Transition from Laminar to Turbulent Flow in Magneto-Fluid Mechanic Channels

PAUL S. LYKOUDIS
School of Aeronautical Engineering, Purdue University, Lafayette, Indiana

1. INTRODUCTION

ONE of the earliest problems in the field of magneto-fluid mechanics was the channel flow of an electrically conducting fluid in the presence of a magnetic field acting perpendicularly to the flow. Although the first theoretical and experimental attempt to solve the problem was made in 1937 with the pioneering work of Hartmann and Lazarus,[1,2] this apparently simple flow is still attracting attention.[1–10] The case of laminar flow, although far from being solved in the most general cases (consideration of magnetic and dynamic entrance effects, generalized cross-sectional geometry, wall-surface effects, effect of high magnetic Reynolds number, etc.) is at a relatively advanced stage of understanding both from an experimental and theoretical point of view. The turbulent case still seems to defy a rigorous mathematical analysis even for the simple situation when the magneto-fluid mechanic effects are absent. Some experimental work is available[1–3] in which essentially skin-friction coefficients have been calculated from measurement in liquid metals in the presence of a transverse magnetic field, but there is a complete lack even of the rudiments of an empirical theory that accounts for the known semiempirical results of the nonmagnetic case. Here an attempt is made to shed some light on this problem.

At this point one must distinguish between two different classes of magnetic turbulent boundary layers:

(a) In the first, the induced current lines are closed loops and, therefore, in the absence of an applied electric field there is no accumulation of electric charge; in this case, the induced electric field is zero and the only force acting is the one due to the cross product of the velocity vector and the vector of the intensity of the magnetic field. Such a situation presents itself, for example, in the magnetic flow over an axisymmetric body carrying a coil with an axis parallel to the axis of the body (for the laminar case, see reference 11).

(b) In the second class of magnetic turbulent boundary layers, the induced current lines do not close in on themselves and therefore they produce an accumulation of electric charge, which in turn gives rise to a nonzero electric field. This is the case of channel flow.[12]

Here an attempt is made to establish a theoretical criterion for the transition from laminar to turbulent flow in the case of channel flow when a magnetic field is acting perpendicularly to the flow. In the course of these calculations the correct nondimensional parameters upon which the problem depends are presented and the structure of the turbulent boundary layer is discussed. The theoretical results are compared with available experimental information.

2. LAMINAR SUBLAYER

If one postulates the existence of a laminar sublayer in a channel, then it is legitimate to assume the validity of the laminar solution found by Hartmann and Lazarus. In terms of the present terminology, this solution is given as follows[12]:

$$u_{\text{lam}}^*=\frac{1}{\lambda}\left\{\frac{\cosh M-\cosh M[1-(y/L)]}{\sinh M}\right\}, \tag{1}$$

where

$$M=\text{Hartmann number}^{13}=BL(\sigma/\mu)^{\frac{1}{2}}, \tag{2}$$

$$\lambda^2=\sigma\mu B^2/\rho\tau_w, \tag{3}$$

$$u^*=u/(\tau_w/\rho)^{\frac{1}{2}},\quad y^*=(\rho/\mu)(\tau_w/\rho)^{\frac{1}{2}}y. \tag{4}$$

The parameter λ is independent of any characteristic length, but it is convenient to introduce one artificially into both the numerator and denominator so that λ may be expressed in terms of other known nondimensional parameters. It is easy to show that

$$\lambda=M/\text{Re}(c_f/2)^{\frac{1}{2}}, \tag{5}$$

where

$$\text{Re}=\text{Reynolds number}=u_{\text{av}}L/\nu, \tag{6}$$

$$c_f=\text{skin-friction coefficient}=\tau_w/\tfrac{1}{2}\rho u_{\text{av}}^2. \tag{7}$$

[1] J. Hartmann, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. **15**, No. 6 (1937).
[2] J. Hartmann and F. Lazarus, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. **15**, No. 7 (1937).
[3] W. Murgatroyd, Phil. Mag. **44**, 1348 (1953).
[4] J. A. Shercliff, Proc. Cambridge Phil. Soc. **49**, 136 (1953).
[5] J. A. Shercliff, Proc. Roy. Soc. (London) **A233**, 396 (1955).
[6] J. A. Shercliff, Proc. Cambridge Phil. Soc. **52**, 573 (1956).
[7] J. A. Shercliff, J. Nuclear Energy **3**, 305 (1956).
[8] J. A. Shercliff, J. Fluid Mech. **1**, 644 (1956).
[9] P. S. Lykoudis, Purdue University, School of Aeronautical Engineering, Rept. A-59-4 (March, 1959).
[10] C. C. Chang and T. S. Lundgren, *Heat Transfer and Fluid Mechanics Institute* (Stanford University Press, Stanford, California, 1959), p. 41.
[11] P. S. Lykoudis, Proc. 9th Intern. Astronaut. Congr., Amsterdam, 1958, 168.
[12] T. G. Cowling, *Magnetohydrodynamics* (Interscience Publishers, Inc., New York, 1957), pp. 13–17.
[13] We here use the conventional definition of the Hartmann number; the square of this number would have been a more natural definition since it is a measure of the magnetic forces over the viscous forces.

L is a characteristic length such as the diameter D of a pipe or the width of a channel.

The laminar velocity profile given by Eq. (1) may be simplified by observing that for typical flow conditions with liquid metals, the Hartmann number M is well above the value of three and therefore we may substitute the quantity tanh$M$ with the value one. With this substitution one may show that

$$u_{\mathrm{lam}}^* = \frac{1}{\lambda}(1-e^{-\lambda y^*}). \qquad (8)$$

This formula is the same as the one obtained in a laminar sublayer with mass sucking; in the case of sucking the quantity λ is defined as follows[14]:

$$\lambda = v/u_\delta(c_f/2)^{\frac{1}{2}} = (\mathrm{Re})_v/(\mathrm{Re})_u(c_f/2)^{\frac{1}{2}}, \qquad (9)$$

where the subscript δ refers to the conditions prevailing at the outer edge of the boundary layer. It is evident by comparing (9) with (5) that the Hartmann number has exactly the same effect on the laminar layer as sucking does when measured by the Reynolds number $(\mathrm{Re})_v$.

Equation (8) shows that the magnetic field enters into the solution only through the parameter λ which must eventually be the factor generalizing the universal velocity profiles in the magnetic case.

At this point, the question arises as to where the laminar sublayer terminates. The criterion that may be adopted for transition[14] is

$$\rho(\tau/\rho)^{\frac{1}{2}}(y_l/\mu) = 11.3. \qquad (10)$$

The subscript l refers to the edge of the sublayer.[15] On calculating the shear stress τ from Eq. (8) and substituting in Eq. (10), we obtain

$$(1+\lambda u_l^*)^{\frac{1}{2}} y_l^* = 11.3. \qquad (11)$$

By substitution of Eq. (8) into Eq. (11), one shows

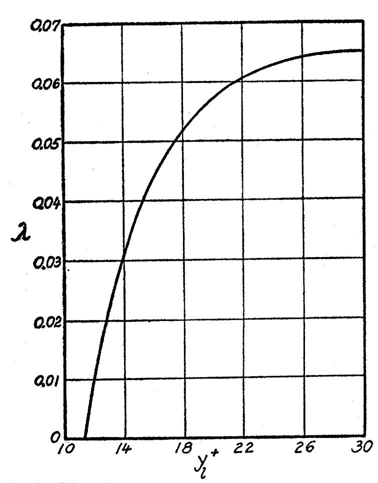

FIG. 1. Thickness y_l^* of the laminar sublayer as a function of the magnetic parameter λ.

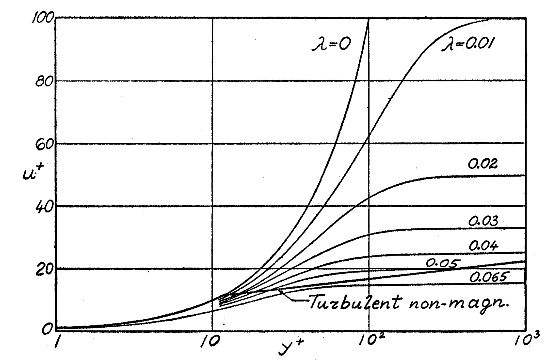

FIG. 2. Structure of the laminar sublayer in magnetic channel turbulent flow.

that the following relation is also valid:

$$\lambda = 2\ln(y_l^*/11.3)/y_l^*. \qquad (12)$$

This relation is plotted in Fig. 1 and indicates that by increasing the values of λ (increasing magnetic fields) the thickness of the laminar sublayer also increases up to a certain value after which an increasing y_l^* yields lower values of λ. It is clear that this maximum value of λ determines a maximum magnetic field above which turbulence cannot be sustained. An analytic determination of this maximum yields the following values:

$$(y_l^*)_{\max} = 30.75, \quad \lambda_{\max} = 0.06511. \qquad (13)$$

In view of the lack of experimental evidence on the validity of this criterion, it is not hard to show by another argument that these numerical values are physically correct. For this purpose, it is sufficient to plot Eq. (8) for different values of λ. For increasing values of y^*, the universal velocity u^* reaches asymptotically the value $1/\lambda$. If on the same graph one plots the velocity profile valid for the nonmagnetic case in the turbulent region, it becomes apparent that there is a value of λ after which the laminar profile always falls under the turbulent one. The value of λ for which the two curves are tangent determines the maximum λ above which the whole boundary layer is laminar. This is shown in Fig. 2. The value $\lambda_{\max}$ thus calculated is found to be

$$\lambda_{\max} \cong 0.0628. \qquad (14)$$

The difference between this value and the value of Eq. (13) is only 3.5%.

3. CRITERION FOR TRANSITION FROM LAMINAR TO TURBULENT FLOW

Although there are no experimental velocity profiles for the turbulent magnetic case, one can use the available skin friction coefficients as an indication of the transition from laminar to turbulent flow. Such measurements[3] indicate that transition occurs for a value

---

[14] E. R. van Driest, Z. angew. Math. u. Phys. **9b**, 233 (1958).
[15] The value 11.3 is the quantity y_l^* at the intersection of the laminar nonmagnetic profile given by $u^*=y^*$ and the fully turbulent relation given by $u^*=5.5+2.5\ln y^*$.

$\text{Re}/M \cong 225$.[16] When the skin-friction coefficient is plotted vs the ratio M/Re, it is known that for the laminar case, the result is a straight line[13]:

$$c_f \cong 2(M/\text{Re}). \tag{15}$$

Deviation of the data from this straight line indicates that turbulence is present.

The theoretical criterion given by relation (10) is in perfect agreement with the empirical criterion previously cited.[3] The proof is as follows: By substituting the value of c_f from Eq. (15) into Eq. (5), defining λ, and using the maximum value of λ as indicated in Eq. (13), one calculates that

$$\text{Re}/M = \lambda^{-2} = (0.0651)^{-2} \simeq 236.$$

In fact, closer examination of Fig. 3 of reference 3 shows that this value is closer to the data than the given 225.[17]

4. TURBULENT REGION

Having calculated u_l^* and y_l^*, let us look into the turbulent region close to the wall.

In fully established turbulent flow the forces per unit volume acting in the direction of the flow x are due to the shear contribution and the ponderomotive force given by the product $\mathbf{J}\times\mathbf{B}$. By neglecting the molecular contribution to the shear force (a valid assumption away from the laminar sublayer) and by assuming the validity of Ohm's law with a constant magnetic field $\mathbf{B}$ always perpendicular to the flow, the current is calculated as follows:

$$J = \sigma(E - uB). \tag{16}$$

If τ is the turbulent contribution to the shear force, then the equation of conservation of momentum in the direction x close to the wall is

$$0 = d\tau/dy - \sigma B^2 u + \sigma BE. \tag{17}$$

The preceding equation indicates that the second term is a retarding force and the last, an accelerating one. It is well established that in the laminar case a good assumption simplifying the problem is to calculate the induced electric field by making the total current $\int J_z dy$ flowing in the plane perpendicular to the flow equal to zero. In this fashion E represents a uniform average electric field contributing a constant accelerating ponderomotive force which predominates in the vicinity of the wall where the component $-\sigma B^2 u$ is rather small. From this argument one expects to see the flow accelerating near the wall and decelerating at the center of the channel with the velocity profile becoming flatter there.

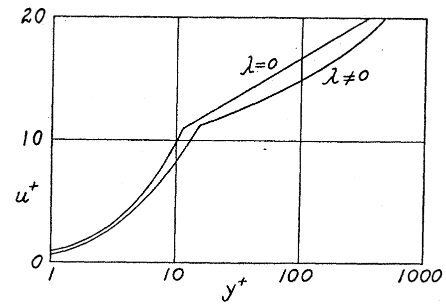

FIG. 3. Nature of the turbulent velocity profiles in magnetic channel turbulent flow.

With these clarifications on the nature of the forces involved, the equation of motion is written in nondimensional form as follows:

$$0 = d\bar{\tau}/dy^* - \lambda^2 u^* + \lambda^2[E/B(\tau_w/\rho)^{\frac{1}{2}}], \tag{18}$$

where

$$\bar{\tau} = \tau/\tau_w. \tag{19}$$

The third term of the momentum equation apart from the quantity λ^2 must now be understood as the average velocity u_{av}^* by virtue of the assumption previously made on the meaning of the uniform electric field E. Hence close to the wall,

$$0 = d\bar{\tau}/dy^* - \lambda^2 u^* + \lambda^2 u_{av}^*. \tag{20}$$

A phenomenological relation is evidently necessary between the shear stress $\bar{\tau}$ and the velocity gradient.

Unfortunately, even if such a relation could be written, such an integration cannot be performed in a universal form. Because of the presence of the term $\lambda^2 u_{av}^*$ it is evident that the velocity u^* depends not only on the universal parameters λ and y^* but also on the value y_e^* taken at the edge of the turbulent boundary layer. This means that universal solutions do not exist in the magnetic case.

Work in progress at Purdue based on different simplifying assumptions has shown that the velocity profile in the turbulent region is of the same phenomenological behavior as the one described for sucking.[14] Preliminary calculations of the skin-friction coefficient based on this model have been performed and have shown good agreement with experiments.[3] The trend and the relative location of a magnetic profile with respect to a nonmagnetic one is shown in Fig. 3. These results will be given in another publication as soon as they are completed.

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation to Dr. Edward R. van Driest, Director of the Aero-Space Laboratories of North American Aviation, Inc., for bringing to his attention the work cited in reference 14, and also for a number of interesting discussions on the subject during the summer of 1959.

[16] The actual value given in reference 3 is 900; the difference (a factor of four) in the two numbers arises from the different definitions used for the characteristic length.

[17] More experiments may well show some deviation from these numbers, but it is worth noting the excellent agreement of the present simple theory with the only available experimental information.