Dynamical Equations and Transport Relationships for a Thermal Plasma

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1. INTRODUCTION

THE general theory for a multicomponent gas mixture has been developed by several authors.[1–3] On the assumption that the behavior of a plasma can be adequately described in terms of binary collisions, the results can be applied directly to a thermal plasma in a magnetic field. Unfortunately, the relevant equations are extremely complicated, and for this reason their value is somewhat questionable. The prime object here is to present these equations in a summary form in which the differences between these and those of magnetohydrodynamics are emphasized.

In the next section the basis of the general theory is briefly outlined. In principle, the method for obtaining dynamical equations for a gas mixture is simple. Equations of change for a finite number of macroscopic variables are generated and then closed by using an approximation for the velocity distribution function. In this, and the papers referred to heretofore, a particular approximation due to Grad,[4] namely, the 13-moment approximation, is used. Equations derived in this manner for a binary (completely ionized) plasma are given in Sec. 3. From these, transport relations can be generated by a method of successive approximation. Expressions of this nature are given in Sec. 4. The significance of relaxation and other characteristic times is also briefly discussed. In Sec. 5 comments are made on the validity of the 13-moment approximation and its relationship to other types of approximations. Except for minor differences, which are explained when introduced, the notation is the same as that of Chapman and Cowling,[5] and where applicable rationalized mks units are used.

2. GENERAL THEORY

2.1. Closure of the Equations of Change

The general equation of change of a dynamical variable $\psi_j(\mathbf{w}_j,\mathbf{r},t)$, referred to a frame moving with the mean mass velocity $\mathbf{v}$, is[6]

$$\frac{dn_j\langle\psi_j\rangle}{dt}+n_j\langle\psi_j\rangle\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{v}+\frac{\partial}{\partial\mathbf{r}}\cdot n_j\langle\psi_j\mathbf{w}_j\rangle$$
$$-n_j\left\{\left\langle\frac{d\psi_j}{dt}\right\rangle+\left\langle\mathbf{w}_j\cdot\frac{\partial\psi_j}{\partial\mathbf{r}}\right\rangle+\left\langle\mathbf{F}_j\cdot\frac{\partial\psi_j}{\partial\mathbf{w}_j}\right\rangle\right.$$
$$\left.+\mathbf{b}_j\cdot\left\langle\frac{\partial\psi_j}{\partial\mathbf{w}_j}\times\mathbf{w}_j\right\rangle-\left\langle\frac{\partial\psi_j}{\partial\mathbf{w}_j}\mathbf{w}_j\right\rangle:\frac{\partial\mathbf{v}}{\partial\mathbf{r}}\right\}=\sum_k I_{jk}(\psi_j). \quad (1)$$

$\mathbf{w}_j$ is the peculiar velocity, while j and k denote particle types. In writing (1) the following abbreviations have been used:

$$d/dt\equiv\partial/\partial t+\mathbf{v}\cdot\partial/\partial\mathbf{r}, \quad (2)$$

$$\mathbf{F}_j\equiv\mathbf{f}_j+\mathbf{v}\times\mathbf{b}_j-d\mathbf{v}/dt. \quad (3)$$

In (3), $\mathbf{f}_j+\mathbf{v}\times\mathbf{b}_j$ is the acceleration due to fields of a macroscopic nature. For instance, $\mathbf{f}_j=(e_j/m_j)\mathbf{E}$, $\mathbf{b}_j=(e_j/m_j)\mathbf{B}$, where $\mathbf{E}$ is an electric field vector and $\mathbf{B}$ a magnetic induction vector. e_j and m_j are, respectively, the charge and mass of a particle type j. n_j is the particle number density, and the average value over velocity space, $\langle\psi_j\rangle$, is defined by

$$\langle\psi_j\rangle=\frac{1}{n_j}\int f_j\psi_j d\mathbf{w}_j. \quad (4)$$

f_j is the velocity distribution function of particles type j.

The collision integrals on the right-hand side of Eq. (1) are

$$I_{jk}=\int f_jf_k\Delta_{jk}(\psi_j)g_{jk}bdbd\epsilon d\mathbf{w}_jd\mathbf{w}_k, \quad (5)$$

where $g_{jk}=|\mathbf{w}_j-\mathbf{w}_k|$, b is the impact parameter, ϵ the azimuthal angle, and $\Delta_{jk}(\psi_j)$ is the change in ψ_j on collision of a particle of type j with one of type k.

In the theory of binary elastic collisions, there are three summational invariants ψ_j for which

$$\sum_j\sum_k I_{jk}(\psi_j)=0. \quad (6)$$

These are m_j, $m_j\mathbf{w}_j$, and $\frac{1}{2}m_jw_j^2$, which on being substituted in (1) give the continuity, momentum, and thermal energy equations, respectively. In view of (6)

[1] R. Herdan and B. S. Liley, A.E.I. Research Rept. No. A.1004 (1959) (to be published).
[2] R. Herdan and B. S. Liley, A.E.I. Research Rept. No. A.1005 (1959) (to be published).
[3] I. Kolodner, doctoral dissertation, New York University, 1950.
[4] H. Grad, Communs. Pure Appl. Math. 2, 331 (1949).
[5] S. Chapman and T. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1952), Chap. 1.
[6] S. Chapman and T. Cowling, reference 5, Chap. 18.

these equations are of special interest. However, as is well known, they do not form a closed set since the two higher moments $\mathbf{P}_j$ the stress tensor, and $\mathbf{q}_j$ the heat flux vector, are introduced. Taking ψ_j equal to $m_j\mathbf{w}_j\mathbf{w}_j$ and $\frac{1}{2}m_j\mathbf{w}_jw_j^2$ in (1), equations for $\mathbf{P}_j$ and $\mathbf{q}_j$ could be obtained, but again two higher moments of the form $n_j\langle\psi_j\mathbf{w}_j\rangle$ would be introduced. This process of generating equations for the higher moments could be extended indefinitely depending solely on how many primary variables one is prepared to introduce.

However, if at any level the distribution function is known, or can be approximated to, in terms of primary variables for which the equations of change have already been generated, then this system of equations can be closed. Of immediate interest and applicability to a thermal plasma is Grad's 13-moment approximation. On introducing the dimensionless velocity

$$\boldsymbol{\xi}_j=(2\alpha_j)^{\frac{1}{2}}\mathbf{w}_j;\quad \alpha_j=m_j/2kT_j, \tag{7}$$

where k is Boltzmann's constant and T_j the kinetic temperature of particles type j (referred to the mean mass velocity $\mathbf{v}$), this expression for f_j is[4]

$$f_j=f_j^{(0)}\{1+\mathbf{a}_j^{(1)}\cdot\boldsymbol{\xi}_j+\tfrac{1}{2}\mathbf{a}^{(2)}:\boldsymbol{\xi}_j\boldsymbol{\xi}_j+\tfrac{1}{10}\mathbf{a}_j^{(3)}\cdot\boldsymbol{\xi}_j(\xi_j^2-5)\}, \tag{8}$$

where

$$f_j^{(0)}=n_j(\alpha_j/\pi)^{\frac{3}{2}}\exp(-\tfrac{1}{2}\xi_j^2) \tag{9}$$

is the usual Maxwellian velocity distribution function. The coefficients $a^{(n)}$ are given by

$$\mathbf{a}_j^{(1)}=(2\alpha_j)^{\frac{1}{2}}\mathbf{u}_j, \tag{10}$$

$$\mathbf{a}_j^{(2)}=(1/p_j)\{\mathbf{P}_j-p_j\mathbf{U}\}\equiv(1/p_j)\mathbf{P}_j^{\circ}, \tag{11}$$

$$\mathbf{a}_j^{(3)}=(2/p_j)(2\alpha_j)^{\frac{1}{2}}\{\mathbf{q}_j-\tfrac{5}{2}p_j\mathbf{u}_j\}\equiv(2/p_j)(2\alpha_j)^{\frac{1}{2}}\mathbf{R}_j. \tag{12}$$

In these $\mathbf{u}_j$ (i.e., $\langle\mathbf{w}_j\rangle$) is the drift velocity of particles type j with respect to a frame moving with the mean velocity $\mathbf{v}$. p_j is the "hydrostatic" pressure, $\mathbf{U}$ the second-order unit tensor and $\mathbf{P}_j^{\circ}$, defined by (11), is the "nonhydrostatic" component of the stress tensor. $\mathbf{R}_j$ is an associated heat-flux vector defined by (12).

More is said about this approximation to f_j in Sec. 5. For the moment, though, it is clear that if equations of change for $\mathbf{u}_j$, T_j, $\mathbf{P}_j^{\circ}$ (or $\mathbf{P}_j$), and $\mathbf{R}_j$ (or $\mathbf{q}_j$) are generated, the higher moments $n_j\langle\psi_j\mathbf{w}_j\rangle$, which inevitably appear, can, using (8) in the defining equation (4), be expressed in terms of these same variables. Furthermore, knowing the dynamics of a collision the integrals can be evaluated and a closed system of equations obtained.

2.2. Collision Integrals

These have been discussed in detail elsewhere[2,3] and only those points relevant to the accuracy of the expressions given in the next section for a thermal plasma are commented upon here.

Equation (8) can be written in the abbreviated form

$$f_j=f_j^{(0)}(1+\Delta_j).$$

In reference 2 the collision integrals have been evaluated only to the accuracy involved in taking

$$f_jf_k=f_j^{(0)}f_k^{(0)}(1+\Delta_j+\Delta_k) \tag{13}$$

in expressions (5). The quadratic terms have been considered in detail by Kolodner,[3,7] but they are in most practical cases unimportant, and no such terms are given here.

To within the approximation (13), the general result is

$$I_{jk}(\psi_j)=\delta_{jk}^{(0)}+\delta_{jk}^{(1)}\mathbf{u}_j+\delta_{jk}^{(2)}\mathbf{u}_k+\delta_{jk}^{(3)}\mathbf{P}_j^{\circ}+\delta_{jk}^{(4)}\mathbf{P}_k^{\circ}+\delta_{jk}^{(5)}\mathbf{R}_j+\delta_{jk}^{(6)}\mathbf{R}_k. \tag{14}$$

The $\delta_{jk}^{(n)}(\neq\delta_{kj}^{(n)})$ are scalars, being functions of n_j, n_k, T_j, and T_k; the exact dependence on these parameters being determined by the nature of ψ_j $(\mathbf{w}_j)$. The fact that Eq. (14) is of nonuniform tensorial rank implies that in the case of each particular ψ_j certain of the δ's are necessarily zero.

In general, the $\delta^{(n)}$'s are extremely complex, only simplifying in the case of equal temperatures. However, in a binary electron plasma, in which $m_e\ll m_i$, they do reduce to something manageable provided

$$T_i\ll(m_i/m_e)T_e. \tag{15}$$

(Here as elsewhere in this paper j and k refer to general particle types, while the subscripts i and e refer explicitly to ions and electrons, respectively.)

In evaluating the integrals (5) for a plasma an upper limit has had to be placed on the impact parameter b. This has been taken to be the Debye length h and the coefficients of the next section, corresponding to the $\delta^{(n)}$'s of (14), involve as expected the ratio h/r_{jk}. r_{jk} is the distance of closest approach of two particles, types j and k, which possess the mean relative energy of these particle types (in the center of mass frame of the colliding particles). Following Spitzer,[8] the ratio h/r_{jk} is denoted by Λ_{jk}. This, in general, is given by[2]

$$\Lambda_{jk}=\frac{3}{2}h\left(\frac{\alpha_k+\alpha_j}{\alpha_k\alpha_j}\right)\frac{m_km_j}{m_k+m_j}\frac{4\pi\epsilon_0}{|e_je_k|}. \tag{16}$$

α_k and α_j are as defined in Eq. (7) and ϵ_0 is the permittivity of free space. $\ln\Lambda_{ie}$ is tabulated in reference 8. In particular, since $m_i\gg m_e$ and $\alpha_i\gg\alpha_e$ [consistent with Eq. (15)], it can be easily verified that

$$\ln\Lambda_{ii}=\ln\Lambda_{ie}+\ln(T_e/T_iZ_i)\simeq\ln\Lambda_{ie}, \tag{17}$$

$$\ln\Lambda_{ee}=\ln\Lambda_{ie}+\ln Z_i\simeq\ln\Lambda_{ie}. \tag{18}$$

[7] I. Kolodner, Institute of Mathematical Sciences, New York University, NYO-7980 (1957).

[8] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), Chap. V.

Z_i is the ionic charge, while the approximations follow from the fact that $\ln\Lambda_{ie}$ is usually within the range 10 to 20.

3. DYNAMICAL EQUATIONS FOR A BINARY PLASMA

In the following subsections, equations corresponding to the dynamical equations for n_j, $n_jm_j\mathbf{u}_j$, p_j, $\mathbf{P}_j^\circ$, and $\mathbf{R}_j$ are given in summary form. The detailed derivation can be found in reference 1. For notational convenience, $\mathbf{q}_j$ and $\mathbf{P}_j$ also appear in these equations, ot being understood that these are related to $\mathbf{P}_j^\circ$ and $\mathbf{R}_j$ by Eqs. (11) and (12). In cases where no confusion can arise, $\mathbf{j}$, the conduction current density, is used in preference to the $\mathbf{u}_j$. This permits a ready comparison of the equations of this section with those of magnetohydrodynamics. To be explicit,

$$\mathbf{j}=\sum_j n_je_j\mathbf{u}_j, \qquad (19)$$

but since by definition

$$\rho_i\mathbf{u}_i+\rho_e\mathbf{u}_e=0, \qquad (20)$$

where

$$\rho_i\equiv n_im_i; \quad \rho_e\equiv n_em_e,$$

it follows that

$$\mathbf{j}\simeq n_ee_e\mathbf{u}_e. \qquad (21)$$

Although the following equations apply to a completely ionized plasma, use of the approximate equality $n_e|e_e|\simeq n_ie_i$ has only been made in simplifying the collision integrals and in neglecting terms in which the ratio σ/n_ee_e occurs explicitly. σ is the charge density defined by

$$\sigma=\sum_j n_je_j. \qquad (22)$$

3.1. Maxwell's Equations

In order to emphasize the distinction between conduction, convection, and displacement currents, the relevant Maxwell equations are included, namely,

$$\mathrm{curl}\mathbf{B}=\mu_0\mathbf{j}+\mu_0\sigma\mathbf{v}+\mu_0\epsilon_0\partial\mathbf{E}/\partial t, \qquad (23)$$

$$\mathrm{curl}\mathbf{E}=-(\partial\mathbf{B}/\partial t). \qquad (24)$$

μ_0 is the permeability and ϵ_0 the permittivity of free space.

3.2. Continuity Equations

The equations of change for n_e and n_i may be combined to give the continuity equations for mass and charge. These are

$$\partial\sigma/\partial t+(\partial/\partial\mathbf{r})\cdot(\mathbf{j}+\sigma\mathbf{v})=0, \qquad (25)$$

$$\partial\rho/\partial t+(\partial/\partial\mathbf{r})\cdot\rho\mathbf{v}=0, \qquad (26)$$

where $\rho=\rho_i+\rho_e$. Since the plasma is completely ionized, the collision terms are zero.

3.3. Momentum Equations

The equations of change for $\rho_e\mathbf{u}_e$ and $\rho_i\mathbf{u}_i$, on being added, lead to the total momentum equation

$$\rho d\mathbf{v}/dt=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})+\rho\mathbf{g}+\mathbf{j}\times\mathbf{B}-(\partial/\partial\mathbf{r})\cdot\mathbf{P}, \quad (27)$$

where $\mathbf{P}\equiv\mathbf{P}_i+\mathbf{P}_e$, and $\rho\mathbf{g}$ is any nonelectromagnetic body force such as gravitation.

Again, if these equations of change are first multiplied throughout by e_i/m_i and e_e/m_e, respectively, and then added, the "generalized Ohm's law" is obtained, namely,

$$\frac{m_e}{n_ee^2}\left\{\frac{d\mathbf{j}}{dt}+\mathbf{j}\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{v}+\mathbf{j}\cdot\frac{\partial}{\partial\mathbf{r}}\mathbf{v}\right\}-\frac{1}{n_ee}\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{P}_e+\frac{1}{n_ee}\mathbf{j}\times\mathbf{B}$$
$$-(\mathbf{E}+\mathbf{v}\times\mathbf{B})=-\eta\mathbf{j}-\alpha\left(\mathbf{R}_e-\frac{m_e}{m_i}Z_i\mathbf{R}_i\right). \quad (28)$$

$e(\equiv|e_e|)$ is the charge on a proton. The coefficient η is the same as that given by Spitzer[8] for the resistivity perpendicular to a strong magnetic field in the absence of Hall currents [these are associated with the $\mathbf{j}\times\mathbf{B}$ term in (28)], pressure gradients, and inertial terms. Numerically,

$$\eta=1.29\times10^2T_e^{-\frac{3}{2}}Z_i\ln\Lambda_{ie}\ \text{ohm-m}. \qquad (29)$$

The second coefficient α is related to η by

$$\alpha=\tfrac{3}{5}(\eta e/kT_e). \qquad (30)$$

It is shown in Sec. 4 that the coupling between the electrical and thermal properties is responsible for the well-known anisotropic resistivity of a plasma (ignoring Hall currents). Owing to the mass ratio m_e/m_i, the term in $\mathbf{R}_i$ in (28) can usually be neglected.

It is convenient to define a time τ_D by the relation

$$\tau_D=\mu_0l_e^2/\eta, \qquad (31)$$

where

$$l_e^2=m_e/\mu_0n_ee^2. \qquad (32)$$

Inspection of (28) shows that τ_D is simply the electron collision time for momentum exchange. It has, however, another interesting interpretation: l_e is the penetration depth in a stationary collisionless plasma. Hence τ_D is of the order of the time for an electromagnetic field to penetrate this distance when collisions are dominant. Numerically,

$$\tau_D=(2.72\times10^5/Z_i\ln\Lambda_{ie})(T_e^{\frac{3}{2}}/n_e)\ \text{sec}, \qquad (33)$$

where it must be remembered that n_e is number per cubic meter and T_e is in degrees Kelvin.

3.4. Energy Equations

The thermal energy equations for both ions and electrons have the general form

$$\frac{d\frac{3}{2}p_j}{dt}+\frac{3}{2}p_j\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{v}+\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{q}_j-\rho_j\mathbf{F}_j\cdot\mathbf{u}_j+\mathbf{P}_j:\frac{\partial \mathbf{v}}{\partial \mathbf{r}}=-n_jn_k\nu_{jk}(T_j-T_k),\tag{34}$$

where

$$\mathbf{F}_j\equiv e_j/m_j(\mathbf{E}+\mathbf{v}\times\mathbf{B})+\mathbf{g}-d\mathbf{v}/dt,\tag{35}$$

and

$$\nu_{ie}=\nu_{ei}=3k(m_e/m_i)(1/n_i\tau_D),\tag{36}$$

τ_D being defined by Eq. (31) or (32).

On adding the equations for the two components, the total thermal energy can be obtained. This is

$$\frac{d\frac{3}{2}p}{dt}+\frac{3}{2}p\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{v}+\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{q}+\mathbf{P}:\frac{\partial\mathbf{v}}{\partial\mathbf{r}}=\mathbf{j}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B}),\tag{37}$$

where

$$p=p_i+p_e;\quad \mathbf{q}=\mathbf{q}_i+\mathbf{q}_e;\quad \mathbf{P}=\mathbf{P}_i+\mathbf{P}_e.$$

Again, taking the scalar product of (27) with v and adding the result to (37) gives

$$\frac{\partial(\frac{3}{2}p+\frac{1}{2}\rho v^2)}{\partial t}+\frac{\partial}{\partial\mathbf{r}}\cdot\{\mathbf{q}+\mathbf{P}\cdot\mathbf{v}+\tfrac{3}{2}p\mathbf{v}+\tfrac{1}{2}\rho v^2\mathbf{v}\}=\mathbf{E}\cdot(\mathbf{j}+\sigma\mathbf{v})+\rho\mathbf{g}\cdot\mathbf{v}.\tag{38}$$

This is simply the equation for total energy balance.

3.5. Stress Equations

The equations for the "nonhydrostatic" components of the stress tensors, written in general form, are

$$\frac{d\mathbf{P}_j^\circ}{dt}+\mathbf{P}_j^\circ\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{v}+\tfrac{4}{5}\langle\langle(\partial/\partial\mathbf{r})\mathbf{q}_j\rangle\rangle^\circ-2\rho_j\langle\langle\mathbf{F}_j\mathbf{u}_j\rangle\rangle^\circ-2\frac{e_j}{m_j}\langle\langle\mathbf{P}_j^\circ\times\mathbf{B}\rangle\rangle^\circ+2\langle\langle\mathbf{P}_j\cdot\partial\mathbf{v}/\partial\mathbf{r}\rangle\rangle^\circ=-\frac{\beta_{jk}}{\tau_{jk}}\mathbf{P}_k^\circ-\frac{\beta_{jj}}{\tau_{jj}}\mathbf{P}_j^\circ.\tag{39}$$

In these equations $\mathbf{F}_j$ is as given by (35), while the cross product between a tensor and a vector is defined to be

$$(\mathbf{P}^\circ\times\mathbf{B})_{\lambda\mu}\equiv P_{\sigma\lambda}{}^\circ\epsilon_{\mu\sigma\tau}B_\tau.\tag{40}$$

$\epsilon_{\mu\sigma\tau}$ is the permutation tensor. If **A** and **B** are two vectors,

$$(\langle\langle\mathbf{AB}\rangle\rangle^\circ)_{\lambda\mu}\equiv\tfrac{1}{2}\{A_\lambda B_\mu+B_\lambda A_\mu-\tfrac{2}{3}\delta_{\lambda\mu}A_\sigma B_\sigma\},\tag{41}$$

where **A** could be the vector operators $\partial/\partial\mathbf{r}$ or $\mathbf{P}\cdot\partial/\partial\mathbf{r}$. $\delta_{\lambda\mu}$ is the Kronecker δ.

In (40) and (41) the double suffix summation convention applies to Greek indices. However, in (39) and elsewhere, no such convention applies to the Roman indices j and k (or i and e). It is important, furthermore, to note the ordering of the subscripts j and k in (39) since $\beta_{jk}\neq\beta_{kj}$ and $\tau_{jk}\neq\tau_{kj}$, as can be seen from the following explicit forms for these coefficients:

$$\tau_{ie}=\frac{-\frac{2}{9}\left(\frac{m_iT_e}{m_eT_i}\right)^2}{\left[1-\frac{2}{5}\left(\frac{T_e}{T_i}\right)-\frac{4}{45\ln\Lambda_{ie}}\left(\frac{T_e}{T_i}\right)^2\right]}\tau_D,\tag{42}$$

$$\tau_{ii}=\frac{m_i}{3Z_im_e\left[1+\frac{4}{15\sqrt{2}}Z_i\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\left(\frac{T_e}{T_i}\right)^{\frac{3}{2}}\frac{\ln\Lambda_{ii}}{\ln\Lambda_{ie}}\right]}\tau_D,\tag{43}$$

$$\tau_{ei}=\frac{10}{27}\frac{1}{Z_i}\frac{m_i}{m_e}\tau_D,\tag{44}$$

$$\tau_{ee}=\frac{10}{13}\frac{1}{\left[1+\frac{8}{13\sqrt{2}}\frac{1}{Z_i}\frac{\ln\Lambda_{ee}}{\ln\Lambda_{ie}}\right]}\tau_D,\tag{45}$$

$$\beta_{ie}=\frac{\frac{4}{15}\left(\frac{m_iT_e}{m_eT_i}\right)\left(1-\frac{T_e}{3T_i}\right)}{\left[1-\frac{2}{5}\left(\frac{T_e}{T_i}\right)-\frac{4}{45\ln\Lambda_{ie}}\left(\frac{T_e}{T_i}\right)^2\right]},\tag{46}$$

$$\beta_{ii}=\frac{2}{3}\frac{\left[1+\frac{3\sqrt{2}}{10}Z_i\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\left(\frac{T_e}{T_i}\right)^{\frac{3}{2}}\frac{\ln\Lambda_{ii}}{\ln\Lambda_{ie}}\right]}{\left[1+\frac{4}{\sqrt{2}15}Z_i\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\left(\frac{T_e}{T_i}\right)^{\frac{3}{2}}\frac{\ln\Lambda_{ii}}{\ln\Lambda_{ie}}\right]}\simeq\frac{3}{2},\tag{47}$$

$$\beta_{ee}=\frac{12}{13}\frac{\left[1+\frac{1}{\sqrt{2}Z_i}\frac{\ln\Lambda_{ee}}{\ln\Lambda_{ie}}\right]}{\left[1+\frac{8}{13\sqrt{2}}\frac{1}{Z_i}\frac{\ln\Lambda_{ee}}{\ln\Lambda_{ie}}\right]}\simeq1,\tag{48}$$

$$\beta_{ei}=8/27\simeq\tfrac{1}{3}.\tag{49}$$

These expressions for the coefficients in (39) are exact to within an error of the order of $1/2\ln\Lambda_{ie}$.

The times τ_{ii} and τ_{ee} should not be confused with ion and electron self-collision times. Contributions from self-encounters are, however, included, being easily identified since they are those terms which involve the ratios

$$\ln\Lambda_{ii}/\ln\Lambda_{ie}\quad\text{and}\quad\ln\Lambda_{ee}/\ln\Lambda_{ie}.$$

Since collision times add according to the law

$$1/\tau_C=1/\tau_A+1/\tau_B,$$

it is seen from (45), for example, that the electron self-collision time is

$$[(5\sqrt{2}/4)Z_i][(\ln\Lambda_{ie}/\ln\Lambda_{ee})\tau_D]. \quad (50)$$

This is independent of Z_i since τ_D [Eq. (33)] is inversely proportional to Z_i.

At first sight the coefficients (42) to (49) appear disappointingly complicated. However, owing to the mass ratios, the crossterms involving the τ_{jk} can usually be neglected. That is, the right-hand side of (39) can be approximated to by

$$[(\beta_{jk}/\tau_{jk})\mathbf{P}_k{}^\circ]+[(\beta_{jj}/\tau_{jj})\mathbf{P}_j{}^\circ]\simeq(\beta_{jj}/\tau_{jj})\mathbf{P}_j{}^\circ. \quad (51)$$

Similarly, the corresponding terms in the heat-flux equations (see next subsection) can also be ignored. Trial solutions indicate that these are valid approximations provided

$$(m_e/m_i)T_e \ll T_i \ll T_e(m_i/m_e)^{\frac{1}{2}}, \quad (52)$$

the error involved being of the order of $(T_i/T_e)(m_e/m_i)^{\frac{1}{2}}$ near the upper limit. Furthermore, within the range indicated by (52), only the self-collision terms are important in τ_{ii}, while from (17) and (18),

$$\ln\Lambda_{ii}/\ln\Lambda_{ie}\simeq\ln\Lambda_{ee}/\ln\Lambda_{ie}\simeq 1. \quad (53)$$

Therefore, to within the limits imposed by (52) and (53), the collision terms simplify considerably. The crossterms can be neglected and the remaining ones are relatively uncomplicated.

In one sense the thermal-energy equations are superfluous, since they could be combined with (39) to give dynamical equations for $\mathbf{P}_j$ rather than $\mathbf{P}_j{}^\circ$. This follows from the fact that $3p_j$ is, by definition, simply the contraction of $\mathbf{P}_j$. However, $\frac{1}{2}m_jw_j{}^2$ is a summational invariant and is therefore of special interest. Thus, Eq. (34) has been retained in explicit form.

3.6. Heat-Flux Equations

These are the most cumbersome of the set, being given by

$$\frac{d\mathbf{R}_j}{dt}+\mathbf{R}_j\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{v}+\frac{k}{m_j}\left(\frac{7}{2}\mathbf{P}_j{}^\circ\cdot\frac{\partial}{\partial\mathbf{r}}T_j+\frac{5}{2}p_j\frac{\partial T}{\partial\mathbf{r}}+T_j\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{P}_j{}^\circ\right)$$
$$+\frac{5}{2}n_jk\mathbf{u}_j\frac{dT_j}{dt}-\mathbf{P}_j{}^\circ\cdot\mathbf{F}_j-\frac{e_j}{m_j}\mathbf{R}_j\times\mathbf{B}+\mathbf{R}_j\cdot\frac{\partial\mathbf{v}}{\partial\mathbf{r}}$$
$$+\frac{2}{5}\left(\mathbf{q}_j\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{v}+\mathbf{q}_j\cdot\frac{\partial\mathbf{v}}{\partial\mathbf{r}}+\frac{\partial\mathbf{v}}{\partial\mathbf{r}}\cdot\mathbf{q}_j\right)$$
$$=-(1/\tau_{jk})\mathbf{R}_k-(1/\tau_{jj})\mathbf{R}_j-\gamma_j\mathbf{j}. \quad (54)$$

The τ's are the same as those in Sec. 3.5 while the γ's are

$$\gamma_e=\tfrac{3}{2}(kT_e/e)(1/\tau_D), \quad (55)$$

$$\gamma_i=-\frac{9}{2}\frac{kT_e}{e}\frac{1}{\tau_D}\left(\frac{m_eT_i}{m_iT_e}\right)^2\left[1-\frac{16}{9}\left(\frac{T_e}{T_i}\right)+\frac{10}{9}\left(\frac{T_e}{T_i}\right)^2\right]. \quad (56)$$

As pointedout in Sec. 3.5, the crossterms involving the times $\tau_{jk}(j\neq k)$ can, subject to (52), be neglected. Furthermore, since $\gamma_i\sim(m_e/m_i)^2$, the term $\gamma_i\mathbf{j}$ can be ignored in the equations for the ions. This, however, is not true of the corresponding term $\gamma_e\mathbf{j}$, which is of major importance in coupling the thermal and electrical properties of a plasma.

4. TRANSPORT RELATIONSHIPS

4.1. General

The equations of the preceding section are a set of quasi-linear differential equations which must be solved subject to certain initial and boundary conditions. This is, in general, a difficult problem. However, in certain cases it is possible to develop expressions of a transport nature which lead to a reduction in the number of variables and thus an over-all simplification.

In particular, in slowly varying flows, it is possible to express the components of $\mathbf{j}$ (or $\mathbf{u}_j-\mathbf{u}_k$), $\mathbf{P}_j{}^\circ$, and $\mathbf{R}_j$ in terms of the n_j (or σ and ρ), T_j and the components of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{g}$, and $\mathbf{v}$. The latter of these two sets of variables is called primary, the former secondary, and in what follows are denoted by P_μ and S_μ, respectively, irrespective of tensorial rank or particle type concerned.

It is convenient to consider the problem generally and write the component equations of (28), (39), and (54) in the generalized form

$$\frac{\partial S_\mu}{\partial t}+a_{\mu\lambda}{}^\kappa\frac{\partial S_\lambda}{\partial x^\kappa}+d_{\mu\nu}{}^\kappa\frac{\partial P_\nu}{\partial x^\kappa}+S_\lambda\left(b_{\mu\lambda\nu}\frac{dP_\nu}{dt}+c_{\mu\lambda\nu}{}^\kappa\frac{\partial P_\nu}{\partial x^\kappa}\right)=\gamma_{\mu\lambda}S_\lambda, \quad (57)$$

where, as before, the double suffix summation convention applies to the Greek indices. Inspection of Eqs. (28), (39), and (54) shows that many of the coefficients a, b, c, d, and γ are zero, but when this is not so they are peculiar in that they are functions of the P_ν only. In particular, after making allowances for dimensional differences, the $\gamma_{\mu\lambda}$'s are basically the reciprocals of the collision and Larmor times (and one other which is discussed in Sec. 4.2.3), while the $a_{\mu\lambda}{}^\kappa$ and $d_{\mu\lambda}{}^\kappa$ are the thermal and mean mass velocities. Again, it is observed that the

$$dP_\nu/dt \quad \text{may be} \quad dT_j/dt \quad \text{or} \quad dv_\tau/dt, \quad (58)$$

where v_τ is the τth component of the mean mass velocity. On the other hand, in nearly all instances,

$$c_{\mu\lambda\nu}{}^\kappa\partial P_\nu/\partial x^\kappa \quad \text{are equivalent to} \quad N_{\mu\lambda\nu}{}^\kappa\partial V_\tau/\partial x^\kappa, \quad (59)$$

the N's being merely numbers. In slowly varying flows the terms in S_λ on the left-hand side of (57) may be ignored in comparison with those on the right. Hence, first approximations to the S_λ are given by

$$\gamma_{\mu\lambda}\ {}_{(1)}S_\lambda = d_{\mu\nu}{}^{\kappa}\partial P_\nu/\partial x^\kappa \tag{60}$$

or

$${}_{(1)}S_\lambda = \Gamma_{\lambda\mu} d_{\mu\nu}{}^{\kappa}\partial P_\nu/\partial x^\kappa. \tag{61}$$

The $\Gamma_{\lambda\mu}$ are the reduced cofactors of the determinant $|\gamma_{\mu\lambda}|$ (which is assumed nonzero). A second approximation can now be found by putting the terms in S_μ (and S_λ) on the left-hand side of (57) equal to the ${}_{(1)}S_\lambda$ of (16) and those on the right equal to ${}_{(2)}S_\lambda$; or, in general, the $(n+1)$th approximation is

$${}_{(n+1)}S_\lambda = \Gamma_{\lambda\mu}\left\{\frac{\partial\, {}_{(n)}S_\mu}{\partial t} + a_{\mu\delta}{}^{\kappa}\frac{\partial\, {}_{(n)}S_\delta}{\partial x^\kappa} + {}_{(n)}S_\delta\left(b_{\mu\delta\nu}\frac{dP_\nu}{dt} + c_{\mu\delta\nu}{}^{\kappa}\frac{\partial P_\nu}{\partial x^\kappa}\right)\right\} + \Gamma_{\lambda\mu} d_{\mu\nu}{}^{\kappa}\frac{\partial P_\nu}{\partial x^\kappa}. \tag{62}$$

In this manner "solutions" to any degree of approximation may be obtained, subject to the flow being slowly varying.

An alternative form of this process of successive approximation emphasizes the question of convergence. If the S_λ are expanded in the form

$$S_\lambda = S_\lambda{}^{(0)} + S_\lambda{}^{(1)} + S_\lambda{}^{(2)} + \cdots S_\lambda{}^{(n)} + \cdots, \tag{63}$$

where

$${}_{(n+1)}S_\lambda \equiv S_\lambda{}^{(0)} + S_\lambda{}^{(1)} + \cdots S_\lambda{}^{(n+1)}, \tag{64}$$

then (61) and (62) are equivalent to the series of equations

$$S_\lambda{}^{(0)} = 0, \tag{65}$$

$$S_\lambda{}^{(1)} = \Gamma_{\lambda\mu} d_{\mu\nu}{}^{\kappa}\partial P_\nu/\partial x^\kappa, \tag{66}$$

and for $n \geq 1$,

$$S_\lambda{}^{(n+1)} = \Gamma_{\lambda\mu}\left\{\frac{\partial S_\mu{}^{(n)}}{\partial t} + a_{\mu\delta}{}^{\kappa}\frac{\partial S_\delta{}^{(n)}}{\partial x^\kappa} + S_\delta{}^{(n)}\left(b_{\mu\delta\nu}\frac{dP_\nu}{dt} + c_{\mu\delta\nu}{}^{\kappa}\frac{\partial P_\nu}{\partial x^\kappa}\right)\right\}. \tag{67}$$

The expansion (63) should be compared with the Chapman-Enskog expansion of the velocity distribution function. Furthermore, the recursive nature of (67) implies that (63) is an expansion in terms of the reduced cofactors $\Gamma_{\lambda\mu}$ or the $\Gamma_{\lambda\mu}a_{\mu\delta}{}^{\kappa}$, etc. Remembering the nature of the $\gamma_{\lambda\mu}$'s, $a_{\mu\delta}{}^{\kappa}$'s, etc., this is tantamount to an expansion in terms of the collision, Larmor, or hybrid times or mean free paths and Larmor radii. Hence, the rapid convergence of (63) would appear to require that the primary variables change little in times and distances comparable with these characteristic quantities. Certain explicit examples are considered in more detail in Sec. 3.3.

By using (23)–(27) and (34), it is possible to eliminate the time derivative from (63) and thus obtain the S_λ in terms of the P_ν and their spatial derivatives only. Unfortunately, the resulting expressions are so unmanageable that they are of little value. The fact that this can be done, however, is of importance in showing the relationship between the 13-moment approximation and the direct Chapman-Enskog solution of the Boltzmann equation.

If the flows are not slowly varying, at least to the extent implied previously, it may still be possible to use (66) as a first approximation provided the γ's are replaced by the coefficients of the S_λ on the left-hand side of (57). These involve dP_ν/dt and $\partial V_\tau/\partial x^\kappa$. However, the convergence of a series of the form (63) for this case has not been investigated by the authors of this paper.

In the next subsection the first approximations to (28), (39), and (54) are considered in detail. Although these are basically similar to those of Chapman and Cowling,[6] they do differ by the inclusion of extra terms.

4.2. First Approximations

For simplicity it is assumed that the ion temperature lies within the limits stipulated by (52). Therefore, $\mathbf{j}^{(1)}$, $\mathbf{P}_j{}^{\circ(1)}$, and $\mathbf{R}_j{}^{(1)}$ satisfy the equations [compare with (28), (39), and (54)]

$$-\frac{1}{n_e e_e}\frac{\partial}{\partial \mathbf{r}}p_e + \frac{1}{n_e e_e}\mathbf{j}\times\mathbf{B} + (\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \eta\mathbf{j} + \alpha\mathbf{R}_e, \tag{68}$$

$$-2p_j\langle\langle\partial\mathbf{v}/\partial\mathbf{r}\rangle\rangle^\circ + 2\frac{e_j}{m_j}\langle\langle\mathbf{P}_j{}^\circ\times\mathbf{B}\rangle\rangle^\circ + 2n_je_j\langle\langle\mathbf{u}_j(\mathbf{E}+\mathbf{v}\times\mathbf{B}+m_j\mathbf{g}/e_j)\rangle\rangle^\circ = \frac{\beta_j}{\tau_j}\mathbf{P}_j{}^\circ, \tag{69}$$

$$-\frac{5}{2}p_j\frac{k}{m_j}\frac{\partial T_j}{\partial \mathbf{r}} + \frac{e_j}{m_j}\mathbf{R}_j\times\mathbf{B} + \frac{e_j}{m_j}\mathbf{P}_j{}^\circ\cdot\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}+\frac{m_j}{e_j}\mathbf{g}\right) = \frac{\mathbf{R}_j}{\tau_j} + \gamma_j\mathbf{j}. \tag{70}$$

The double suffix notation on the β's and τ's has been dropped and the superscripts (1) denoting the first approximations have been left out.

From (20) and (21),

$$n_e e_e \mathbf{u}_e \simeq \mathbf{j},$$

$$n_i e_i \mathbf{u}_i \simeq \frac{Z_i m_e}{m_i}\mathbf{j},$$

hence due to the factor m_e/m_i the third term on the left-hand side of (69) can be ignored in the case of the ions. Therefore, the ion and electron equations "decouple" in the first and all subsequent approximations.

Although this in itself introduces a considerable simplification, the determinant for the electrons, corresponding to the $|\gamma_{\lambda\mu}|$ of Eq. (60), is still 12×12 (six components of the stress tensor: three for $\mathbf{j}$, and three for $\mathbf{R}_e$). Hence, rather than attempt to solve the set (68) to (70) *en bloc*, it is much simpler to proceed in steps. This is done in the next three subsections.

4.2.1. Stress Tensor

It is preferable to start with Eq. (69) since this involves no terms in $\mathbf{R}_j$. This can be written in the abbreviated form,

$$-2p_j\mathbf{e}_j^\circ+2\langle\langle \mathbf{P}_j^\circ\times\boldsymbol{\omega}_j\rangle\rangle^\circ=\frac{\beta_j}{\tau_j}\mathbf{P}_j^\circ, \qquad (71)$$

where $\boldsymbol{\omega}_j\equiv e_j/m_j\mathbf{B}$. In all subsequent equations it should be noted that $\omega_j=e_j/m_j|\mathbf{B}|\neq|\boldsymbol{\omega}_j|$. In particular, $\omega_e=-e/m_e|\mathbf{B}|$, while in general $|\boldsymbol{\omega}_j|$ is the cyclotron frequency.

$$\mathbf{e}_e^\circ\equiv\langle\langle\partial\mathbf{v}/\partial\mathbf{r}\rangle\rangle^\circ-\frac{1}{p_e}\langle\langle\mathbf{j}[\mathbf{E}+\mathbf{v}\times\mathbf{B}+m_e\mathbf{g}/e_e]\rangle\rangle^\circ,$$

$$\mathbf{e}_i^\circ\equiv\langle\langle\partial\mathbf{v}/\partial\mathbf{r}\rangle\rangle^\circ,$$

while $\beta_i\simeq\frac{3}{2}$; $\beta_e\simeq1$ [compare with Eqs. (47) and (48)].

If a Cartesian reference frame is so orientated that the magnetic field is in the x direction, then the solution of (71) is formally the same as that given by Chapman and Cowling.[6] On dropping the subscripts j, this is

$$P_{xx}^\circ=-2\mu e_{xx}^\circ,$$

$$P_{yy}^\circ=-\frac{2\mu}{1+(4\omega^2\tau^2/\beta^2)}\times\left\{e_{yy}^\circ+\tfrac{1}{2}(e_{yy}^\circ+e_{zz}^\circ)\frac{4\omega^2\tau^2}{\beta^2}+e_{yz}^\circ\frac{2\omega\tau}{\beta}\right\},$$

$$P_{zz}^\circ=-\frac{2\mu}{1+(4\omega^2\tau^2/\beta^2)}\times\left\{e_{zz}^\circ+\tfrac{1}{2}(e_{yy}^\circ+e_{zz}^\circ)\frac{4\omega^2\tau^2}{\beta^2}-e_{yz}^\circ\frac{2\omega\tau}{\beta}\right\},$$

$$P_{yz}^\circ=-\frac{2\mu}{1+(4\omega^2\tau^2/\beta^2)}\left\{e_{yz}^\circ+\tfrac{1}{2}(e_{zz}^\circ-e_{yy}^\circ)\frac{2\omega\tau}{\beta}\right\}=P_{zy}^\circ,$$

$$P_{xy}^\circ=P_{yx}^\circ=-\frac{2\mu}{1+(\omega^2\tau^2/\beta^2)}\left\{e_{xy}^\circ+\frac{\omega\tau}{\beta}e_{xz}^\circ\right\},$$

$$P_{xz}^\circ=P_{zx}^\circ=-\frac{2\mu}{1+(\omega^2\tau^2/\beta^2)}\left\{e_{xz}^\circ-\frac{\omega\tau}{\beta}e_{xy}^\circ\right\}.$$

μ is the coefficient of viscosity defined to be

$$\mu=p\tau/\beta. \qquad (72)$$

The limiting forms of these expressions when $\omega\tau\gg1$ are of special interest. Those for the nondiagonal components are still valid whether this corresponds to large ω or large τ (see the next section for $1/\tau=0$). However, as $\tau\rightarrow\infty$ (a collisionless plasma) the expressions for the diagonal components become meaningless and the exact Eqs. (39) must be used. In this case, subject to certain other assumptions, these equations when combined with the thermal energy equations give the well-known double adiabatic law.[9]

4.2.2. Heat-Flux Vector

As for the stress tensor it is convenient to write the equation for $\mathbf{R}_j$ in the abbreviated form

$$-\frac{5}{2}p_j\frac{k}{m_j}\mathbf{D}_j+\mathbf{R}_j\times\boldsymbol{\omega}_j=\frac{1}{\tau_j}\mathbf{R}_j, \qquad (73)$$

where

$$\mathbf{D}_i\equiv\frac{\partial T_i}{\partial\mathbf{r}}-\frac{e_i}{m_i}\frac{2m_i}{5p_ik}\mathbf{P}_i^\circ\cdot\left[\mathbf{E}+\mathbf{v}\times\mathbf{B}+\frac{m_i}{e_i}\mathbf{g}\right],$$

$$\mathbf{D}_e\equiv\frac{\partial T_e}{\partial\mathbf{r}}+\frac{2m_e}{5p_ek}\gamma_e\mathbf{j}-\frac{e_e}{m_e}\frac{2m_e}{5p_ek}\mathbf{P}_e^\circ\cdot\left[\mathbf{E}+\mathbf{v}\times\mathbf{B}+\frac{m_e}{e_e}\mathbf{g}\right].$$

Using the vector identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B}), \qquad (74)$$

the solution of (73) is straightforward. On ignoring the subscript j, the result is

$$\mathbf{R}=-[\lambda/(1+\omega^2\tau^2)]\{\mathbf{D}+\tau^2\boldsymbol{\omega}\cdot\mathbf{D}\boldsymbol{\omega}-\tau\boldsymbol{\omega}\times\mathbf{D}\}. \quad (75)$$

Contained in this solution are the usual properties of heat conduction parallel and perpendicular to a magnetic field, while the Righi-Leduc and Ettingshausen effects are accounted for by the transverse terms $\boldsymbol{\omega}\times\mathbf{D}$. λ is the coefficient of thermal conductivity defined by

$$\lambda=\tfrac{5}{2}p(k/m)\tau. \qquad (76)$$

The expression for (75) is relatively straightforward except for those components of $\mathbf{D}$ which involve $\mathbf{P}^\circ$. These could be eliminated by using the results of the previous subsection. There is, however, little value in doing this except in explicit cases. A simple example along these lines is given in the next subsection. Equation (75) is a vector equation, but for $\omega\tau\gg1$, the only significant components are

$$\mathbf{R}_{\parallel}=-\lambda\mathbf{D}_{\parallel}, \qquad (77)$$

$$\mathbf{R}_{\perp}=\tfrac{5}{2}p(k/m)(\boldsymbol{\omega}\times\mathbf{D}/\omega^2). \qquad (78)$$

As for the diagonal stress components, if $1/\tau\rightarrow0$, the expression for $\mathbf{R}_{\parallel}$ becomes meaningless and the more exact equations must be used. However, the expression for $\mathbf{R}_{\perp}$ is still valid, provided it is regarded as an average value over a Larmor period. This is easily seen by considering the more exact equation [compare with (54)]

[9] G. F. Chew, M. L. Goldberger, and F. E. Low, Proc. Roy. Soc. (London) **A236**, 112 (1956).

for $1/\tau=0$, namely,

$$(d\mathbf{R}/dt)-\mathbf{R}\times\boldsymbol{\omega}=\mathbf{D}.$$

If it is assumed that $\mathbf{v}=0$, then $\mathbf{D}$ is independent of $\mathbf{R}$. Let the magnetic field be in the x direction and $\mathbf{D}$ in the y direction, then provided D is essentially constant over a Larmor period,

$$R_z=-(D_y/\omega)+K\cos(\omega t+\epsilon),$$
$$R_y=-K\sin(\omega t+\epsilon),$$

where K and ϵ are constants. Therefore, $\mathbf{R}$, has a steady component in the z direction upon which there is superimposed another component which rotates with a frequency ω. This is analogous to the guiding center of a particle moving with a drift velocity $\mathbf{E}\times\mathbf{B}/B^2$. The effect of collisions is to damp out the periodic component.

4.2.3. Ohm's Law

The coupling between $\mathbf{j}$ and $\mathbf{R}_e$ in (68) is of prime importance. To illustrate the effects of this, two special cases are considered. In the first, the term in $\mathbf{P}_e^\circ$ in (70) is neglected; when this is done, substitution of the expression for $\mathbf{D}_e$ in (75) yields [after a certain amount of manipulation in which the vector identity (74) is again used]

$$\mathbf{R}_e=-\gamma_e\tau_e\mathbf{j}-\tau_e\boldsymbol{\omega}_e\times\mathbf{R}_D+\mathbf{R}_D-\frac{\lambda_e\tau_e^2}{1+\omega_e^2\tau_e^2}\boldsymbol{\omega}_e\cdot\frac{\partial T_e}{\partial\mathbf{r}}\boldsymbol{\omega}_e, \quad (79)$$

where

$$\mathbf{R}_D\equiv\frac{-\gamma_e\tau_e}{1+\omega_e^2\tau_e^2}\left\{\mathbf{j}\times\boldsymbol{\omega}_e\tau_e+\frac{\lambda_e}{\gamma_e\tau_e}\frac{\partial T_e}{\partial\mathbf{r}}\right\}. \quad (80)$$

Use of this expression for $\mathbf{R}_e$ in (68) gives the Ohm's law,

$$\eta_{\shortparallel}\mathbf{j}=\mathbf{A}+\mathbf{K}\times\mathbf{B}, \quad (81)$$

where

$$\mathbf{A}\equiv-\frac{1}{n_e e_e}\frac{\partial}{\partial\mathbf{r}}p_e+\mathbf{E}-\alpha\mathbf{R}_D+\frac{\kappa_T\tau_e^2}{1+\omega_e^2\tau_e^2}\boldsymbol{\omega}_e\cdot\frac{\partial T_e}{\partial\mathbf{r}}\boldsymbol{\omega}_e, \quad (82)$$

$$\mathbf{K}\equiv\frac{1}{n_e e_e}\mathbf{j}+\mathbf{v}-\alpha\left(\frac{\omega_e\tau_e}{B}\right)\mathbf{R}_D. \quad (83)$$

The use of both $\mathbf{B}$ and $\boldsymbol{\omega}_e$ is a matter of convenience, while it should be noted that $e_e=-e$ (the charge on an electron) and $\omega_e=-e/m_eB$. $\eta_{\shortparallel}$ is defined by

$$\eta_{\shortparallel}=\eta[1-(\alpha\gamma_e\tau_e/\eta)], \quad (84)$$

and as implied by the notation is the resistivity parallel to a magnetic field. That this is so can be seen by taking the scalar product of (81) with $\mathbf{B}$, namely,

$$\eta_{\shortparallel}=\mathbf{A}\cdot\mathbf{B}/\mathbf{j}\cdot\mathbf{B}, \quad (85)$$

where

$$\mathbf{A}\cdot\mathbf{B}=[-(1/n_e e_e)(\partial p_e/\partial\mathbf{r})+\mathbf{E}+\kappa_T(\partial T_e/\partial\mathbf{r})]. \quad (86)$$

From the definition (84) and the previously defined values of α, γ_e, τ_e, and η [Eqs. (30), (45), (55), and (29)], it may be shown that

$$\frac{\eta_{\shortparallel}}{\eta}=\frac{4(\sqrt{2}Z_i\ln\Lambda_{ie}+2\ln\Lambda_{ee})}{13\sqrt{2}Z_i\ln\Lambda_{ie}+8\ln\Lambda_{ee}}. \quad (87)$$

In particular for $Z_i=1$, $\eta/\eta_{\shortparallel}\simeq2$, while as $Z_i\to\infty$, $\eta/\eta_{\shortparallel}\to3.25$, these results being consistent with those of Spitzer.[8] The coefficient κ_T in (86) is called the thermal diffusion emf coefficient and is defined as

$$\kappa_T\equiv\alpha\lambda_e=\frac{15}{13}\frac{k}{e}\left[1+\frac{8}{13\sqrt{2}}\frac{1}{Z_i}\frac{\ln\Lambda_{ee}}{\ln\Lambda_{ie}}\right]^{-1}\simeq\frac{k}{e}. \quad (88)$$

If pressure and temperature gradients can be neglected, and there is no current flow in the direction of $\mathbf{j}\times\mathbf{B}$ (that is, no Hall currents), then η is the resistivity perpendicular to a strong magnetic field. On using (80), (82), and (83), subject to these restrictions, (81) becomes

$$\eta_{\shortparallel}\mathbf{j}=\mathbf{E}+\mathbf{v}\times\mathbf{B}+\frac{\alpha\gamma_e\tau_e^2}{1+\omega_e^2\tau_e^2}\left(\frac{\omega_e\tau_e}{B}\right)(\mathbf{j}\times\boldsymbol{\omega}_e)\times\mathbf{B}.$$

For $(\omega_e\tau_e)^2\gg1$, the identity (74), now gives

$$(\eta_{\shortparallel}+\alpha_e\gamma_e\tau_e)\mathbf{j}\equiv\eta\mathbf{j}=\mathbf{E}+\mathbf{v}\times\mathbf{B}+\frac{\alpha\gamma_e\tau_e}{\omega_e^2}(\mathbf{j}\cdot\boldsymbol{\omega}_e)\boldsymbol{\omega}_e,$$

which proves the preceding statement.

In general, the $\mathbf{R}_D$ term in $\mathbf{K}$, is responsible for the para- and diamagnetic properties of a plasma (the temperature gradient term in this expression is the Nernst effect).

Coming now to the second case, it is assumed that $\mathbf{B}=0$ and that all space derivatives are zero. Then Eqs. (68)–(70) are

$$\mathbf{E}=\eta\mathbf{j}+\alpha\mathbf{R}_e, \quad (89)$$

$$2\langle\langle\mathbf{Ej}\rangle\rangle^\circ=(1/\tau_e)\mathbf{P}_e^\circ, \quad (90)$$

$$(e_e/m_e)\mathbf{P}_e^\circ\cdot\mathbf{E}=(1/\tau_e)\mathbf{R}_e+\gamma_e\mathbf{j}. \quad (91)$$

For $\mathbf{E}\equiv(E_x,0,0)$, the solution of these equations is

$$j_x=\frac{E_x}{\eta_{\shortparallel}(1-\zeta^2E_x^2)};\quad j_y=j_z=0, \quad (92)$$

$$P_{xx}^\circ=\frac{\frac{4}{3}E_x^2\tau_e}{\eta_{\shortparallel}(1-\zeta^2E_x^2)};\quad P_{yy}^\circ=P_{zz}^\circ=-\tfrac{1}{2}P_{xx}^\circ;$$
$$P_{xy}^\circ=P_{yz}^\circ=P_{zx}^\circ=0, \quad (93)$$

$$R_x=\frac{-\gamma\tau_eE_x[1+(e/m_e)\frac{4}{3}(\tau_e/\gamma_e)E_x^2]}{\eta_{\shortparallel}(1-\zeta^2E_x^2)};$$
$$R_y=R_z=0, \quad (94)$$

where

$$\zeta^2\equiv\frac{4}{3}\frac{\alpha e}{\eta_{\shortparallel}m_e}\tau_e^2=\frac{4}{5}\frac{m_e}{kT_e}\frac{e^2}{m_e^2}\frac{\eta}{\eta_{\shortparallel}}\tau_e^2. \quad (95)$$

It is now convenient to define an acceleration time τ_a by the relationship

$$\tfrac{3}{2}kT_e=\tfrac{1}{2}(e^2/m_e)E_x^2\tau_a^2. \quad (96)$$

That is, τ_a is the time to accelerate an electron, in the absence of collisions, to an energy equal to $\frac{3}{2}kT_e$. (It should be noted that by definition T_e is measured with respect to the mean mass velocity. The true thermal temperature T_e' of the electrons is related to T_e by $\frac{3}{2}kT_e=\frac{3}{2}kT_e'+\frac{1}{2}m_eu_e^2$.)

By using (87) and (45), the term $(1-\zeta^2E_x^2)$ can now be written as

$$1-\zeta^2E_x^2=1-\theta^2(\tau_D^2/\tau_a^2),$$

where

$$\theta^2\equiv\frac{120}{\left[\left\{\sqrt{2}+\dfrac{2}{Z_i}\dfrac{\ln\Lambda_{ee}}{\ln\Lambda_{ie}}\right\}\left\{13\sqrt{2}+\dfrac{8}{Z_i}\dfrac{\ln\Lambda_{ee}}{\ln\Lambda_{ie}}\right\}\right]},$$

and τ_D is as defined by (31) or (33). For $Z_i=1$; $\theta^2\simeq\frac{4}{3}$, while for $Z_i\to\infty$, $\theta^2\to 60/13$. Therefore, according to (92), the effective resistivity tends to zero as $\tau_a\to\theta\tau_D$ and does, in fact, eventually become negative. This implies that the first approximations are no longer adequate, and the "inertial" terms, in particular the time derivatives, in Eqs. (28), (39), and (54), must be taken into account. This suggests, for this particular problem, that an adequate criterion for the complete "runaway" of electrons is

$$\theta^2(\tau_D^2/\tau_a^2)\geq 1,$$

or, numerically,

$$2.25\times 10^{23}(E_x^2T_e^2/Z_i^2n_e^2)\theta^2\geq 1 \quad (97)$$

(corresponding to $\ln\Lambda_{ie}\simeq 15$). E_x is in v/m and n_e is number/m^3. Comparison with Secs. 4.2.1 and 4.2.2 shows that (92)–(94) and the subsequent discussion still apply in the case of a magnetic field provided it is parallel to **E** and there are no space gradients.

4.3. Higher Approximations

In general, the higher approximations are extremely complicated. However, to illustrate what is involved, two relatively simple examples of possible practical value are considered. In both of these, it is assumed that the mean mass velocity, time derivatives, forces, and currents are all zero. With these assumptions, Eqs. (39) and (54) are

$$\tfrac{4}{5}\langle\langle(\partial/\partial\mathbf{r})\mathbf{R}\rangle\rangle^\circ-2\langle\langle\mathbf{P}^\circ\times\boldsymbol{\omega}\rangle\rangle^\circ=-(\beta/\tau)\mathbf{P}^\circ, \quad (98)$$

$$\frac{7}{2}\frac{k}{m}\mathbf{P}^\circ\cdot\frac{\partial T}{\partial\mathbf{r}}+\frac{k}{m}T\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{P}^\circ+\frac{5}{2}\frac{k}{m}P\frac{\partial T}{\partial\mathbf{r}}-\mathbf{R}\times\boldsymbol{\omega}=-\frac{\mathbf{R}}{\tau}. \quad (99)$$

The first approximation to $\mathbf{P}^\circ$ is zero, while $\mathbf{R}^{(1)}$ is given by (75). The higher approximations satisfy the equations [compare with (67)]

$$-\frac{\beta}{\tau}\mathbf{P}^{\circ(n+1)}+2\langle\langle\mathbf{P}^{\circ(n+1)}\times\boldsymbol{\omega}\rangle\rangle^\circ=\tfrac{4}{5}\langle\langle(\partial/\partial\mathbf{r})\mathbf{R}^{(n)}\rangle\rangle^\circ, \quad (100)$$

$$-\frac{1}{\tau}\mathbf{R}^{(n+1)}+\mathbf{R}^{(n+1)}\times\boldsymbol{\omega}=\frac{7}{2}\frac{k}{m}\mathbf{P}^{\circ(n)}\cdot\frac{\partial T}{\partial\mathbf{r}}+\frac{k}{m}T\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{P}^{\circ(n)}. \quad (101)$$

Since $\mathbf{P}^{\circ(1)}$ is zero, it follows from (100) and (101) that

$$\mathbf{P}^{\circ(n)}=0 \text{ for } n \text{ odd};\quad \mathbf{R}^{(n)}=0 \text{ for } n \text{ even}.$$

The first case to be considered is

$$\partial T/\partial\mathbf{r}\equiv(\partial T/\partial x,0,0);\quad \boldsymbol{\omega}\equiv(0,0,0).$$

The only nonzero components are R_x, P_{xx}°, P_{yy}°, and P_{zz}°. At all levels

$$P_{yy}^\circ=P_{zz}^\circ=-\tfrac{1}{2}P_{xx}^\circ.$$

By using (75), (100), and (101), it may be confirmed that to the third approximation,

$$P_{xx}^{\circ(1)}=0;\quad P_{xx}^{\circ(2)}=-\frac{4}{3}\frac{k}{m}\frac{\tau}{\beta}\frac{\partial}{\partial x}\left(p\tau\frac{\partial T}{\partial x}\right);\quad P_{xx}^{\circ(3)}=0, \quad (102)$$

$$R_x^{(1)}=-\frac{5}{2}\frac{k}{m}p\tau\frac{\partial T}{\partial x};\quad R_x^{(2)}=0,$$

$$R_x^{(3)}=\frac{14}{3}\left(\frac{k}{m}\right)^2\frac{\tau^2}{\beta}\frac{\partial}{\partial x}\left(p\tau\frac{\partial T}{\partial x}\right)\frac{\partial T}{\partial x}+\frac{4}{3}\left(\frac{k}{m}\right)^2\frac{\tau}{\beta}T\frac{\partial}{\partial x}\left\{\tau\frac{\partial}{\partial x}\left(p\tau\frac{\partial T}{\partial x}\right)\right\}. \quad (103)$$

Since τ and p are simple algebraic functions of T, it is obvious that P_{xx}° and R_x can be expressed as a power series in τ. $R_{xx}^{(3)}$ involves derivatives of T up to the order and degree of three. In particular, one such term is

$$(14/3)(k/m)^2(\tau^2/\beta)p(\partial T/\partial x)^2\partial\tau/\partial x,$$

which, since $\tau\sim T^{\frac{3}{2}}/n$, includes

$$[R_x^{(3)}]_3=7(k/m)^2(\tau^3/\beta)(p/T)(\partial T/\partial x)^3. \quad (104)$$

For rapid convergence $R_x^{(3)}$ should be small compared with $R_x^{(1)}$. Since (104) is likely to be a dominant term in (103), consider the ratio of $[R_x^{(3)}]_3$ to $R_x^{(1)}$. This is

$$[R_x^{(3)}]_3/R_x^{(1)}=(28/15)(l^2/T^2)(\partial T/\partial x)^2, \quad (105)$$

where $l^2=[\frac{3}{2}(kT/m)]^{\frac{1}{2}}\tau$ is the mean free path. Thus, rapid convergence requires little change over a mean free path.

The second case to be considered is for

$$\partial T/\partial \mathbf{r} \equiv (0, \partial T/\partial y, 0),$$
$$\boldsymbol{\omega} \equiv (\omega_x, 0, 0).$$

From the first approximation (75), it is seen that the "expansion" times are of a hybrid nature. However, for $(\omega\tau) \gg 1$, the only significant component of $\mathbf{R}^{(1)}$ is

$$R_z^{(1)} = \tfrac{5}{2}(k/m)p(1/\omega_x)\partial T/\partial y.$$

On using this expression for $\mathbf{R}^{(1)}$ in (100), the only terms of $\mathbf{P}^{\circ(2)}$ of any magnitude are (compare with Sec. 4.2.1)

$$P_{yy}^{\circ(2)} = -\frac{k}{2m}\frac{1}{\omega_x}\frac{\partial}{\partial y}\left(\frac{p}{\omega_x}\frac{\partial T}{\partial y}\right) = -P_{zz}^{\circ(2)}.$$

Consequently, the solution for $R_z^{(3)}$ is given by (103) when τ is replaced by $1/\omega_x$, the expression multiplied by $-\frac{3}{8}$ and $\beta = 1$. Since $p \propto T$, a convergence criterion similar to (105) follows, namely,

$$1 \gg (7/15)(r^2/T^2)(\partial T/\partial y)^2,$$

where $r = [\frac{3}{2}(kT/m)]^{\frac{1}{2}}(1/\omega_x)$ is the Larmor radius. In these higher approximations the gradient of the magnetic field also appears and this may be dominant. For instance, such a term in $R_z^{(3)}$ is

$$\frac{1}{2}\frac{k}{m}\frac{1}{\omega_x}\frac{kT}{m}p\frac{\partial T}{\partial y}\left(\frac{\partial}{\partial y}\frac{1}{\omega_x}\right)^2.$$

The ratio of this term to $R_z^{(1)}$ is

$$(2/15)(r^2/\omega_x^2)(\partial\omega_x/\partial y)^2.$$

That is, as anticipated, the magnetic field should vary little over a Larmor radius. Similar arguments apply to density gradients and all such primary variables, while time dependence can be taken into account in a like manner.

5. DISTRIBUTION FUNCTION

Expression (8) for the distribution function is an approximation to the complete expansion in terms of multidimensional Hermite polynomials. By considering more terms in this series, greater accuracy can be achieved, but only at the expense of greater complexity. It might be expected that the equations deduced by using (8) only apply near true equilibrium states, hence the inclusion of "thermal" in the title of this paper. There is, however, no rigorous mathematical argument by which the range of validity of these equations can be judged and a certain amount of physical reasoning is required.

With regard to the distribution function itself, it is clear that it must be finite and nonnegative, at least over a considerable range of ξ. Take, for example, a problem in which there is an axis of symmetry in the x direction and along which there is applied a force [compare with Eqs. (89)–(91)]. The only nonzero components of the vectors $\mathbf{u}$, $\mathbf{R}$, and the tensor $\mathbf{P}^\circ$ are $\mathbf{u}_x$, R_x, P_{xx}°, P_{yy}°, and P_{zz}°, while again from the symmetry

$$P_{yy}^\circ = P_{zz}^\circ = -\tfrac{1}{2}P_{xx}^\circ.$$

In this case, expression (8) becomes,

$$f = f^{(0)}\{1 + (2\alpha)^{\frac{1}{2}}u_x\xi_x + (1/2p)P_{xx}^\circ(\xi_x^2 - \tfrac{1}{2}\xi_y^2 - \tfrac{1}{2}\xi_z^2) + (2/p)(2\alpha)^{\frac{1}{2}}R_x\xi_x(\xi^2 - 5)\}. \quad (106)$$

Therefore, in particular, when $\xi_x = 0$,

$$f = f^{(0)}\{1 - (1/4p)P_{xx}^\circ(\xi_y^2 + \xi_z^2)\},$$

which is valid only if

$$1 > (1/4p)P_{xx}^\circ(\xi_y^2 + \xi_z^2).$$

Similar arguments apply to other values of ξ_x.

If now (106) is expressed in terms of the peculiar velocity $\mathbf{w}$, instead of ξ, a useful and interesting variant for this distribution function can be obtained. If w_x is put equal to $w\cos\theta$, where θ is the polar angle in velocity space, this is

$$f = f^{(0)}\{P_0(\cos\theta) + \psi P_1(\cos\theta) + \chi P_2(\cos\theta)\}. \quad (107)$$

The P_n are Legendre polynomials in $\cos\theta$, while

$$\psi \equiv w[2\alpha u_x + \tfrac{4}{5}(\alpha/p)R_x(\alpha w^2 - \tfrac{5}{2})],$$
$$\chi \equiv (\alpha/p)P_{xx}^\circ w^2$$

could be regarded as the first term in, say, expansions of Sonine polynomials. The "variant," (107), is particularly useful for determining average values over θ within the shell w, $w+dw$. It may, for instance, be readily confirmed that

$$\langle w_x\rangle_\theta = \tfrac{1}{3}w\psi.$$

In view of (107) and the discussion given in Sec. 4, the relationship between the 13-moment approximation of Grad and other types of expansions and approximations in common use can be understood.

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DISCUSSION

Session Reporter: W. B. RIESENFELD

B. Lehnert, *The Royal Institute of Technology, Stockholm, Sweden:* The second of your momentum equations corresponds to Ohm's law. Do you get in this equation a term which is due to thermal conduction as well?

B. S. Liley: The second moment equation involves the total heat flux vector, which in turn can be related to temperature gradients.

J. M. Burgers, *University of Maryland, College Park, Maryland:* It is the type of approximation assumed for the distribution function which brings the necessity of having the heat-flow terms go with the diffusion terms on which the electric current depends.

J. E. McCune, *Aeronautical Research Associates, Princeton, New Jersey:* At what point did your discussion stop treating a multicomponent gas and start dealing with a fully ionized gas?

B. S. Liley: I intended all the explicit moment equations which I presented to describe a fully ionized gas. Actually the formalism for incomplete ionization is much the same. The nature of the numerical coefficients changes, however.