Hydromagnetic Flow due to an Oscillating Plane

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1. INTRODUCTION

IN order to understand the effect of viscosity in modifying the motion of a fluid in contact with vibrating solids, Stokes examined a particularly simple case (see Rayleigh,[1] 1896, p. 317). He supposed that an infinite plane located at $z=0$ executes harmonic vibration in a direction (x, say) parallel to itself, and that the fluid in contact with this plane at $z=0$ occupies the whole of the region $z>0$ and is at rest at very large values of z. Assuming that ν, the coefficient of kinematical viscosity, is constant, and that no slip occurs between the fluid and the vibrating surface, he showed that if the velocity of the vibrating plane is ($U_0 \cos\omega t$, 0, 0), where t denotes time, and U_0 and ω, which are assumed to be constant, are, respectively, the velocity, amplitude, and angular frequency of the vibration, the velocity at any point in the fluid $\mathbf{u}=(u_x,u_y,u_z)$, is given by

$$u_x=U_0 \exp[-(\omega/2\nu)^{\frac{1}{2}}z]\cos[\omega t-(\omega/2\nu)^{\frac{1}{2}}z], \quad u_y=u_z=0. \tag{1}$$

According to this expression the velocity amplitude falls off exponentially with distance from the plate, having dropped to $e^{-1}U_0=0.3679U_0$ at a distance $z=\Delta$, where

$$\Delta\equiv(2\nu/\omega)^{\frac{1}{2}}. \tag{2}$$

In addition to this variation of amplitude with z, there is also a variation in phase, due to the inertia of the fluid. The wavelength associated with the variation in phase is $2\pi\Delta$, at which distance from the plate the velocity amplitude is

$$e^{-2\pi}U_0=0.0018U_0.$$

The flow generated by the vibrating plane can be regarded as a heavily damped plane shear wave, the coupling between different layers being due to viscous friction. The tangential force per unit area at any level can be obtained from the stress tensor

$$p_{ij}=p_{ji}=\rho\nu(\partial u_i/\partial x_j+\partial u_j/\partial x_i), \tag{3}$$

where ρ is the density of the fluid, also assumed to be constant. This gives

$$p_{xz}=\frac{-\rho\nu U_0}{\Delta}\exp\left(-\frac{z}{\Delta}\right)\left[\cos\left(\omega t-\frac{z}{\Delta}\right)-\sin\left(\omega t-\frac{z}{\Delta}\right)\right], \quad p_{yz}=0, \tag{4}$$

so that the tangential force per unit area acting on the plane is

$$p_{xz}(0,t)=(\rho\nu U_0/\Delta)(\sin\omega t-\cos\omega t). \tag{5}$$

The negative of the second term in this bracket is in phase with the motion of the plane and corresponds to a dissipative force tending to stop the motion. The other term is $\pi/2$ out of phase and represents an effective increase in the inertia of the vibrating body due to the presence of the fluid.

The rate at which internal stresses do work at any surface parallel to the vibrating plane, in general $p_{zx}u_x+p_{zy}u_y+p_{zz}u_z$ (see Goldstein,[2] p. 98), reduces to $p_{zx}u_x$ in this problem. Denoting this quantity by R, by Eqs. (1) and (4), we find that

$$R=R(z,t)=\frac{-\rho\nu U_0^2}{\Delta\sqrt{2}}\exp\left(-\frac{2z}{\Delta}\right)\left[\frac{1}{\sqrt{2}}+\cos\left\{2\omega t-\frac{2z}{\Delta}+\frac{\pi}{4}\right\}\right], \tag{6}$$

which oscillates at double the frequency of the vibration about a nonzero average value

$$\bar{R}(z)=(-\rho\nu U_0^2/2\Delta)\exp(-2z/\Delta). \tag{7}$$

The power input per unit area required to maintain the motion, which is denoted by P, must equal $-R(0,t)$ so that

$$P=P(t)=\frac{\rho\nu U_0^2}{\Delta\sqrt{2}}\left[\frac{1}{\sqrt{2}}+\cos\left(2\omega t+\frac{\pi}{4}\right)\right]. \tag{8}$$

The direction of energy transfer between the vibrating plane and the fluid depends on the sign of P, which alternates, changing twice each half-cycle. However, the average value of P, given by

$$\bar{P}=\rho\nu U_0^2/2\Delta, \tag{9}$$

is essentially positive, corresponding to a net energy transfer from the vibrating plane to the fluid, where it is dissipated by viscous friction.

Rayleigh[1] made use of Stokes' result for the drag on the plane [see Eq. (5)] in his investigation of the effect of the boundary layer on the propagation of sound in tubes. The present writers became interested in the main problem discussed here as a result of a suggestion that it may be possible to measure in the laboratory the effect of a magnetic field on the propagation of sound in a tube containing mercury. The obvious extension of Stokes' problem was to the case of a conducting

[1] Lord Rayleigh (J. W. Strutt), *The Theory of Sound* (1896) (reprinted by Dover Publications, New York, 1945).

[2] S. Goldstein, *Modern Developments in Fluid Dynamics* (Clarendon Press, Oxford, England, 1938), Vol. I.

TABLE I. Typical values of α and β.

	Temperature °K	Kinematic viscosity ν ($m^2\ sec^{-1}$)	Electromagnetic viscosity ($m^2\ sec^{-1}$) $\lambda=(\mu\sigma)^{-1}$	Density ρ ($kg\ m^{-3}$)	$\beta=\lambda/\nu$	B_0 ($vs\ m^{-2}$) ($\times 10^4$ for emu)	Alfvén velocity $(B_0H_0/\rho)^{\frac{1}{2}}$ ($m\ sec^{-1}$)	$\alpha\omega$ (sec^{-1})	$\alpha\omega/\beta$ (sec^{-1})
Mercury	293	1.14×10^{-7}	7.8×10^{-1}	1.35×10^{4}	6×10^{6}	1	6	3×10^{8}	50
Sodium	373	6.31×10^{-7}	7.7×10^{-2}	9.28×10^{2}	1.2×10^{5}	1	20	6×10^{9}	5×10^{4}
Ionized hydrogen[a]	10^{5}	10	15	10^{-7}	1.5	10^{-1}	2×10^{5}	4×10^{9}	3×10^{9}
Earth's interior	10^{4}	10^{-6}?	1	10^{4}	10^{6}?	10^{-3}	6×10^{-3}	36?	3.6×10^{7}
Sunspots	4×10^{3}	10^{-2}	20	10^{-1}	2×10^{3}	2×10^{-1}	360	10^{7}	5×10^{3}
Solar granulation	6×10^{3}	10	10^{2}	10^{-4}	10	10^{-2}	600	3.6×10^{4}	3.6×10^{3}
Magnetic variable stars	10^{6}	3×10^{-7}	1	10^{3}?	3×10^{6}	1	20	10^{9}	3×10^{2}
Interstellar space	10^{4}	10^{17}	10^{3}?	10^{-21}	10^{-14}	10^{-9}?	2×10^{4}	4×10^{-9}	4×10^{5}
Interplanetary space	10^{5}	10^{16}	10?	10^{-20}	10^{-15}	10^{-9}	5×10^{3}	2.5×10^{-9}	2.5×10^{6}
Solar corona	10^{6}	10^{16}	1?	10^{-15}	10^{-16}	10^{-4}	10^{6}	10^{-4}	10^{12}

[a] For ionized hydrogen: $\nu \propto \rho^{-1}T^{5/2}$; $\lambda \propto T^{-3/2}$; $\beta \propto \rho T^{-4}$; $\alpha \propto \rho T^{-5/2}$; $\alpha/\beta \propto T^{3/2}$. $\mu_0 = 4\pi\times10^{-7}$ newton amp^{-2}. $\sigma_{copper} = 5\times10^{7}\Omega^{-1}\cdot m^{-1}$.

fluid (electrical conductivity σ) in the presence of an impressed uniform magnetic field of strength B_0 in the z direction, the formal solution of which is presented in Sec. 2 and discussed in Sec. 3. Rationalized mks units are used throughout.

The over-all behavior of the system now depends on three parameters

$$\alpha \equiv \frac{H_0 B_0}{\rho\omega\nu}; \tag{10a}$$

$$\beta \equiv \frac{1}{\mu\sigma\nu} = \frac{\lambda}{\nu}; \tag{10b}$$

$$\gamma \equiv \frac{U_0}{(\omega\nu)^{\frac{1}{2}}}; \tag{10c}$$

where μ denotes magnetic permeability ($B_0=\mu H_0$) and $\lambda\equiv(\mu\sigma)^{-1}$ is sometimes called "electromagnetic viscosity" because it has the same dimensions as ν. Some typical values of α and β are listed in Table I. α, β, and γ measure, in suitable units, the magnetic-field energy, the electrical resistivity, and the velocity amplitude of the motion of the plane, respectively. From their definitions we can relate α, β, and γ to the more familiar dimensionless parameters of magneto-fluid dynamics provided we base these parameters on the characteristic length $L=(\nu/\omega)^{\frac{1}{2}}$, which is of the order of the boundary-layer thickness Δ in Stokes' problem. Thus we find the Reynolds number

$$\text{Re} \equiv U_0 L/\nu = \gamma, \tag{11}$$

the magnetic Reynolds number

$$\text{Rm} \equiv U_0 L/\lambda = \gamma/\beta, \tag{12}$$

the Hartmann number

$$\text{Ha} \equiv B_0 L(\sigma/\rho\nu)^{\frac{1}{2}} = (\alpha/\beta)^{\frac{1}{2}}, \tag{13}$$

and the Lundquist number[3]

$$\text{Lu} \equiv B_0 L\sigma(\mu/\rho)^{\frac{1}{2}} = \alpha^{\frac{1}{2}}/\beta. \tag{14}$$

The fluid motion is no longer of the form given by Eq. (1). In the presence of the magnetic field, u_x consists of two parts characterized by different attenuation and phase factors. The mathematical form of these parts suggests that they should be termed "velocity" mode and "magnetic" mode. The relative amplitudes of these modes, their associated attenuation and phase factors, and the induced magnetic and electric fields depend on α, β, and γ. These quantities also depend on the electromagnetic boundary conditions, which in turn are determined by the electrical properties of the region $z<0$ not occupied by the fluid and also on the conditions prevailing at $y=\pm\infty$. We have chosen to restrict attention to the case when $z<0$ is filled by an insulator and there are insulating surfaces, on which electric charges are about to accumulate, at $y=\pm\infty$. There are no additional difficulties associated with the other cases when the region $z<0$ is a conductor or when charges are not able to accumulate at $y=\pm\infty$.

Explicit solutions have been found in a sufficient number of cases, corresponding to different limiting values of α and β, to cover most situations of physical interest. The results demonstrate quantitatively the complicated interplay between hydromagnetic and viscous effects (see Sec. 3).

Now we digress before taking up the next part of the problem, in order to explain how it arose. It is now generally accepted that hydromagnetic flow in the earth's liquid core is somehow responsible for the main geomagnetic field, and therefore, an entirely satisfactory theory of the earth's magnetism requires, in the first instance, a theory of the dynamics of core motions. For detailed discussions of this geophysical problem see Hide[4] and Hide and Roberts.[5] Suffice it to say here that no one is certain as to the cause of core motions, and only very general indications of the flow pattern are

[3] T. G. Cowling, *Magnetohydrodynamics* (Interscience Publishers, Inc., New York, 1957).

[4] R. Hide, "Hydrodynamics of the earth's core," in *Physics and Chemistry of the Earth*, L. H. Ahrens, K. Rankama, and S. K. Runcorn, Editors (Pergamon Press, New York, 1956), Vol. 1.

[5] R. Hide and P. H. Roberts, "The origin of the earth's magnetic field," in *Physics and Chemistry of the Earth*, L. H. Ahrens, K. Rankama, and S. K. Runcorn, Editors (Pergamon Press, New York, 1960), Vol. 4.

revealed by geomagnetic data. However, it has been suggested that the near coincidence between the geomagnetic and geographic poles is the result of the strong influence of Coriolis forces, due to the earth's rotation, on motions in the core.

Rough order-of-magnitude estimates show that Coriolis forces are very much stronger than inertial and viscous forces. They also suggest that Coriolis and hydromagnetic forces are comparable with one another in strength, and one wonders why this should be. As the magnetic energy in the core exceeds the kinetic energy of fluid motion relative to the rotating earth by a large factor, we cannot invoke equipartition of energy to account for this state of affairs. Thus, in the absence of any obvious general reason why a system should adjust itself until hydromagnetic and Coriolis forces are of comparable magnitude we have to take the alternative approach and seek a clue to the solution of this problem by considering specific fluid systems in which both forces play a part.

In spite of the importance in cosmical fluid dynamics of understanding the effects of rotation and magnetic fields on flow phenomena, the simultaneous action of these agencies seems to have been studied in only a few specific cases. Lehnert[6] has considered the effect of Coriolis forces on plane hydromagnetic waves in a perfectly conducting inviscid fluid and thus showed how the propagation of these waves can, in some circumstances, be radically affected by rotation. However, the assumption of perfect conductivity restricts the value of Lehnert's results in the present context. Chandrasekhar[7] has examined the theory of the onset of thermal convection in a thin horizontal layer of a conducting fluid which rotates about a vertical axis, in the presence of a magnetic field. He found a number of unexpected results, the interpretation of which is far from simple, owing to the essentially three-dimensional nature of the problem.

We have considered the effect of uniform rotation and a uniform magnetic field acting simultaneously on the flow of a conducting fluid due to the oscillation of a rigid plane with which it is in contact, as a simple extension of the first problem dealt with here. For simplicity the angular velocity vector $\mathbf{\Omega}$ is taken to be parallel to the z axis, and therefore to the magnetic field. The formal discussion of this case is given in Sec. 2, where it is shown that a further parameter,

$$\delta \equiv 2\Omega/\omega, \tag{15}$$

which measures Ω in suitable units, enters the problem. It follows from its definition[4] that the so-called Rossby number,

$$\mathrm{Ro} \equiv \omega/2\Omega = \delta^{-1}, \tag{16}$$

and that based on the characteristic length $L=(\nu/\omega)^{\frac{1}{2}}$, the Taylor number,[4]

$$\mathrm{Ta} \equiv 4\Omega^2 L^4/\nu^2 = \delta^2. \tag{17}$$

Although it was expected initially that owing to the symmetry of the system the mathematical difficulties involved would not be prohibitive, an unanticipated feature of the solution of the rotating case is that the equations satisfied by the phase and amplitude factors are formally identical with the corresponding equations of the nonrotating case. As a result of this feature, the properties of the flow can be found directly from the discussion of the nonrotating case (see Sec. 4). The interpretation of these properties will be dealt with in the full account of this work, which will be published elsewhere.

2. FORMAL SOLUTION

The hydromagnetic flow set up by the plane is governed by the equation of hydrodynamics,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\mathrm{grad}\,\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} = -\rho^{-1}\,\mathrm{grad}\,p + \rho^{-1}\mathbf{j}\times\mathbf{B} + \nu\nabla^2\mathbf{u}; \tag{18}$$

the equation of continuity,

$$\mathrm{div}\,\mathbf{u} = 0; \tag{19}$$

Maxwell's equations (neglecting displacement currents),

$$\mathrm{curl}\,\mathbf{B} = \mu\mathbf{j}, \tag{20}$$

$$\mathrm{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{21}$$

$$\mathrm{div}\,\mathbf{B} = 0; \tag{22}$$

and Ohm's law for a moving conductor,

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}). \tag{23}$$

In these equations p denotes the hydrostatic pressure; $\mathbf{E}$, the electric field; $\mathbf{j}$, the electric current density.

We suppose that $\mathbf{B}$ consists of two parts

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \mathbf{B}_0 = (0,0,B_0), \tag{24}$$

where B_0 is constant and $\mathbf{b}$ is the induced field.

The boundary conditions necessary to determine the unique solution to these equations are the following. We must require that the fluid be at rest at large distances from the plane and that no slip should occur between the fluid and plane, whence

$$\mathbf{u}(z\rightarrow\infty) = 0, \quad \mathbf{u}(z=0) = (U_0\cos\omega t, 0, 0). \tag{25}$$

In addition, there are boundary conditions on the magnetic field. Because we are assuming $z<0$ to be an insulator, $\mathbf{j}$ must vanish there. It must also vanish as $z\rightarrow\infty$. By (20) and (22), together with the fact that because we are assuming that the fluid is bounded by insulating surfaces at $y=\pm\infty$ the total current must

[6] B. Lehnert, Astrophys. J. **119**, 647 (1954); **121**, 481 (1955).

[7] S. Chandrasekhar, Proc. Roy. Soc. (London) **A225**, 173 (1954); **A237**, 476 (1956); Proc. Am. Acad. Arts Sci. U. S. **86**, 372 (1957).

vanish, these conditions require that

$$\mathbf{b}(z\to\infty)=0,\quad \mathbf{b}(z=0)=0. \tag{26}$$

The fields set up by the plate depend on z and t alone and, in consequence, it is found that $\mathbf{u}$ and $\mathbf{b}$ are, without approximation, governed by the *linear* equations

$$\left[\frac{\partial}{\partial t}-\frac{1}{\sigma\mu}\frac{\partial^2}{\partial z^2}\right]\mathbf{b}=B_0\frac{\partial\mathbf{u}}{\partial z}, \tag{27}$$

$$\left[\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial z^2}\right]\mathbf{u}+2\mathbf{\Omega}\times\mathbf{u}=\frac{B_0}{\mu\rho}\frac{\partial\mathbf{b}}{\partial z}, \tag{28}$$

and that u_z and b_z are zero everywhere. On eliminating $\mathbf{b}$ or $\mathbf{u}$ from Eqs. (27) and (28), it may be shown that

$$\left\{\left(\left[\frac{\partial}{\partial t}-\frac{1}{\sigma\mu}\frac{\partial^2}{\partial z^2}\right]\left[\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial z^2}\right]-\frac{B_0^2}{\mu\rho}\frac{\partial^2}{\partial z^2}\right)^2 +4\Omega^2\left[\frac{\partial}{\partial t}-\frac{1}{\sigma\mu}\frac{\partial^2}{\partial z^2}\right]^2\right\}(\mathbf{u},\mathbf{b})=0. \tag{29}$$

As Eqs. (27) and (28) are linear, we can, by analogy with Stokes' result (1), seek solutions of the form

$$(\mathbf{u},\mathbf{b})\propto e^{i\omega t-qz/L},\quad [L=(\nu/\omega)^{\frac{1}{2}}],$$

where $\mathcal{R}(q)>0$, by (25) and (26). On substituting in Eq. (29) and introducing the dimensionless parameters α, β, γ, δ defined in (10) and (15), we find

$$[(i-q^2)(i-\beta q^2)-\alpha q^2]^2=-\delta^2(i-\beta q^2)^2. \tag{30}$$

When $\alpha=0$, $\delta=0$, the roots of this equation are

$$q_1=(1/\sqrt{2})(1+i),\quad q_2=[1/(2\beta)^{\frac{1}{2}}](1+i). \tag{31}$$

q_2 corresponds to the magnetic mode which is of no interest in this case since $\alpha=0$. q_1 corresponds to Stokes' solution, Eq. (1).

In the nonrotating case ($\delta=0$), Eq. (30) reduces to

$$(i-q^2)(i-\beta q^2)-\alpha q^2=0, \tag{32}$$

which has two roots q_1, q_2 satisfying (25) and (26). These we term the velocity mode and the magnetic mode, respectively. As $\alpha\to 0$, these roots tend uniformly to (31).

In the nonmagnetic case $\alpha=0$, the magnetic mode is [as in the Stokes' case, (31)] redundant, but there are two velocity modes q_+ and q_- given by

$$[q^2-(1+\delta)i][q^2-(1-\delta)i]=0, \tag{33}$$

the positive and negative subscripts referring to the first and second brackets, respectively.

In the general case there are four roots of interest, q_{+1}, q_{+2}, q_{-1}, q_{-2}, corresponding, respectively, to the positive velocity mode, the positive magnetic mode, the negative velocity mode, and the negative magnetic mode. q_{+1}, q_{+2} are the relevant roots of

$$\beta q^4-[\alpha+i\{1+\beta(1+\delta)\}]q^2-(1+\delta)=0, \tag{34}$$

and q_{-1}, q_{-2} are the relevant roots of

$$\beta q^4-[\alpha+i\{1+\beta(1-\delta)\}]q^2-(1-\delta)=0. \tag{35}$$

Observe that the last two equations are the same as Eq. (32), provided we make a suitable change of scale: In Eq. (34), write

$$\beta_+=(1+\delta)\beta,\quad q_+=(1+\delta)^{\frac{1}{2}}q, \tag{36}$$

and in Eq. (35) write

$$\begin{aligned}\beta_-&=(1-\delta)\beta,\quad q_-=(1-\delta)^{\frac{1}{2}}q,\quad (\delta<1),\\ \beta_-&=(1-\delta)\beta,\quad q_-=-i(\delta-1)^{\frac{1}{2}}q,\quad (\delta>1).\end{aligned} \tag{37}$$

In the remainder of this section we describe the results obtained for the nonrotating case, returning only briefly in Sec. 4 to the analysis of the rotating case. In the nonrotating case, u_y and b_y are uncoupled to u_x and b_x, so that Eqs. (25) and (26) prove that u_y, b_y, j_x, and E_x are zero everywhere.

Let

$$\begin{aligned}u_x&=U_0e^{i\omega t}[f_1e^{-q_1z/L}+f_2e^{-q_2z/L}],\\ b_x&=B_0e^{i\omega t}[g_1e^{-q_1z/L}+g_2e^{-q_2z/L}].\end{aligned} \tag{38}$$

By Eq. (27), g_j is related to f_j ($j=1, 2$) by

$$(i-\beta q_j^2)g_j=-\gamma q_jf_j,\quad (j=1,2). \tag{39}$$

Also, by the conditions (25) and (26), we have

$$f_1+f_2=1, \tag{40}$$

$$g_1+g_2=0. \tag{41}$$

From (32) it follows that

$$q_1q_2=i/\beta^{\frac{1}{2}}. \tag{42}$$

On solving Eqs. (39)–(41) for f_j and g_j and using (42) as a means of simplifying the final results, we find

$$f_1=\frac{\beta^{\frac{1}{2}}q_1-q_2}{[1+\beta^{\frac{1}{2}}](q_1-q_2)},\quad f_2=\frac{\beta^{\frac{1}{2}}q_2-q_1}{[1+\beta^{\frac{1}{2}}](q_2-q_1)}, \tag{43}$$

$$g_1=-g_2=\{\beta^{\frac{1}{2}}[1+\beta^{\frac{1}{2}}](q_1-q_2)\}^{-1}. \tag{44}$$

From these results and Eq. (20), we find

$$j_y=-\left(\frac{B_0}{\mu L}\right)\frac{\gamma e^{i\omega t}}{\beta^{\frac{1}{2}}[1+\beta^{\frac{1}{2}}](q_1-q_2)}\times[q_1e^{-q_1z/L}-q_2e^{-q_2z/L}], \tag{45}$$

and, from Eqs. (20) and (23), we find

$$E_y=-(U_0B_0)\frac{e^{i\omega t}}{[1+\beta^{\frac{1}{2}}](q_1-q_2)}\times[q_2e^{-q_1z/L}-q_1e^{-q_2z/L}]. \tag{46}$$

Note that, on the plane itself,

$$j_y(z=0)=-(\beta_0/\mu L)(\gamma e^{i\omega t}/\beta^{\frac{1}{2}}[1+\beta^{\frac{1}{2}}]), \quad (47)$$

and

$$E_y(z=0)=(U_0B_0)(e^{i\omega t}/[1+\beta^{\frac{1}{2}}]). \quad (48)$$

Thus, in the limit $\beta\rightarrow 0$, there is a surface current on the plane, while the electric field there is given by

$$\mathbf{E}=-\mathbf{U}_0\times\mathbf{B}_0,$$

[cf. Eq. (23)]. In the limit $\beta\rightarrow\infty$, both current and electric field tend to zero. Both these results are physically reasonable.

The mean rate of working $\bar{P}$ of the force driving the plate is the average, over a cycle, of

$$P=-\nu\rho[u_x\partial u_x/\partial z]_{z=0},$$

and, by (38),

$$u_x(z=0)=\tfrac{1}{2}U_0e^{i\omega t}+\tfrac{1}{2}U_0e^{-i\omega t},$$

$$-\frac{\partial u_x}{\partial z}(z=0)=\frac{1}{2L}U_0e^{i\omega t}(f_1q_1+f_2q_2)+\frac{1}{2L}U_0e^{-i\omega t}(f_1q_1+f_2q_2)^*,$$

so that

$$P=(\rho\nu U_0^2/L)\tfrac{1}{2}\Re(f_1q_1+f_2q_2)+\text{periodic terms},$$

or, using Eqs. (42) and (43) and averaging,

$$\bar{P}=(\rho\nu U_0^2/L)\tfrac{1}{2}(\beta^{\frac{1}{2}}/[1+\beta^{\frac{1}{2}}])\Re(q_1+q_2). \quad (49)$$

3. DISCUSSION OF SOME LIMITING CASES

Having given the formal solution we now present the results in a number of limiting cases. We are interested in low, moderate, and high conductivity ($\beta\gg1$, $\beta=1$, $\beta\ll1$), and these cases are designated A, B, and C, respectively. In each of these cases we must consider first the effect of a weak magnetic field, and then the effect of a strong field.

The results for case A are summarized in Table II. Observe that α/β turns out to be the appropriate measure of the magnetic field. In the weak-field case, although the magnetic mode of u_x is associated with a slow fall-off with z, its amplitude is only a small fraction, $\sim\alpha/\beta^{\frac{3}{2}}$, of that of the velocity mode. In the presence of a strong field, the phase and amplitude factors of the velocity mode now depend strongly on σ and B_0, the amplitude of this mode of u_x being only slightly less than in the absence of a magnetic field. The magnetic mode of u_x is weak, the amplitude at $z=0$ being $\sim\beta^{-\frac{1}{2}}$.

TABLE II. Case A: $\beta\gg1$: $[\sigma\ll(\mu\nu)^{-1}]$; low conductivity. ($\zeta\equiv z/L$.)

	$\alpha/\beta\ll1\,[B_0\ll(\omega\rho/\sigma)^{1/2}]$ weak field	$\alpha/\beta\gg1\,[B_0\gg(\omega\rho/\sigma)^{1/2}]$ strong field
q_1	$\frac{1}{\sqrt{2}}\left\{1+\frac{\alpha}{2\beta}+i\left(1-\frac{\alpha}{2\beta}\right)\right\}$	$\left(\frac{\alpha}{\beta}\right)^{1/2}+\frac{i}{2}\left(\frac{\beta}{\alpha}\right)^{1/2}$
$q_1\left(\frac{\omega}{\nu}\right)^{1/2}$		$\left(\frac{H_0B_0}{\rho\lambda\nu}\right)^{1/2}+\frac{i\omega}{2}\left(\frac{\rho\lambda}{H_0B_0\nu}\right)^{1/2}$
q_2	$\frac{1}{(2\beta)^{1/2}}\left\{1-\frac{\alpha}{2\beta}+i\left(1+\frac{\alpha}{2\beta}\right)\right\}$	$\frac{\beta}{2\alpha^{3/2}}+\frac{i}{\alpha^{1/2}}$
$q_2\left(\frac{\omega}{\nu}\right)^{1/2}$		$\frac{\lambda\omega^2}{2}\left(\frac{\rho}{H_0B_0}\right)^{3/2}+i\omega\left(\frac{\rho}{H_0B_0}\right)^{1/2}$
$\frac{u_x}{U_0e^{i\omega t}}$	$\left(1+\frac{i\alpha}{\beta^{3/2}}\right)e^{-q_1\zeta}-\frac{i\alpha}{\beta^{3/2}}e^{-q_2\zeta}$	$\left(1-\frac{1}{\beta^{1/2}}\right)e^{-q_1\zeta}+\frac{1}{\beta^{1/2}}e^{-q_2\zeta}$
$\frac{b_x}{B_0e^{i\omega t}}$	$\frac{\gamma}{\beta\sqrt{2}}(1-i)[e^{-q_1\zeta}-e^{-q_2\zeta}]$	$\frac{\gamma}{(\alpha\beta)^{1/2}}\{e^{-q_1\zeta}-e^{-q_2\zeta}\}$
$\frac{E_y}{U_0B_0e^{i\omega t}}$	$\frac{1}{\beta}e^{-q_1\zeta}+\frac{1}{\beta^{1/2}}e^{-q_2\zeta}$	$\frac{i}{\alpha}e^{-q_1\zeta}+\frac{1}{\beta}e^{-q_2\zeta}$
$\frac{j_y(\nu/\omega)^{1/2}}{H_0e^{i\omega t}}$	$\frac{\gamma(1-i)}{\beta\sqrt{2}}[-q_1e^{-q_1\zeta}+q_2e^{-q_2\zeta}]$	$\frac{\gamma}{(\alpha\beta)^{1/2}}[-q_1e^{-q_1\zeta}+q_2e^{-q_2\zeta}]$
$\bar{P}\left(\frac{(\nu/\omega)^{1/2}}{\rho\nu U_0^2}\right)$	$\frac{1}{2\sqrt{2}}\left(1+\frac{\alpha}{2\beta}\cdots\right)$	$\frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{1/2}$

TABLE III. Case $B: \beta=1$ $[\sigma=(\mu\nu)^{-1}]$; moderate conductivity. ($\zeta\equiv z/L$.)

	$\alpha\ll 1[B_0\ll(\omega\nu\mu\rho)^{1/2}]$ weak field	$\alpha\gg 1[B_0\gg(\omega\nu\mu\rho)^{1/2}]$ strong field
q_1	$\frac{1}{\sqrt{2}}\left\{1+\left(\frac{\alpha}{2}\right)^{1/2}+\frac{\alpha}{8}+i\left(1-\frac{\alpha}{8}\right)\right\}$	$\alpha^{1/2}+\frac{i}{\alpha^{1/2}}$
$q_1\left(\frac{\omega}{\nu}\right)^{1/2}$		$\left(\frac{H_0B_0}{\rho\lambda\nu}\right)^{1/2}+i\omega\left(\frac{\rho}{H_0B_0}\right)^{1/2}$
q_2	$\frac{1}{\sqrt{2}}\left\{1-\left(\frac{\alpha}{2}\right)^{1/2}+\frac{\alpha}{8}+i\left(1-\frac{\alpha}{8}\right)\right\}$	$\frac{1}{\alpha^{3/2}}+\frac{i}{\alpha^{1/2}}$
$q_2\left(\frac{\omega}{\nu}\right)^{1/2}$		$\omega^2(\lambda\nu)^{1/2}\left(\frac{\rho}{H_0B_0}\right)^{3/2}+i\omega\left(\frac{\rho}{H_0B_0}\right)^{1/2}$
$\frac{u_x}{U_0e^{i\omega t}}$	$\frac{1}{2}e^{-q_1\zeta}+\frac{1}{2}e^{-q_2\zeta}$	$\frac{1}{2}e^{-q_1\zeta}+\frac{1}{2}e^{-q_2\zeta}$
$\frac{b_x}{B_0e^{i\omega t}}$	$\frac{\gamma}{2\alpha^{1/2}}[e^{-q_1\zeta}-e^{-q_2\zeta}]$	$\frac{\gamma}{2\alpha^{1/2}}[e^{-q_1\zeta}-e^{-q_2\zeta}]$
$\frac{E_y}{U_0B_0e^{i\omega t}}$	$\frac{1+i}{(2\alpha)^{1/2}}\{-\frac{1}{2}e^{-q_1\zeta}+\frac{1}{2}e^{-q_2\zeta}\}$	$\frac{-i}{2\alpha}e^{-q_1\zeta}+\left(\frac{1}{2}+\frac{i}{2\alpha}\right)e^{-q_2\zeta}$
$\frac{j_y(\nu/\omega)^{1/2}}{e^{i\omega t}H_0}$	$\frac{\gamma}{2\alpha^{1/2}}\{-q_1e^{-q_1\zeta}+q_2e^{-q_2\zeta}\}$	$\frac{\gamma}{2\alpha^{1/2}}\{-q_1e^{-q_1\zeta}+q_2e^{-q_2\zeta}\}$
$\bar{P}\left(\frac{(\nu/\omega)^{1/2}}{\rho\nu U_0^2}\right)$	$\frac{1}{2\sqrt{2}}\left[1+\frac{\alpha}{8}\cdots\right]$	$\frac{\alpha^{1/2}}{4}$

According to the form of q_2, this mode corresponds to an Alfvén wave damped by electrical resistance (see Alfvén,[8] p. 81).

Now consider case B, that of moderate conductivity, corresponding to $\beta=1$. The results are summarized in Table III. Now it is α that measures the impressed magnetic field. Observe that in the presence of a weak field, in contrast with cases A and C, q_1 and q_2 contain terms of order $\alpha^{\frac{1}{2}}$. The amplitude factors of each mode of u_x are the same, namely, 0.5.

In the strong-field case, the phase factors of each mode correspond to an Alfvén wave. The velocity mode is much more rapidly attenuated than the magnetic mode, and at moderate distances from the plane, the magnetic mode dominates. The form of the attenuation factor of this mode shows that viscosity and electrical resistivity play equal parts in dissipating the energy of the wave.

Finally, we consider the third case, C, that of high conductivity ($\beta\ll 1$). The results are summarized in Table IV.

When the magnetic field is weak, the velocity field is only slightly modified by it. The magnetic mode of u_x is weak, having a small amplitude at $z=0$ and a high attenuation factor, of order $\beta^{-\frac{1}{2}}$. There is no term in $\alpha^{\frac{1}{2}}$ in the expressions for q_1 and q_2.

In the presence of a strong field, the velocity mode of u_x is characterized by a small amplitude at $z=0$ and rapid attenuation, the motion consisting almost entirely of an Alfvén wave, which, according to the form of q_2, is damped by viscosity.

In all cases, the mean power $\bar{P}$ required to maintain the vibration has to be increased in the presence of a magnetic field; in the case of a strong magnetic field $\bar{P}$ is proportional to B_0.

The possibility of detecting the effect of a magnetic field on the propagation of sound in a tube of mercury was mentioned in Sec. 1. According to Table I, $\beta\gg 1$, so that α/β is the appropriate measure of B_0. As $\alpha/\beta\sim 50/\omega$, frequencies of vibration as low as a few cycles per second would be needed to produce any marked effect. In the kilocycle region the sound speed would be reduced by about 1%. The position should be rather more favorable if liquid sodium were used, because then it would be possible to work at much higher frequencies.

4. ROTATING CASE

We have shown in Sec. 2 how the roots q_{+1}, q_{+2}, q_{-1}, q_{-2} for the rotating case can be obtained in pairs

[8] H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, England, 1950).

TABLE IV. Case C: $\beta\ll 1[\sigma\gg(\mu\nu)^{-1}]$; high conductivity. ($\zeta\equiv z/L$.)

	$\alpha\ll 1[B_0\ll(\nu\omega\mu\rho)^{1/2}]$ weak field	$\alpha\gg 1[B_0\gg(\nu\omega\mu\rho)^{1/2}]$ strong field
q_1	$\frac{1}{\sqrt{2}}\left\{1-\frac{\alpha}{2}+i\left(1+\frac{\alpha}{2}\right)\right\}$	$\left(\frac{\alpha}{\beta}\right)^{1/2}+\frac{i}{2(\alpha\beta)^{1/2}}$
$q_1\left(\frac{\omega}{\nu}\right)^{1/2}$		$\left(\frac{H_0B_0}{\rho\nu\lambda}\right)^{1/2}+\frac{i\omega}{2}\left(\frac{\rho}{H_0B_0}\frac{\nu}{\lambda}\right)^{1/2}$
q_2	$\frac{1}{(2\beta)^{1/2}}\left\{1+\frac{\alpha}{2}+i\left(1-\frac{\alpha}{2}\right)\right\}$	$\frac{1}{2\alpha^{3/2}}+\frac{i}{\alpha^{1/2}}$
$q_2\left(\frac{\omega}{\nu}\right)^{1/2}$		$\frac{\nu\omega^2}{2}\left(\frac{\rho}{H_0B_0}\right)^{3/2}+i\omega\left(\frac{\rho}{H_0B_0}\right)^{1/2}$
$\frac{u_x}{U_0e^{i\omega t}}$	$(1+i\alpha\beta^{1/2})e^{-q_1\zeta}-i\alpha\beta^{1/2}e^{-q_2\zeta}$	$\beta^{1/2}e^{-q_1\zeta}+(1-\beta^{1/2})e^{-q_2\zeta}$
$\frac{b_x}{B_0e^{i\omega t}}$	$\frac{\gamma}{\sqrt{2}}\{e^{-q_1\zeta}-e^{-q_2\zeta}\}$	$\frac{\gamma}{\alpha^{1/2}}\{e^{-q_1\zeta}-e^{-q_2\zeta}\}$
$\frac{E_y}{U_0B_0e^{i\omega t}}$	$(1-i\alpha)e^{-q_1\zeta}-\beta^{1/2}e^{-q_2\zeta}$	$-i\beta^{1/2}e^{-q_1\zeta}+(1-\beta^{1/2})e^{-q_2\zeta}$
$\frac{j_y(\nu/\omega)^{1/2}}{H_0e^{i\omega t}}$	$\frac{\gamma}{\sqrt{2}}\{-q_1e^{-q_1\zeta}+q_2e^{-q_2\zeta}\}$	$\frac{\gamma}{\alpha^{1/2}}\{-q_1e^{-q_1\zeta}+q_2e^{-q_2\zeta}\}$
$\bar{P}\left[\frac{(\nu/\omega)^{1/2}}{\rho\nu U_0^2}\right]$	$\frac{1}{2\sqrt{2}}\left(1+\frac{\alpha}{2}\cdots\right)$	$\frac{\alpha^{1/2}}{2}$

from the nonrotating case by suitable changes of scale. In this section, we show that the same is true of the corresponding coefficients f_{+1}, f_{+2}, f_{-1}, f_{-2}, and g_{+1}, g_{+2}, g_{-1}, g_{-2} for the velocity and magnetic fields.

If we write [cf. Eq. (38)]

$$
\begin{aligned}
u_x&=U_0e^{i\omega t}[f_{+1}e^{-q_{+1}z/L}+f_{+2}e^{-q_{+2}z/L}\\
&\quad+f_{-1}e^{-q_{-1}z/L}+f_{-2}e^{-q_{-2}z/L}],\\
b_x&=B_0e^{i\omega t}[g_{+1}e^{-q_{+1}z/L}+g_{+2}e^{-q_{+2}z/L}\\
&\quad+g_{-1}e^{-q_{-1}z/L}+g_{-2}e^{-q_{-2}z/L}],\\
u_y&=U_0e^{i\omega t}[k_{+1}e^{-q_{+1}z/L}+k_{+2}e^{-q_{+2}z/L}\\
&\quad+k_{-1}e^{-q_{-1}z/L}+k_2e^{-q_{-2}z/L}],\\
b_y&=B_0e^{i\omega t}[l_{+1}e^{-q_{+1}z/L}+l_{+2}e^{-q_{+2}z/L}\\
&\quad+l_{-1}e^{-q_{-1}z/L}+l_2e^{-q_{-2}z/L}],
\end{aligned}
\tag{50}
$$

we find that Eq. (39) is still valid and that a similar relation holds for the y components:

$$(i-\beta q^2)l=-\gamma qg. \tag{51}$$

These equations, together with the simplifying relationships

$$
\begin{aligned}
q_{+1}q_{+2}&=i[(1+\delta)/\beta]^{\frac{1}{2}},\\
q_{-1}q_{-2}&=\begin{cases} i[(1-\delta)/\beta]^{\frac{1}{2}}, & \delta<1,\\ [(\delta-1)/\beta]^{\frac{1}{2}}, & \delta>1,\end{cases}
\end{aligned}
\tag{52}
$$

and the boundary conditions (25) and (26), lead to

$$
\begin{aligned}
2f_{+1}&=\frac{2k_{+1}}{\delta}=\frac{\beta_+^{\frac{1}{2}}q_{+1}-q_{+2}}{[1+\beta_+^{\frac{1}{2}}](q_{+1}-q_{+2})},\\
2f_{+2}&=\frac{2k_{+2}}{\delta}=\frac{\beta_+^{\frac{1}{2}}q_{+2}-q_{+1}}{[1+\beta_+^{\frac{1}{2}}](q_{+2}-q_{+1})},
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
2g_{+1}=-2g_{+2}&=\frac{2l_{+1}}{\delta}=-\frac{2l_{+2}}{\delta}\\
&=\frac{\gamma(1+\delta)}{\beta_+^{\frac{1}{2}}[1+\beta_+^{\frac{1}{2}}](q_{+1}-q_{+2})},
\end{aligned}
\tag{54}
$$

where

$$\beta_+^{\frac{1}{2}}=(1+\delta)^{\frac{1}{2}}\beta^{\frac{1}{2}} \tag{55}$$

and to

$$
\begin{aligned}
2f_{-1}&=-\frac{2k_{-1}}{\delta}=\frac{\beta_-^{\frac{1}{2}}q_{-1}-q_{-2}}{[1+\beta_-^{\frac{1}{2}}](q_{-1}-q_{-2})},\\
2f_{-2}&=-\frac{2k_{-2}}{\delta}=\frac{\beta_-^{\frac{1}{2}}q_{-2}-q_{-1}}{[1+\beta_-^{\frac{1}{2}}](q_{-2}-q_{-1})},
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
2g_{-1}=-2g_{-2}&=-\frac{2l_{-1}}{\delta}=\frac{2l_{-2}}{\delta}\\
&=\frac{\gamma(1-\delta)}{\beta_-^{\frac{1}{2}}[1+\beta_-^{\frac{1}{2}}](q_{-1}-q_{-2})},
\end{aligned}
\tag{57}
$$

where

$$\beta_-^{\frac{1}{2}}=\begin{cases}(1-\delta)^{\frac{1}{2}}\beta^{\frac{1}{2}}, & \delta<1,\\ -i(\delta-1)^{\frac{1}{2}}\beta^{\frac{1}{2}}, & \delta>1.\end{cases} \tag{58}$$

By comparing the results (53), (54) and (56), (57) with the results (47) and (44), we see how complete the division between the two modes is; on writing

$$u_x=\tfrac{1}{2}(u_{x+}+u_{x-}),\quad b_x=\tfrac{1}{2}(b_{x+}+b_{x-}),$$

it follows that

$$u_z=\tfrac{1}{2}\delta(u_{x+}-u_{x-}),\quad b_z=\tfrac{1}{2}\delta(b_{x+}-b_{x-})$$

and that each of the modes (u_{x+},b_{x+}) and (u_{x-},b_{x-}) can be calculated independently of the other and directly from the nonrotating case by means of the changes of scale introduced in Sec. 2.

After our investigation had been completed, our attention was drawn to a number of papers bearing on some of the topics with which we have dealt.[9–11]

[9] J. A. Shercliff, thesis, Cambridge, 1955 (unpublished).
[10] T. Kakutani, Proc. Phys. Soc. Japan **13**, 1504 (1958).
[11] J. A. Steketee, University of Toronto Institute of Aerophysics Rept. No. 63 (1959).

DISCUSSION

Session Reporter: W. H. Reid

In reply to a question by Dr. **J. E. McCune** (*Aeronautical Research Associates of Princeton, Princeton, New Jersey*) concerning the relationship between rotation and oscillation for the case in which the whole system is in rotation, the speaker remarked that for problems of this type in which one is primarily interested in understanding the effects of Coriolis forces one could, following the standard procedure in such cases, ignore centrifugal forces and, hence, need not define an axis of rotation.

Attention was also drawn to a number of related papers, the references to which are given previously. Professor **C. C. Lin** (*Institute for Advanced Study, Princeton, New Jersey*) mentioned his generalization to magneto-fluid dynamics of many of the exact solutions of fluid dynamics.[a]

[a] C. C. Lin, Arch. Ratl. Mech. Anal. **1**, 391 (1958).