# On Turbulent Magneto-Fluid Dynamic Boundary Layers*

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## INTRODUCTION

THE study of the behavior and characteristics of boundary layers over flat plates when the fluid is electrically conducting and a magnetic field is present has both theoretical attractiveness and practical importance. Rossow has solved the laminar case: it is the aim here to investigate the turbulent case.

The main steps of the present analysis can be enumerated as:

(1) deriving the basic equations for the turbulent flow of a constant property, electrically conducting fluid;

(2) postulating a suitable model for the turbulent boundary layers;

(3) obtaining quantitative results based on the postulated model.

The analysis takes advantage of both the technique and results of ordinary turbulent boundary layers (such as have been recently presented, for instance, by Townsend,[1] Clauser,[2] and Ferrari[3] and of the results of the elementary theory of hydromagnetic turbulence developed by Chandrasekhar.[4]

In Sec. 1 the basic magneto-fluid dynamic equations, linearized with respect to the induced magnetic field, are presented. In Sec. 2 their "turbulent" form is derived and the balances of mean and turbulent kinetic energy are formulated and discussed. In Sec. 3 a suitable model of the turbulent boundary layer is deduced.

Based on this model, the law of the wall and the skin-friction law in their modified forms, valid for linearized magneto-fluid dynamic turbulence, are derived in Sec. 4 in terms of two constants. In Sec. 5 the evaluation of these two constants is carried out by means of a technique closely patterned after the Clauser-Ferrari method.[2,3] Results are analyzed and discussed in Sec. 6.

## 1. LINEARIZED MAGNETO-FLUID DYNAMIC EQUATIONS

The basic nondimensional equations for the magneto-fluid dynamics of a constant property fluid are

$$\nabla\cdot\mathbf{v}=0,$$

$$\partial\mathbf{v}/\partial t+\nabla\cdot[\mathbf{vv}+(1/\gamma M_0^2)p\mathbf{U}+(\boldsymbol{\tau}/\mathrm{Re})]=R_mP_m\mathbf{J}\times\mathbf{H},$$

$$\nabla\times\mathbf{E}=-\partial\mathbf{H}/\partial t;\quad \nabla\cdot\mathbf{E}=0, \tag{1}$$

$$\nabla\times\mathbf{H}=4\pi R_m\mathbf{J}+\partial\mathbf{E}/\partial t;\quad \nabla\cdot\mathbf{H}=0,$$

$$\mathbf{J}=R_m(\mathbf{v}\times\mathbf{H}),$$

where Re is the Reynolds number $=V_0L/\nu$, $R_m$ is the magnetic Reynolds number $=\sigma V_0\mu L$, and $P_m$ the magnetic pressure number $=\mu H_0^2/\rho V_0^2$. Subscripts 0 indicate reference conditions and all the other symbols have their usual meanings.

Equations (1) are consistent with the following assumptions:

(i) The fluid is incompressible and with constant properties.

(ii) Electric free charges density and displacement currents are negligible.

(iii) The total electric field and the convective electric current are of the same order of magnitude as those originated by the magneto-fluid dynamic interaction. Moreover, any applied electric field is absent.

(iv) The "continuum" hypothesis holds throughout. Justifications for these hypothesis as well as their implications and range of validity can be found in several appropriate references.

By eliminating the electric field **E** and the electric current **J** and by specializing the viscous stress tensor $\boldsymbol{\tau}$ to the present case, the system (1) is reduced to the following two differential equations in the two unknowns **v** and **H**:

$$\nabla\cdot\left[\mathbf{vv}+\frac{p\mathbf{U}}{\gamma M_0^2}-\frac{\boldsymbol{\tau}}{\mathrm{Re}}+\frac{P_m}{4\pi}(\tfrac{1}{2}\mathbf{H}\cdot\mathbf{HU}-\mathbf{HH})\right]+\frac{\partial\mathbf{v}}{\partial t}=0,$$
$$\nabla\cdot\left[R_m(\mathbf{Hv}-\mathbf{vH})-\frac{1}{4\pi}\nabla\mathbf{H}\right]+R_m\frac{\partial\mathbf{H}}{\partial t}=0, \tag{2}$$

and, naturally, both vectors **v** and **H** have zero divergence.

It may be proper here to point out that the Maxwell equations for the electromagnetic field in the absence of the fluid-dynamic field are linear. It is the coupling of the two fields which introduces nonlinear terms in the equation for the magnetic field and additional nonlinear terms in the momentum equation. This consideration bears a certain importance, as is seen later, in the analysis of the turbulent field.

The complicated nature of Eqs. (2) usually requires

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[1] A. A. Townsend, *The Structure of Turbulent Shear Flow* (Cambridge University Press, New York, 1956).

[2] F. H. Clauser, Advances in Appl. Mech. 4, 118 (1956).

[3] C. Ferrari, "Turbolenza di parete," Teoria della turbolenza, C.I.M.E., Varenna, September 1–10, 1957.

[4] S. Chandrasekhar, Proc. Roy. Soc. (London) **A233**, 330 (1955).

that some additional simplifications be made before attempts towards a solution can be afforded. One possibility is the linearization with respect to the magnetic field.

Let

$$\mathbf{H}=\mathbf{n}+\epsilon\mathbf{h}+O(\epsilon^2), \tag{3}$$

where $\mathbf{n}$ is the unit vector $\mathbf{H}_0/|\mathbf{H}_0|$ ($H_0$ is the applied magnetic field) and $\epsilon$ is a quantity of the order of magnitude of $R_m$. Since a large variety of cases of interest for the aerodynamicist are concerned with small values of $R_m$, it is assumed that terms of order higher than the first in $\epsilon$ can be neglected. Then, if the applied magnetic field is assumed to be constant both in magnitude and direction, substitution of (3) into Eqs. (2) yields

$$\nabla\cdot\left\{\mathbf{vv}+\frac{p\mathbf{U}}{\gamma M_0^2}-\frac{\boldsymbol{\tau}}{\mathrm{Re}}+\frac{\epsilon P_m}{4\pi}[(\mathbf{n}\cdot\mathbf{h})\mathbf{U}-(\mathbf{nh}+\mathbf{hn})]\right\}+\frac{\partial\mathbf{v}}{\partial t}=0, \tag{4}$$

$$\nabla\cdot[(\mathbf{nv}-\mathbf{vn})-(1/4\pi)\nabla\mathbf{h}]=0. \tag{5}$$

From these equations it appears that, in order to preserve the magnetic term in the momentum equation, $\epsilon P_m$ must be of order one.[5]

A simple vectorial algorithm indicates that Eq. (5) can be integrated once to obtain

$$\nabla\times\mathbf{h}=4\pi(\mathbf{v}\times\mathbf{n}) \tag{6}$$

and that

$$\nabla\cdot[(\mathbf{n}\cdot\mathbf{h})\mathbf{U}-(\mathbf{nh}+\mathbf{hn})]=\mathbf{n}\times(\nabla\times\mathbf{h}), \tag{7}$$

since both $\mathbf{n}$ and $\mathbf{h}$ have zero divergence and, in addition, $\nabla\times\mathbf{n}=0$. Combining Eqs. (6) and (7) results in

$$\nabla\cdot[(\mathbf{n}\cdot\mathbf{h})\mathbf{U}-(\mathbf{nh}+\mathbf{hn})]=4\pi\mathbf{n}\times(\mathbf{v}\times\mathbf{n})=4\pi\mathbf{v}_n,$$

where $\mathbf{v}_n$ is the component of $\mathbf{v}$ normal to the applied magnetic field.

The linearized equations of magneto-fluid dynamics thus read

$$\begin{gathered}\nabla\cdot\left\{\mathbf{vv}+\frac{p\mathbf{U}}{\gamma M_0^2}\right\}-\frac{\nabla^2\mathbf{v}}{\mathrm{Re}}+P_mR_m\mathbf{v}_n+\frac{\partial\mathbf{v}}{\partial t}=0,\\ \nabla\cdot\mathbf{v}=0.\end{gathered} \tag{8}$$

The linearization process thus eliminates the additional nonlinear terms and uncouples the equation of motion from that of the magnetic field.

[5] Other possibilities may be of interest but they are outside the scope of this paper. Actually, besides the unsteady term, the momentum equation contains pressure, viscous, and magnetic terms whose relative importance with respect to the convective term is measured by the numbers $|\gamma M_0^2|$, Re, and $\epsilon P_m$, respectively. The order of magnitude of these numbers determines whether or not the corresponding terms are negligible and affords several simplified forms of the equation itself.

## 2. LINEARIZED MAGNETO-FLUID DYNAMIC EQUATIONS FOR THE TURBULENT FIELD. ENERGY BALANCES

The equations for the turbulent field are derived with the usual assumption that Eqs. (8) still hold for the instantaneous values of the quantities involved.

Let $\mathbf{v}=\mathbf{V}+v'$ and $p=\langle p\rangle+p'$, where $V$ and $\langle p\rangle$ are average values of the velocity vector and of the pressure; and $v'$, $p'$ the corresponding fluctuating parts. Angular brackets $\langle\ \rangle$ indicate an average. Substitution into Eqs. (8) yields

$$\begin{gathered}\nabla\cdot(\mathbf{V}+\mathbf{v}')=0\\ \nabla\cdot\left\{\mathbf{VV}+\mathbf{v}'\mathbf{v}'+\mathbf{Vv}'+\mathbf{v}'\mathbf{V}+\frac{(\langle p\rangle+p')\mathbf{U}}{\gamma M_0^2}\right\}\\ -\frac{\nabla^2}{\mathrm{Re}}(\mathbf{V}+\mathbf{v}')+R_mP_m(\mathbf{V}_n+\mathbf{v}_n')+\frac{\partial}{\partial t}(\mathbf{V}+\mathbf{v}')=0.\end{gathered} \tag{9}$$

Averaging these equations and indicating by angular brackets $\langle\ \rangle$ the averaged values result in

$$\begin{gathered}\nabla\cdot\mathbf{V}=0\\ \nabla\cdot\left\{\mathbf{VV}+\langle\mathbf{v}'\mathbf{v}'\rangle+\langle p\rangle\frac{\mathbf{U}}{\gamma M_0^2}\right\}\\ -\frac{\nabla^2\mathbf{V}}{\mathrm{Re}}+R_mP_m\mathbf{V}_n+\frac{\partial\mathbf{V}}{\partial t}=0.\end{gathered} \tag{10}$$

The turbulent linearized magneto-fluid dynamic equations have just the same expression as the fluid-dynamic equations except, naturally, for the added magnetic terms which, however, depend only on the mean velocity field. This was naturally to be expected since, as noticed before, there is no additional nonlinear term involved and thus there cannot be any additional term involving fluctuations.

It is now rather simple to extend to Eqs. (10) the well-known boundary-layer order of magnitude analysis. The result is, in terms of dimensional quantities and for isobaric outer flow,

$$\begin{gathered}u_x+v_y=0,\quad p_y\equiv O(\delta),\\ uu_x+vu_y+mu=\nu u_{yy}-\langle u'v'\rangle,\end{gathered} \tag{11}$$

where mean values are to be intended for all quantities and where

$$m=\sigma B_0^2/\rho \tag{12}$$

depends on the applied magnetic field $H_0$, on the fluid electric conductivity $\sigma$, and has the dimension $(\text{time})^{-1}$. The term $\langle u'v'\rangle$ represents the familiar Reynolds stress due to the fluctuation of velocity.

For future reference, it is useful to write down the equations expressing the balance of mean kinetic energy $\frac{1}{2}V^2$ and of turbulent energy $E_t=\langle\frac{1}{2}v'^2\rangle$.

Multiply Eqs. (9) scalarly by $\mathbf{v}'$ and take the mean to obtain

$$\langle \mathbf{v}'\cdot\nabla\cdot[\mathbf{V}\mathbf{v}'+\mathbf{v}'\mathbf{V}]\rangle+\langle \mathbf{v}'\cdot\nabla\cdot(\mathbf{v}'\mathbf{v}')\rangle +\frac{1}{\gamma M_0^2}\langle \mathbf{v}'\cdot\nabla p'\rangle-\frac{1}{\mathrm{Re}}\langle \mathbf{v}'\cdot\nabla^2\mathbf{v}'\rangle +R_mP_m\langle \mathbf{v}'\cdot\mathbf{v}_n'\rangle+\left(\mathbf{v}'\cdot\frac{\partial \mathbf{v}'}{\partial t}\right)=0.$$

It is easily verified that this equation can be transformed into

$$\mathbf{V}\cdot\nabla E_t+\nabla\cdot[\langle \mathbf{v}'(p'+\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}')\rangle]+\langle \mathbf{v}'\mathbf{v}'\rangle:\nabla\mathbf{V} -\mathrm{Re}^{-1}[\nabla^2E_t+\textstyle\sum_i\sum_j\langle\partial \mathrm{v}_i'/\partial x_j\rangle^2] +R_mP_m\langle(v_n')^2\rangle=0, \quad (13)$$

thus giving the turbulent energy balance in a steady state. The first four terms contributing to this energy balance are well known.[1,3] They are the convection term, the diffusive term, the production term, and the dissipation term, respectively. One more term is now present, the last one. It is due to the presence of the magnetic field and represents the additional dissipation of turbulent energy caused by the magnetic field. Physically, this is due to the fact that the line of forces of the magnetic field tend to oppose any crossing of conductive fluid, thus some of the available turbulent kinetic energy must be spent to do work against the magnetic field.

To obtain the balance of the mean kinetic energy $\frac{1}{2}V^2=E_m$, Eq. (9) is multiplied scalarly by $\mathbf{V}$ to obtain

$$\mathbf{V}\cdot\nabla\cdot\left\{\mathbf{V}\mathbf{V}+\langle\mathbf{v}'\mathbf{v}'\rangle+\frac{1}{\gamma M_0^2}\langle p\rangle\mathbf{U}\right\}-\mathbf{V}\cdot\frac{\nabla^2\mathbf{V}}{\mathrm{Re}} +R_mP_m\mathbf{V}_n\cdot\mathbf{V}+\mathbf{V}_n\cdot\frac{\partial\mathbf{V}}{\partial t}=0,$$

or, in the steady case and through the usual algorithm

$$\mathbf{V}\cdot\nabla E_m+\nabla\cdot(\langle\mathbf{v}'\mathbf{v}'\rangle\cdot\mathbf{V})+\frac{1}{\gamma M_0^2}\nabla\cdot(p\mathbf{V}) -\langle\mathbf{v}'\mathbf{v}'\rangle:\nabla\mathbf{V}-\mathrm{Re}^{-1}\left[\nabla^2E_m-\sum_i\sum_j\left(\frac{\partial V_i}{\partial x_j}\right)^2\right] +R_mP_m\mathbf{V}_n\cdot\mathbf{V}=0. \quad (14)$$

Again, all the features of the corresponding energy balance in a purely fluid-dynamic field are preserved. In addition, there is again an additional term due to the exchange of energy between the velocity and the magnetic fields.

We revert now to Eqs. (11) valid for the turbulent boundary layer in linearized magneto-fluid dynamics. Within this linear approximation, the only fluctuating term is that corresponding to the Reynolds stress. To relate this fluctuating term to properties of the mean flow field, it is necessary first to get a clearer idea of what are the essential physical features of the subject problem and how, and to what extent, they differ from the familiar ones of ordinary turbulence. This analysis is carried out in the next section wherein it is also formulated a suitable "model" of the subject dissipative region so as to make subsequent quantitative analysis possible.

## 3. DEDUCTION OF A MODEL FOR THE TURBULENT LAYER

In the current view about turbulence, a fundamental physical idea is that of the cascade of energy from the larger to the smaller eddies.[1,3] This process has been recently generalized by Chandrasekhar[4] to hydromagnetic turbulence.

In ordinary turbulence the accepted supposition is as follows. Energy is first supplied to the Fourier components of the velocity field which are of the smallest wave numbers (largest eddies), it is then transmitted to the higher wave numbers and it is finally dissipated into heat by viscosity at the highest wave number. The energy transfer from one wave number to another can only occur because of the nonlinear coupling between the different Fourier components of the velocity field (this nonlinearity character is contained in the inertial terms of the equation of motion). The rate at which the kinetic energy per unit volume and per unit wave number interval at the wave number $k'$ is transformed into kinetic energy at the wave number $k''(>k')$ is given by the transition probability $Q_1(v,k';v,k'')$. At any number $k$, the rate of change of energy density is equal to the balance between the energy gained from the lower wave numbers, the energy lost to the higher wave numbers, and the energy directly dissipated by viscosity.

According to Chandrasekhar, this process can be generalized to hydromagnetic turbulence as follows. First of all one must consider that there is another type of turbulent energy, the energy of the turbulent magnetic field, so that at each wave number, one has to deal now with two energy balances. Secondly, other nonlinear terms beside the inertial terms appear in the equations of motion and in that of the magnetic field so that additional couplings are possible among the Fourier's components: kinetic and magnetic energies can be transformed into one another. No direct transformation of magnetic energy between two wave numbers is, however, possible, owing to the linearity of the Maxwell's equation. Chandrasekhar thus introduced two more transition probabilities: $Q_3(v,k';h,k'')$ (rate at which kinetic energy per unit volume and unit wave number interval at the wave number $k'$ is transformed into magnetic energy at the wave number $k''>k'$); and $Q_2(h,k';v,k'')$ (rate at which magnetic energy per unit volume and unit wave number interval at the wave number $k'$ is transformed into magnetic energy at the wave number $k''>k'$). At any wave number $k$ the net rate of change of kinetic energy density is now expressed by the balance of five terms: (1) kinetic energy gained from lower wave numbers, (2) kinetic energy lost to higher wave numbers; (3) magnetic energy from lower wave number transformed into kinetic energy at $k$ (gain); (4) kinetic energy at $k$ transformed into magnetic energy at higher wave numbers (loss); and (5) energy dissipation by viscosity. The corresponding net rate of change of magnetic energy at a wave number $k$ is expressed by the balance of three terms: (i) kinetic energy at lower wave numbers transformed into magnetic energy at $k$ (gain); (ii) magnetic energy at $k$ transformed into kinetic energy at higher wave numbers; and (iii) dissipation of magnetic energy due to the electrical conductivity (Joule heat). Chandrasekhar shows then that it is plausible to assume for the transition probabilities the following relations:

$$Q_s(k',k'')=\text{const}\left[\frac{k'}{k''^{\frac{3}{2}}}\right] \times[iF(k')F^{\frac{1}{2}}(k'')+jG(k')G^{\frac{1}{2}}(k'')], \quad (15)$$

where $F(k)$ and $G(k)$ are the turbulent kinetic energy and magnetic energy spectra, respectively, and $i=1$; $j=0$ for $s=1$; $i=0$, $j=1$ for $s=2$; $i=1$; $j=1$ for $s=3$.

From these expressions and from the results of Chandrasekhar, it is possible to deduce the following statements for the case of linearized magneto-fluid dynamic turbulence:

(1) The intensity of the turbulent magnetic energy spectrum is, at any wave number, one order of magnitude smaller than that of the turbulent kinetic energy.

(2) As a consequence, from Eq. (15) it follows that the probability of exchange between turbulent kinetic and magnetic energies is negligible compared to the probability of transition of turbulent kinetic energy from one wave number to another.

(3) At any wave number the intensity of the kinetic energy spectrum is lowered by the presence of the magnetic field.

The first two statements are almost obvious and relate back to the already noticed absence of any other nonlinear term, beside the inertial ones, in the basic equations. The third one could be rigorously demonstrated by subjecting the Chandrasekhar theory to a linearization process.[6] For an heuristic proof, one can consider that the balance equation from which the energy spectrum is to be calculated must necessarily have a structure similar to that of Eqs. (8). It therefore contains the additional dissipative term due to the Joule effect and leads, consequently, to lower values of the energy spectrum.

[6] It is our intention, indeed, to do this in the immediate future.

In conclusion, the energy cascade process in linearized magneto-fluid dynamic turbulence can be pictured as follows. Transfer can only occur between turbulent kinetic energies of different wave numbers, all the other possible exchanges having much lower probabilities. At any wave number $k$, the rate of change of energy density is equal to the balance between the energy gained from the lower wave numbers, the energy lost to the higher wave numbers, the energy directly dissipated by viscosity, and that dissipated by Joule effect.

Let us now proceed further and examine the turbulent flow along bodies. When the fluid is nonconducting there exist two regions with different "characteristic times of adjustment": the inner region and the outer region. In the first one there is a form of statistically determined local quasi-equilibrium while in the second one conditions of "dynamic equilibrium" prevail which lead to self-preserving flows. Energy is being extracted from the mean flow by the Reynolds stress gradient, transferred to the inner layer, and converted into turbulent energy by the working of the mean flow against the Reynolds stresses. In the inner layer this turbulent energy input is nearly all dissipated by viscous action, the small surplus being fed back into the outer layer.

When the fluid is conducting, the effect of the magnetic field on the above described energy transfer scheme as deduced from equations of energy balances (13) and (14) is twofold. For one thing, the exchange of work between mean velocity field and magnetic field modifies the input of energy into the inner layer. Secondly, the already modified turbulent energy input is dissipated by both viscous and magnetic action so that the viscous dissipation at the wall is now a smaller fraction of the energy input.

The turbulent energy input to the inner layer is reduced since the interaction between mean flow and magnetic field in the outer layer leads to a dissipation of mean kinetic energy. The reduction of local skin-friction coefficient is thus due to two concomitant factors: magnetic dissipation of mean kinetic energy in the outer layer and magnetic dissipation of turbulent kinetic energy in the inner layer.

The equation giving the turbulent energy balance makes it possible to formulate another important conclusion which proves essential in the further development of a quantitative analysis. The additional magnetic term comparing in the equation is multiplied by the number $R_m P_m$ which, by hypothesis, is of order of magnitude one. Then the additional "characteristic time" connected with the magnetic field is of the same order of magnitude as that prevailing in the absence of the magnetic field. This fact justifies then the fundamental assumption that even in the subject case the inner layer is in a state of statistically determined quasi-equilibrium. Quite obviously this equilibrium is now determined by the balance of turbulent energy production, viscous dissipation, and magnetic dissipation.

To conclude: In linearized magneto-fluid dynamics it is justified to extend the concept of the two-layer model of the dissipative turbulent region and to assume that equilibrium conditions and wake-type characters still prevail in the inner and outer regions, respectively. This accepted model for linearized hydromagnetic turbulence forms the basis for the development of the quantitative analysis which is carried out in the subsequent sections.

## 4. LAW OF THE WALL AND SKIN-FRICTION LAW

In this section are derived the modified forms of the law of the wall and of the skin-friction law with the help of the two-layer model discussed in the preceding section.

Since in the inner region there is statistically determined local quasi-equilibrium, the velocity $u$ can depend only upon the quantities $\rho$, $\nu$, $y$, $m$, and $\tau_0$ with which the following nondimensional ratios can be formed:

$$\zeta = u^+y/\nu; \quad \chi = my/u^+. \tag{16}$$

Here, $u^+ = (\tau_0/\rho)^{\frac{1}{2}}$ and $\tau_0$ is the shear stress at the wall. Thus, in the inner region, the velocity profile is expressible as

$$u/u^+ = F(\zeta,\chi). \tag{17}$$

$F$ is still an undetermined function, and to get an explicit relationship for it one must consider the velocity profiles in the outer region first.

As said, this region is characterized by the fact that, for sufficiently large Reynolds numbers, molecular transport effects can be neglected. The equation of motion indicates then that the Reynolds stress $\langle u'v'\rangle$ is essentially a function only of the two independent variables $\eta = y/l_0(x)$ and $\sigma = mx/U_0$, where $l_0(x)$ is an adequate scale for the lengths. By accepting the usual approximation that an analogous functional dependence for the velocity profiles follows from the momentum and continuity equations,[1–3] we infer that in the outer region it is

$$u - U_1 = u_0 G(\eta,\sigma) \quad [G(+\infty,\sigma) = 0], \tag{18}$$

where $U_1 = U_0(1-\sigma)$ is the free-stream velocity; $u_0(x)$ is a suitable velocity scale. To determine the functional expression for $F$, one now postulates the existence of an overlapping region wherein both Eqs. (17) and (18) are valid and, wherein, accordingly, it must be

$$u^+F(\zeta,\chi) = U_1 + u_0 G(\eta,\sigma). \tag{19}$$

A necessary but not sufficient condition for this equality, is that some definite relationships exist between the two couples of independent variables $(\zeta,\chi)$ and $(\eta,\sigma)$.

By assuming the scale length proportional to the boundary layer thickness $\delta$ and $u_0(x) = u^+$, one obtains

$$\zeta = R\eta; \quad \chi = \sigma\eta/H, \tag{20}$$

where

$$R = u^+\delta/\nu = R(x); \quad H = u^+x/U_0\delta = H(x), \tag{21}$$

and Eq. (19) becomes

$$F(R\eta,\sigma\eta/H) = f(R,H) + G(\eta,\sigma)$$

with $U_1/u^+ = f(R,H)$. It has been found impossible to satisfy this relation with a suitable expression for $F$ which would reduce to the familiar one for $\sigma = 0$ (nonconducting fluid) unless $\sigma$ itself can be considered small. In this case, by neglecting the higher powers of $\sigma$, it is possible to write, for the inner and outer regions, respectively,

$$\begin{aligned} u/u^+ &= F_0(\zeta_0) + \chi_0 F_1(\zeta_0), \\ (u-U_1)/u^+ &= G_0(\eta_0) + \sigma G_1(\eta_0) \end{aligned} \tag{22}$$

with

$$\zeta_0 = u_0^+y/\nu; \quad \chi_0 = my/u_0^+; \quad \eta_0 = \eta^+y/\delta_0, \tag{23}$$

where the subscript zero indicates the values for $\sigma = 0$ $(m=0)$ and $\eta^+$ is the value of $\eta$ for $y = \delta_0$. It is pointed out that once it is assumed that $\sigma$ is much less than 1, $\chi_0$ is also very small, since, as it is seen in the following the maximum value of $\chi_0$ is equal to about $(1/25)\sigma$.

Consider now that, to the same degree of approximation in $\sigma$, we have

$$\frac{U_1}{u^+} = \frac{U_0}{u_0^+}\left[1 - \sigma\left(1 + \frac{u_1^+}{u_0^+}\right)\right], \tag{24}$$

where, as it is always possible to do formally,

$$u^+ = u_0^+(1 + \sigma u_1^+/u_0^+).$$

The requirement that the two different expressions for the velocity profile coincide within the overlapping region is then equivalent to the following two equations:

$$\begin{aligned} F_0(\zeta_0) &= (U_0/u_0^+) + G_0(\eta_0), \\ F_1(\zeta_0) &= \frac{H_0}{\eta_0}\left[G_1(\eta_0) - \frac{U_0}{u_0^+}\left(1 + \frac{u_1^+}{u_0^+}\right)\right], \end{aligned} \tag{25}$$

wherein $H_0 = u_0^+x/U_0\delta_0$ is now a constant. The first equation yields the well-known relations

$$F_0 = k^{-1}[\ln\zeta_0 + A]; \quad G_0 = k^{-1}[\ln\eta_0 + A']. \tag{26}$$

The second equation can be satisfied if, and only if,

$$\begin{aligned} F_1(\zeta_0) &= \text{const} = B, \\ (U_0/u_0^+)[1 + (u_1^+/u_0^+)] &= \text{const} = k_2, \end{aligned} \tag{27}$$

from which it follows that

$$G_1(\eta_0) = (B/H_0)\eta_0 + k_2. \tag{28}$$

It should be pointed out that the second of equations (27), following *a fortiori* from the hypothesis of the existence of an overlapping region, is the crucial point of the present approach. It is needless to say that it cannot be justified theoretically but can be proved valid only through comparison with experiments.

By properly combining the preceding equations, we can write the law of the wall and the skin-friction law in their modified forms valid for $\sigma\ll 1$ as follows:

Law of the wall:

$$\frac{u}{u^+}=\frac{1}{k}\left[\ln\frac{u_0^+y}{\nu}+A\right]+B\frac{my}{u_0^+}\quad (B=\text{const})\tag{29}$$

Skin-friction law:

$$\frac{U_1}{u^+}=\frac{1}{k}\left[\ln\frac{u_0^+\delta_0}{\nu}+A-A'\right]-k_2\frac{mx}{U_0}\quad (k_2=\text{const}),\tag{30}$$

where the constants $k$, $A$, and $A'$ are those of the familiar laws of fluid dynamics.

The constants $B$ and $k_2$ are still undetermined. Their evaluation is taken up in the next section. Before this, however, we show how the law of the wall, (29), is in agreement with that derivable through the expression of the energy balance close to the wall.

The balance of turbulent kinetic energy near the wall is expressed by

$$\langle u'v'\rangle U_y+\nu\sum_i\sum_j(\langle\partial u_i'/\partial x_j\rangle)^2+m\langle u_i'^2\rangle=0,\tag{31}$$

where the first term gives the production of turbulent kinetic energy $E_t$ and the last two the viscous and "magnetic" dissipation, respectively.

It looks plausible to let

$$\langle u'v'\rangle=-\alpha_1E_t;\quad \langle u'^2\rangle=\alpha_2E_t$$
$$\sum_i\sum_j\left\langle\frac{\partial u_i'}{\partial x_j}\right\rangle^2=\frac{\alpha_3E_t}{\lambda^2},\tag{32}$$

where the $\alpha_i$ are positive and $\lambda$ has the dimension of a length (the so-called microscale of turbulence).

In the case of a nonconducting fluid the quantity $\lambda^2/(\nu y/u^+)$ is a constant. In the present case it is logical to assume that it is a function of the nondimensional parameter $\chi=my/u^+$ so that, for $\chi$ small, one can let

$$\lambda^2=\alpha_4(y\nu/u^+)[1+\alpha_5(my/u^+)].\tag{33}$$

Substitution of Eqs. (32) and (33) into Eq. (31) yields

$$\frac{u}{u^+}=\frac{\alpha_3\alpha_4}{\alpha_1}\frac{1}{y}+\frac{\alpha_2-\alpha_3\alpha_4\alpha_5}{\alpha_1}\frac{m}{u^+},$$

from which, upon integration, Eq. (29) follows with

$$B=(\alpha_2-\alpha_3\alpha_4\alpha_5)/\alpha_1.\tag{34}$$

There seems to be, therefore, an inner consistency in the relations found for the law of the wall and the skin-friction law.

## 5. DETERMINATION OF THE CONSTANTS APPEARING IN THE MODIFIED LAWS

To evaluate the two constants $k_2$ and $B$ appearing in the law of the wall and in the skin-friction law, an extension of the Clauser-Ferrari technique is used.[2,3] Essentially, the method consists in determining a series of "laminar" velocity profiles, with nonzero velocity at the wall, for the outer region and matching them with the velocity distribution valid for the inner region by imposing at the matching point continuity in the velocity, the velocity derivative, and the eddy coefficient of kinematic viscosity.

Consider the inner region first. The velocity profile is given by Eq. (39). To compute the eddy coefficient $\epsilon=\tau/u_y$ one must first determine the shear stress $\tau$. From the momentum equation written in the form

$$uu_x+vu_y+mu=(\tau/\rho)_y,$$

it is inferred that in the inner region, if the terms containing the second and higher powers of the distance from the wall are neglected, the shear stress is still constant and equal to its value at the wall $\tau_0$. Consequently, it is

$$\epsilon=\tau/u_y=\tau_0/u_y=k(yu^+/\nu)[1-kB(my/u_0^+)].\tag{35}$$

Consider now the outer region wherein, by hypothesis, molecular transport effects are neglected. The basic equations are

$$uu_x+vu_y+mu=\epsilon u_{yy},\quad u_x+v_y=0,\tag{36}$$

where the eddy coefficient $\epsilon=\tau/u_y$ is now assumed to be a function only of $x$. The functional dependence for $\epsilon$ is to be determined from the requirement that this region retains a certain degree of similarity, i.e., that the velocity profiles could be described by a series of universal functions in terms of the magnetic number $\sigma=mx/U_0$. It is shown that this implies the following expression for $\epsilon$:

$$\epsilon=h\rho U_0\delta_0^+[1+\sigma a]\quad (a=\text{const}),\tag{37}$$

where $\delta_0^+$ is the displacement thickness in the absence of the magnetic field and $h$ is a constant.

Let, indeed, the stream function $\psi$ defined by $\psi_y=u$ and $\psi_x=-v$ be given by

$$\psi=l_0(x)U_0[f_0(\eta_0)+\sigma f_1(\eta_0)+\cdots]\tag{38}$$

with

$$\eta_0=y/l_0(x),\tag{39}$$

where $l_0(x)$ is the length scale. Substituting the expressions for the velocity components derived from Eq. (38), grouping the terms in the like powers of $\sigma$, and neglecting terms in $\sigma^2$ yield the following two equations for $f_0$ and $f_1$:

$$\frac{\epsilon_0x}{U_0l_0^2}f_0'''+\frac{l_0'x}{l_0}f_0f_0''=0,$$
$$f_1'f_0'-\frac{l_0'x}{l_0}[f_0f_1''+f_1f_0'']-f_1f_0''+f_0'=\frac{\epsilon_0x}{U_0l_0^2}f_1'''+\frac{\epsilon_1x}{U_0l_0^2}f_0''',\tag{40}$$

where primes indicate differentiation with respect to the pertinent variable and where, for the moment, we have put $\epsilon=\epsilon_0+\sigma\epsilon_1$.

For Eqs. (40) to be consistent with Eq. (38), it is necessary that their coefficients be constant. The first of Eqs. (40) determines the relationship between $\epsilon_0$ and $l_0$. Since $\epsilon_0$ is the value of the eddy coefficient in the absence of the magnetic field, one can let (cf. Clauser[2])

$$\epsilon_0=\tfrac{1}{2}U_0(l_0^2)'=h\rho U_0\delta_0^+ \tag{41}$$

with $h=0.018$. The similarity requirements on the second of Eqs. (40) imposes that $l_0'x/l_0=\text{constant}=a_1$ and that $\epsilon_1$ as indicated in Eq. (37) be proportional to $\epsilon_0$. The first requirement is the most stringent one since it can be met only if the Reynolds number $U_0x/\nu$ is sufficiently large, for then it is $\delta_0^+\simeq x^n$ and $a_1=\frac{1}{2}(n+1)$.

In conclusion, the characteristics of the outer region can be found from the solution of the following system of equations:

$$f_0'''+f_0f_0''=0, \tag{42a}$$

$$\frac{2}{n+1}[f_1'f_0'-f_1f_0''+f_0']-f_0f_1''-f_1f_0''=f_1'''+\frac{2a}{n+1}f_0'''. \tag{42b}$$

The boundary conditions are

$$\begin{aligned} f_0'(+\infty)&=1, & f_1'(+\infty)&=-1,\\ f_0(0)&=0, & f_1(0)&=0,\\ f_0'(0)&=\alpha_0, & f_1'(0)&=\alpha_1. \end{aligned} \tag{43}$$

The first two boundary conditions are dictated by the joining of the outer region with the nonviscous flow. The last one prescribes a velocity at the wall different from zero.

The velocity profiles are given by

$$\frac{u}{U_0}=f'(\eta_0)+\sigma f_1'(\eta_0) \tag{44}$$

and are functions of the three parameters $\alpha_0$, $\alpha_1$, and $P=2a/(n+1)$. The latter one, together with the constants still unknown in the expression for the velocity in the inner region and in the skin-friction law are to be determined by imposing a certain number of continuity conditions at the point where the profile given by Eq. (44) is matched with that holding in the inner region.

The point where the matching occurs is also unknown *a priori*. Let $y=y_0+\sigma y_1$ be the ordinate of the point $Q$ where the matching takes place; $y_0$ being the ordinate of the point where the matching would occur if the magnetic field were absent. The point $Q$, by hypothesis, lies within the overlapping region.

When $Q$ is though of as belonging to the inner region, it is characterized by the value $\zeta_0+\sigma\zeta_1$ of $\zeta$ with

$$\zeta_0=u_0^+y_0/\nu;\quad \zeta_1=u_0^+y_1/\nu, \tag{45}$$

and the velocity, its gradient $u_y$, and the eddy viscosity coefficient at the point $Q$ are equal, respectively, to

$$\begin{gathered} \frac{u_0^+}{k}[\ln\zeta_0+A]+\sigma\left\{\frac{u_1^+}{k}(\ln\zeta_0+A)+\frac{u_1^+\zeta_1}{k\zeta_0}+B\frac{u_0^+y_0}{H_0\delta_0}\right\},\\ \frac{u_0^{+2}}{k\nu\zeta_0}+\sigma\left\{\frac{u_1^+u_0^+}{k\nu\zeta_0}+\frac{Bu_0^+}{H_0\delta_0}-\frac{u_0^{+2}}{\nu\zeta_0k}\frac{\zeta_1}{\zeta_0}\right\},\\ k\zeta_0+\sigma\left\{k\zeta_1-\zeta_0\frac{kB}{H_0}\frac{y_0}{\delta_0}+k\zeta_0\frac{u_1^+}{u_0^+}\right\}, \end{gathered} \tag{46}$$

wherein it has been assumed that: $u^+=u_0^++\sigma u_1^+$, and it has been taken into account that

$$\sigma=mx/U_0=H_0(my_0/u_0^+)(\delta_0/y_0). \tag{47}$$

When the point $Q$ is thought of as belonging to the outer region, it is characterized by the value $\eta_0+\sigma\eta_1$ of $\eta$ with

$$\eta_0=\eta^+y_0/\delta_0;\quad \eta_1=\eta^+y_1/\delta_0, \tag{45a}$$

wherein $\eta^+$ is equal to the value of $\eta$ for which $y=\delta_0$. Velocity, velocity gradient $u_y$, and eddy coefficient are, respectively,

$$\begin{gathered} U_0\{f_0'(\eta_0)+\sigma[\eta_1f_0''(\eta_0)+f_1'(\eta_0)]\},\\ \frac{U_0\eta^+}{\delta_0}\{f_0''(\eta_0)+\sigma[\eta_1f_0'''(\eta_0)+f_1''(\eta_0)]\},\\ \frac{hU_0\delta_0^+}{\nu}\{1+\sigma P\}. \end{gathered} \tag{46a}$$

At the matching point $Q$ the values given by Eqs. (46) and (46a) must coincide for any value of $\sigma$. This leads to the following two systems of equations:

$$\begin{aligned} \zeta_0&=u_0^+y_0/\nu=(C_0)^{\frac{1}{2}}R_0\eta_0/K_0,\\ \ln\zeta_0+A&=kf_0'(\eta_0)/(C_0)^{\frac{1}{2}},\\ (k\zeta_0)^{-1}&=C_0^{-1}(K_0/R_0)f_0''(\eta_0),\\ k\zeta_0&=hR_0, \end{aligned} \tag{48}$$

with

$$\begin{gathered} (C_0)^{\frac{1}{2}}=\left(\frac{C_{f,0}}{2}\right)^{\frac{1}{2}}=\frac{u_0^+}{U_0};\quad R_0=\frac{U_0\delta_0^+}{\nu};\\ K_0=\eta^+\frac{\delta_0^+}{\delta_0}=\int_0^\infty(1-f_0')d\eta \end{gathered} \tag{49}$$

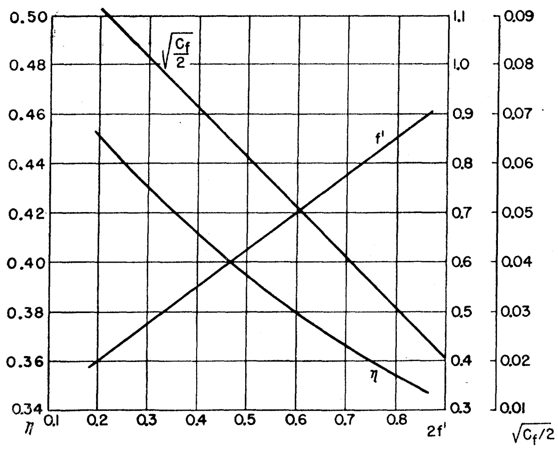


FIG. 1. Characteristics of the nonconducting turbulent boundary layer.

and

$$\zeta_1=\zeta_0\eta_1/\eta_0,$$

$$\frac{u_1^+}{kU_0}(\ln\zeta_0+A)+\frac{1}{k}\frac{\zeta_1}{\zeta_0}+\frac{B}{H_0}\frac{\eta_0}{\eta^+} = (C_0)^{-\frac{1}{2}}\left[\eta_0\frac{\zeta_1}{\zeta_0}f_0''(\eta_0)+f_1'(\eta_0)\right],$$

$$\frac{u_1^+}{u_0^+\zeta_0}+\frac{Bk}{H_0R_0}-\frac{1}{\zeta_0}\frac{\zeta_1}{\zeta_0}=\frac{kK_0}{C_0R_0}[\eta_1 f_0'''(\eta_0)+f_1''(\eta_0)], \qquad (50)$$

$$\frac{k\zeta_1}{\zeta_0}-\frac{kB}{H_0}\frac{\eta_0}{\eta^+}+k\frac{u_1^+}{u_0^+}=\frac{hR_0}{\zeta_0}P.$$

The system of equations (48) governs the matching in the case of nonconducting fluids and it is identical with that given by Clauser.[2] The four unknowns are $\zeta_0$, $\eta_0$, $(C_0)^{\frac{1}{2}}$, and $R_0$. The parameter free is the value $f'(0)=\alpha_0$ of the velocity at the wall. By varying $\alpha_0$ one can obtain the relation $(C_0)^{\frac{1}{2}}=g(R_0)$. Clauser has shown that the relation thus obtained is almost identical with the following well-known law:

$$(C_0)^{-\frac{1}{2}}=5.6\log R_0+4.3. \qquad (51)$$

We have first solved the system of equations (48) using for $f_0$ numerical solutions obtained by means of an electronic computing machine. Some of the essential quantities thus obtained are shown in Fig. 1.

For the solution of the system of equations (50), all the dependent variables are needed in terms of $\alpha_0$. Therefore, we have sought another solution by taking for $f_0$ the expressions obtained by integral methods (see Appendix A). All the results thus obtained exhibited errors which never exceeded 1 or 2%. The quantities essential to the analysis are given by the following relations:

$$\begin{aligned} &(C_0)^{\frac{1}{2}}=0.103313(1-\alpha_0),\\ &f_0'(\eta_0)=1-0.749627(1-\alpha_0),\\ &f_0''(\eta_0)=(1.975902/\eta^+)(1-\alpha_0),\\ &\eta_0/\eta^+=y_0/\delta_0=0.127110,\\ &K_0/\eta^+=0.3(1-\alpha_0).\end{aligned} \qquad (52)$$

The diagram of $(C_0)^{\frac{1}{2}}$ against $\log R_0$ as computed from the preceding relations is compared, in Fig. 2, with that obtained from Eq. (51). It is seen that the agreement is satisfactory for our purposes.

Consider now the system of equations (50). The functions $f_1'$ and $f_1''$ have been obtained by means of an integral method (see Appendix A) and can be expressed as

$$\begin{aligned} f_1'(\eta_0)&=\Sigma_0+(1+\alpha_1)\Sigma_1+P\Sigma_2,\\ f_1''(\eta_0)&=\Sigma_0'+(1+\alpha_1)\Sigma_1'+P\Sigma_2',\end{aligned} \qquad (53)$$

where the $\Sigma_i$ and $\Sigma_i'$ are functions of $\alpha_0$ given in Appendix A.

By taking Eqs. (48) and (52) into account and by recalling that $u_1^+/u_0^+=k_2(C_0)^{\frac{1}{2}}-1$, the system of equations (50) can be written as

$$\begin{aligned} &k_2+A_1B+A_2P=G_{10}+(1+\alpha_1)G_{11}+0.749627k_2(1-\alpha_0),\\ &A_3B+A_4P=G_{20}+(1+\alpha_1)G_{21}+ZG_{22}-0.103313k_2(1-\alpha_0),\\ &-A_3B-P=1-Z-0.103313(1-\alpha_0),\end{aligned} \qquad (54)$$

where the $A_i$'s are constants, $Z=\zeta_1/\zeta_0$, and the $G_{ij}$ are functions of $(1-\alpha_0)$ reported in Appendix A.

The fundamental unknowns of Eqs. (54) are the three constants $k_2$, $B$, and $P$. The quantities $\alpha_1=f'(0)$ and $Z=\zeta_1/\zeta_0$ are some as yet undetermined functions of $\alpha_0$. These functions must be such as to make the right-hand side of Eqs. (54) constant.

Let

$$(1+\alpha_1)G_{11}=-G_{10}-0.749627(1-\alpha_0)+D_4, \qquad (55a)$$

$$Z=-0.103313k_2(1-\alpha_0)+D_3, \qquad (55b)$$

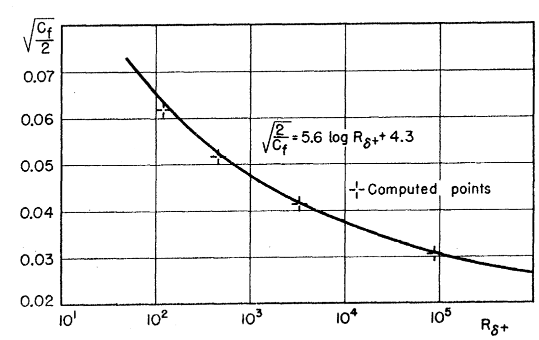


FIG. 2. Comparison between experimental skin-friction law and computed points.

$$G_{20}+\frac{G_{21}}{G_{11}}[-G_{10}-0.749627k_2(1-\alpha_0)+D_4]$$
$$+G_{22}[0.137819D_1(1-\alpha_0)+D_3] \qquad (55c)$$
$$-0.137819D_1(1-\alpha_0)=D_2,$$
$$D_1=-0.739726k_2,$$

so that the system of equations (54) becomes

$$k_2+A_1B+A_2P=D_4,$$
$$A_3B+A_4P=D_2, \qquad (56)$$
$$A_3B+P=D_3-1,$$

where all the $D_i$'s are constants. Equations (56a) and (56b) can always be satisfied since they merely serve to define the variation with $\alpha_0$ of the free parameters $\alpha_1$ and $Z$. It remains to see whether the variations thus defined are such as to render constant the left-hand side of Eq. (55c). Since this relation contains three arbitrary constants $D_1$, $D_2$, and $D_4$, it is always possible to satisfy it for three values of $\alpha_0$. Thus the actual solution has been accomplished by determining first $D_4$ and $D_3$ as functions of $k_2$ from Eq. (55c) written for three different values of $\alpha_0$ and then by solving Eq. (56) for the remaining unknowns $B$, $P$, and $k_2$. That the constants $D_i$ thus determined would actually satisfy Eq. (55c) for any value of $\alpha_0$ was checked *a posteriori* for several values of $\alpha_0$ and proved to be true within less than 1%.

This procedure yielded the following values for the required quantities:

$$B=-56.94, \quad P=0.4222, \quad k_2=1.867,$$
$$Z=\zeta_1/\zeta_0=0.3973-0.1929(1-\alpha_0), \qquad (57)$$
$$(1+\alpha_1)G_{11}=-G_{10}-1.3997(1-\alpha_0)-0.3061.$$

This completes the solution of the problem since now we know both qualitatively and quantitatively the characteristics of the turbulent boundary layer. They are discussed in the next section.

## 6. ANALYSIS OF THE RESULTS AND CONCLUDING REMARKS

Let us summarize first the results obtained in the present investigation. When an electrically conducting fluid is in turbulent flow along a flat plate in the presence of a magnetic field, the law of the wall and the skin-friction law assume the following modified forms:

$$u/u^+=5.6\log(u_0^+y/\nu)+4.9$$
$$-56.9(\sigma/R_{\delta_0^+})(u_0^+y/\nu), \qquad (58a)$$
$$U_1/u^+=5.6\log(u_0^+\delta_0/\nu)+7.543-1.87\sigma, \qquad (58b)$$

where $\sigma=mx/U_0$, $R_{\delta_0^+}=U_0\delta_0^+/\nu$, and the subscript zero refers to conditions for $\sigma=0$ (absence of magnetic field or nonconducting fluid). Equations (58) hold for small values of the magnetic Reynolds number and are valid to within terms of order $\sigma$. The velocity $U_1$ is the free-stream velocity equal to $U_0(1-\sigma)$.

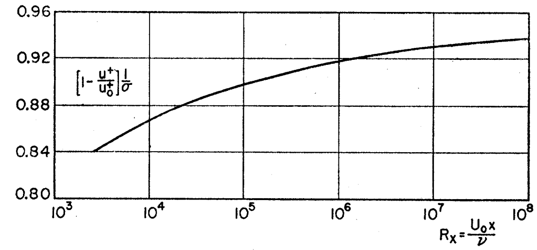


FIG. 3. Rate of decrease of friction velocity per unit magnetic parameter $\sigma$.

The friction velocity $u^+=(\tau_0/\rho)^{\frac{1}{2}}$ is given by

$$u^+=u_0^+[1+\sigma(1.867(\tfrac{1}{2}C_{f_0})^{\frac{1}{2}}-1)], \qquad (59)$$

where $(\frac{1}{2}C_{f_0})^{\frac{1}{2}}=u_0^+/u_0$ is the skin-friction coefficient for $\sigma=0$. With the help of the preceding three relations we can now discuss the effects of the magnetic field on the characteristics of the turbulent layer.

Considering that $(\frac{1}{2}C_{f_0})^{\frac{1}{2}}$ is a function of $R_x=U_0x/\nu$, Eq. (59) indicates that, for a given $R_x$, the friction velocity at the wall is decreased by the action of the magnetic field. In Fig. 3 the quantity

$$[1-(u^+/u_0^+)]\sigma^{-1}=1-1.867(\tfrac{1}{2}C_{f_0})^{\frac{1}{2}},$$

yielding the rate of this decrease per unit $\sigma$, is plotted against the Reynolds number $R_x$. It appears that the rate increases as $R_x$ increases and that it is quite sizable. For a Reynolds number of $10^5$ the decrease in the friction velocity (i.e., the shear stress at the wall) is already of the order of 10% for $\sigma$ as low as 0.1.

This decrease in shear stress at the wall for a given $R_x$ is also discernible from the expression for the microscale of turbulence given by Eq. (33). It appears indeed, for the value of the constant that we have computed, that everything else being constant (that is, for $R_x=$const), the presence of the magnetic field tends to increase the microscale $\lambda$ and thus decreases the percentage of turbulent kinetic energy dissipated through the viscous action of the fluid.

This consideration leads to another important remark which is connected with the previous discussion of the energy transfer within the inner layer. We have seen that, for a given initial energy level of the free stream, the turbulent kinetic energy input into the inner layer is decreased by the presence of a magnetic field fixed with respect to the plate due to "magnetic" dissipation of mean kinetic energy (Joule dissipation) in the outer layer. This reduced energy input is to be dissipated in the inner layer by both viscous and "magnetic" action so that one expects the shear stress at the wall to be decreased for two reasons. If one considers that the increase of the microscale of turbulence could be ascribed to the reduced energy input, one sees, from Eq. (34) and from the numerical value found for $B$, that the reduction of shear stress at the wall is mainly due to the reduced input rather than to the additional Joule dissipation within the inner layer.

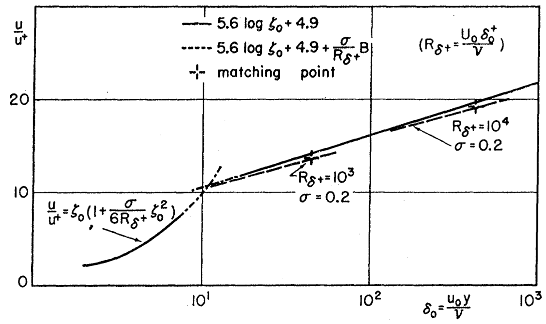


FIG. 4. Velocity profiles in the wall region for turbulent magneto-fluid dynamic boundary layers.

Equation (58b) shows that, for a given Reynolds number $R_x$, the ratio $(U_1/u^+)(u_0^+/u_0)$ is always slightly less than one: that is, the skin-friction coefficients $\frac{1}{2}C_f$ referred to the free-stream velocity $U_1$ is slightly larger than $\frac{1}{2}C_{f0}$. This is an immediate consequence of the fact that, as seen again from Eq. (59), the free-stream velocity rate of decrease with $\sigma$ is slightly larger than that of $u^+$.

In the laminar case the presence of the magnetic field, with its corresponding introduction of an additional characteristic length, destroys the possibility of having a strictly similar flow field.

Likewise, in the turbulent case, the presence of the magnetic fields destroys the validity of an "universal" law of the wall insofar as there appears now an explicit dependence upon the Reynolds number $R_x$ even in the velocity profile of the inner region.

These velocity profiles are plotted against $\zeta_0$ in Fig. 4 for some indicative values of $\sigma$ and of $R_{\delta^+}$. It is seen that the deviation from the classical straight line occurs the earlier the lower the Reynolds number $R_{\delta^+}$, and it is rather sizable even for low value of $\sigma$. In Fig. 4 there is also the profile in the laminar sublayer and it appears that there is practically no influence of the magnetic field on it. This is due to the fact that the corrective terms appearing in the equation giving the "laminar" profile

$$u/u^+=\zeta_0[1+\tfrac{1}{6}(\sigma/R_{\delta^+})\zeta_0^2]$$

becomes appreciable only when $\zeta_0$ lies already in the "turbulent" region.

Although the law of the wall is no longer an "universal law," such is not the case for the skin-friction law. Considering that $U_0=U_0(1-\sigma)$, defining a skin-friction coefficient $C_f$ referred to the initial free-stream velocity $U_0$, and a Reynolds number $R_\delta$ by

$$(\tfrac{1}{2}C_f)^{-1}=(U_0/u^+)^2;\quad R_\delta=(U_0\delta_0^+/\nu)(1-0.2323\sigma),$$

Eq. (58b) could also be written as

$$(1-\sigma)(\tfrac{1}{2}C_f)^{-\frac{1}{2}}=5.6\log[(\tfrac{1}{2}C_f)R_\delta]+7.543. \qquad (60)$$

It is noteworthy pointing out the similarity of the structure of this formula with the conventional skin-friction law. In Fig. 5 it is indeed shown how points computed for several values of $\sigma$, $R_\delta$, and $\alpha_0$ can all be correlated by the straight line given by Eq. (60). On the same figure are also marked some points computed for $\sigma=0.5$. Their position on the diagram clearly indicates the limitations connected with the present results.

## CONCLUSION

Some aspects of turbulent magneto-fluid dynamic boundary layers have been investigated here for the case of low magnetic Reynolds number. The basic equations have been derived together with the balances of mean and turbulent kinetic energy. A suitable model for the dissipative region has been assumed and the modified forms of the law of the wall and of the skin-friction law derived. The principal results are:

The intensity of turbulent magnetic energy spectrum is, at any wave number, at least one order of magnitude smaller than that of the turbulent kinetic energy.

At any wave number the intensity of the kinetic energy spectrum is lowered by the presence of the magnetic field.

The reduction of shear stress at the wall is due to two concomitant reasons. First the work done by the mean flow against the magnetic field in the outer layer tends to reduce the energy input in the inner layer. Second, the already decreased energy input is to be dissipated by both viscous and magnetic action.

Quantitative analysis shows that the first cause of reduction is the more important one.

The law of the wall is no longer an universal law since it depends explicitly on the Reynolds number also.

The skin-friction law can be given an universal form which is structurally similar to that valid in ordinary turbulent fields. This last results as well as the majority of the quantitative analysis is valid only for a small value of the parameter $\sigma$ (the Kàrmàn number referred to the streamwise distance $x$ and to the initial free-stream velocity $U_0$).

## APPENDIX A. SOLUTION OF THE BASIC EQUATIONS

*Equation (42a)*. The equation to be solved is

$$f_0'''+f_0f_0''=0\quad(\eta_0=\eta^+y/\delta_0), \qquad (A1)$$

$$f_0(0)=0;\quad f_0'(0)=\alpha_0;\quad f_0'(+\infty)=1. \qquad (A2)$$

By letting $z=\eta^0/\eta^+$ and by integrating Eq. (A1) from 0 to $+\infty$, the solution which satisfies the boundary

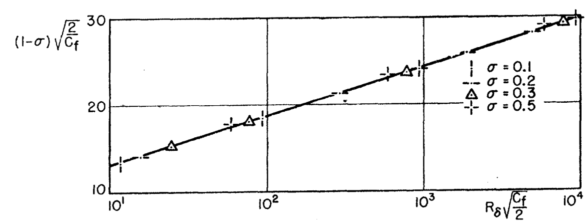


FIG. 5. Skin-friction law for turbulent magneto-fluid dynamic boundary layers.

condition given by Eq. (A2) and yields

$$f_0'''(0)=f_0''(\infty)=f_0'''(\infty)=0$$

is determined as

$$f_0=\eta^+[z+(1-\alpha_0)T(z)] \tag{A3}$$

with

$$\begin{aligned} &T(z)=-z+z^2-\tfrac{1}{2}z^4+\tfrac{1}{5}z^5,\\ &(\eta^+)^2=2/[0.3-0.182539(1-\alpha_0)]. \end{aligned} \tag{A4}$$

The quantities essential for the analysis of the main body of the paper are to be computed at $\eta_0/\eta^+=z_0=0.12711$ and are given by

$$\begin{aligned} &(u/u_0)(z_0)=f_0'(z_0)=1-0.749627(1-\alpha_0),\\ &\delta_0^+/\delta_0=\textstyle\int_0^1(1-f_0')dz=K_0/\eta^+=0.3(1-\alpha_0),\\ &f_0''(z_0)=(1/\eta^+)1.975902(1-\alpha_0),\\ &f_0(z_0)=\eta^+[0.12711-0.111077(1-\alpha_0)]. \end{aligned} \tag{A5}$$

*Equation (42b)*. The equation to be integrated is

$$\frac{2}{n+1}[f_1'f_0'-f_1f_0''+f_0']-f_1f_0''-f_0f_1''=f_1'''+Pf_0''', \tag{A6}$$

$$f_1(0)=0;\quad f_1'(+\infty)=1;\quad f_1'(0)=\alpha_1 \tag{A7}$$

where $n=\frac{4}{5}$ and $P=2a/(n+1)$; $a$ being the constant defined by Eq. (37). The solution we are interested in has to be given in terms of the two parameters $\alpha_1$ and $P$.

The solution satisfying Eq. (A6) at $z=\eta_0/\eta^+=0$ and the boundary condition given by Eqs. (A7) is

$$f_1'=\Sigma_0+(1+\alpha_1)\Sigma_1+P\Sigma_2 \tag{A8}$$

with

$$\begin{aligned} &\Sigma_0=2H_0S_1-1,\\ &\Sigma_1=1+S_0+2H_1S_1+\frac{\eta^{+2}\alpha_0}{n+1}S_2\\ &\Sigma_2=2H_2S_1, \end{aligned} \tag{A9}$$

where the $S_i$'s are functions only of $z$ and the $H_i$'s are functions only of $\alpha_0$. Their expressions are

$$S_0=-4z^3+3z^4,\quad S_1=z-3z^3+2z^4,\quad S_2=z^2-2z^3+z^4,$$

$$H_0=0.15(1-\alpha_0)/A,$$

$$H_1=\frac{1}{A}\left\{0.3-(1-\alpha_0)\left[0.273811+0.007937\frac{\eta^{+2}\alpha_0}{n+1}\right]+\frac{0.05\eta^{+2}\alpha_0}{3\quad n+1}\right\}, \tag{A10}$$

$$H_2=\frac{1-\alpha_0}{\eta^{+2}A}\frac{n+1}{3+n},$$

$$A=0.091270(1-\alpha_0)-0.15-\frac{n+1}{3+n}\frac{1}{\eta^{+2}}.$$

The expression for $\eta^+$ is that given by Eq. (A4). The values of $S_i$'s and of their first derivatives at $z=z_0$ are

$$\begin{aligned} &S_0=-0.007433,\quad S_1=0.121470,\quad S_2=0.012310,\\ &S_0'=-0.169236,\quad S_1'=0.871019,\quad S_2'=0.165494. \end{aligned} \tag{A11}$$

*Expressions for the functions appearing in the system of equations (54).*

$$\begin{aligned} &G_{10}=(f_0'+\Sigma_0)/(C_0)^{\frac{1}{2}}; &&G_{11}=\Sigma_1/(C_0)^{\frac{1}{2}},\\ &G_{20}=1+[kz_0\Sigma_1'/(C_0)^{\frac{1}{2}}]; &&G_{21}=kz_0\Sigma_1'/(C_0)^{\frac{1}{2}},\\ &G_{22}=(1-\eta_0f_0); &&k=0.4112. \end{aligned} \tag{A12}$$

All the functions are to be evaluated at $z=z_0=0.12711$. The constants appearing in Eqs. (56) are given by

$$\begin{aligned} &A_1=z_0/H_0=0.043772;\\ &A_2=-\Sigma_2/(C_0)^{\frac{1}{2}}=0.755845;\\ &A_3=h=0.018;\\ &A_4=-kz_0\Sigma_2'/(C_0)^{\frac{1}{2}}=0.283287. \end{aligned} \tag{A13}$$

The constant $H_0=u_0^+x/U_0\delta_0$ has been taken equal to 2.9038 by way of the following calculation.

From Eq. (A5) and (52) one derives that

$$\delta_0^+/\delta_0=0.3(1-\alpha_0)=2.9038(C_0)^{\frac{1}{2}}.$$

On the other hand, for large $R_x=U_0x/\nu$ it is, to within an accuracy sufficient for our purposes, $\delta_0^+=xC_0$, so that

$$u_0^+x/U_0\delta_0=2.9038. \tag{A14}$$

## DISCUSSION

Session Reporter: W. H. Reid

Professor **C. C. Lin** (*Institute for Advanced Study, Princeton, New Jersey*) asked for the physical reasons for assuming that the turbulent magnetic energy is much smaller than the turbulent kinetic energy. The author argued that since the magnetic Reynolds number had been assumed to be small, the equations could be linearized. There would then be no transfer of energy from the turbulent velocity field to the turbulent magnetic field or more precisely that $G(k)\ll F(k)$ for *all* values of $k$, where $G(k)$ and $F(k)$ are the energy spectra of the turbulent magnetic and velocity fields, respectively.

A further physical reason for accepting this assumption was suggested by Dr. **S. A. Colgate** (*Lawrence Radiation Laboratory, University of California, Livermore, California*). He remarked that if the magnetic Reynolds number is small, as the author had assumed, then the skin depth would be large. Hence, the vorticity would not be able to stretch or distort the lines of force. In the absence of such a mechanism, the field would not be magnified and so would remain small.