Irreversible Processes in Plasmas in a Strong Magnetic Field

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GENERAL INTRODUCTION

THIS paper discusses systematically irreversible processes in plasma in a magnetic field without attempting laborious numerical calculations. The irreversible processes can be classified with regard to the tensor rank of the irreversible flow under consideration. Thus, in the case of relaxation between ion and electron temperatures the flow is a scalar. This simplest case is treated in Sec. I. For electric and thermal conductions, on the other hand, flows are vectors. These phenomena are discussed in Sec. II. The theory of irreversible processes may be considered from the phenomenological (or macroscopic) point of view and from the molecular (or microscopic) point of view. In Sec. I a microscopic theory of the temperature relaxation in a fully ionized gas is developed; in this case the macroscopic theory is trivial. In Sec. II the discussion is developed from both macroscopic and microscopic points of view.

The phenomenological theory in this paper is an application of the thermodynamics of irreversible processes. In the absence of the magnetic field, or for weak fields at most, the phenomenological coefficients in linear relations for irreversible processes are scalars. This case has been treated by Maecker and Peters[1] and by Kihara.[2] In the presence of a strong magnetic field these coefficients are tensors whose characteristic features are discussed here.

The main purpose of our microscopic theory is an application of the force-correlation formalism to a gaseous plasma. When the mean gyration radius of electrons is larger than the Debye length, the microscopic theory can be developed by use of the Boltzmann equation; in fact, many studies have been made along this line.[3–6] Here the electron gyration radius can be smaller than the Debye length, and the Boltzmann equation is not used.

I. ION-ELECTRON TEMPERATURE RELAXATION

1. Introduction

Since the mass ratio of an ion to an electron is far from unity, energy transfer between ion and electron is much smaller than that between ions or between electrons. Hence, the ion temperature T_1 and the electron temperature T_2 are often different even when the velocity distributions are both uniform and Maxwellian. The rate R of this relaxation is defined by

$$(d/dt)(T_2-T_1)=-(T_2-T_1)R, \tag{1}$$

where t denotes time.

The object of Sec. I is to discuss the rate of relaxation in a fully ionized gas in a magnetic field. Both the gas and the magnetic field are assumed to be uniform in space. We consider a gas which is composed of one type of positive ion and electrons. The particle mass, particle charge, and number density of the ion are denoted by m_1, Ze, and n_1, respectively; those for electrons are expressed by m_2, $-e$, and n_2. (Charge neutrality, $Zn_1=n_2$, is assumed and the inequality $Z^2\ll m_1/m_2$ is used.) Here we consider the case where the mean ion velocity is much smaller than the mean electron velocity:

$$(T_1/m_1)^{\frac{1}{2}}\ll(T_2/m_2)^{\frac{1}{2}}. \tag{2}$$

The long-range many-body interaction which results from the Coulomb potential r^{-1} can be eliminated by the use of a shielded interparticle potential, say, $r^{-1}\exp(-\kappa_{12}r)$. Although the shielding constant κ_{12} can not be determined exactly, it is of order of the Debye characteristic constant l_D^{-1}, l_D being the Debye length. At high temperatures or low densities, i.e., for weak interactions, the rate or relaxation does not depend sensitively on the constant κ_{12}. We treat this case.

2. Formulation in Terms of the Force Correlation

The particle velocities of the ion and electron are denoted by $\mathbf{c}_1$ and $\mathbf{c}_2$, respectively, with $|\mathbf{c}_1|\equiv c_1$ and $|\mathbf{c}_2|\equiv c_2$. The average value of $\frac{1}{3}m_1c_1^2$ is kT_1, and the time rate of change of kT_1 is given by the average value of the time rate of change of $\frac{1}{3}m_1c_1^2$:

$$(d/dt)kT_1=\tfrac{1}{3}m_1(1/\Delta t)\langle|\mathbf{c}_1+\Delta\mathbf{c}_1|^2-c_1^2\rangle_{\text{av}},$$

k being the Boltzmann constant, or

$$(d/dt)kT_1=\tfrac{1}{3}m_1(1/\Delta t)\langle(2\mathbf{c}_1+\Delta\mathbf{c}_1)\cdot\Delta\mathbf{c}_1\rangle_{\text{av}}. \tag{3}$$

Here, $\Delta\mathbf{c}_1$ indicates the increment in ion velocity due to the interaction with surrounding electrons in a time interval Δt; Δt is taken to be much larger than the time characteristic of fluctuation of the interaction,

[1] M. Maecker and Th. Peters, Z. Physik **144**, 586 (1956).
[2] T. Kihara, J. Phys. Soc. Japan **14**, 128 (1959).
[3] T. G. Cowling, Proc. Roy. Soc. (London) **A183**, 453 (1945).
[4] R. Landshoff, Phys. Rev. **76**, 904 (1949).
[5] E. S. Fradkin, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 1176 (1957).
[6] S. I. Braginskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 459 (1957).

yet it must be much smaller than both the ion gyration period and the relaxation time R^{-1}. $\langle\quad\rangle_{av}$ denotes the average with respect to the ion velocity and the interaction with surrounding electrons.

When the ion temperature T_1 is much smaller than the electron temperature T_2, the average value of $\mathbf{c}_1\cdot\Delta\mathbf{c}_1$ can be neglected in comparison with the average value of $|\Delta\mathbf{c}_1|^2$; hence,

$$(d/dt)kT_1=\tfrac{1}{3}m_1(1/\Delta t)\langle|\Delta\mathbf{c}_1|^2\rangle_{av}\quad\text{for}\quad T_1\ll T_2. \tag{4}$$

This is essentially diffusion due to the Brownian motion in the velocity space.[7]

The relation (4) can be generalized into the form

$$(d/dt)kT_1=\tfrac{1}{3}m_1(1/\Delta t)\langle|\Delta\mathbf{c}_1|^2\rangle_{av}[(T_2-T_1)/T_2], \tag{5}$$

which holds for T_1 satisfying (2). The quantity $|\Delta\mathbf{c}_1|^2$ averaged with respect to the interaction with surrounding electrons is independent of $\mathbf{c}_1$; but the average value of $\Delta\mathbf{c}_1$ depends linearly upon $\mathbf{c}_1$. For constant T_2, therefore, $\langle|\Delta\mathbf{c}_1|^2\rangle_{av}$ is independent of T_1, and $\langle\mathbf{c}_1\cdot\Delta\mathbf{c}_1\rangle_{av}$ is proportional to T_1. On taking account of the fact that $\langle(2\mathbf{c}_1+\Delta\mathbf{c}_1)\cdot\Delta\mathbf{c}_1\rangle_{av}$ vanishes for $T_1=T_2$, we obtain

$$2\langle\mathbf{c}_1\cdot\Delta\mathbf{c}_1\rangle=-(T_1/T_2)\langle|\Delta\mathbf{c}_1|^2\rangle_{av}.$$

Then (5) follows from (3). (Under our assumption, the dependence of the shielding constant on the ion temperature has been neglected.)

By use of the relationship

$$(d/dt)(n_1T_1+n_2T_2)=0,$$

the rate of relaxation R defined by (1) is calculated to

$$R=\tfrac{1}{3}(n_1+n_2)(m_1/kT_2n_2)(1/\Delta t)\langle|\Delta\mathbf{c}_1|^2\rangle_{av}. \tag{6}$$

The increment in ion velocity during Δt is given by

$$\Delta\mathbf{c}_1=\frac{1}{m_1}\int_0^{\Delta t}\mathfrak{F}(\tau)d\tau,$$

where $\mathfrak{F}(\tau)$ is the force acting on the ion at time τ. Hence,

$$|\Delta\mathbf{c}_1|^2=\frac{1}{m_1^2}\int_0^{\Delta t}\int_0^{\Delta t}\mathfrak{F}(\tau)\cdot\mathfrak{F}(\tau')d\tau d\tau'$$
$$=\frac{1}{m_1^2}\int_0^{\Delta t}\int_{-\tau}^{\Delta t-\tau}\mathfrak{F}(\tau)\cdot\mathfrak{F}(\tau+t)dtd\tau.$$

Since Δt is much larger than the continuance of the force correlation, we have

$$\frac{1}{\Delta t}\langle|\Delta\mathbf{c}_1|^2\rangle_{av}=\frac{1}{m_1^2}\int_{-\infty}^{\infty}\langle\mathfrak{F}(0)\cdot\mathfrak{F}(t)\rangle_{av}dt. \tag{7}$$

The force acting on an ion is a sum of contributions from all the surrounding electrons

$$\mathfrak{F}(t)=\textstyle\sum_i \mathbf{F}^i(t),$$

$\mathbf{F}^i$ indicating the force due to ith electron. The long-range many-body interaction has been eliminated by the use of a shielded interparticle potential, and the force due to an electron is independent of force due to other electrons, namely,

$$\langle\mathfrak{F}(0)\cdot\mathfrak{F}(t)\rangle_{av}=\langle\textstyle\sum_i\mathbf{F}^i(0)\cdot\mathbf{F}^i(t)\rangle_{av},$$

where the crossterms vanish. We have, therefore,

$$\int_{-\infty}^{\infty}\langle\mathfrak{F}(0)\cdot\mathfrak{F}(t)\rangle_{av}dt=\int\int\int\mathbf{F}(0)\cdot\mathbf{F}(t)f_2d\mathbf{c}_2d\mathbf{r}dt, \tag{8}$$

where $\mathbf{F}(t)$ is force due to an electron at time t, and f_2 is the velocity distribution function for electrons,

$$f_2=n_2(m_2/2\pi kT_2)^{\frac{3}{2}}\exp(-m_2c_2^2/2kT_2). \tag{9}$$

Here $\mathbf{r}$ and $\mathbf{c}_2$ are position and velocity of the electron at time 0 in the absence of the interaction.

Thus, we obtain, from (6)–(8), the basic relation

$$R=\frac{1}{3}\frac{n_1+n_2}{kT_2n_2}\frac{1}{m_1}\int\int\int\mathbf{F}(0)\cdot\mathbf{F}(t)f_2d\mathbf{c}_2d\mathbf{r}dt. \tag{10}$$

3. Correspondence with the Kinetic Theory

This section is devoted to the transformation of (10), in the absence of a magnetic field, into a form which is familiar in the kinetic theory of gases.

By taking the origin of coordinates at the position of ion and taking z axis in the direction of initial velocity $\mathbf{c}_2$ of an encountering electron, let us introduce the cylindrical coordinate (b, ϕ, z); then

$$d\mathbf{r}\equiv bdbd\phi dz\ (0\le b,\ 0\le\phi<2\pi,\ -\infty<z<\infty),$$

where b is the impact parameter. It follows from (10) that

$$R=\frac{2\pi}{3}\frac{n_1+n_2}{kT_2n_2}\frac{1}{m_1}\int\int\int\mathbf{F}(0)\cdot\mathbf{F}(t)dtdzbdbf_2d\mathbf{c}_2. \tag{11}$$

Now, $-\mathbf{F}(t)$ is the force acting on the encountering electron, and we have

$$\left|\int_{-\infty}^{\infty}\mathbf{F}(t)dt\right|=2m_2c_2\sin(\theta/2),$$

θ being the angle of diffraction of the electron orbit. Further,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{F}(0)\cdot\mathbf{F}(t)dtdz=c_2\left|\int_{-\infty}^{\infty}\mathbf{F}(t)dt\right|^2$$
$$=2m_2^2c^3{}_2(1-\cos\theta).$$

[7] For example, S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

Hence,

$$R=\frac{(4\pi)^2}{3}\frac{n_1+n_2}{kT_2n_2}\frac{m_2{}^2}{m_1}\int_0^\infty\int_0^\infty(1-\cos\theta)c_2{}^5f_2bdbdc_2. \quad (12)$$

The expression (12) can also be derived directly from (6). Moreover, (12) can be obtained from the Boltzmann equation[8] without using the basic relation (6).

For the effective interparticle potential

$$Ze^2r^{-1}\exp(-\kappa_{12}r) \quad (13)$$

between an ion and an electron, the integral in (12) has been calculated by Kihara[8] and Liboff.[9] With such results, we can write

$$R=R^0\ln(3kT_2/4Ze^2\kappa_{12}), \quad (14)$$

where

$$R^0\equiv(8/3)(n_1+n_2)(m_2/m_1)(2\pi kT_2/m_2)^{\frac{1}{2}}(Ze^2/kT_2)^2. \quad (15)$$

In (14) an irrational number close to $\frac{3}{4}$ has been replaced by $\frac{3}{4}$ in the argument of the logarithm.

The expression (14) is exact as a weak-interaction asymptote for $kT_2/Ze^2\kappa_{12}\gg1$, so far as the effective potential (13) is assumed. When the plasma is sufficiently hot and dilute, then $\ln(kT_2/Ze^2\kappa_{12})$ is much larger than unity and the formula (14) can be approximated by the well-known expression

$$R=R^0\ln(kT_2l_D/Ze^2), \quad (16)$$

in which l_D is the Debye length. To obtain the limiting expression (16), it is sufficient to make integration of (11) or (12) over the domain of "weak interaction without correlation" $Ze^2/kT_2\lesssim b\lesssim l_D$. The integral $\int\int\mathbf{F}(0)\cdot\mathbf{F}(t)dtdz$ in this domain becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{b}{(b^2+z^2)^{\frac{3}{2}}}\frac{b}{[b^2+(z+c_2t)^2]^{\frac{3}{2}}}dtdz=\frac{4}{b^2c_2},$$

and, after integration with respect to $\mathbf{c}_2$, (11) is reduced to

$$R=R^0\int_{Ze^2/kT_2}^{l_D}\frac{db}{b},$$

which is equal to (16). Here, this kind of approximation is used throughout.

4. Relaxation in a Magnetic Field

Returning to (10) let us discuss the rate of relaxation in a magnetic field. We assume that the mean electron gyration radius r_g is larger (but not necessarily much larger) than the limit of weak interaction Ze^2/kT_2. For our purpose it is sufficient to integrate (10) over the domain of "weak interaction without correlation." This domain is characterized by

$$t_s\lesssim|t|\lesssim t_p,$$

where t_p is the period of plasma oscillation, and t_s is the mean time of passage through a sphere of strong interaction, namely,

$$t_p\approx\left(\frac{m_2}{kT_2}\right)^{\frac{1}{2}}l_D \quad\text{and}\quad t_s\approx\left(\frac{m_2}{kT_2}\right)^{\frac{1}{2}}\frac{Ze^2}{kT_2}. \quad (17)$$

Thus, when coordinate origin is taken at position of the ion, $\mathbf{F}(0)$ is simply equal to $Ze^2\mathbf{r}/|\mathbf{r}|^3$, and $\mathbf{F}(t)$ is given by

$$\mathbf{F}(t)=Ze^2[\mathbf{r}+\mathbf{s}(t,\mathbf{c}_2)]/|\mathbf{r}+\mathbf{s}(t,\mathbf{c}_2)|^3,$$

where $\mathbf{s}(t,\mathbf{c}_2)$ is the increment in the position vector during time t when the electron velocity is $\mathbf{c}_2$ at $t=0$:

$$\mathbf{s}(0,\mathbf{c}_2)\equiv0, \quad [d\mathbf{s}(t,\mathbf{c}_2)/dt]_{t=0}\equiv\mathbf{c}_2.$$

First, the relation

$$\int\frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\frac{\mathbf{r}+\mathbf{s}}{|\mathbf{r}+\mathbf{s}|^3}d\mathbf{r}=\frac{4\pi}{|\mathbf{s}|} \quad (18)$$

is proved. The volume integral on the left-hand side of (18),

$$-\int\frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\nabla\frac{1}{|\mathbf{r}+\mathbf{s}|}d\mathbf{r}$$

can be transformed into an integral

$$-\int\frac{\mathbf{r}\cdot\mathbf{n}}{|\mathbf{r}|^3|\mathbf{r}+\mathbf{s}|}ds$$

over three spherical surfaces—a small sphere at $\mathbf{r}=0$, a small sphere at $\mathbf{r}=-\mathbf{s}$, and a large sphere with the center near the origin; $\mathbf{n}$ is the normal on the surfaces from the domain of original volume integral. The integration over the first small spherical surface gives the right-hand side of (18), those over the other two vanishing.

On denoting by $c_{\parallel}$ and $c_{\perp}$ the parallel and perpendicular components of the electron velocity with respect to the magnetic field, and by ω_2 the gyration frequency of the electron, we have

$$|\mathbf{s}(t,\mathbf{c}_2)|=[2(c_{\perp}/\omega_2)^2(1-\cos\omega_2t)+c_{\parallel}{}^2t^2]^{\frac{1}{2}}$$

and

$$\int\frac{1}{|\mathbf{s}(t,\mathbf{c}_2)|}f_2d\mathbf{c}_2=2n_2\left(\frac{m_2}{2\pi kT_2}\right)^{\frac{1}{2}}\frac{\tanh^{-1}Y}{Y}\frac{1}{|t|}, \quad (19)$$

where

$$Y\equiv\left[1-\frac{2(1-\cos y)}{y^2}\right]^{\frac{1}{2}}, \quad y\equiv\omega_2t. \quad (20)$$

[8] T. Kihara, J. Phys. Soc. Japan 14, 402 (1959).
[9] R. L. Liboff, Phys. Fluids 2, 40 (1959).

Thus, (10) is calculated to be, with R^0 defined by (15),

$$R=R^0\left[\ln(t_p/t_s)+\int_0^{\omega_2 t_p}\left(\frac{\tanh^{-1}Y}{Y}-1\right)\frac{dy}{y}\right], \quad (21)$$

where the second term in the bracket is an increasing function of $\omega_2 t_p$. The first term, $\ln(t_p/t_s)$, gives the same result as (16).

The integral in (21) takes the asymptotic value $\frac{1}{2}[\ln(\omega_2 t_p)]^2$ for large $\omega_2 t_p$, and the expression (21) is essentially equivalent to

$$\frac{R}{R^0}=\begin{cases}\ln(t_p/t_s) & \text{for} \quad \omega_2 t_p \lesssim 1 \\ \ln(t_p/t_s)+\frac{1}{2}[\ln(\omega_2 t_p)]^2 & \text{for} \quad \omega_2 t_p \gtrsim 1.\end{cases} \quad (22)$$

Here $\omega_2 t_p$ is equal to the ratio of the Debye shielding length to the mean electron gyration radius. The limiting value at a very strong field,

$$R=R^0\tfrac{1}{2}[\ln(\omega_2 t_p)]^2,$$

was previously obtained by one of the present authors[10] in an elementary manner.

In conclusion, the rate of relaxation increases with increasing magnetic field according to (21) or (22).

II. ELECTRIC AND THERMAL CONDUCTIONS

5. Phenomenological Foundation

The thermodynamics of irreversible processes, which has been developed since Onsager-Casimir's reciprocity theorem, is applied, in Secs. 5 and 6, to plasmas in a strong magnetic field. A fundamental assumption is that the system is not far from thermal equilibrium. We consider plasmas composed of several species of component fluids in which neither ionization nor recombination occurs nor radiation plays any role. We suppose in Sec. II that no temperature difference exists between plasma components and that the viscosity is negligible.

Let n_j and $\mathbf{v}_j$ be, respectively, the number density and the flow velocity of jth component. Then the equation of continuity for the component is

$$\partial n_j/\partial t+\operatorname{div}(n_j\mathbf{v}_j)=0. \quad (23)$$

Denoting by $n_j m_j$ and $n_j e_j$ the mass and the charge densities of the jth component, we adopt the equation of motion

$$\sum n_j m_j\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\text{grad}\right)\mathbf{v} =\sum n_j\left[e_j\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_j\times\mathbf{B}\right)+m_j\mathbf{g}\right]-\text{grad}P$$

for the plasma as a whole. Here $\mathbf{v}$ is the average flow

$$\mathbf{v}=\sum n_j m_j \mathbf{v}_j/\sum n_j m_j \quad (\sum\equiv\sum_j),$$

$\mathbf{E}$ and $\mathbf{B}$ the electric and the magnetic fields, $\mathbf{g}$ the gravitational acceleration, and P the pressure. (The Gaussian system of units is used, and $c=3\times10^{10}$ cm/sec.) The equation of motion can be written in the form

$$\sum n_j\mathbf{F}_j^*-\text{grad}P=0, \quad (24)$$

where

$$\mathbf{F}_j^*=e_j(\mathbf{E}+c^{-1}\mathbf{v}_j\times\mathbf{B})+m_j[\mathbf{g}-(\partial/\partial t+\mathbf{v}\cdot\text{grad})\mathbf{v}]. \quad (25)$$

Following Prigogine[11] we "decompose" the pressure gradient into

$$\text{grad}P=\sum n_j(\text{grad}\mu_j)_T.$$

Here, μ_j is the chemical potential per particle of the jth species and

$$(\text{grad}\mu_j)_T\equiv\text{grad}\mu_j-(\partial\mu_j/\partial T)_{P,\,\text{composition}}\ \text{grad}T,$$

T being the temperature. The equation of motion (24) then becomes

$$\sum n_j[\mathbf{F}_j^*-(\text{grad}\mu_j)_T]=0. \quad (26)$$

On assuming that the plasma is not very far from thermal equilibrium, we can adopt the thermodynamics of irreversible processes, which gives the rate of entropy production $(\partial s/\partial t)_{\text{irr}}$ due to irreversible processes in the form[2]

$$T(\partial s/\partial t)_{\text{irr}}=-\mathbf{q}\cdot\text{grad}\ln T+\sum n_j\mathbf{v}_j\cdot[\mathbf{F}_j^*-(\text{grad}\mu_j)_T]. \quad (27)$$

Here $\mathbf{q}$ is the heat flow, sometimes called reduced heat flow, due to conduction.

By virtue of (26), $\mathbf{v}_j$ and $\mathbf{F}_j^*$ in (27) can be replaced, respectively, by $\mathbf{v}_j-\mathbf{v}$ and

$$\mathbf{F}_j\equiv e_j(\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B})+m_j[\mathbf{g}-(\partial/\partial t+\mathbf{v}\cdot\text{grad})\mathbf{v}], \quad (28)$$

namely,

$$T(\partial s/\partial t)_{\text{irr}}=-\mathbf{q}\cdot\text{grad}\ln T+\sum n_j(\mathbf{v}_j-\mathbf{v})\cdot[\mathbf{F}_j-(\text{grad}\mu_j)_T]. \quad (29)$$

The entropy production (27) or (29) indicates that forces conjugate to fluxes $\mathbf{q}$ and $\mathbf{v}_j$ are, respectively,

$$-\text{grad}\ln T \quad \text{and} \quad n_j[\mathbf{F}_j^*-(\text{grad}\mu_j)_T].$$

With forces $\mathbf{X}_\alpha$ conjugate to fluxes $\mathbf{Y}_\alpha$, the entropy production (27) or (29) is thus expressed in the form

$$T(\partial s/\partial t)_{\text{irr}}=\sum_\alpha \mathbf{X}_\alpha\mathbf{Y}_\alpha. \quad (30)$$

When a linear relation

$$\mathbf{X}_\alpha=\sum_\beta L_{\alpha\beta}\mathbf{Y}_\beta \quad (31)$$

is assumed among forces and fluxes, the so-called phenomenological coefficients $L_{\alpha\beta}$ are tensors. For the

[10] T. Kihara, J. Phys. Soc. Japan 14, 1751 (1959).

[11] I. Prigogine, *Étude Thermodynamique des Phénomènes irréversibles* (Liège, Ed. Desoer, 1947), p. 101.

choice (31) of the linear relation the Onsager-Casimir reciprocity[12] takes the simplest form

$$L_{\alpha\beta}(\mathbf{B})=L_{\beta\alpha}^{\dagger}(-\mathbf{B}). \tag{32}$$

This relation means that the tensor $L_{\alpha\beta}$ in the magnetic field $\mathbf{B}$ is equal to the transposed tensor $L_{\beta\alpha}^{\dagger}$ in the field $-\mathbf{B}$.

Let us take in the plasma a rectangular coordinate system with the z axis in the direction of $\mathbf{B}$. Then each tensor is of the form

$$L_{\alpha\beta}=\begin{pmatrix} L_{\alpha\beta}^{\mathrm{I}} & -L_{\alpha\beta}^{\mathrm{II}} & 0 \\ L_{\alpha\beta}^{\mathrm{II}} & L_{\alpha\beta}^{\mathrm{I}} & 0 \\ 0 & 0 & L_{\alpha\beta}^{\mathrm{III}} \end{pmatrix}, \tag{33}$$

because the tensor must be invariant with respect to rotations about the z axis. Furthermore, (33) must be invariant with respect to the transformation

$$\mathbf{B}\rightarrow-\mathbf{B},\quad z\rightarrow -z,\quad y\rightarrow -y,$$

hence

$$L_{\alpha\beta}^{\mathrm{I}}(\mathbf{B})=L_{\alpha\beta}^{\mathrm{I}}(-\mathbf{B}),\quad L_{\alpha\beta}^{\mathrm{II}}(\mathbf{B})=-L_{\alpha\beta}^{\mathrm{II}}(-\mathbf{B}),\quad L_{\alpha\beta}^{\mathrm{III}}(\mathbf{B})=L_{\alpha\beta}^{\mathrm{III}}(-\mathbf{B})$$

or

$$L_{\alpha\beta}(\mathbf{B})=L_{\alpha\beta}^{\dagger}(-\mathbf{B}). \tag{34}$$

From (32) and (34), we obtain therefore

$$L_{\alpha\beta}(\mathbf{B})=L_{\beta\alpha}(\mathbf{B}) \tag{35}$$

for all α and β.

In a weak magnetic field the phenomenological coefficients $L_{\alpha\beta}$ are scalars independent of the magnetic field for which the Onsager-Casimir reciprocity is simply $L_{\alpha\beta}=L_{\beta\alpha}$. This case has been treated by Maecker and Peters.[1] It is a characteristic feature of (otherwise isotropic) plasmas in a magnetic field that the Onsager-Casimir reciprocity (32) reduces to the simple reciprocal relation (35) between tensors.

If, instead of (31), we express the linear relation in such a "mixed" form as

$$\mathbf{Y}_0=K_{00}\mathbf{X}_0+\textstyle\sum_i K_{0i}\mathbf{Y}_i$$
$$\mathbf{X}_j=K_{j0}\mathbf{X}_0+\textstyle\sum_i K_{ji}\mathbf{Y}_i\quad (j=1,2,\cdots), \tag{36}$$

then the reciprocity relation becomes

$$K_{0j}=-K_{j0},\quad K_{ji}=K_{ij}. \tag{37}$$

6. Linear Relations

(a) *Equation of Motion for Components*

From (27), the linear relation

$$\mathbf{q}=-K_{00}\,\mathrm{grad}\ln T+\textstyle\sum_i K_{0i}\mathbf{v}_i$$
$$n_j[\mathbf{F}_j^{*}-(\mathrm{grad}\mu_j)_T]=-K_{j0}\,\mathrm{grad}\ln T+\textstyle\sum_i K_{ji}\mathbf{v}_i \tag{38}$$

of the type (36) follows, with coefficient tensors satisfying the reciprocity (37). As the entropy production is always positive, the quadratic form for the coefficients is positive definite; in particular, tensors K_{00} and K_{jj} are positive, the thermal conductivity $\lambda\equiv K_{00}/T$ being positive. By virtue of (26) we have the relations

$$\textstyle\sum_j K_{j0}=\sum_j K_{0j}=0,\quad \sum_j K_{ji}=\sum_j K_{ij}=0. \tag{39}$$

The second set of equations (38) gives the equation of motion of each component which can be transformed, by use of (39), into

$$n_j[\mathbf{F}_j^{*}-(\mathrm{grad}\mu_j)_T]+K_{j0}\,\mathrm{grad}\ln T+\textstyle\sum_i(-K_{ji})(\mathbf{v}_i-\mathbf{v}_j)=0. \tag{40}$$

The term with K_{j0} indicates an inner force due to thermal diffusion and the last term is the friction due to relative motion. For two-component plasmas the coefficient of friction $-K_{12}$ is positive definite.

(b) *Transport Coefficients*

In the following we neglect the gravitation and inertia:

$$\mathbf{g}+[(\partial/\partial t)+\mathbf{v}\cdot\mathrm{grad}]\mathbf{v}=0.$$

Then (29) becomes

$$T\left(\frac{\partial s}{\partial t}\right)_{\mathrm{irr}}=-\mathbf{q}\cdot\mathrm{grad}\ln T+\textstyle\sum n_j(\mathbf{v}_j-\mathbf{v})\cdot\left[e_j\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)-(\mathrm{grad}\mu_j)_T\right]. \tag{41}$$

In the absence of the mutual diffusion, i.e., when

$$\textstyle\sum n_j(\mathbf{v}_j-\mathbf{v})\cdot(\mathrm{grad}\mu_j)_T=0,$$

(41) reduces to

$$T(\partial s/\partial t)_{\mathrm{irr}}=-\mathbf{q}\cdot\mathrm{grad}\ln T+\mathbf{J}\cdot(\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B}) \tag{42}$$

in terms of the conduction current

$$\mathbf{J}=\textstyle\sum n_je_j(\mathbf{v}_j-\mathbf{v}). \tag{43}$$

From (42), the linear relation

$$\mathbf{q}=-T\lambda\,\mathrm{grad}\ln T+\alpha\mathbf{J}$$
$$\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B}=\alpha\,\mathrm{grad}\ln T+\eta\mathbf{J}, \tag{44}$$

follows, λ, η, α being tensors of the type (33):

$$\lambda=\begin{pmatrix} \lambda^{\mathrm{I}} & -\lambda^{\mathrm{II}} & 0 \\ \lambda^{\mathrm{II}} & \lambda^{\mathrm{I}} & 0 \\ 0 & 0 & \lambda^{\mathrm{III}} \end{pmatrix},\ \text{etc.}$$

The λ is the thermal conductivity and η is the electric resistivity, both being positive definite.

On returning to (41), we next consider a neutral two-component plasma under uniform pressure

$$n_1e_1+n_2e_2=0,$$
$$n_1(\mathrm{grad}\mu_1)_T+n_2(\mathrm{grad}\mu_2)_T=0.$$

[12] See, for example, S. R. de Groot, *Thermodynamics of Irreversible Processes* (North-Holland Publishing Company, Amsterdam, 1952); or J. Meixner und H. G. Reik, *Handbuch der Physik* (Springer-Verlag, Berlin, 1959), Vol. III2.

Then it follows that

$$e_1^{-1}(\mathrm{grad}\mu_1)_T = e_2^{-1}(\mathrm{grad}\mu_2)_T = (e_1-e_2)^{-1}[(\mathrm{grad}\mu_1)_T - (\mathrm{grad}\mu_2)_T].$$

Then (41) can be transformed into

$$T(\partial s/\partial t)_{\mathrm{irr}} = -\mathbf{q}\cdot\mathrm{grad}\,\ln T + \mathbf{J}\cdot\{\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B} - (e_1-e_2)^{-1}[(\mathrm{grad}\mu_1)_T - (\mathrm{grad}\mu_2)_T]\}, \quad (45)$$

the corresponding linear relation being

$$\mathbf{q} = -T\lambda\,\mathrm{grad}\,\ln T + \alpha\mathbf{J}$$
$$\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B} - (e_1-e_2)^{-1}[(\mathrm{grad}\mu_1)_T - (\mathrm{grad}\mu_2)_T] = \alpha\,\mathrm{grad}\,\ln T + \eta\mathbf{J}. \quad (46)$$

Here we introduce the electric conductivity σ, which is the inverse η^{-1} to the electric resistivity, or

$$\sigma^{\mathrm{I}} = \frac{\eta^{\mathrm{I}}}{(\eta^{\mathrm{I}})^2+(\eta^{\mathrm{II}})^2}, \quad \sigma^{\mathrm{II}} = \frac{-\eta^{\mathrm{II}}}{(\eta^{\mathrm{I}})^2+(\eta^{\mathrm{II}})^2}, \quad \sigma^{\mathrm{III}} = \frac{1}{\eta^{\mathrm{III}}} \quad (47)$$

in terms of the components

$$\eta = \begin{pmatrix} \eta^{\mathrm{I}} & -\eta^{\mathrm{II}} & 0 \\ \eta^{\mathrm{II}} & \eta^{\mathrm{I}} & 0 \\ 0 & 0 & \eta^{\mathrm{III}} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma^{\mathrm{I}} & -\sigma^{\mathrm{II}} & 0 \\ \sigma^{\mathrm{II}} & \sigma^{\mathrm{I}} & 0 \\ 0 & 0 & \sigma^{\mathrm{III}} \end{pmatrix}$$

concerning the rectangular coordinate system with the z axis in the direction of $\mathbf{B}$. Then we have, for $\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B}=0$, the equation of diffusion

$$\mathbf{v}_1-\mathbf{v}_2 = \mathbf{J}/n_1e_1 = -\sigma(n_1e_1^2+n_2e_2^2)^{-1}[(\mathrm{grad}\mu_1)_T - (\mathrm{grad}\mu_2)_T] - \sigma\alpha(n_1e_1)^{-1}\,\mathrm{grad}\,\ln T.$$

On introducing the "modified" diffusion coefficient D^* and the thermal diffusion ratio $\mathrm{k_T}$ by

$$\mathbf{v}_1-\mathbf{v}_2 = -D^*\left\{\frac{1}{kT}[(\mathrm{grad}\mu_1)_T - (\mathrm{grad}\mu_2)_T] + \frac{(n_1+n_2)^2}{n_1n_2}\mathrm{k_T}\,\mathrm{grad}\,\ln T\right\} \quad (48)$$

with the Boltzmann constant k, we obtain finally

$$D^*/kT = \sigma/(n_1e_1^2+n_2e_2^2), \quad (49)$$

$$[(n_1+n_2)^2/n_1n_2]\mathrm{k_T} = [(n_1e_1^2+n_2e_2^2)/n_1e_1](\alpha/kT). \quad (50)$$

In the following we consider a dilute gaseous plasma for which the chemical potential μ_j of jth species is related to the partial pressure P_j of the same species simply by

$$n_j(\mathrm{grad}\mu_j)_T = \mathrm{grad}P_j.$$

The modified diffusion coefficient D^* then reduces to the conventional diffusion coefficient

$$D = \begin{pmatrix} D^{\mathrm{I}} & -D^{\mathrm{II}} & 0 \\ D^{\mathrm{II}} & D^{\mathrm{I}} & 0 \\ 0 & 0 & D^{\mathrm{III}} \end{pmatrix},$$

defined by

$$\mathbf{v}_1-\mathbf{v}_2 = -D(n_1^{-1}\,\mathrm{grad}n_1 - n_2^{-1}\,\mathrm{grad}n_2), \quad (51)$$

and (49) reduces to the Einstein-type relation

$$D/kT = \sigma/(n_1e_1^2+n_2e_2^2). \quad (52)$$

7. Diffusion and Electric Conduction[13]

In this and the following sections transport coefficients *across* a strong magnetic field are calculated, forces and fluxes being supposed to be perpendicular to the magnetic field. We consider a fully ionized gas composed of one type of positive ion, as species 1, and electrons, as species 2. The electric charge of an ion will be denoted by $e_1=Ze$ and that of an electron by $e_2=-e$; charge neutrality $Zn_1=n_2$ (or quasi-neutrality in case of diffusion) and the condition $Z^2\ll m_1/m_2$ are assumed.

In this section we assume that the electrons make many gyrations in a time interval between effective collisions (while in the next section such an assumption is made both for electrons and ions).

First, it is proved that under these conditions interference terms with coefficient α can be ignored. When electrons do not collide in a period of gyration they have a drift velocity $cB^{-1}\mathbf{E}^0\times\mathbf{b}$ ($\mathbf{b}\equiv\mathbf{B}/B$) with respect to the coordinate system moving with the flow $\mathbf{v}_1$ of ions or the mean flow $\mathbf{v}$. Here $\mathbf{E}^0$ is the electric field in this coordinate system, $\mathbf{E}^0=\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B}$. This electron drift does not cause any conduction of heat. We have, therefore,

$$\mathbf{J} = n_2ecB^{-1}\mathbf{b}\times(\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B}) \quad \text{and} \quad \mathbf{q}=0 \quad (53)$$

for $\mathrm{grad}T=0$. ($\mathbf{q}$ is the reduced heat flow excluding the convection heat.) Hence, $\alpha=0$ at the limit of free gyration of electrons; and, for weak interactions, the terms with α can be neglected, the tensor α^2 being sufficiently smaller than the tensor $T\lambda\eta$.

(a) D^{II} and σ^{II}

At the limit of vanishing interaction D^{II} takes a definite value, which will be calculated by choosing a coordinate system where the flow $\mathbf{v}_1$ of ions (and therefore $\mathbf{v}$ also) vanishes.

When the electric field does not exist in this coordinate system, $\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B}=0$, the flow $\mathbf{v}_2$ of electrons is simply

$$\mathbf{v}_2 = (ckT/B)(-en_2)^{-1}\mathbf{b}\times\mathrm{grad}n_2 \quad (54)$$

at the limit of vanishing interaction, a proof being given in the following. The component D^{II}, which is defined by the relation

$$\mathbf{v}_2-\mathbf{v}_1 = -D^{\mathrm{II}}\mathbf{b}\times(n_2^{-1}\,\mathrm{grad}n_2 - n_1^{-1}\,\mathrm{grad}n_1) = -[(n_1+n_2)/n_1n_2]D^{\mathrm{II}}\mathbf{b}\times\mathrm{grad}n_2,$$

[13] Sections 7 and 8 were prepared with the assistance of Dr. S. Kaneko.

is therefore

$$D^{\mathrm{II}}=[n_1/(n_1+n_2)](ckT/eB). \quad (55)$$

Proof of (54). In terms of the distribution function $f_2(\mathbf{c}_2,\mathbf{r})$ of the electron velocity $\mathbf{c}_2$ at a position $\mathbf{r}$, the flow $\mathbf{v}_2$ is determined by

$$\mathbf{v}_2=\frac{1}{n_2}\int \mathbf{c}_2 f_2(\mathbf{c}_2,\mathbf{r})d\mathbf{c}_2. \quad (56)$$

When electrons gyrate many times during a time interval between successive collisions, the function f_2 is equal to a Maxwellian distribution with respect to the position of the guiding center,

$$\mathbf{r}+\mathbf{a}_2, \quad \text{where} \quad \mathbf{a}_2\equiv\omega_2^{-1}\mathbf{c}_2\times\mathbf{b}.$$

Thus,

$$\begin{aligned}f_2(\mathbf{c}_2,\mathbf{r})&=f_2^0(\mathbf{c}_2,\mathbf{r}+\mathbf{a}_2)\\&=f_2^0(\mathbf{c}_2,\mathbf{r})[1+\mathbf{a}_2\cdot\mathrm{grad}\ln n_2]\\&=f_2^0(\mathbf{c}_2,\mathbf{r})[1+\omega_2^{-1}\mathbf{c}_2\cdot\mathbf{b}\times\mathrm{grad}\ln n_2], \quad (57)\end{aligned}$$

where

$$f_2^0(\mathbf{c}_2,\mathbf{r})=n_2(m_2/2\pi kT)^{\frac{3}{2}}\exp(-m_2c_2^2/2kT), \quad (58)$$

n_2 being the number density of guiding centers of electrons at the position $\mathbf{r}$. On substituting (57) into (56), we obtain

$$\begin{aligned}\mathbf{v}_2&=\frac{\mathbf{b}\times\mathrm{grad}\ln n_2}{\omega_2}\cdot\int\mathbf{c}_2\mathbf{c}_2\frac{f_2^0}{n_2}d\mathbf{c}_2\\&=\frac{\mathbf{b}\times\mathrm{grad}\ln n_2}{\omega_2}\int_0^\infty\frac{1}{3}c_2^2\left(\frac{m_2}{2\pi kT}\right)^{\frac{3}{2}}\times\exp\left(-\frac{m_2c_2^2}{2kT}\right)4\pi c_2^2dc_2\\&=\frac{\mathbf{b}\times\mathrm{grad}\ln n_2}{\omega_2}\frac{kT}{m_2},\end{aligned}$$

the final expression being equal to (54).

The nondiagonal element of the electric conductivity is given by the relation (52):

$$\sigma^{\mathrm{II}}=(Ze^2/kT)(n_1+n_2)D^{\mathrm{II}}=en_2c/B. \quad (59)$$

This result agrees with (53).

(b) D^{I} and σ^{I}

The diagonal element D^{I} of the coefficient of diffusion perpendicular to the magnetic field appears when there is an interaction between an ion and electron.

The guiding center $\mathbf{r}_g$ of an ion with the velocity $\mathbf{c}_1$ at the position $\mathbf{r}$ is given by

$$\mathbf{r}_g=\mathbf{r}+\omega_1^{-1}(\mathbf{c}_1\times\mathbf{b}),$$

where ω_1 is the gyration frequency of the ions:

$$\omega_1=ZeB/m_1c.$$

For an increment in ion velocity $\Delta\mathbf{c}_1$ due to the interaction with surrounding electrons in a time interval Δt, the displacement of the guiding center is given by

$$\Delta\mathbf{r}_g=\omega_1^{-1}(\Delta\mathbf{c}_1\times\mathbf{b}).$$

Here Δt is taken to be much larger than the time characteristic of fluctuation of the interaction, but it must be much smaller than the ion gyration period. Taking the average in regard to the ion velocity, and the interaction with surrounding electrons, we have

$$(\Delta t)^{-1}\langle|\Delta\mathbf{r}_g|^2\rangle_{\mathrm{av}}=(\omega_1^2)^{-1}(\Delta t)^{-1}\langle|\Delta\mathbf{c}_1\times\mathbf{b}|^2\rangle_{\mathrm{av}}. \quad (60)$$

Now (51) can be written in the form

$$\mathbf{v}_1-\mathbf{v}_2=-[(n_1+n_2)/n_2]Dn_1^{-1}\,\mathrm{grad}n_1$$

and the diagonal element of the coefficient $(n_1+n_2)n_2^{-1}D$ is given by[7]

$$[(n_1+n_2)/n_2]D^{\mathrm{I}}=\tfrac{1}{4}(\Delta t)^{-1}\langle|\Delta\mathbf{r}_g|^2\rangle_{\mathrm{av}}.$$

Hence we obtain

$$D^{\mathrm{I}}=\tfrac{1}{4}[n_2/(n_1+n_2)]\omega_1^{-2}(\Delta t)^{-1}\langle|\Delta\mathbf{c}_1\times\mathbf{b}|^2\rangle_{\mathrm{av}}. \quad (61)$$

The diagonal element σ^{I} of the electric conductivity is calculated, by use of (52), to be

$$\begin{aligned}\sigma^{\mathrm{I}}&=Ze^2(kT)^{-1}(n_1+n_2)D^{\mathrm{I}}\\&=\tfrac{1}{4}(n_1/kT)(Ze/\omega_1)^2(\Delta t)^{-1}\langle|\Delta\mathbf{c}_1\times\mathbf{b}|^2\rangle_{\mathrm{av}}. \quad (62)\end{aligned}$$

The factor $(\Delta t)^{-1}\langle|\Delta\mathbf{c}_1\times\mathbf{b}|^2\rangle_{\mathrm{av}}$ can be transformed, as in Sec. 2, to an integral of force correlation:

$$\frac{1}{\Delta t}\langle|\Delta\mathbf{c}_1\times\mathbf{b}|^2\rangle_{\mathrm{av}}=\frac{1}{m_1^2}\int\int\int[\mathbf{F}(0)\times\mathbf{b}]\cdot[\mathbf{F}(t)\times\mathbf{b}]f_2d\mathbf{c}_2d\mathbf{r}dt. \quad (63)$$

The integration can be performed along a line similar to Sec. 4, the equation corresponding to (18) being

$$\int\left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\times\mathbf{b}\right)\cdot\left(\frac{\mathbf{r}+\mathbf{s}}{|\mathbf{r}+\mathbf{s}|^3}\times\mathbf{b}\right)d\mathbf{r}=\frac{8\pi}{3}\frac{1}{|\mathbf{s}|}[1+\tfrac{1}{2}P_2(\cos\theta)].$$

Here P_2 is the Legendre polynomial of the order two, and $\cos\theta=\mathbf{b}\cdot\mathbf{s}/|\mathbf{s}|$. The result is

$$\frac{1}{\Delta t}\langle|\Delta\mathbf{c}_1\times\mathbf{b}|^2\rangle_{\mathrm{av}}\left[\frac{16}{3}n_2kT\frac{m_2}{m_1^2}\left(\frac{2\pi kT}{m_2}\right)^{\frac{1}{2}}\left(\frac{Ze^2}{kT}\right)^2\right]^{-1}=\ln(t_p/t_s)+\int_0^{\omega_2t_p}\left[\frac{\tanh^{-1}Y}{Y}-1+\frac{1}{4Y^2}\left(\frac{3-Y^2}{Y}\tanh^{-1}Y-3\right)\right]\frac{dy}{y},$$

where Y is defined by (20); the right-hand side is equivalent to

$$\ln(t_p/t_s) \quad \text{for} \quad \omega_2 t_p \lesssim 1$$
$$\ln(t_p/t_s)+\tfrac{3}{4}[\ln(\omega_2 t_p)]^2 \quad \text{for} \quad \omega_2 t_p \gtrsim 1. \tag{64}$$

Here t_p and t_s are the same as in (17). The limiting expression $\frac{3}{4}[\ln(\omega_2 t_p)]^2$ can be derived in an elementary way.

(c) η^I and η^{II}

As regards electric resistivity, (47) reduces to

$$\eta^{\rm I}=\sigma^{\rm I}(\sigma^{\rm II})^{-2}, \quad \eta^{\rm II}=-(\sigma^{\rm II})^{-1},$$

since $(\sigma^{\rm I})^2 \ll (\sigma^{\rm II})^2$ for weak interactions.

8. Thermal Conduction

The thermal conductivity λ is treated under the assumption that both electrons and positive ions make many gyrations between effective collisions.

(a) Expression for λ^{II}

When collisions can be neglected entirely the reduced heat flow $\mathbf{q}$ is given by

$$\mathbf{q}=\sum_{j=1}^{2}\tfrac{1}{2}m_j\int c_j^2\mathbf{c}_j f_j(\mathbf{c}_j,\mathbf{r})d\mathbf{c}_j \tag{65}$$

in terms of the distribution function $f_j(\mathbf{c}_j\cdot\mathbf{r})$. Here the function f_j is equal to a Maxwellian distribution f_j^0 with respect to position of the guiding center $\mathbf{r}+\mathbf{a}_j$, where

$$\mathbf{a}_j=\omega_j^{-1}\mathbf{c}_j\times b,$$

ω_j being the gyration frequency. Thus,

$$\begin{aligned} f_j(\mathbf{c}_j,\mathbf{r})&=f_j^0(\mathbf{c}_j,\ \mathbf{r}+\mathbf{a}_j)\\ &=f_j^0(\mathbf{c}_j,\mathbf{r})\{1+[(m_jc_j^2/2kT)-\tfrac{5}{2}]\mathbf{a}_j\\ &\qquad\cdot\operatorname{grad}\ln T\}, \end{aligned} \tag{66}$$

where it has been taken into account that the number density n_j is inversely proportional to the temperature T. On substituting (66) into (65), we have

$$\mathbf{q}=\frac{5}{2}\sum_j\frac{n_j(kT)^2}{m_j\omega_j}\mathbf{b}\times\operatorname{grad}\ln T. \tag{67}$$

Since $\lambda^{\rm II}$ is defined by

$$\mathbf{q}=\lambda^{\rm II}\mathbf{b}\times(-\operatorname{grad}T),$$

we finally obtain

$$\lambda^{\rm II}=-\tfrac{5}{2}k^2T\sum_j\frac{n_j}{m_j\omega_j}. \tag{68}$$

It is important to note that all $\mathbf{v}_j$ vanish,

$$\mathbf{v}_j=\frac{1}{n_j}\int\mathbf{c}_jf_j(\mathbf{c}_j,\mathbf{r})d\mathbf{c}_j=0,$$

as it should since there is no interference between thermal and electric conductions (cf. the third paragraph of Sec. 7).

(b) Expression for λ^I

When the magnetic field is so strong that the electrons and ions make many free gyrations, only collisions between ions determine the heat conduction in the direction of the temperature gradient which is assumed to be perpendicular to the strong magnetic field. [Because an effective collision between ions causes an energy transfer proportional to the square of ion gyration radius, which is nearly m_1/m_2 times the square of the electron gyration radius. The collision frequency between electrons or between ion and electron is only $(m_1/m_2)^{\frac{1}{2}}$ times the collision frequency between ions.] The ion gyration radius is usually larger than the Debye shielding length, and the collisions between ions can be treated in a way similar to collision integrals in the kinetic theory of gases. In fact, the coefficient of thermal conduction in this case has been obtained by Fradkin,[5] who solved the Boltzmann equation. Here we give a short discussion without using the Boltzmann equation.

Let $\mathbf{e}$ denote the unit vector in the direction of grad$T$. Then

$$\tfrac{1}{2}m_1(c_1'\mathbf{a}_1'+c_1^{0\prime 2}\mathbf{a}_1^{0\prime}-c_1^2\mathbf{a}_1-c_1^{02}\mathbf{a}_1^0)\cdot\mathbf{e}$$

or

$$(m_1/2\omega_1)(c_1'^2\mathbf{c}_1'+c_1^{0\prime 2}\mathbf{c}_1^{0\prime}-c_1^2\mathbf{c}_1-c_1^{02}\mathbf{c}_1^0)\cdot(\mathbf{b}\times\mathbf{e})$$

indicates energy transferred in the direction of $\mathbf{e}$ by a collision with initial velocities $\mathbf{c}_1$, $\mathbf{c}_1^0$ and final velocities $\mathbf{c}_1'$, $\mathbf{c}_1^{0\prime}$. The component q of the heat flow in the direction of $\mathbf{e}$ is then given by

$$\begin{aligned} q=\frac{1}{2}\frac{m_1}{2\omega_1}\int\int\int\int&(c_1'^2\mathbf{c}_1'+c_1^{0\prime 2}\mathbf{c}_1^{0\prime}-c_1^2\mathbf{c}_1-c_1^{02}\mathbf{c}_1^0)\\ &\cdot(\mathbf{b}\times\mathbf{e})f_1(\mathbf{c}_1,\mathbf{r})f_1(\mathbf{c}_1^0,\mathbf{r})gbdbd\epsilon d\mathbf{c}_1d\mathbf{c}_1^0, \end{aligned} \tag{69}$$

where g is the relative speed $|\mathbf{c}_1-\mathbf{c}_1|$, b is the impact parameter, and ϵ is the azimuthal angle of the orbit of relative motion.

The expression (69) can be transformed, by virtue of (66) for $j=1$, into

$$\begin{aligned} q=|\operatorname{grad}\ln T|&\frac{m_1}{12\omega_1^2}\\ &\times\int\int\int\int(c_1'^2\mathbf{c}_1'+c_1^{0\prime 2}\mathbf{c}_1^{0\prime}-c_1^2\mathbf{c}_1-c_1^{02}\mathbf{c}_1^0)\\ &\cdot\left[\mathbf{c}_1\left(\frac{m_1c_1^2}{2kT}-\frac{5}{2}\right)+\mathbf{c}_1^0\left(\frac{m_1c_1^{02}}{2kT}-\frac{5}{2}\right)\right]\\ &\times f_1^0(\mathbf{c}_1)f_1^0(\mathbf{c}_1^0)gbdbd\epsilon d\mathbf{c}_1d\mathbf{c}_1^0. \end{aligned} \tag{70}$$

On introducing the center-of-mass velocity $\mathbf{G}=\frac{1}{2}(\mathbf{c}_1+\mathbf{c}_1^0)$ and the relative velocity $\mathbf{g}=\mathbf{c}_1-\mathbf{c}_1^0$, and integrating with respect to $\mathbf{G}$, we can reduce (70) to

$$q=-|\text{grad}\ \ln T|(m_1/6)(n_1^2/\omega_1^2)(m_1/4kT)^{\frac{3}{2}}$$

$$\times\pi^{\frac{1}{2}}\int\int g^6\exp(-m_1g^2/4kT)(1-\cos^2\theta)gbdbdg, \quad (71)$$

where θ is the angle of diffraction in the orbit of relative motion. On evaluating the integral in (71), we obtain the thermal conductivity across a strong magnetic field.

$$\lambda^{\text{I}}=\frac{8}{3}k\frac{kT}{\omega_1^2m_1}n_1^2\left(\frac{\pi kT}{m_1}\right)^{\frac{1}{2}}\left(\frac{Z^2e^2}{kT}\right)^2\ln\frac{kTl_D}{Z^2e^2}, \quad (72)$$

which agrees with Fradkin's[5] expression.

According to (72) λ^{I}/k is equal to the square of ion gyration radius, $kT/m_1\omega_1^2$, multiplied by the ion-ion collision frequency per unit volume, as it should be.

CONCLUSION

Irreversible processes in plasmas in a strong magnetic field are discussed from both the phenomenological and the microscopic points of view. It is not assumed that the gyration radius of electrons is larger than the Debye length. The rate of relaxation between ion and electron temperatures in a fully ionized gas increases with the magnetic field. When the gyration radius r_g of the electrons is shorter than the Debye length l_D the rate of relaxation is proportional to

$$\ln(kTl_D/Ze^2)+\tfrac{1}{2}[\ln(l_D/r_g)]^2,$$

where Ze and $-e$ are the charges of an ion and an electron, respectively. The thermodynamics of irreversible processes is applied in general and it is shown that the Onsager-Casimir reciprocity relation takes a symmetrical form for plasmas in a magnetic field. For a two-component fully ionized gas where the electrons make free gyrations interference between electrical and thermal conductions vanishes. In this case the diagonal elements of coefficient tensors for the electric conduction and diffusion perpendicular to the magnetic field are proportional to

$$\ln(kTl_D/Ze^2)+\tfrac{3}{4}[\ln(l_D/r_g)]^2,$$

when the gyration radius r_g of the electrons is shorter than the Debye length l_D.

DISCUSSION

Session Reporter: W. B. Riesenfeld

L. Spitzer, Jr., *Matterhorn Project, Princeton University, Princeton, New Jersey:* What is the physical reason for the modification of the rate of equipartition? Why should the magnetic field affect this process?

T. Kihara: In the presence of a strong magnetic field the mean interaction between an ion and an electron is modified by the tightly spiraling orbits. The time of interaction is longer in the mean than in the absence of fields; the relaxation rate increases.

W. B. Thompson. *Atomic Energy Research Establishment, Harwell, Berkshire, England:* But, if the orbits are tight spirals, the electrons do not appear to be scattered; instead the guiding center is rotated in coordinate space, and I do not see why scattering should be increased.

T. Kihara: In dealing with this problem, intuition is not a very powerful method. That is why I calculated analytically.