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Some Remarks about Flow past Bodies

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THESE remarks are confined to one special class of problems: steady flow past solid bodies in the presence of external magnetic fields.

Even within this limited category, various investigators have studied problems falling into a number of subcategories. Some have made approximations appropriate for small electrical conductivity, others for large conductivity, and still others have been able to treat cases where the conductivity may have arbitrary values. The fluid medium is sometimes assumed to be incompressible and, in other studies, compressible. The viscosity may or may not be neglected. In some cases the magnetic field strength has been taken to be very small or very large. The geometry of bodies and fields may be such as to produce flow of special symmetry, such as plane or axisymmetric. Finally, although most investigations have been made for constant, scalar (Ohmic) conductivity, there have been some studies in which realistic variation of this property with temperature has been introduced, and a few in which anisotropy of electrical conductivity in the presence of a magnetic field has been considered.

It is difficult (and may be impossible) to make generalizations that are correct for all of these cases. Nevertheless, certain common features seem to be recognizable in the results of a number of studies, and these are the subjects of the present lecture. Broadly speaking, these are features which appear to distinguish flows of low conductivity from those of high conductivity, or more accurately, flows of low magnetic Reynolds number R_m from those of high R_m. There are unmistakable analogies between the influences of the magnetic and the actual Reynolds numbers, i.e., between electrical resistance and viscosity in their effects upon flow patterns. Thus the features that distinguish between low-R_m and high-R_m flows are often analogous to those that distinguish between flows of small and large Reynolds numbers.

FLOWS OF SMALL MAGNETIC REYNOLDS NUMBERS

Flow patterns in this category are analogous to flows of small Reynolds numbers, such as those treated in the approximations of Stokes and Oseen. The analogy is not only mathematical but arises from the fact that in both situations the *diffusion* process (i.e., the diffusion of vorticity by electrical resistance and by viscosity, respectively) is rapid compared to the convective process, and in extreme cases may dominate the situation completely. Thus the flow past a solid body results in a large vortical wake, for the disturbance due to the body diffuses rapidly into the fluid as it is carried downstream (Fig. 1).

But in the magnetohydrodynamical analog a most remarkable new phenomenon appears, for, if the magnetic field strength is sufficiently large, the "wake" may extend *upstream* instead of downstream (Fig. 2). It seems that the magnetohydrodynamic effect has reversed the convection, so that now the vorticity diffuses outward from the body as it is transported upstream rather than downstream. To be sure, the physical nature of this kind of transport can be recognized: It is not convection but the mechanism of Alfvén-wave propagation, which for sufficiently strong field carries flow disturbances along the field lines at a speed exceeding the flow speed. These effects are clearly seen in the results presented by H. Hasimoto at this conference,[1] in which both magnetohydrodynamic and viscous wakes appear, extending, in general, in different directions.

In general, low-R_m flows are characterized by large "wakes" of vorticity and electric current. Diffusion is a dominant process, and wavelike disturbances, excepting sound waves, are completely absent. Nevertheless, the presence of the Alfvén-wave mechanism is felt indirectly, for by this means convection is overwhelmed and apparently reversed when magnetic-field strength is large.

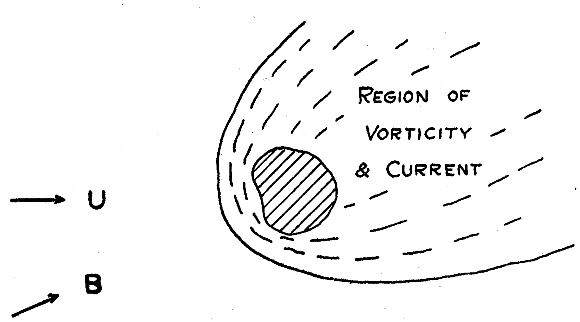

FIG. 1. Sketch showing steady flow past body at low R_m. U denotes the flow vector and B the magnetic-field vector. Small B.

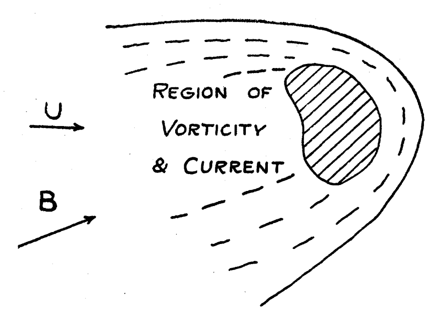

FIG. 2. Same as Fig. 1, except B is large.

[1] H. Hasimoto, Revs. Modern Phys. 32, 860 (1960), this issue.

The analogy between Stokes flow and magnetohydrodynamic flow at very low R_m also implies an analogous mathematical difficulty. It is well known that the Stokes approximation, in which the inertial terms are neglected in comparison with viscous terms, must fail at large distances from a disturbance. The same is true of the analogous approximation in which the diffusion term is taken to be the only significant one in the equation of electric-current distribution. One result is the failure of a very simple-looking process of successive approximations in which terms of successive powers of R_m are sought. This process consists of neglecting the magnetohydrodynamic effects entirely in calculating, to first order in R_m, the current-density distribution. In principle, this distribution could be inserted into the momentum equations to find the first approximation to the magnetohydrodynamic effect on the flow field; this in turn yields a second approximation to the current distribution, and so on. The failure of this simple procedure has apparently been corrected by Murray and Ludford.

FLOWS OF LARGE MAGNETIC REYNOLDS NUMBERS

For flows about slender obstacles, at least, the equations of motion linearized in the fashion of Oseen's approximation are adequate to describe the flow pattern for a wide range of R_m. In certain problems of this class, namely, those of "aligned fields," in which the unperturbed flow and magnetic vectors are parallel to one another, it is possible to construct flow patterns by distribution of singular solutions having, locally, the character of sources, vortices, and doublets. These singular solutions involve vortical wakes extending downstream (for relatively small field strength) or upstream (for large field strength). The lateral extent of these wakes is determined by the speed of diffusion in relation to the effective transport speed, which is the speed of Alfvén-wave propagation relative to the moving medium as mentioned in the foregoing. We suggest that for other orientations of the fields the results would be analogous. At moderate values of R_m, there must be vortical wakes extending outward from solid bodies in directions determined by the effective convection velocity, which is the resultant of the true convection and Alfvén propagation. As R_m is increased, such wakes must become narrow, for at sufficiently large R_m the regions of vortical current-carrying flow lose the character of wakes and become narrow diffusion zones lying along wavelike disturbances (Fig. 3). These disturbances are, in fact, *standing Alfvén waves*, their directions determined by the resultant of Alfvén propagation and convection by the moving fluid.

This is the character of large-R_m steady flow: The picture is dominated by standing Alfvén waves. The effect of electrical resistance is only to diffuse these waves, and thus to cause them to attenuate at a distance from the solid obstacle. The analogy with the damping of sound waves by viscosity is accurate.

The special case of "aligned fields" in connection with small-perturbation flow has been mentioned. More generally this name might be applied to any case in which the magnetic and velocity vectors are locally parallel to one another. This is a singular case in which the wave character of the flow pattern is less obvious, since Alfvén propagation occurs along streamlines, upstream or downstream, and the result of diffusion again appears as a (narrow) wake rather than a diffuse wave. In particular, the standing Alfvén waves now lie along the body surface and form there boundary layers of rotational flow.

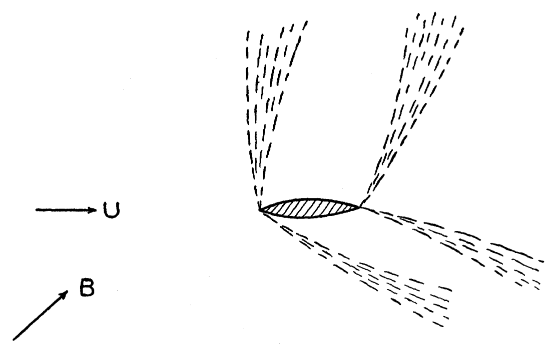

FIG. 3. Sketch showing steady flow past body at high R_m. The shaded regions denote diffuse standing Alfvén waves.

These wave-dominated flow patterns are rather complicated, in general, because there are several families of waves. The discussion is clarified by use of certain wave-propagation diagrams that have appeared in studies of unsteady magnetohydrodynamic flow.

Friedrichs Diagrams

Studies of small-disturbance propagation in a conducting gas (or of the real characteristics of the differential equations of magneto-gas dynamics) have been made by several investigators, notably van de Hulst, Herlofsen, and Friedrichs. Actually these were studies of perfect conductors, but as has already been mentioned in the foregoing the effect of small resistance is principally to diffuse the waves at a distance from the disturbance source and for the present qualitative discussion this effect can be overlooked.

Friedrichs used two diagrams in discussing these acoustic-magnetohydrodynamic waves, the first of which (Fig. 4) is a hodograph of the propagation velocities of plane waves. Each point of the curves drawn in Fig. 4 represents the propagation speed in still gas of a plane wave and the direction of its wave normal. What is most interesting is that there are both "fast" and "slow" waves. In Fig. 4 the notation adopted is as follows: a = speed of sound in absence of mhd effects; A = speed of Alfvén waves for incompressible medium. Thus, for example, there are four possible propagation speeds for plane waves normal to the magnetic field vector $\mathbf{B}$; two of these travel at the speed of sound a, and two at the Alfvén speed A. For propagation of waves at an arbitrary angle to the field vector there are generally four possible speeds, but these speeds are not equal to a or A. For the special case of a wave parallel to **B** there are but two speeds, namely, $\pm(a^2+A^2)^{\frac{1}{2}}$.

Figure 4 has been simplified by omission of still another curve, representing what may be called "intermediate" waves. These are characterized by velocity components that are normal to the plane of **B** and the wave normal, i.e., the plane of Fig. 4; thus they are not produced by motions of bodies in either two-dimensional or axisymmetric geometries.

From this wave-velocity diagram the second Friedrichs diagram can be constructed. It is the shape of the disturbance that propagates from a point disturbance at the origin. It is, in other words, a picture of the *pulse* that propagates from a point with self-preserving geometry. Since it is self-similar, the diagram is also a velocity diagram; i.e., every point represents in direction and magnitude the velocity of propagation of a part of the pulse. But since it is composed of elementary plane waves, every part of the pulse must propagate normal to itself at the speed and direction given by a point of Fig. 4. This tells how the pulse shape is to be constructed (Fig. 5): if a normal $\alpha\beta$ is drawn to any vector of Fig. 4, a corresponding element of the pulse may occur anywhere along $\alpha\beta$, provided the element is tangent to $\alpha\beta$. One simply draws such normals for all points of the fast and slow waves of Fig. 4; the result is the pulse shape of Fig. 5. Particular attention is called to the points P, Q, R, and S of this diagram. The propagation

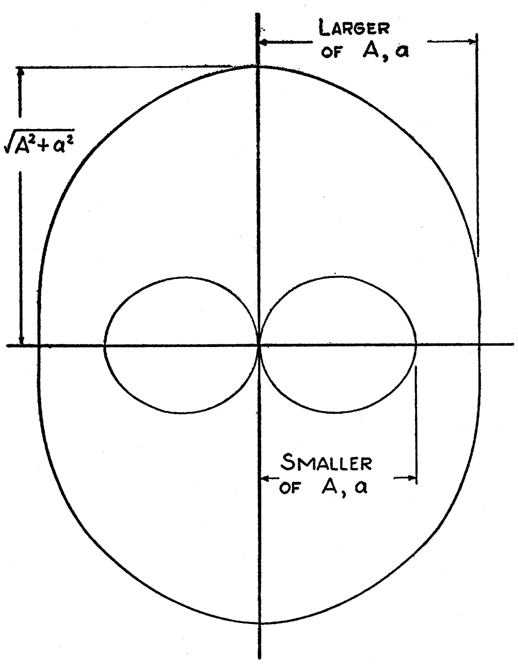

FIG. 4. Friedrichs diagram showing speed of propagation of plane acoustimagnetohydrodynamic waves as function of the direction of the wave normal. The horizontal axis is in the direction of the magnetic-field vector. A is the speed of Alfvén waves and a the speed of sound in the absence of magnetohydrodynamic effects. Here $A/a=\sqrt{2}$ or $1/\sqrt{2}$.

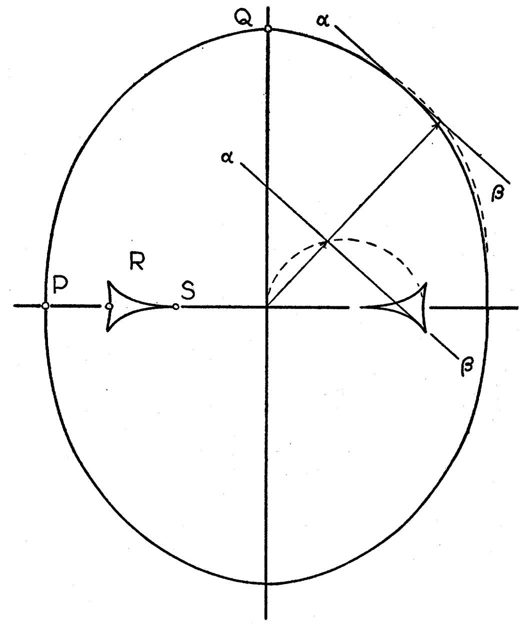

FIG. 5. Friedrichs diagram showing shape of self-similar pulse propagating from a disturbance (heavy lines). The light lines in the diagram show how this figure is constructed from Fig. 4. The points P, Q, R, S are defined in the text. $A/a=\sqrt{2}$ or $1/\sqrt{2}$.

speeds of these points are as follows:

P: a or A, whichever is greater;

Q: $(a^2+A^2)^{\frac{1}{2}}$;

R: A or a, whichever is less;

S: $\{a^{-1}+A^{-1}\}^{-1}$.

This pulse consists of a wave front of oval shape, behind which lies a region of disturbed flow; this region contains, in particular two "cusped triangles" formed by the envelopes of "slow waves." These are not fronts, but might be called crests, i.e., discontinuities in the disturbance pattern. The standing-wave patterns of steady magneto-gas dynamics may be formed by the envelopes of both the wave fronts and the wave crests of this diagram.

Before proceeding to draw conclusions regarding steady flow from these diagrams, let us remark on the degeneracy of Figs. 4 and 5 in the case of incompressible fluid. This is the case $a\rightarrow\infty$, so that the "fast waves" travel at infinite speed. It is clear that the propagation speeds at the several points identified before are P: ∞; Q: ∞; R and S: A. Thus, the diagram degenerates into two points (actually two cusped triangles of vanishing size) moving in the directions along the magnetic lines with the Alfvén speed [Fig. 6(b)]. Correspondingly the wave-speed diagram, Fig. 4, consists simply of two circles [Fig. 6(a)].

Standing Waves of Steady Flow

The differential equations of plane steady flow at large R_m reveal a complicated situation of flow patterns,

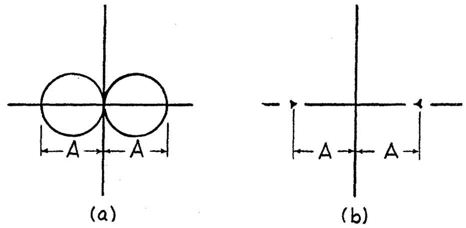

FIG. 6. Degenerate Friedrichs diagram for incompressible fluid.

sometimes doubly hyperbolic and sometimes elliptic-hyperbolic depending upon the relative magnitudes of flow speed, sound speed, and Alfvén speed, and also upon the relative orientation of the flow and field vectors. The diagrams constructed in the foregoing are sufficient to predict and explain these various regimes. The principle involved is that standing waves are envelopes of the pulses that a moving body produces at all points of its trajectory, and therefore, given any point in the plane of Fig. 5 representing the body's velocity vector, such standing waves must appear as tangents drawn from this point to the pulse diagram. It is easy to verify that this construction is equivalent to drawing a series of self-similar pulses from successive points of the body's trajectory and finding their envelopes.

This construction has been used in Fig. 7 to illustrate two doubly hyperbolic and two elliptic-hyperbolic flow situations. These have been chosen in particular to show how *waves inclined upstream* are formed.

These standing waves can also be found by a simple construction involving Fig. 4, for the waves are actually propagating plane waves relative to the fluid and must therefore propagate at speeds given by Fig. 4. But the construction based on the pulse shape has the important advantage of distinguishing between the upstream and downstream branches of any particular plane wave in these cases where the waves are produced by body motion.

This construction and the determination of the number of waves involved in the pattern are *locally* correct, where the flow and field vectors are the local ones; in other words, we are determining the real characteristics of the flow at any point. In linearized small-perturbation flow the characteristics are everywhere those of the undisturbed fields. The small sketches of flow about bodies in Fig. 7 have been drawn for this special case for clarity.

Let us now consider the degenerate case of *aligned fields*, i.e., flow and field vectors parallel to one another. The construction reveals only two symmetric families of waves [Figs. 8(a) and 8(c)] or none at all [Figs. 8(b) and 8(d)]. The other families have degenerated into the horizontal axis, i.e., they coincide with the streamlines. But such a characteristic permits the introduction of discontinuities along streamlines in order to satisfy boundary conditions; the result is a surface-current layer at a boundary surface. Investigation shows that the number of available functions always coincides with the number of boundary conditions to be satisfied, with the interpretation that surface currents are permitted only when a characteristic direction has degenerated into the flow direction.

It is interesting to note that incompressible flow is always elliptic-hyperbolic except for aligned fields, for in such flow we are always in the situation of Figs. 7(b) and 7(d). On recalling that the two "Alfvén points" of Fig. 6(b) are really tiny cusped-triangles, we recognize that a tangent can be drawn to each of them from any point of the plane. For aligned fields the standing waves degenerate into the streamlines and the field is purely elliptic with surface currents.

Effects of Finite Conductivity

As already mentioned, the effects of small electrical resistance of the fluid medium is to damp the standing Alfvén waves, and to expand the surface currents of the aligned-field flows into magnetic boundary layers. We return to this point to emphasize that the appearance of waves extending upstream from a solid body does not render the resulting flow pattern physically impossible or uninteresting. In actual experiments, because the electrical resistance is appreciable, the experimenter probably finds that upstream waves are damped to extinction in a few body lengths and have little if any

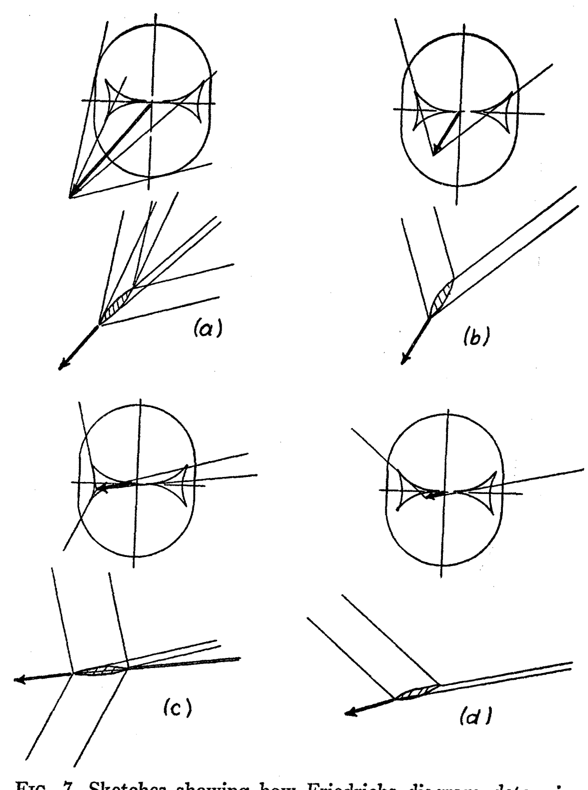

FIG. 7. Sketches showing how Friedrichs diagram determines wave pattern produced by body in steady flow. The magnetic field is horizontal. The diagrams are not to scale.

effect on the upstream flow conditions in a wind tunnel or shock tube.

Similarly, the upstream wakes mentioned before attenuate gradually with distance upstream of the body and therefore do not necessarily involve any conflict with experimental observation. Since these wakes are due to effects that occur at the body surface and are propagated upstream in an attenuating fashion, we believe that these flows can be produced by causing a uniform stream to flow past the body. When steady flow is established the wake extends forward—perhaps even into the nozzle of a wind tunnel—but does not necessarily affect the flow pattern appreciably.

In those aligned-field flows for which the wake extends upstream of the body it is found that the magnetic boundary layer that replaces a current layer is thinnest at the rear parts of the body, increases in thickness toward the front, and there joins the upstream wake. It seems particularly interesting that the "pulse diagram" of Friedrichs provides information concerning the direction of growth of the magnetic boundary layer and the direction (upstream or downstream) of the wake. When both of the degenerate, horizontal tangents must be drawn downstream from the body-speed vector, as in Figs. 8(a)–8(c), the wake lies behind the body. When one degenerate wave extends forward, however, as in Fig. 8(d), it is always found that the wake lies ahead of the body. (Since in the last case there is also a degenerate characteristic wave lying downstream along the streamline, it is not clear why the wake should not extend in both directions. Nevertheless, the differential equations seem to yield the unambiguous result stated here.)

Thus the speed at which the change-over from upstream to downstream wake occurs is the speed of point S (Fig. 5); i.e., the magnetohydrodynamic wake extends upstream when

$$\text{flow speed} < (a^{-1}+A^{-1})^{-1}$$

and downstream when

$$\text{flow speed} > (a^{-1}+A^{-1})^{-1}.$$

Since previous work was concerned with incompressible flow, the change of behavior was attributed to the change from sub-Alfvénic to super-Alfvénic flow, i.e., from flow speed $<A$ to flow speed $>A$. We now recognize that this requires modification for compressible fluids.

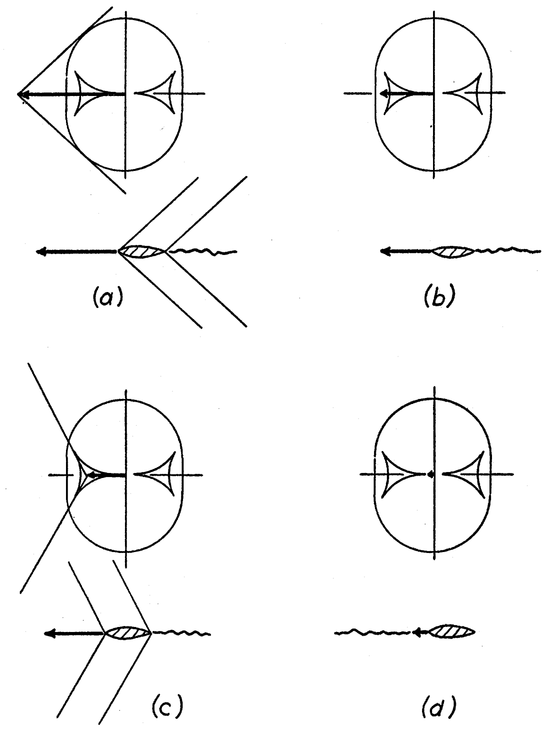

Fig. 8. Same as Fig. 7 but for cases of "aligned fields." The wavy lines in the flow sketches denote magnetohydrodynamic "wakes."

CONCLUSION

At the present writing, actual observation of most of these phenomena characterizing magnetohydrodynamic flow about solid bodies is still to be achieved. Our studies suggest that some of the most intriguing effects will be difficult to see at laboratory scale because of the small available values of R_m. One hopes, nevertheless, that upon the occasion of another international symposium on the subject experimental proof or disproof of the generalizations of this paper will be available.

DISCUSSION

Session Reporter: G. Kuerti

S. I. Pai, *University of Maryland, College Park, Maryland:* I would like to add, as a case of practical interest, the case of a strongly magnetized thin body in a uniform flow of a compressible fluid of small conductivity. In this case

$$(1-M^2)\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 Q}{\partial y^2}+\frac{\partial^2 Q}{\partial z^2}=R_M F_Q. \qquad (1)$$

Here the undisturbed flow is in x direction (Mach number M) and Q denotes any of the perturbed velocity components or the perturbed density; F_Q is a known function of the magnetic field of the body; $R_M=R_\sigma R_H$ with the magnetic Reynolds number $R_\sigma \ll 1$, and the magnetic pressure number $R_H=(\mu_e H)^2/\rho U^2 \cong 1$. The general solution of (1) is $Q=Q_0+R_M Q_1$, where Q_0 is the solution of the corresponding gas-dynamical problem.