop the right-hand side; however, we choose to keep the equally small $\mathbf{u}_0\cdot\nabla$ terms on the left simply to preserve the correct characteristics of the problem. We now conclude that $\mathbf{u}'=\pm\mathbf{A}'$ to this order; i.e., $\mathbf{u}'$ is small as well as $\mathbf{A}'$.

Since $\mathbf{u}'=0$, the remaining component of $\mathbf{u}$, parallel to $\mathbf{A}_0$, takes the boundary condition set by the given component normal to the airfoil. From (95) and (96) we can solve for $\mathbf{A}_0\cdot\mathbf{u}$ and σ. The domain of dependence is two-dimensional and is explicit in the characteristic form

$$[a_0(\mathbf{A}_0\cdot\nabla)\pm A_0(\mathbf{u}_0\cdot\nabla)](\mathbf{A}_0\cdot\mathbf{u}\pm a_0A_0\sigma)=0. \quad (98)$$

We can summarize this solution as follows. The perturbation of the magnetic field is zero. The velocity perturbation is parallel to $\mathbf{A}_0$. The magnitudes of $\mathbf{A}_0\cdot\mathbf{u}$ and σ at a point are simply expressed in terms of the velocity boundary value at that point on the airfoil which is reached by tracing back the appropriate characteristic from the given point. The domain of dependence does not widen conically. Exactly the same solution holds for a wavy "airfoil" which is not plane but which is any part of a cylindrical surface ruled by the straight lines $\mathbf{u}_0$.

The solution which we have obtained is not unique. This can be seen most easily by examination of the two-dimensional problem. It was seen in Sec. 6 that, superposed on the plane solution, there can be a transverse wave produced by a constant magnetic field which is trapped in the airfoil. In the three-dimensional problem, such a trapped field is also possible. For example, even in a fluid at rest, one can "freeze" a perturbation magnetic field into the nonconducting airfoil. One can probably say that the solution to the flow problem would be determined if one stated from what equilibrium configuration the airfoil was accelerated. The point in our previous argument at which the trapped field was excluded was in the statement that $\mathbf{A}_0\cdot\text{curl}\mathbf{A}$ is zero globally. In the presence of a trapped field there can be a singularity in $\mathbf{A}_0\cdot\text{curl}\mathbf{A}$ at the edge of the airfoil. Consequently, $\mathbf{A}$ is a harmonic vector except for possible current sheets which lie on the backward characteristic surfaces through the edges of the airfoil. We do not consider these solutions further.

For the case in which u_0 is not necessarily small compared to A_0, we can verify that the solution just given with $\mathbf{A}=0$ and $\mathbf{u}'=0$ satisfies the equations. First we replace the system (90) by an equivalent system valid in the limit $A_0\gg a_0$:

$$\begin{aligned}
&(\mathbf{u}_0\cdot\nabla)\sigma+\text{div}\mathbf{u}=0\\
&(\mathbf{u}_0\cdot\nabla)\mathbf{A}_0\cdot\mathbf{u}+a_0^2(\mathbf{A}_0\cdot\nabla)\sigma=0\\
&(\mathbf{u}_0\cdot\nabla)\mathbf{u}'=\text{curl}\mathbf{A}\times\mathbf{A}_0\\
&(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0\cdot\mathbf{u}-(\mathbf{u}_0\cdot\nabla)\mathbf{A}_0\cdot\mathbf{A}-A_0^2\,\text{div}\mathbf{u}=0\\
&(\mathbf{A}_0\cdot\nabla)\mathbf{u}'-(\mathbf{u}_0\cdot\nabla)\mathbf{A}'=0, \quad \text{div}\mathbf{A}=0.
\end{aligned} \quad (99)$$

The only approximation that has been made is the elimination of the term involving a_0^2 in taking the perpendicular component of the second line of (90). On inserting $\mathbf{A}=0$ and $\mathbf{u}'=0$, we find that all equations are satisfied identically except

$$\begin{aligned}
&(\mathbf{u}_0\cdot\nabla)\sigma+\text{div}\mathbf{u}=0\\
&(\mathbf{u}_0\cdot\nabla)\mathbf{A}_0\cdot\mathbf{u}+a_0^2(\mathbf{A}_0\cdot\nabla)\sigma=0\\
&(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0\cdot\mathbf{u}-A_0^2\,\text{div}\mathbf{u}=0,
\end{aligned} \quad (100)$$

which is exactly what was found before [Eqs. (95) and (96)].

APPENDIX 1

Two-Dimensional Flow around a Perfectly Conducting Body

Consider the two-dimensional nonlinear equations for the flow around a perfectly conducting body. On some portion of the surface it is assumed that $B_n\neq 0$, and where this is so we have the boundary condition $\mathbf{u}=0$.

From the magnetic flux equation we conclude $\mathbf{u}\times\mathbf{B}=-\mathbf{E}$, where $\mathbf{E}$ is a constant vector in the direction of the ignorable coordinate. The boundary condition tells us that this constant is zero, so $\mathbf{u}\times\mathbf{B}=0$ in the entire flow. This is a parallel flow studied in Sec. 5. We have $\mathbf{B}=\lambda\mathbf{u}$, where λ is a scalar and $\lambda=\alpha\rho$, where α is constant on each streamline. It is more convenient to write $\alpha=1/\beta$; we have

$$\beta\mathbf{B}=\rho\mathbf{u},$$

where β is constant on a streamline. At a point on the boundary, $\mathbf{u}=0$ and $\mathbf{B}\neq 0$; consequently, either $\beta=0$ or ρ approaches infinity near the boundary. However, from Bernoulli's law [Eq. (34)], we see that $h(\rho)+\frac{1}{2}u^2$ is constant on a streamline and ρ cannot approach infinity. Hence β (and therefore $\mathbf{u}$) is identically zero along every magnetic line which intersects the boundary. It may be possible to construct nontrivial flows which pass around both the body and a rigid mass of fluid at rest with the body, but in such a flow it might be more appropriate to consider the body together with the stagnant fluid as the rigid body (with $B_n=0$) around which the flow takes place.

APPENDIX 2

Boundary Condition $J_n=0$

We wish to compare the perfectly conducting transverse flow around a nonconducting object with a slightly resistive transverse flow around the same object. First, it is necessary to complete the perfectly conducting solution and obtain the interior solution, inside the body. This is very simple, since the only harmonic (i.e., vacuum) unidirectional magnetic field is a constant field. Since the magnetic field in the fluid is not constant in general (it is proportional to the value of ρ on the streamline which coincides with the boundary), there must be a surface current in the fluid layer adjacent to the conductor. The plane vector $\mathbf{J}=(1/\mu_0)\,\text{curl}\mathbf{B}$ is equal to ∇B in magnitude but is perpendicular to it. Thus, J_n does not vanish at the boundary unless B (i.e., ρ) is constant there.

We now turn to the case of a finite but small resistivity, R, and estimate the solution using methods which, although not rigorous, are very similar to more conventional boundary layer analyses, and which can probably be made just as precise. The flux equation now reads

$$\partial B/\partial t + \mathrm{div}(\mathbf{u}B) = (R/\mu_0)\Delta B.$$

This is a conventional heat equation with a convection term. We consider only the steady-flow problem and construct a boundary-layer solution in several steps.

First, assuming that $\mathbf{u}$ is given as the solution of the perfectly conducting problem, we solve this heat equation subject to the boundary condition B = constant exactly as we would the conventional convection heat-transfer problem. The constant value of B is determined by the condition that the line integral of $\mathbf{E}$ around the body vanish,

$$\oint \mathbf{E}\cdot d\mathbf{x} = \oint R\mathbf{J}\cdot d\mathbf{x} = 0.$$

But this is exactly the condition that there be no net heat flow into the body in the heat flow analog. It is very easy to verify by properly "stretching" the boundary layer that the locally plane boundary layer solution is consistent with p_*=constant and $\mathbf{u}$=constant. We interpret this to mean that, in the limit of small R, both $\mathbf{u}$ and p_* approach their perfectly conducting values, and we compute p in the boundary layer as $p_* - B^2/2\mu_0$.

The resistive solution has the property that $J_n=0$ at the boundary. However, there are very large, mostly tangential, currents in a thin boundary layer, and these merge into the finite currents of the perfectly conducting solution (with $J_n \neq 0$) just outside the boundary layer. From this analysis one can predict with confidence that a rigorous solution of the resistive problem converges in the limit of vanishing R to the *weak* solution of the perfectly conducting problem including surface currents.

The boundary condition $J_n=0$ is, in the resistive case, an alternative way of stating the boundary condition B=constant, but this boundary condition (in either version) is lost in the limit.

APPENDIX 3

At the symposium, an exact fluid analog was presented by I. Imai for a parallel flow which is two-dimensional, isentropic, and irrotational. This analog can be extended as is indicated in the following. However, the existence of this analog does not reduce to a conventional fluid problem the transonic difficulties of the parallel channel-flow problem discussed in Sec. 5. The reason is that the analog is not global; the transformation is singular at $M=1$, and a given fluid flow is directly applicable to only one of the two regions $M<1$ or $M>1$. Furthermore, the induced equations of state are not compatible with thermodynamic restrictions in all regions and must be interpreted with great care.

We wish to introduce $p_\dagger$, $\rho_\dagger$, $\mathbf{u}_\dagger$ in such a way that the fluid system

$$\mathrm{div}(\rho_\dagger\mathbf{u}_\dagger)=0, \quad \rho_\dagger(\mathbf{u}_\dagger\cdot\nabla)\mathbf{u}_\dagger+\nabla p_\dagger=0 \tag{A1}$$

is equivalent to the parallel-flow equations (33). Following Imai, we take

$$p_\dagger = p + B^2/2\mu_0, \tag{A2}$$

and impose the restriction

$$\rho_\dagger\mathbf{u}_\dagger = \rho\mathbf{u}. \tag{A3}$$

In order for the two sets of equations of motion to be equivalent, we must have

$$\rho_\dagger(\mathbf{u}_\dagger\cdot\nabla)\mathbf{u}_\dagger = -\nabla p_\dagger = \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - (1/\mu_0)(\mathbf{B}\cdot\nabla)\mathbf{B}.$$

From this (recalling that $\mathbf{u}$ is parallel to both $\mathbf{u}_\dagger$ and $\mathbf{B}$ while α is constant on a streamline), we obtain

$$\rho(\mathbf{u}\cdot\nabla)[\mathbf{u} - (\alpha^2/\mu_0)\rho\mathbf{u} - \mathbf{u}_\dagger] = 0.$$

This suggests that we complete the analog by specifying

$$\begin{aligned} \mathbf{u}_\dagger &= (1-\alpha^2\rho/\mu_0)\mathbf{u} = (1-1/M^2)\mathbf{u} \\ \rho_\dagger &= (1-\alpha^2\rho/\mu_0)^{-1}\rho = (1-1/M^2)^{-1}\rho. \end{aligned} \tag{A4}$$

We now introduce the notation

$$\begin{aligned} A_\dagger^2 &= B^2/\mu_0\rho_\dagger = (1-1/M^2)A^2 \\ M_\dagger^2 &= u_\dagger^2/A_\dagger^2 = M^2-1 \end{aligned} \tag{A5}$$

and note the identity

$$p + \tfrac{1}{2}\rho u^2 = p_\dagger + \tfrac{1}{2}\rho_\dagger u_\dagger^2. \tag{A6}$$

In order to complete the system (A1), we still need an equation of state connecting $p_\dagger$ and $\rho_\dagger$. This is obtained with the aid of Bernoulli's law (34):

$$\begin{aligned} p_\dagger(\rho_\dagger;\eta,\alpha,h_0) &= p(\rho,\eta) + B^2/2\mu_0 \\ &= p(\rho,\eta) + \alpha^2\rho^2u^2/2\mu_0 \\ &= p(\rho,\eta) + \alpha^2\rho^2[h_0 - h(\rho,\eta)]/\mu_0 \end{aligned} \tag{A7}$$

after replacement of ρ in terms of $\rho_\dagger$; h_0 is the stagnation enthalpy or Bernoulli constant on a given streamline. This equation of state, together with the system (A1) and the supplementary adiabatic relations

$$(\mathbf{u}_\dagger\cdot\nabla)\eta = (\mathbf{u}_\dagger\cdot\nabla)\alpha = (\mathbf{u}_\dagger\cdot\nabla)h_0 = 0 \tag{A8}$$

complete the system.

On using the thermodynamic relations (along a streamline),

$$dh = dp/\rho, \quad dh_\dagger = dp_\dagger/\rho_\dagger$$
$$a^2 = \partial p/\partial\rho, \quad a_\dagger^2 = \partial p_\dagger/\partial\rho_\dagger,$$

we compute

$$\begin{aligned} h_\dagger &= h + A^2(1-\tfrac{1}{2}M^{-2}) = h + A_\dagger^2(1+\tfrac{1}{2}M_\dagger^{-2}) \\ a_\dagger^2 &= (1-1/M^2)^2[a^2(1-1/M^2)+A^2] \end{aligned} \tag{A9}$$

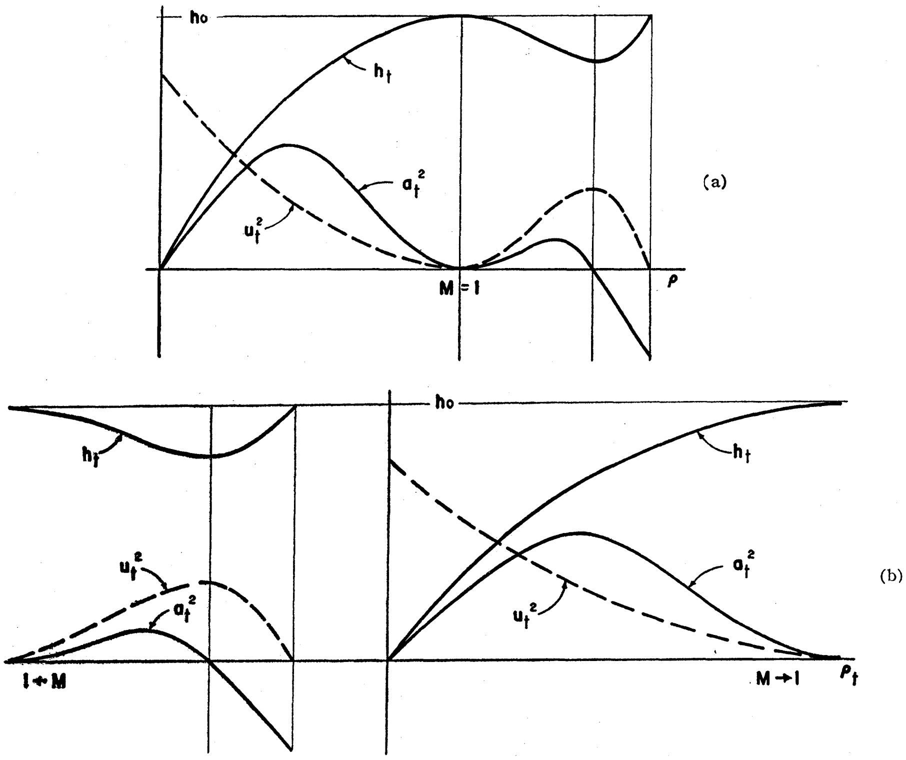

FIG. 5. Parallel flow analog.

and verify that

$$h_\dagger+\tfrac{1}{2}u_\dagger^2=h+\tfrac{1}{2}u^2=h_0. \tag{A10}$$

If the quantities α, η, and h_0 are taken to be constant in the entire flow, then the circulation is conserved for the analog velocity $\mathbf{u}_\dagger$, and one can adopt the constraint $\mathrm{curl}\,\mathbf{u}_\dagger=0$ as done by Imai.

On a given streamline, using the entropy and Bernoulli constants, one can solve for all the quantities $p_\dagger$, $h_\dagger$, $a_\dagger^2$, $u_\dagger^2$ in terms of, say, $\rho_\dagger$. In a conventional fluid, one finds that p, h, a^2, ρ are monotone and vary in the direction opposite to u^2. The equation of state given by this analog implies a much more complicated variation. First we note that $\mathbf{u}_\dagger$ and $\rho_\dagger$ change sign across $M=1$; in particular, this implies the thermodynamically unusual result that $\rho_\dagger$ is negative for $M<1$. The variation of $h_\dagger$, $a_\dagger^2$, and $u_\dagger^2$ as functions of ρ and of $\rho_\dagger$ are indicated schematically in Fig. 5 for the case $a<A$ and the stagnation density larger than $\rho=\mu_0/\alpha^2$.

Two sonic transitions, $u_\dagger=a_\dagger$, are found at $u=a$ and $u=A$. The third transition from hyperbolic to elliptic, at $u^2=a^2A^2/(a^2+A^2)$, occurs where $a_\dagger^2$ changes sign. The transition at $u=a$ is similar to the conventional fluid one. From the fact that $\rho_\dagger$ becomes infinite as u approaches A, one might be tempted to conclude that flow ends before reaching this state. But this analog flow is unusual in that $h_\dagger$ remains finite as $\rho_\dagger$ becomes infinite, thereby allowing the flow to be continued up to $M=1$. It would seem that the transition problem across $M=1$ is less singular in the original variables than in terms of the fluid analog.

The elliptic "fluid" system with $a_\dagger^2<0$ and $\rho_\dagger<0$ seems to be quite unconventional. In the hyperbolic case where $\rho_\dagger<0$, we can formally replace $\rho_\dagger$ by $-\rho_\dagger$ in the equation of motion (keeping $a_\dagger^2$ unaltered), after which we obtain a conventional system including a fairly conventional equation of state. The case $\rho_\dagger>0$ $(M>1)$ is conventional whether elliptic or hyperbolic.

It is tempting to associate the change in sign of $\mathbf{u}_\dagger$ across $M=1$ with the altered domain of dependence as discussed before; in the analog variable $\mathbf{u}_\dagger$, one would choose forward characteristics in both hyperbolic regions.

This fluid analog can evidently be extended to the parallel-transverse flows since they were shown to be analogs of the simple parallel flow. The results are contained in the formulas

$$\begin{aligned}
p_\dagger&=p_*+B^2/2\mu_0=p+(\rho/M^2)(\tfrac{1}{2}\delta^2+h_0-h)\\
h_\dagger&=h_*+A^2(1-\tfrac{1}{2}M^{-2})\\
&=h+(M^2-\tfrac{1}{2})\{[\delta^2/(M^2-1)^2]+(A^2/M^2)\},\\
a_\dagger^2&=(1-1/M^2)^2[a_*^2(1-1/M^2)+A^2]\\
&=(1-1/M^2)^2\{(1-1/M^2)a^2\\
&\qquad+[\delta^2M^2/(M^2-1)^2]+A^2\},
\end{aligned} \tag{A11}$$

which we present without discussion.

DISCUSSION

Session Reporter: G. S. S. Ludford

W. R. Sears, *Cornell University, Ithaca, New York:* I want to make one short remark about Fig. 2. For **u** parallel to **B**, the equations are hyperbolic, elliptic, hyperbolic again, elliptic again, successively, as we go to zero speed. The last one is the really interesting one, because it has no resemblance to anything in ordinary aerodynamics; it is a hypersonic transition from hyperbolic to elliptic, the speed being given by

$$1/u^2=1/a^2+1/A^2.$$

For this speed the propagation velocity of the Friedrichs-van de Hulst-Herlofson-Grad waves goes to zero. As u decreases to $aA(a^2+A^2)^{-\frac{1}{2}}$, the waves, which are forward facing, become hypersonic, i.e., they lie down on the (reverse) flow direction. If the flow goes just a little bit slower they are gone entirely, so that this is a transition between hypersonic and elliptic.

H. Grad: Just a remark about the second transition point. In the nozzle problem, assuming $a>A$, first there is the transition through $u=aA(a^2+A^2)^{-\frac{1}{2}}$, then one through A, and finally there is the transition through the ordinary sound speed a. Thus you have elliptic, hyperbolic, elliptic, and hyperbolic regions successively. The transition through $u=aA(a^2+A^2)^{-\frac{1}{2}}$ is very unusual, as mentioned by Professor Sears. But the transition through $u=A$ is also unconventional. For there is an argument in ordinary gas dynamics which indicates that if you have a transition from supersonic to subsonic, it is unstable without shocks. The argument goes something like this. Consider an ordinary nozzle with supersonic conditions in front. The domain of dependence tells you that you should take the forward characteristics. A little disturbance at the wall is carried forward and tends to pile up at the throat since the characteristics steepen and never pass the throat. Now you can very easily show that this argument does not apply here, because for a disturbance at the wall the correct characteristic to take is the *upstream* one which passes the disturbance safely away from the sonic section. So this particular argument does not disqualify the solution. Whether there are others which do, I do not know.

G. A. Lyubimov, *Moscow University, Moscow, U.S.S.R.:* I would like to say a few words about M. N. Kogan's researches on magneto-fluid dynamics in the U.S.S.R. Kogan investigated, for $\sigma=\infty$, the characteristics both for the case where the magnetic field is parallel to the flow and for the case of arbitrary orientation of the magnetic lines. The first is investigated in a paper which has already appeared,[a] the second will be published this year.[b] He found that the system of equations is hyperbolic in the two regions, 1 and 2, marked in Diagram 1, whene $N=A/a$ and $A=H^2/4\pi\rho$ has been assumed

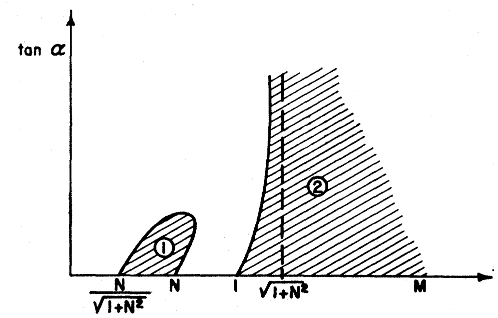

Diagram 1.

to be smaller than a; elsewhere it is elliptic-hyperbolic. Only one hyperbolic range for M remains if the angle α between the magnetic field and the velocity exceeds a certain value. As the magnetic field tends to zero ($N\rightarrow 0$) the region 1 is contracted to a point, and therefore the flow properties which are typical of this region do not have any analogies in conventional gas dynamics. Proceeding from these investigations, Kogan has calculated the linear approximation to flow around airfoils for arbitrary orientation of the magnetic field.

J. A. Shercliff, *University of Cambridge, Cambridge, England:* There is a very easy way of showing that this compressible parallel case is exactly analogous to ordinary gas dynamics. This has been done by Cowley,[c] who simply puts

$$\mathbf{u}^*=\mathbf{u}(1-A^2/u^2).$$

It all comes out, without any linearization, with equations exactly analogous to those in ordinary gas dynamics, and the relevant speed of sound is given by

$$a^{*2}=dp^*/d\rho^*,\ p^*=\text{Grad's } p_*=p+B^2/2\mu_0$$

but with ρ^* different from Grad's ρ. In fact, $\rho^*=\rho(1-A^2/u^2)^{-1}$. In addition, you can feel confident in taking these forward-facing waves in the subsonic hyperbolic case, because this is the case where the factor $(1-A^2/u^2)$ is negative and the real flow is in the opposite direction to the fictitious nonconducting gas flow.

H. Grad: It would seem impossible, from this analogy with ordinary fluid dynamics, to cross from one side to the other.

J. A. Shercliff: But the fictitious gas in the analogy sometimes has to have negative density and therefore flow at different parts of the same "streamline" could be in different directions.

[a] M. N. Kogan, Priklad. Mat. Mech. **23**, 70 (1959) [English translation: Priklad. Mat. Mech. **23**, 92 (1959)].
[b] M. N. Kogan, Kriklad. Mat. Mech. (to be published).

[c] M. D. Cowley, Jet Propulsion **30**, 271 (1960); see also I. Imai, Revs. Modern Phys. **32**, 992 (1960), this issue.