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Local Conservation Laws in Generally Covariant Theories

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I. INTRODUCTION

History of Conservation Laws

CONSERVATION laws in physics are almost as old as the Newtonian revolution. Originally conceived when physics dealt only with particles, their usefulness continued after the advent of field physics, of special relativity, and even of quantum mechanics. It is therefore not surprising that considerable attention has been paid to the question of the application of conservation laws to theories, such as general relativity, which are covariant under general coordinate transformations. It has been hoped that such laws would be just as beneficial in the application of and the understanding of such theories as they have been elsewhere. In particular, great interest has centered around the problem of finding the laws analogous to the conservation of momentum, of angular momentum, and especially of energy, which are familiar in theories (such as special relativity) which are not generally covariant. Today, nearly fifty years after the advent of general relativity, no general agreement has been reached on this problem; the appropriate form that each of these laws should take is still a subject of controversy.

The original suggestion as to an energy-momentum conservation law for general relativity was made by Einstein when the notion of general relativity was scarcely a year old (21, 22).† As was immediately noted, there are two curious features of Einstein's law, features not present in Lorentz-covariant theories. First, the conserved quantity does not constitute a tensor nor a tensor density. This enabled Schroedinger (50) and Bauer (5) to show that the value of the total energy as derived from Einstein's law could be radically altered merely by performing a coordinate transformation. Second, the values for the total energy and momentum, defined as integrals over a three-dimensional initial surface, could be rewritten as integrals over the two-dimensional boundary of that surface.[1] It was discovered that these two features were inherent in the nature of the theory.[2]

Einstein's energy-momentum law was generally accepted as *the* energy-momentum law for general relativity for over thirty years. During this period, however, alternative expressions for that law were advanced such as those of Klein (36) and Lorentz (40). Then about 1950, Landau and Lifschitz (39, p. 318) suggested a new expression which had the advantage of being "symmetrized." The existence of more than one expression supposedly representing the energy was a cause of some disturbance. This disturbance became chaos in 1958 when the number of alternative expressions suddenly increased to infinity. This was due largely to the efforts of Goldberg, who listed an infinite number in a single paper (32). Expressions suggested by Møller (43), Bergmann (11), Komar (37), and Dirac (19) also added to the general confusion. This rapid multiplication of energy expressions motivates the work in the present paper.

Statement of Purpose

The present paper examines the situation created by the presence of so many proposed energy laws and attempts to understand it. The connection between these various laws is illuminated. An inquiry is made into the *uses* of conservation laws. Such an inquiry is obviously much needed, for no other way is evident to decide which, if any, of the proposed laws is the correct energy law and of what significance the other laws might be.[3]

Conservation laws may be put to two general kinds of uses. First, the numerical values of the conserved quantities may be used in various ways. For example, they may be used as a "check-up service": Two sets of physical data are given, one representing the initial state of a physical system, the other an alleged final state of the system. One calculates the total energy represented by each set of data; if they are not the same in value, one concludes that the two sets of data do not, as alleged, describe the same physical system. Second, conservation laws may be put to mathematical uses. Such uses depend on the specific way in which the conservation law is written; in such uses the conservation laws are regarded as interesting rewritings of the equations of motion, rewritings which may provide new physical insights. For example, the Einstein-Infeld-Hoffman treatment of the motion of sources of the gravitational field depends upon the field equations being rewritten in certain ways which are suggested from the consideration of conservation laws.

The conclusion of this paper is that the uses of the

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† References in parentheses will be found in alphabetical order at end of article.

[1] The explicit expression for the quantity to be integrated over the two-surface was first given by Freud (29). See also Zatzkis (58).

[2] See Secs. III and VI.

[3] Similar sentiments have been expressed by Eddington (20) and more recently by Dirac (19).

values of conserved quantities in generally covariant theories are far more limited than in other theories. Some uses cannot in fact be made at all. On the other hand, mathematical uses are quite appropriate in generally covariant theories and have a value which has not as yet been fully appreciated.

Outline of Subsequent Sections

The remainder is devoted to showing the relation between the various conservation laws in generally covariant theories and to examining the usefulness of these laws. Sections II through VI are concerned with various concepts and theorems relating to conservation laws. Section II reviews some of the basic facts about the kinds of theories under discussion. It introduces much of the notation and describes some particular theories that are used as examples. The Lagrange formalism is briefly reviewed.

Section III introduces and describes the notion of a local conservation law, pointing out the contrasts between strong and weak laws. Covariant properties of conservation laws are discussed. Section IV discusses the idea of symmetry operations of a Lagrangian, reviewing those important infinitesimal symmetry operations, the infinitesimal coordinate and similarity transformations.

Section V ties together the two preceding sections by showing how conservation laws may be derived from a knowledge of the infinitesimal symmetry operations of the Lagrangian. Examples of this procedure are given. The converse problem of whether all conservation laws follow from symmetries of the Lagrangian is discussed in greater detail than has been done previously. The section concludes with the description of a second method of obtaining conservation laws which proves of value in Sec. VIII.

Section VI proves Noether's theorem (44), which states that all the conservation laws characteristic of generally covariant theories are strong laws. It explicitly displays the superpotentials in greater generality than has been done before.

Section VII draws upon the results of Secs. II through VI and uses them to establish the conclusions. It casts light upon the connection between the various conservation laws in generally covariant theories, and shows the limitations upon the usefulness of the values of conserved quantities. Specifically, it shows the limitations upon the use of these values as first integrals of the equations of motion and as definitions of experimental quantities. The most suitable use of these values is found to be as indicators of the Schwarzschild mass, a use possible only in certain limited cases. The conservation laws suitable for this use are explicitly indicated. It then considers mathematical uses of conservation laws. Their uses in relation to the Hamiltonian and the problem of motion are mentioned briefly. Then in Sec. VIII three new mathematical uses of conservation laws are described—their use in finding the source terms of the gravitational field, in showing a connection between gauges and constraints, and in the Schwinger formalism for quantum mechanics.

II. BASIC CONCEPTS

Nature of the Theories to be Discussed

The methods used here are sufficiently general to be applicable to any physical theory likely to be encountered at the present time. But, in order to make things definite, it is now stated just what sort of a theory will be considered. First, the theory employs a d-dimensional, locally Euclidean, topological space, called *physical space*. This is just the sort of space that is most familiar; Euclidean, Minkowskian, in fact any Riemannian spaces are common examples. The points in this space are labeled by a set of real coordinates x^μ, $\mu=1, \cdots, d$ which vary continuously from point to point. Certain theorems of topology show, in general, that one nonsingular coordinate system does not suffice for an entire space; instead, several coordinate "patches" have to be employed. For example, a space having the topology of the surface of a sphere or of a Euclidean space with "handles" (or "worm-holes") requires at least two such patches. However, in the important case of a space having the topology of Euclidean space, one coordinate system is sufficient.

Second, the theory employs a set of quantities, the *field variables* χ^A which are defined at every point of physical space. When necessary for clarity, dependence of the χ^A on the coordinates is explicitly shown: $\chi^A(x^\mu)$. Although much of what is said is more widely applicable, the discussion is limited specifically to classical (nonquantum) theories for which the values which the $\chi^A(x^\mu)$ may assume are real or complex numbers.

Finally, the theory distinguishes certain of the field variables as being the *fundamental field variables* Φ^A, $A=1, \cdots, F$, and employs a certain specified function of these fundamental variables and their derivatives (with respect to the coordinate labels), the *Lagrangian*

$$L\equiv L(x^\mu)\equiv L[\Phi^A(x^\mu),x^\mu],$$

to obtain in a familiar manner[4] (which is reviewed briefly below) *equations of motion*, which are differential equations relating the Φ^A. In general, no assumptions are made regarding the orders of the derivatives appearing in either the Lagrangian or the equations of motion.

Riemannian Space

In order to illustrate many of the remarks made in this paper, certain particular theories are chosen as examples. All these theories employ a four-dimensional Riemanian physical space with metric signature $(1,1,1,-1)$. The next few paragraphs review some pertinent facts about Riemanian spaces,[5] introducing

[4] See, for example, Goldstein (33), especially Chap. II.

[5] There are many references on the subject of Riemannian geometry. See, for example, Synge and Schild (55).

the notations used. The role of spinors is given emphasis so as to make it clear which of the several spinor formalisms in general use today is used here.[6] The discussion stresses the similarity between the treatment of the derivatives of tensors and the derivatives of spinors.

Every object in a Riemannian space with spinors has both a tensor character and a spin character. The tensor character of an object determines its transformation properties under coordinate transformations; its spin character determines its behavior under similarity transformations. As regards spin character, most important are nonspinors, which are unchanged by similarity transformation; ordinary column spinors ψ, which under similarity transformation by a matrix S go to $S\psi$; ordinary row spinors $\bar{\psi}$, which go to $\bar{\psi}S^{-1}$; ordinary matrices M, which go to SMS^{-1}; and spin densities ϕ of weight W', which go to $(\det S)^{W'}\phi$.

The basic geometric objects are the (nonspinor) metric tensor $g_{\mu\nu}$ and the (ordinary matrix) metric vector (the Dirac matrices) γ_μ, which are related by the anticommutation relation,

$$\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}I, \tag{2.1}$$

where I is the unit matrix. Indices are lowered and raised in the usual fashion by $g_{\mu\nu}$ and its contravariant form $g^{\mu\nu}$. The determinant of $g_{\mu\nu}$ is called g. The summation convention for repeated indices is used.

Since the partial derivative [symbolized with a subscript comma (,)] of a nonspinor tensor with respect to a coordinate label does not, in general, transform like a tensor, a covariant derivative [symbolized with a subscript semicolon (;)], which so transforms, is defined thus:

For a scalar density of weight W,

$$D_{;\mu}=D_{,\mu}-WD\Gamma_{\mu\lambda}{}^{\lambda}. \tag{2.2a}$$

For a covariant vector,

$$V_{\nu;\mu}=V_{\nu,\mu}-V_\lambda\Gamma_{\nu\mu}{}^{\lambda}. \tag{2.2b}$$

For a contravariant vector,

$$V^{\nu}{}_{;\mu}=V^{\nu}{}_{,\mu}+V^{\lambda}\Gamma_{\lambda\mu}{}^{\nu}. \tag{2.2c}$$

For a general tensor density, the appropriate formula can be obtained by use of these formulas, and of the fact that covariant differentiation is linear and distributive when applied to a sum of products. The object $\Gamma_{\nu\mu}{}^{\lambda}$ is the affine connection (Christoffel symbol) and is determined by the requirements that it be symmetric in $\nu\mu$ and that

$$g_{\mu\nu;\lambda}\equiv g_{\mu\nu,\lambda}-g_{\mu\kappa}\Gamma_{\nu\lambda}{}^{\kappa}-g_{\nu\kappa}\Gamma_{\mu\lambda}{}^{\kappa}=0. \tag{2.3}$$

This implies that

$$\Gamma_{\mu\nu}{}^{\lambda}=\tfrac{1}{2}g^{\lambda\kappa}(g_{\kappa\nu,\mu}+g_{\kappa\mu,\nu}-g_{\mu\nu,\kappa}). \tag{2.4}$$

[6] This treatment is given by Bargmann (4).

Since the covariant derivative of an ordinary spinor does not, in general, transform like an ordinary spinor, a total derivative [symbolized with a subscript stroke (|)], which does so transform, is defined thus:

For an ordinary column spinor (of arbitrary tensor character),

$$\psi_{|\mu}=\psi_{;\mu}-\Gamma_\mu\psi. \tag{2.5a}$$

For an ordinary row spinor,

$$\bar{\psi}_{|\mu}=\bar{\psi}_{;\mu}+\bar{\psi}\Gamma_\mu. \tag{2.5b}$$

For an ordinary matrix,

$$M_{|\mu}=M_{;\mu}-[\Gamma_\mu,M]. \tag{2.5c}$$

For a spin density of weight W',

$$\phi_{|\mu}=\phi_{;\mu}-W'(\mathrm{Tr}\Gamma_\mu)\phi. \tag{2.5d}$$

The second of these relations follows from the first, and the third from the first two, in consequence of the requirements that total differentiation is linear and distributive when applied to a sum of products and that the total derivative of a nonspinor equals its covariant derivative. The object Γ_μ is the affine spinor connection (spin-Christoffel symbol) and satisfies

$$\gamma_{\nu|\mu}\equiv\gamma_{\nu;\mu}-[\Gamma_\mu,\gamma_\nu]=0. \tag{2.6}$$

It is assumed here, in order to simplify the discussion, that an irreducible representation of the γ_μ is being employed. Then (2.6) determines Γ_μ entirely, except that an arbitrary multiple of the unit matrix may be added to it. The explicit expression for Γ_μ is now given without proof.[7]

First, it is necessary to define an operator ξ which acts on matrices:

$$\xi M\equiv\tfrac{1}{2}(dM-\gamma_\mu M\gamma^\mu), \tag{2.7a}$$

where d (as usual) is the dimension of the Riemannian space. It can be shown that, for all matrices M,

$$\prod_{k=0}^{d}(\xi-k)M=0. \tag{2.7b}$$

By employing this fact, one may find a unique polynomial of degree $(d-1)$ in the operator ξ (symbolized as $1/\xi$) such that

$$(1/\xi)(\xi M)=M-(\mathrm{Tr}M/\mathrm{Tr}I)I. \tag{2.7c}$$

One may now write the expression for Γ_μ:

$$\Gamma_\mu=\tfrac{1}{2}(1/\xi)(\gamma_{\nu;\mu}\gamma^\nu)+ieA_\mu I. \tag{2.8}$$

The vector A_μ may be identified as the vector potential of the electromagnetic field (which thus enters the theory in a natural way) and e as the charge of the field characterized by the spinors $\bar{\psi}$ and ψ.

Some other important geometrical quantities are:

[7] Proofs are given by Fletcher (28).

the Riemann curvature tensor,

$$R^{\kappa}{}_{\lambda\mu\nu}=\Gamma_{\lambda\nu}{}^{\kappa}{}_{,\mu}-\Gamma_{\lambda\mu}{}^{\kappa}{}_{,\nu}+\Gamma_{\iota\mu}{}^{\kappa}\Gamma_{\lambda\nu}{}^{\iota}-\Gamma_{\iota\nu}{}^{\kappa}\Gamma_{\lambda\mu}{}^{\iota}; \quad (2.9a)$$

the Ricci tensor,[8]

$$R_{\mu\nu}=R^{\lambda}{}_{\mu\lambda\nu}; \quad (2.9b)$$

the scalar curvature,

$$R=g^{\mu\nu}R_{\mu\nu}; \quad (2.9c)$$

the electromagnetic field tensor,

$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}. \quad (2.9d)$$

Typical Theories

The first general kind of theory to be used as an example is a Lorentz-covariant theory (Riemannian space is flat) in which no higher than first derivatives of the fundamental variables appear in the Lagrangian. Such a theory is called an ordinary Lorentz theory. Specific examples are:

(1) The electromagnetic (or Maxwell) field. The Lagrangian is

$$L_E=-\tfrac{1}{4}(-g)^{\frac{1}{2}}F_{\mu\nu}F^{\mu\nu}. \quad (2.10a)$$

The fundamental field variables are the covariant components of the vector potential A_μ.

(2) The lepton (or Dirac) field. The Lagrangian is

$$L_D=(-g)^{\frac{1}{2}}(-\tfrac{1}{2}\bar{\psi}\gamma^\mu\psi_{|\mu}+\tfrac{1}{2}\bar{\psi}_{|\mu}\gamma^\mu\psi-m\bar{\psi}\psi). \quad (2.10b)$$

The fundamental variables are the components of $\bar{\psi}$ and ψ. The quantity m is a constant, the mass of the field. If L_D is used alone, the vector potential must be viewed as a given, fixed quantity. On the other hand, L_D and L_E may be added to give the Lagrangian of an interacting Maxwell-Dirac field.

(3) The neutral scalar meson (or Klein-Gordon) field. The Lagrangian is

$$L_K=-\tfrac{1}{2}(-g)^{\frac{1}{2}}(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}+\mu^2\phi^2). \quad (2.10c)$$

The fundamental field variable is ϕ. The quantity μ is a constant, the mass of the field. In the theories mentioned in this paragraph, $g_{\mu\nu}$ and γ_μ are given fixed quantities.

The second general kind of theory to be used as an example is a generally covariant theory with the Lagrangian

$$L_G=(-g)^{\frac{1}{2}}R+L_M. \quad (2.10d)$$

The term L_M is the Lagrangian of matter. It may assume different forms depending upon the exact theory under consideration. It may, for example, be equal to zero; one then has the pure gravitational (or Einstein) field. It may equal L_E; one then has the Rainich field. In fact it may be any one, the sum of any two, or the sum of all three of L_E, L_D, and L_K, as well as many other possibilities. The fundamental variables are the fundamental variables appropriate to the Lagrangians making up L_M plus the components of the contravariant metric. In addition, if a Lagrangian involving spinors (such as L_D) appears in L_M, suitable independent variables describing the Dirac matrices must be included among the fundamental field variables; the components of the Dirac matrices may themselves be so used, provided suitable precautions are made for taking into account the fact that they are not all independent of one another nor of the metric tensor.

These Lagrangians have all been expressed in a system of units so selected that the numerical values of the velocity of light (c), Planck's constant divided by 2π ($\hbar$), and 16π times Newton's gravitational constant ($16\pi G$) are all equal to one. This means that the length unit is 1.1455×10^{-32} cm, the time unit is 3.821×10^{-43} sec, and the mass unit is 3.071×10^{-6} g.[9]

Lagrangian Formalism

Let an infinitesimal change or *variation* in the Φ^A as functions of the x^μ be considered. This variation is symbolized by δ. A way to view this symbol is as follows: The Φ^A are imagined to be functions of a parameter α as well as of the x^μ; then $\delta\Phi^A\equiv(\partial\Phi^A/\partial\alpha)d\alpha$. Clearly, the variation of the Lagrangian will be linear in the variations of the Φ^A and of the various derivatives of the Φ^A; hence, by repeated use of the expression for the derivative of a product, it may be written as[10]

$$\delta L=\Delta^\nu{}_{,\nu}-M_A\delta\Phi^A. \quad (2.11)$$

The condition that $-M_A\delta\Phi^A$ vanish for all $\delta\Phi^A$ determines the equations of motion. If the $\delta\Phi^A$ are all independent, this implies that all $M_A=0$. If the $\delta\Phi^A$ are not all independent but have certain algebraic identities among them (as, for example, is the case for $g^{\mu\nu}$ and γ^μ), the equations of motion are still uniquely determined and can be found, for example, with the use of Lagrange multipliers.

Two facts should be noted regarding (2.11). First, changing L by adding to it a term of the form $L^\nu{}_{,\nu}$ does not change the equations of motion but merely adds a term δL^ν to Δ^ν. Second, Δ^ν is not determined uniquely by (2.11) but only up to an arbitrary additive term of the form $\Delta^{[\nu\mu]}{}_{,\mu}$. [A pair of square brackets enclosing a number of indices means that the indicated expression should be completely skew-symmetrized in the enclosed indices, i.e., should be replaced by the sum of all expressions obtained from the indicated one by even permutation of the indices minus the sum of all ex-

[8] The convention as to the sign of $R_{\mu\nu}$ has been chosen so that the curvature R of a d-sphere is positive. This is the convention employed by Landau and Lifschitz (39) and Misner and Wheeler (41). It differs from that employed by Bergmann (9), Einstein (24), Synge and Schild (55), and Møller (42). Misner and Wheeler, and Møller, incidentally, choose the sign of $R^{\kappa}{}_{\lambda\mu\nu}$ to be opposite to that given here.

[9] These computations are based on the values of c and $\hbar$ given by Strominger, Hollander, and Seaborg (54) and on the value of G given in *Handbook of Chemistry and Physics* (59).

[10] The mathematics of Lagrangians with higher than first derivatives of the field variables is discussed in some detail by Chang (16).

pressions obtained by odd permutation of the indices all divided by the number of such expressions (equals the factorial of the number of indices). Similarly, a pair of parentheses enclosing a number of indices means that the indicated expression should be completely symmetrized in the enclosed indices, an operation differing from skew-symmetrization only in that the sum of the odd permutations is added to, rather than subtracted from, the sum of the even permutations.]

In the case in which L equals L_G (2.10d) with L_M equal to $L_E+L_D+L_K$, the result for Δ^ν [11] is

$$\begin{aligned}\Delta_G{}^\nu=&(-g)^{\frac{1}{2}}g^{\kappa\lambda}\delta(\Gamma_{\kappa\lambda}{}^\nu-\tfrac{1}{2}\delta_\kappa{}^\nu\Gamma_{\lambda\mu}{}^\mu-\tfrac{1}{2}\delta_\lambda{}^\nu\Gamma_{\kappa\mu}{}^\mu)\\&+(-g)^{\frac{1}{2}}F^{\mu\nu}\delta A_\mu-\tfrac{1}{2}(-g)^{\frac{1}{2}}\bar{\psi}\gamma^\nu\delta\psi+\tfrac{1}{2}(-g)^{\frac{1}{2}}\delta\bar{\psi}\gamma^\nu\psi\\&-(-g)^{\frac{1}{2}}g^{\nu\mu}\phi_{,\mu}\delta\phi+\tfrac{1}{8}(-g)^{\frac{1}{2}}\bar{\psi}\\&\times\{\gamma^\nu,(1/\xi)[\delta\gamma_\mu,\gamma^\mu]\}\psi,\end{aligned}\quad(2.12)$$

while for $M_A\delta\Phi^A$ it is

$$\begin{aligned}M_A{}^G&\delta\Phi^A\\=&\tfrac{1}{2}(-g)^{\frac{1}{2}}[-2(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)+(F_{\mu\lambda}F_\nu{}^\lambda-\tfrac{1}{4}g_{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda})\\&+g_{\mu\nu}(-\tfrac{1}{2}\bar{\psi}\gamma^\lambda\psi_{|\lambda}+\tfrac{1}{2}\bar{\psi}_{|\lambda}\gamma^\lambda\psi-m\bar{\psi}\psi)\\&+(\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}g^{\kappa\lambda}\phi_{,\kappa}\phi_{,\lambda}-\tfrac{1}{2}g_{\mu\nu}\mu^2\phi^2)]\delta g^{\mu\nu}\\&+(-g)^{\frac{1}{2}}[F^{\nu\mu}{}_{;\mu}-ie\bar{\psi}\gamma^\nu\psi]\delta A_\nu\\&+(-g)^{\frac{1}{2}}[-\bar{\psi}_{|\mu}\gamma^\mu+m\bar{\psi}]\delta\psi+(-g)^{\frac{1}{2}}\delta\bar{\psi}[\gamma^\mu\psi_{|\mu}+m\psi]\\&+\tfrac{1}{2}(-g)^{\frac{1}{2}}[\bar{\psi}\delta\gamma^\nu\psi_{|\nu}+\tfrac{1}{4}\bar{\psi}\{\gamma^\mu,(1/\xi)[\delta\gamma^\nu,\gamma_\nu]\}\psi_{|\mu}\\&-\bar{\psi}_{|\nu}\delta\gamma^\nu\psi+\tfrac{1}{4}\bar{\psi}_{|\mu}\{\gamma^\mu,(1/\xi)[\delta\gamma^\nu,\gamma_\nu]\}\psi]\\&+(-g)^{\frac{1}{2}}[-g^{\mu\nu}\phi_{;\mu\nu}+\mu^2\phi]\delta\phi.\end{aligned}\quad(2.13)$$

It is clear which portions of these expressions are contributed by each portion of L_G; thus, in effect, the Δ^ν and $M_A\delta\Phi^A$ appropriate to each Lagrangian (2.10) have been indicated. The equations of motion corresponding to δA_ν, $\delta\bar{\psi}$, $\delta\psi$, and $\delta\phi$ are obvious from (2.13). This is not so for the equations corresponding to $\delta g^{\mu\nu}$ and $\delta\gamma^\nu$, since these quantities are not independent[12]; however, if L_D were absent, the equations corresponding to $\delta g^{\mu\nu}$ *would* be obvious. As noted above, (2.12) is not unique. For example, one might add $(A^\mu\delta A^\nu-A^\nu\delta A^\mu)_{,\mu}$ to it. To do so would certainly render Δ_E into an unfamiliar form, but it would be a form *no less valid* than the one given.

The most general form Δ^ν can take is clearly

$$\Delta^\nu=p^B\mathfrak{A}_B{}^\nu{}_C\delta q^C,\quad(2.14)$$

where the $\mathfrak{A}_B{}^\nu{}_C$ are a set of constant quantities and p^B (called canonical momenta) and q^C (called canonical coordinates) are functions of Φ^A and their derivatives. In the special case of an ordinary Lorentz Lagrangian, one finds that $q^C=\Phi^C$ and $p^B\mathfrak{A}_B{}^\nu{}_C=(\delta L/\delta\Phi^C{}_{,\nu})$, but these relations do not hold in general. The p^A, q^A, and Φ^A are all special instances of the field variables χ^A. Hence, a list of χ^A, $A=1,\cdots,{}^*F$, may be prepared, which includes all p^A, all q^A, and independent algebraic functions of all the Φ^A. Equation (2.11) may now be rewritten as

$$\delta L=(\chi^B\mathfrak{A}_B{}^\nu{}_C\delta\chi^C)_{,\nu}-M_A\delta\chi^A,\quad(2.15)$$

where appropriate elements in the now enlarged matrix $\mathfrak{A}_B{}^\nu{}_C$ vanish as do appropriate linear combinations of the M_A. If one now defines the *pseudo-Hamiltonian* as

$$H=\chi^B\mathfrak{A}_B{}^\nu{}_C\chi^C{}_{,\nu}-L,\quad(2.16)$$

then it follows that

$$\delta H=\delta\chi^B[2\mathfrak{A}_{[B}{}^\nu{}_{C]}\chi^C{}_{,\nu}+M_B].\quad(2.17)$$

This last expression shows that H is a function of the χ^A only and not their derivatives. By applying the condition $M_B\delta\chi^B=0$, one obtains Hamilton's equations of motion which are satisfied if and only if the Lagrange equations are satisfied. In particular, if all the $\delta\Phi^A$ were independent, Hamilton's equations are

$$N_B\equiv2\mathfrak{A}_{[B}{}^\nu{}_{C]}\chi^C{}_{,\nu}-\partial H/\partial\chi^B=0.\quad(2.18)$$

Hamilton's equations may be derived as the Lagrange equations corresponding to

$${}^*L=\chi^B\mathfrak{A}_B{}^\nu{}_C\chi^C{}_{,\nu}-H(\chi^A),\quad(2.19)$$

with the χ^A as fundamental field variables. By adding the term $-(\chi^B\mathfrak{A}_{(B}{}^\nu{}_{C)}\chi^C)_{,\nu}$ to ${}^*L$, a new Lagrangian is produced which yields the same equations of motion but for which $\mathfrak{A}_B{}^\nu{}_C=\mathfrak{A}_{[B}{}^\nu{}_{C]}$; such a Lagrangian is called a Schwinger (51, 52) Lagrangian. It has thus been shown that any set of equations of motion derived from a Lagrangian can be derived from a Schwinger Lagrangian.

The Hamiltonian variables χ^A appropriate to the Lorentz covariant Lagrangians L_E, L_D, and L_K (2.10a, b, c) are, respectively, $F^{\mu\nu}$ and A_μ, $\bar{\psi}$ and ψ, and ϕ and $p^\nu=-g^{\mu\nu}\phi_{,\mu}$. The associated Schwinger Lagrangians are, respectively,

$${}^*L_E=\tfrac{1}{2}F^{\mu\nu}A_{\mu,\nu}-\tfrac{1}{2}A_\mu F^{\mu\nu}{}_{,\nu}+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu},\quad(2.20a)$$

$${}^*L_D=-\tfrac{1}{2}\bar{\psi}\gamma^\nu\psi_{|\nu}+\tfrac{1}{2}\bar{\psi}_{|\nu}\gamma^\nu\psi-m\bar{\psi}\psi,\quad(2.20b)$$

$${}^*L_K=\tfrac{1}{2}p^\nu\phi_{,\nu}-\tfrac{1}{2}p^\nu{}_{,\nu}\phi+\tfrac{1}{2}p^\mu p_\mu-\tfrac{1}{2}\mu^2\phi^2.\quad(2.20c)$$

Note that ${}^*L_D$ is identical with L_D. These have not been written in a form suitable for immediate generalization to generally covariant theories since the Schwinger Lagrangian corresponding to L_G is given here only for $L_M=0$. The reader may consider for himself other cases; when L_M includes spinors, the situation becomes quite complex. For the Einstein Lagrangian the Hamiltonian variables are[13]

$$\mathfrak{g}^{\mu\nu}=(-g)^{\frac{1}{2}}g^{\mu\nu},\quad\mathfrak{G}_{\mu\nu}{}^\lambda=\Gamma_{\mu\nu}{}^\lambda-\tfrac{1}{2}\delta_\mu{}^\lambda\Gamma_{\nu\kappa}{}^\kappa-\tfrac{1}{2}\delta_\nu{}^\lambda\Gamma_{\mu\kappa}{}^\kappa.$$

[11] The only terms in (2.12) and (2.13) which require more than trivial effort to obtain are those involving $\delta g^{\mu\nu}$ and $\delta\gamma^\nu$. In regard to the terms involving $\delta g^{\mu\nu}$, a simplified derivation is given by Landau and Lifschitz (39, p. 297). For the derivation of the terms involving $\delta\gamma^\nu$, a useful relation is

$$\delta\Gamma_\mu=ie\delta A_\mu I+\tfrac{1}{4}\{(1/\xi)[\delta\gamma_\nu,\gamma^\nu]\}_{|\mu}+\tfrac{1}{8}[\gamma^\nu,\gamma_\lambda]\delta\Gamma_{\nu\mu}{}^\lambda.$$

[12] This problem is discussed in Sec. VIII.

[13] The choice of $\mathfrak{G}_{\mu\nu}{}^\lambda$ instead of $\Gamma_{\mu\nu}{}^\lambda$ is found to be more convenient in the sequel (Sec. VIII). It is less convenient from the point of view of familiarity.

The Schwinger Lagrangian is

$$*L_G=\tfrac{1}{2}\mathfrak{g}^{\mu\nu}\mathfrak{G}_{\mu\nu}{}^{\lambda}{}_{,\lambda}-\tfrac{1}{2}\mathfrak{g}^{\mu\nu}{}_{,\lambda}\mathfrak{G}_{\mu\nu}{}^{\lambda}-\mathfrak{g}^{\mu\nu}\mathfrak{G}_{\lambda\mu}{}^{\kappa}\mathfrak{G}_{\kappa\nu}{}^{\lambda}+\tfrac{1}{3}\mathfrak{g}^{\mu\nu}\mathfrak{G}_{\mu\lambda}{}^{\lambda}\mathfrak{G}_{\nu\kappa}{}^{\kappa}. \qquad (2.20\text{d})$$

There is no need to write out the Δ^ν nor the equations of motion corresponding to these Lagrangians here, since they can be obtained by inspection.

III. LOCAL CONSERVATION LAWS

Differential Form

A *local conservation law* is an expression of the form,

$$S^\nu{}_{,\nu}=0,\ddagger \qquad (3.1)$$

where S^ν, like the Lagrangian, is a function of the Φ^A and their derivatives at the point to which S^ν refers. In order that both S^ν and Eq. (3.1) be generally covariant quantities (which is not, however, always required even in generally covariant theories), S^ν must be a totally skew-symmetric contravariant tensor density of weight one with ν as one of its tensor indices. It is a simple matter to construct an unlimited number of quantities S^ν satisfying (3.1). One merely has to pick any function of the Φ^A and their derivatives of the form $S^{[\nu\mu]}$; then the choice

$$S^\nu=S^{[\nu\mu]}{}_{,\mu} \qquad (3.2)$$

will always satisfy (3.1). The quantities $S^{[\nu\mu]}$ are called superpotentials. If S^ν can be written in the form (3.2), then (3.1) is called a *strong conservation law*; otherwise, it is called a *weak conservation law*.[14] If S^ν is a totally skew-symmetric contravariant tensor density of weight one, then $S^{[\nu\mu]}$ can be taken to be a totally skew-symmetric contravariant tensor density of weight one and rank one higher than S^ν.

It might be supposed at first glance that all local conservation laws are strong; this would seem to be a reasonable generalization of the statement that a vector of vanishing divergence can always be written locally as the curl of some other vector. This supposition is, however, erroneous. All S^ν satisfying (3.1) can be written as the divergence of a skew-symmetric quantity, but this quantity is not necessarily a function of the Φ^A and their derivatives at the point of definition of S^ν; it may involve primitive integrals over the Φ^A.

‡ *Note added in proof.*—It is perhaps well to point out that some writers (e.g., Lichnerowicz, Bel) call an expression of the form $S^\nu{}_{;\nu}$ a conservation law. While such expressions are useful, they do not possess a global form, and in any event are not under discussion here.

[14] The terms "strong" and "weak" were originated by Bergmann (10). Their use here differs slightly from that of Bergmann, however. Here a law is called strong if it can be written in the form (3.2) when the equations of motion are satisfied, while Bergmann seems to distinguish between strong and weak forms of writing the laws here called strong laws, the strong form always holding, the weak form holding only when the equations of motion are satisfied. This seems an unnecessary distinction to the present author, since a strong law may always be made artificially weak in Bergmann's sense by adding to S^ν a function which vanishes when the equations of motion are satisfied. Examples of strong and weak laws are given in Sec. V.

Integral Form

There are two views of physical space which one may take when applying the equations of motion. First, there is what might be called the "broadside" view. The entire physical space, populated with the field variables, is examined point by point to see whether the field variables satisfy the equations of motion. The pleasant feature of this viewpoint is that it does not destroy any symmetries which may exist among the roles of the different coordinates. However, it is a view of limited practical value. One is not often in the position of being able to look at an entire physical space and "check up" on it point by point to see whether the equations of motion are satisfied. Rather, one usually takes the second or "edge-on" view. A family of $(d-1)$-dimensional surfaces is selected as constituting a distinguished set of surfaces, the *initial*[15] surfaces. Each initial surface divides space into two parts, a *past* and a *future*. One observes the values of the field variables on a particular initial surface and then, by assuming the equations of motion to be satisfied, predicts the values of the field variables in the future (and also infers their values in the past).

Let there be defined now

$$S(\omega)\equiv\int_\omega S^\nu d\sigma_\nu, \qquad (3.3)$$

where ω is a (in general, bounded) $(d-1)$-dimensional surface in physical space with differential extension $d\sigma_\nu$. *In order that the value of $S(\omega)$ be a generally covariant concept, S^ν must be a contravariant vector density of weight one, and $S(\omega)$ must therefore necessarily be a scalar.*[16] Of the familiar examples of quantities $S(\omega)$, namely, energy, momentum, angular momentum, and charge,[17] only charge is a scalar. Hence, as mentioned in the introduction, energy and momentum in generally covariant theories are doomed to be quantities which are not generally covariant. If S^ν is of the form (3.2), it follows from Gauss' theorem that

$$S(\omega)=\bar{S}(\beta)\equiv\oint_\beta S^{[\nu\mu]}d\sigma_{\nu\mu}, \qquad (3.4)$$

where β is the $(d-2)$-dimensional boundary of ω with differential extension $d\sigma_{\nu\mu}$.

In the usual interpretation of the local conservation laws, (3.1) is integrated over a d-volume bounded by a $(d-1)$-surface ω which consists of three parts—parts of two initial surfaces ω_I and ω_F, and a cylinder-shaped surface ω_L which joins ω_I and ω_F along $(d-2)$-surfaces β_I and β_F. From Gauss' theorem it follows that

$$S(\omega_F)-S(\omega_I)=-S(\omega_L), \qquad (3.5)$$

[15] The objects here called "initial surfaces" also include the objects more conventionally called "final surfaces."

[16] See Synge and Schild (55, p. 276).

[17] These quantities are discussed in Sec. V.

where the sign conventions should be clear. Gauss' theorem can be applied, and therefore (3.5) is true only if S^ν is a contravariant vector density of weight one or if only one coordinate patch is employed throughout the region enclosed by ω (or if some suitable combination of these two conditions is realized). Similar remarks apply to (3.4). The quantities $S(\omega_F)$ and $S(\omega_I)$ are interpreted as the amount of a quantity S (e.g., energy, momentum, angular momentum, charge) contained in ω_F and ω_I and $S(\omega_L)$ as the amount which escapes through the surface ω_L. If (3.4) can be applied, (3.5) becomes quite trivial.

Many important uses of (3.5) occur when ω_F and ω_I constitute entire initial surfaces. When this is so, three cases may be distinguished.

Case 1.—The surfaces ω_I and ω_F are closed. (β_I, β_F, and ω_L do not exist.) Then (3.5) shows that $S(\omega_F) = S(\omega_I)$; i.e., S is *conserved.* This is the *global conservation law* derived from the local conservation law (3.1). If, in addition, the conservation law is strong [and hence (3.4) applies], then S is not only conserved, it vanishes as well.

Case 2.—The surfaces ω_I and ω_F are infinite in extent but coincide as infinity is approached. (β_I and β_F are identical; ω_L does not exist.) In this case S is also conserved. However, in this case, when the conservation law is strong, S does not necessarily vanish.

Case 3.—The surfaces ω_I and ω_F are infinite in extent and distinct at infinity. In this case S is conserved if and only if $S(\omega_L)$ vanishes as ω_L recedes to infinity; otherwise, $S(\omega_L)$ represents an escape of S "to infinity." If the conservation law is strong in this case, then whether or not S is conserved can be determined solely by studying the behavior of $S^{[\nu\mu]}$ in the infinite reaches of ω_I and ω_F. It is also possible that a combination of the conditions of two or all three of the above cases may occur, but such a situation leads to no essentially new considerations.

It is often convenient to consider the initial surfaces to be surfaces for which one coordinate is constant. (This excludes case 2.) This coordinate (which labels initial surfaces) is then called x^o, the *time*; the remaining coordinates (which label the points in these surfaces) will be indicated by a Latin index, e.g., x^m. Repeated Latin indices are summed over all values except o. With this notation, it is easy to express a limiting case of (3.5) as

$$(d/dx^o)S(\omega)\equiv(d/dx^o)\int_\omega S^o d\sigma_o = -\oint_\beta S^n d\sigma_{on}, \quad (3.6)$$

where the negative of the right-hand side is called the flux of S through β. If the conservation law is strong [and hence S^ν is given by (3.2)], this expression becomes

$$(d/dx^o)S(\omega)\equiv(d/dx^o)\oint_\beta S^{[om]}d\sigma_{om} = -\oint_\beta S^{[no]}{}_{,o}d\sigma_{on}, \quad (3.7)$$

which is rather trivial.

When S^ν is not a contravariant vector density of weight one (as it is not for the energy, momentum, and angular momentum laws), application of the foregoing results is usually limited to spaces of Euclidean topology, for which only one coordinate patch is necessary. Møller (43) has observed, however, that since in usual applications there is no need to "patch" the time coordinate, these results may be applied to the energy law (no matter what the topology of the initial surfaces) provided that the corresponding S^ν is selected so as to behave like a contravariant vector density of weight one under coordinate transformations of the coordinates within the initial surfaces (a behavior not compatible with the characteristics of momentum and angular momentum). He has explicitly displayed such an S^ν.[18]

IV. SYMMETRY OPERATIONS

Definition

Let it be supposed that a set of F functions $\Phi^{A'}(x^\mu) = \Phi^{A'}[\Phi^B(x^\mu)]$ of the field variables is found such that the following relation holds for a suitable choice of the function K^ν and of the $x^{\nu'}$ as functions of the x^μ:

$$L'[\Phi^A(x^{\nu'})]d\Omega' \equiv L[\Phi^{A'}(x^{\nu'})]d\Omega' = L[\Phi^A(x^\mu)]d\Omega + K^\nu[\Phi^A(x^\mu)]_{,\nu}d\Omega, \quad (4.1)$$

where $d\Omega$ and $d\Omega'$ are the differential extensions of physical space in the coordinate systems of x^μ and $x^{\nu'}$, respectively. The $\Phi^{A'}$ are then said to be related to the Φ^A by a *symmetry operation.*

For example, for the Lorentz-covariant Lagrangian of the electromagnetic field (2.10a), there are the following symmetry operations: space and time inversions, (inhomogeneous) restricted Lorentz transformations, and electromagnetic gauge transformations. Among these, an immediate distinction can be made: Space and time inversions are *discrete* operations; the others are *continuous.* In consequence of the existence of discrete symmetry operations, one may deduce certain facts often called conservation laws. Space inversions, for example, lead to the conservation of parity. These conservation laws, however, are, on the classical level, qualitatively different from the conservation laws (even the global ones) described in the last section, for one cannot assign to *every* field configuration, for example, a parity value in the way one can always assign an energy value. Discrete symmetry operations do not pertain to the subject of this paper.

What do pertain are the *infinitesimal symmetry operations* corresponding to the continuous symmetry opera-

[18] See Sec. VI.

tions, i.e., infinitesimal changes $\delta\Phi^A$ in the functional dependence of the Φ^A on the x^μ such that

$$\delta L \equiv L[\Phi^A + \delta\Phi^A] - L[\Phi^A] = K^\nu{}_{,\nu}, \qquad (4.2)$$

where here K^ν is an infinitesimal quantity. Note that K^ν is not uniquely determined by this equation but may have any term of the form $K^{[\nu\mu]}{}_{,\mu}$ added to it. If L is replaced by an L differing from it by a term $L^\nu{}_{,\nu}$, then δL^ν is added to K^ν.

Examples

The infinitesimal symmetry operations of most general interest are undoubtedly the infinitesimal coordinate transformations. For reference, some formulas relating to such transformations are now given. Under an infinitesimal coordinate transformation, a point formerly labeled as x^μ is relabeled as $x^\mu + \xi^\mu$, where ξ^μ is an infinitesimal vector. Let a scalar function Φ be considered; under transformation it becomes a new scalar function Φ' such that

$$\Phi'(x^\mu + \xi^\mu) = \Phi(x^\mu). \qquad (4.3)$$

By expanding the left side in a Taylor series, it follows that

$$\delta\Phi \equiv \Phi' - \Phi = -\Phi_{,\rho}\xi^\rho. \qquad (4.4a)$$

For objects other than scalars, there are terms in the expression for their variation in addition to the term that appears for scalars. These terms arise from their differing covariant transformation properties. For a scalar density of weight W, one finds that

$$\delta D = -D_{,\rho}\xi^\rho - WD\xi^\rho{}_{,\rho}, \qquad (4.4b)$$

since $-\xi^\rho{}_{,\rho}$ is the infinitesimal part of the Jacobian. For a contravariant and covariant vector, it follows, respectively, that

$$\delta V^\mu = -V^\mu{}_{,\rho}\xi^\rho + V^\rho\xi^\mu{}_{,\rho}, \qquad (4.4c)$$

$$\delta V_\mu = -V_{\mu,\rho}\xi^\rho - V_\rho\xi^\rho{}_{,\mu}. \qquad (4.4d)$$

The following may be verified directly: In Eqs. (4.4a, b, c, d) the partial derivatives may be replaced throughout by covariant derivatives. From (4.4b, c, d) it is clear what expression obtains for the variation of a tensor of arbitrary rank and density. In Riemannian space there is one important geometrical object which is not a tensor, namely, the Christoffel symbol. The variation of a Christoffel symbol, unlike the symbol itself, is a tensor[19] as the following shows for the case of an infinitesimal coordinate transformation:

$$\begin{aligned}\delta\Gamma_{\mu\nu}{}^\lambda &= -\Gamma_{\mu\nu}{}^\lambda{}_{,\rho}\xi^\rho + \Gamma_{\mu\nu}{}^\rho\xi^\lambda{}_{,\rho} - \Gamma_{\mu\rho}{}^\lambda\xi^\rho{}_{,\nu} \\ &\qquad - \Gamma_{\rho\nu}{}^\lambda\xi^\rho{}_{,\mu} - \xi^\lambda{}_{,\mu\nu} \\ &= -\xi^\lambda{}_{;\mu\nu} + R^\lambda{}_{\mu\nu\rho}\xi^\rho.\end{aligned} \qquad (4.4e)$$

In a generally covariant theory, infinitesimal coordinate transformations are symmetry operations for arbitrary (coordinate dependent) choices of ξ^ρ; in a Lorentz-covariant theory, however, ξ^ρ must be limited to the form $a^\rho + a_\mu{}^\rho x^\mu$, where a^ρ and $a_\mu{}^\rho$ are coordinate independent and $a_{\rho\mu}$ (the index being lowered with the Lorentz metric) is skew-symmetric in $\rho\mu$.

Another important group of infinitesimal symmetry operations is the infinitesimal similarity transformations of spinors, i.e., similarity transformations by a matrix $I + \Sigma$, where Σ is an infinitesimal ordinary matrix. Under such transformations, ordinary column spinors, row spinors, matrices, and spin-densities of weight W' transform as

$$\delta\psi = \Sigma\psi, \qquad (4.5a)$$

$$\delta\bar{\psi} = -\bar{\psi}\Sigma, \qquad (4.5b)$$

$$\delta M = [\Sigma, M], \qquad (4.5c)$$

$$\delta\phi = W'(\mathrm{Tr}\Sigma)\phi. \qquad (4.5d)$$

Also of interest is the behavior of the spin-Christoffel symbol. In this instance its variation, unlike itself, is an ordinary matrix, namely,

$$\delta\Gamma_\mu = [\Sigma, \Gamma_\mu] + \Sigma_{,\mu} = \Sigma_{|\mu}. \qquad (4.5e)$$

Electromagnetic gauge transformations are included among the similarity transformations as a special case, in fact,[20]

$$\delta A_\mu = (-i/e\ \mathrm{Tr}I)(\mathrm{Tr}\Sigma)_{,\mu} \equiv \Lambda_{,\mu}. \qquad (4.5f)$$

The scalar Λ is defined for convenience in writing some expressions in the sequel; it is the usual parameter for describing electromagnetic gauge transformations. In a generally covariant theory, infinitesimal similarity transformations are symmetry operations for arbitrary (coordinate dependent) choices of Σ; in a Lorentz-covariant theory, however, it is necessary that coordinate transformations and similarity transformations be performed together in such a way that the form of the matrices γ^μ is preserved.[21] Thus, if

$$\xi^\rho = a^\rho + a_\mu{}^\rho x^\mu, \qquad (4.6a)$$

then it necessarily follows that

$$\Sigma = \tfrac{1}{4}a_{[\sigma\rho]}\gamma^\rho\gamma^\sigma + ie\Lambda I, \qquad (4.6b)$$

where Λ is an arbitrary scalar function. Infinitesimal coordinate and similarity transformations are the only particular examples of infinitesimal symmetry operations considered in this paper.

A common situation, which arises for all the particular examples considered here, is that the Lagrangian is unchanged by any of the symmetry operations it possesses, except that under coordinate transformations it behaves like a scalar density of weight one. (This means that the action integral $\int L d\Omega$ is invariant under all symmetry operations irrespective of the boundary conditions on the integral.) Reference to (4.4b) shows that this implies

$$K^\nu = -L\xi^\nu. \qquad (4.7)$$

[19] See Landau and Lifschitz (39, p. 297).

[20] Careful consideration shows that the case $e=0$ causes no difficulty.

[21] See Jauch and Rohrlich (35, p. 52).

V. GENERATION OF LOCAL CONSERVATION LAWS

This section describes two methods by which local conservation laws may be discovered. The first is based upon the considerations of Sec. IV and is therefore called the *symmetry method*. This method has been known for some time[22] and is described here for purposes of review and of emphasizing points important in the sequel.

The symmetry method depends upon Eqs. (2.11) and (4.2). The following two steps are taken:

(1) The variation appearing in (2.11) is selected to be an infinitesimal symmetry operation; hence, δL may be rewritten by the use of (4.2).

(2) The quantity $-M_A\delta\Phi^A$ in (2.11) is assumed to vanish. The only situation of interest in this paper in which this is so is when the equations of motion are satisfied and the $\delta\Phi^A$ are well behaved.

It follows that (2.11) becomes a local conservation law of the form (3.1) with

$$S^\nu=(1/\epsilon)(\Delta^\nu-K^\nu), \tag{5.1}$$

where ϵ is a constant infinitesimal quantity of the order of $\delta\Phi^A$. In other words, corresponding to every infinitesimal symmetry operation (for which the $\delta\Phi^A$ are well behaved), there is a quantity S^ν which is locally conserved when the equations of motion are satisfied. It should be noted, however, that the S^ν of (5.1) is not unique. It is undetermined to within a constant factor due to the lack of precise definition of ϵ; this indeterminacy corresponds to the fact that the units in which the conserved quantity S is to be measured have not been specified. More significant, however, is that S^ν is only determined up to an additive term of the form $S^{[\nu\mu]}{}_{,\mu}$, since Δ^ν and K^ν are similarly undetermined. On the other hand, changing L to a form which yields the same equations of motion by adding to it a term of the form $L^\nu{}_{,\nu}$ has no effect on S^ν, since the resulting change in Δ^ν is exactly compensated by the change in K^ν.

The various infinitesimal symmetry operations possessed by a Lagrangian are characterized by certain parameters ϵp^A. It is therefore convenient to express the dependence of S^ν on these parameters as

$$S^\nu=\sum_{N=0} S(N)^\nu{}_A{}^{\tau\cdots}p^A{}_{,\tau\cdots}, \tag{5.2}$$

where there are N indices $\tau\cdots$ and $S(N)^\nu{}_A{}^{\tau\cdots}=S(N)^\nu{}_A{}^{(\tau\cdots)}$. In practice, only the first few terms of the sum over N are nonvanishing. For infinitesimal coordinate transformations, the ϵp^A are the ξ^ρ; for infinitesimal similarity transformations, they are the components of Σ. For these two cases, (5.2) is written

$$S^\nu=(1/\epsilon)\sum_{N=0}\{S(N)^\nu{}_\rho{}^{\tau\cdots}\xi^\rho{}_{,\tau\cdots}+\mathrm{Tr}[S(N)^\nu{}_\Sigma{}^{\tau\cdots}\Sigma_{,\tau\cdots}]\}, \tag{5.3a}$$

where $S(N)^\nu{}_\Sigma{}^{\tau\cdots}$ is a matrix.

[22] The originator of this method is unknown to the present author. It is discussed, for example, by Pauli (46).

The Lagrangians (2.10) are now considered. The only infinitesimal symmetry operations they possess are infinitesimal coordinate and similarity transformations (of which the infinitesimal electromagnetic gauge transformations are a special case). The conservation laws which can be thereby found are now displayed. Actually, all the nonvanishing $S(N)^\nu{}_\rho{}^{\tau\cdots}$ and $S(N)^\nu{}_\Sigma{}^{\tau\cdots}$ corresponding to the Lagrangian of general relativity, L_G (2.10d), with $L_M=L_E+L_D+L_K$ are displayed; the terms appropriate to the separate Lagrangians (2.10) will then be clear. Equation (2.12) is used to determine Δ^ν. The Lagrangians considered all behave as scalar densities of weight one under the transformations considered; hence, K^ν is determined by (4.7). The expressions resulting from infinitesimal coordinate transformations are

$$\begin{aligned}S(0)^\nu{}_\rho=L_G\delta_\rho{}^\nu+(-g)^{\frac12}(&-g^{\lambda\mu}\Gamma_{\lambda\mu}{}^\nu{}_{,\rho}+g^{\lambda\nu}\Gamma_{\lambda\mu}{}^\mu{}_{,\rho}-F^{\mu\nu}A_{\mu,\rho}\\&-\tfrac12\bar\psi_{,\rho}\gamma^\nu\psi+\tfrac12\bar\psi\gamma^\nu\psi_{,\rho}+g^{\mu\nu}\phi_{,\mu}\phi_{,\rho}\\&-\tfrac18\bar\psi\{\gamma^\nu,(1/\xi)[\gamma_{\mu,\rho},\gamma^\mu]\}\psi);\end{aligned} \tag{5.3b}$$

$$S(1)^\nu{}_\rho{}^\sigma=(-g)^{\frac12}(g^{\lambda\mu}\Gamma_{\lambda\mu}{}^\sigma\delta_\rho{}^\nu-2g^{\mu\sigma}\Gamma_{\mu\rho}{}^\nu+g^{\nu\sigma}\Gamma_{\rho\mu}{}^\mu-F^{\sigma\nu}A_\rho-\tfrac{1}{16}\bar\psi\{\gamma^\nu,[\gamma_\rho,\gamma^\sigma]\}\psi); \tag{5.3c}$$

$$S(2)^\nu{}_\rho{}^{\sigma\tau}=(-g)^{\frac12}(-g^{\sigma\tau}\delta_\rho{}^\nu+\tfrac12g^{\nu\sigma}\delta_\rho{}^\tau+\tfrac12g^{\nu\tau}\delta_\rho{}^\sigma). \tag{5.3d}$$

In the case of the infinitesimal similarity transformations a curious cancellation of terms occurs such that the only quantity of importance is $\mathrm{Tr}\Sigma$; i.e., only electromagnetic gauge transformations are of any consequence. The quantities of interest are

$$S(0)^\nu{}_\Sigma=-(-g)^{\frac12}\bar\psi\gamma^\nu\psi I/(\mathrm{Tr}I); \tag{5.3e}$$

$$S(1)^\nu{}_\Sigma{}^\sigma=(-i/e)(-g)^{\frac12}F^{\sigma\nu}I/(\mathrm{Tr}I). \tag{5.3f}$$

Although it may not appear so at first glance, in a Lorentz-covariant theory, where the similarity transformations are tied to the coordinate transformations, these formulas still apply, as direct computation will show.

In a Lorentz-covariant theory there are, in general, ten conservation laws (corresponding to the ten parameters a_ρ and $a_{[\rho\sigma]}$) which arise from infinitesimal coordinate transformations, namely,

$$S(0)^{\nu\rho}{}_{,\nu}=0, \tag{5.4a}$$

$$[(S(1)^{\nu\rho\sigma}-S(1)^{\nu\sigma\rho})+(S(0)^{\nu\rho}x^\sigma-S(0)^{\nu\sigma}x^\rho)]_{,\nu}=0, \tag{5.4b}$$

which are known, respectively, as the energy-momentum conservation law and the angular momentum conservation law.[23] The angular momentum consists of two terms which are known, respectively, as the "spin" and the "orbital" parts.[24] As has been remarked, however, the quantity S^ν is only determined up to an additive term of the form $S^{[\nu\mu]}{}_{,\mu}$. If a given choice for

[23] There are clearly six angular momentum laws, while one is accustomed to only three. The other three actually are related to conservation of the position of the center of mass; see Lanczos (38).

[24] These terms are used by Papapetrou (45).

S^ν is made and then the following term added to it,

$$S^{[\nu\mu]}{}_{,\mu}=(1/\epsilon)[(S(1)^{[\mu}{}_{\rho}{}^{\nu]}+S(1)_{\rho}{}^{[\mu\nu]}+S(1)^{[\nu\mu]}{}_{\rho})\xi^{\rho}]_{,\mu}, \quad (5.4c)$$

the new $S(0)^{\nu\rho}$ is symmetric in the indices $\nu\rho$ and the "spin" part of the angular momentum vanishes; these requirements are often made in order that the words "energy-momentum" and "angular momentum" refer to unique conservation laws.[25] A Lorentz-covariant theory may also possess the conservation laws arising from the electromagnetic gauge. These are infinite in number due to the complete arbitrariness of $\mathrm{Tr}\Sigma$. The one corresponding to $\mathrm{Tr}\Sigma$ a constant is the familiar law of conservation of charge.

The laws (5.4a, b) are, in general, weak as is now shown in a particular case. Consider a theory for which $S(0)^{oo}$ is positive definite and where x^o is the usual time coordinate of Lorentz-covariant theories. The electromagnetic field is such a theory, as is the neutral scalar meson; this may be easily verified. If (5.4a) were a strong law, the integral of $S(0)^{oo}$ over a time-constant surface could be rewritten by use of (3.4) as an integral over the boundaries of that surface. Hence, if the field variables vanished everywhere outside a certain region, $S(0)^{oo}$ would have to vanish everywhere. It is a simple matter to construct counter-examples to this conclusion. Hence, (5.4a) for $\rho=0$ is weak, at least for some theories. On the other hand, the conservation of charge is a strong law as is shown in Sec. VI.

Symmetry Method in the Converse

It is interesting to consider whether all quantities locally conserved when the equations of motion are satisfied can be found by the symmetry method. That they can in a certain sense is easy to see: Given an S^ν satisfying the conservation equation (3.1) when the equations of motion are satisfied, let the $\delta\Phi^A$ in (2.11) be chosen such that $M_A\delta\Phi^A$ equals $\epsilon S^\nu{}_{,\nu}$. These $\delta\Phi^A$ then describe an infinitesimal symmetry operation with $K^\nu=\Delta^\nu-S^\nu$, an operation which clearly leads to the conservation law $S^\nu{}_{,\nu}=0$.[26] But in general, the $\delta\Phi^A$ so selected are ill behaved when the equations of motion are satisfied, and in general this must be the case.[27] There are, however, many special circumstances in which the $\delta\Phi^A$ that can always be selected to be well behaved when the equations of motion are satisfied. It is now proved that in a certain special circumstance, fulfilled by all the particular examples considered here (as may be easily verified), the $\delta\Phi^A$ can be so selected.

The theorem to be proved is: All conservation laws of a theory may be found by the symmetry method using $\delta\Phi^A$ which are well behaved when the equations of motion are satisfied, provided that the following condition is fulfilled. There exist some linearly independent linear combinations M_A' of all the equations of motion M_B such that

$$M_A'=D^A(\Phi^A)-M_A''(\Phi^B) \quad (5.5a)$$

(A not summed). By D^A is meant a derivative of Φ^A, $n^A(\geq 1)$ times with respect to a single coordinate label $x^{(A)}$. The quantities M_A'' and the coefficients used in forming the linear combinations M_A' contain derivatives of no Φ^B with respect to its corresponding $x^{(B)}$ to an order n^B or higher. The proof follows.

An S^ν is given; all that is known of it is its behavior when the equations of motion are satisfied (namely, that $S^\nu{}_{,\nu}$ then vanishes). Therefore, one is free to use the equations of motion to replace in S^ν every occurrence of every Φ^A which is differentiated at least n^A times with respect to $x^{(A)}$ with M_A'' or a suitable derivative thereof. This process may be continued until no Φ^A is differentiated more than (n^A-1) times with respect to its corresponding $x^{(A)}$; the quantities now remaining in S^ν are unrestricted (at a point) by the equations of motion. Now let $S^\nu{}_{,\nu}$ be computed for cases in which the equations of motion are not necessarily satisfied. Any derivative of any Φ^A with respect to its $x^{(A)}$ of the order n^A must necessarily appear linearly in this expression. These can be rewritten as $M_A'+M_A''$ or a suitable derivative thereof. By continuing the process, $S^\nu{}_{,\nu}$ can be rewritten as a linear sum of the M_A' and their derivatives with coefficients which are unrestricted by the equations of motion plus an additional term unrestricted by the equations of motion:

$$S^\nu{}_{,\nu}=\sum_{N=0} C(N)^{A(\tau\cdots)}M_A'{}_{,\tau\ldots}+C, \quad (5.5b)$$

where there are N indices $\tau\cdots$. Since the M_A' and $S^\nu{}_{,\nu}$ all vanish when the equations of motion are satisfied, C must be identically zero. By use of the expression for the derivative of a product, (5.5b) may be rewritten as

$$[S^\nu+\sum_{N=0} C'(N)^{A\nu(\tau\cdots)}M_A'{}_{,\tau\ldots}]_{,\nu}=C''^AM_A. \quad (5.5c)$$

The $C'(N)^{A\nu(\tau\cdots)}$ and C''^A are not necessarily unrestricted by the equations of motion. They may depend on the equations of motion or their derivatives multilinearly; this means, however, that they are still well behaved when the equations of motion are satisfied. Thus, the choice $\delta\Phi^A=\epsilon C''^A$ describes a symmetry operation which leads to the conservation law represented by the vanishing of the left side of (5.5c). Since the $M_A'{}_{,\tau\ldots}$ all vanish when the equations of motion are satisfied, this is what was required.

[25] Conservation laws in Lorentz-covariant theories are discussed in great detail by Rosenfeld (48) and Belinfante (6, 7, 8).

[26] This proof has been given by Noether (44). Her assumptions specifically exclude the difficulties with which the subsequent proof deals.

[27] Consider a Lagrangian in a one-dimensional physical space (coordinate x^o): $L=q(p_{,o})^2+f(p)$, where the function f need not be specified. Since $(p_{,o})^2=0$ is an equation of motion, it follows that $p_{,o}=0$. Hence $S^o=p$ satisfies the local conservation law (3.1). The symmetry operation leading to this law is $\delta p=0$, $\delta q=(\epsilon/p_{,o})$. When the equations of motion are satisfied, δq is singular. The source of the difficulty is, roughly, that the conservation law being considered equals the square root of an equation of motion instead of a linear combination of equations of motion.

Commutator Method

The second method for arriving at local conservation laws is of relatively recent origin[28]; it is also based on (2.11). Let the field variables appearing in that equation be varied again with respect to a set of parameters different from those characterizing the variation δ; this new variation is symbolized by $\eth$. The result is

$$\eth\delta L = (\eth\Delta^\nu)_{,\nu} - M_A \eth\delta\Phi^A - \eth M_A \delta\Phi^A. \qquad (5.6)$$

The two variations can be performed just as well in the reverse order. That is, first, an equation like (2.11), except that $\eth$ replaces δ, can be obtained and then the variation symbolized by δ performed. The equation so obtained can be subtracted from (5.6) to give

$$(\eth\Delta^\nu - \delta\bar{\Delta}^\nu)_{,\nu} = \eth M_A \delta\Phi^A - \delta M_A \eth\Phi^A, \qquad (5.7)$$

where $\bar{\Delta}^\nu$ is the same function of $\eth\Phi^A$ that Δ^ν is of $\delta\Phi^A$. Now, if the δM_A and $\eth M_A$ all vanish (and the $\delta\Phi^A$ and $\eth\Phi^A$ are well behaved), then

$$S^\nu = (1/\epsilon^2)(\eth\Delta^\nu - \delta\bar{\Delta}^\nu) \qquad (5.8a)$$

$$= (1/\epsilon^2)(\eth p^B \mathfrak{A}_{B}{}^{\nu}{}_{C} \delta q^C - \delta p^B \mathfrak{A}_{B}{}^{\nu}{}_{C} \eth q^C) \qquad (5.8b)$$

$$= (2/\epsilon^2)(\eth\chi^B \mathfrak{A}_{[B}{}^{\nu}{}_{C]} \delta\chi^C), \qquad (5.8c)$$

satisfies the local conservation law (3.1). The vanishing of δM_A and $\eth M_A$ does not necessarily mean that the equations of motion are satisfied. It means, rather, that $\delta\Phi^A$ and $\eth\Phi^A$ *conform* to the equations of motion, i.e., that the right-hand side of the equations of motion are equal to quantities which do not depend on the parameters characterizing δ and $\eth$. Thus, to every pair of infinitesimal changes which preserve the equations of motion, there corresponds a conservation law. These conservation laws are not unique just as the laws (5.1) were not unique. This second method of obtaining conservation laws is called the *commutator method*, the name referring to the crucial step in the derivation.

If one is making use of conservation laws by considering the values of the conserved quantities, the laws obtained by the commutator method are superfluous, since all laws may be obtained by the symmetry method. The commutator method is not superfluous, however, if one is making a mathematical use of conservation laws, for the commutator method and the symmetry method, in general, write the conservation laws in different ways, one of which may be more useful than the other. In Sec. VIII, two uses of the commutator method are given. The results of the next paragraph are of interest in that connection.

Let the Lorentz-covariant interacting Maxwell-Dirac field be considered. The operation δ is taken to be arbitrary (except that it conform to the equations of motion), while the operation $\eth$ is taken to be an infinitesimal electromagnetic gauge transformation. Then one finds [refer to (2.20a, b)]

$$S^\nu = (1/\epsilon^2)(\eth F^{\mu\nu}\delta A_\mu - \delta F^{\mu\nu}\eth A_\mu - \eth\bar{\psi}\gamma^\nu\delta\psi + \delta\bar{\psi}\gamma^\nu\eth\psi) \qquad (5.9a)$$

$$= (1/\epsilon^2)(-\delta F^{\mu\nu}\Lambda_{,\mu} + ie\Lambda\delta[\bar{\psi}\gamma^\nu\psi]), \qquad (5.9b)$$

which is, in a certain sense, the variation of (5.3e, f). As another example, consider the pure gravitational field. Once again let δ be arbitrary, but let $\eth$ be an infinitesimal coordinate transformation [refer to (2.20d)]:

$$S^\nu = (1/\epsilon^2)(\eth \mathfrak{g}^{\lambda\mu}\delta\mathfrak{G}_{\lambda\mu}{}^{\nu} - \eth\mathfrak{G}_{\lambda\mu}{}^{\nu}\delta\mathfrak{g}^{\lambda\mu}) \qquad (5.10a)$$

$$= (1/\epsilon^2)[(\mathfrak{G}_{\lambda\mu}{}^{\nu}{}_{,\rho}\xi^\rho + 2\mathfrak{G}_{\lambda\rho}{}^{\nu}\xi^\rho{}_{,\mu} - \mathfrak{G}_{\lambda\mu}{}^{\rho}\xi^\nu{}_{,\rho} + \xi^\nu{}_{,\lambda\mu} - \delta_\lambda{}^\nu\xi^\rho{}_{,\mu\rho})\delta\mathfrak{g}^{\lambda\mu} + (-\mathfrak{g}^{\lambda\mu}{}_{,\rho}\xi^\rho - \mathfrak{g}^{\lambda\mu}\xi^\rho{}_{,\rho} + 2\mathfrak{g}^{\lambda\rho}\xi^\mu{}_{,\rho})\delta\mathfrak{G}_{\lambda\mu}{}^{\nu}]. \qquad (5.10b)$$

A general variation in χ^B depending upon a set of parameters, may be written

$$\eth\chi^B = \epsilon \sum_{N=0} \chi(N)^B{}_A{}^{(\tau\cdots)} p^A{}_{,\tau\cdots}. \qquad (5.11)$$

Thus, (5.8c) may be written in the form (5.2) with

$$S(N)^\nu{}_A{}^{\tau\cdots} = (2/\epsilon)\chi(N)^B{}_A{}^{(\tau\cdots)}\mathfrak{A}_{[B}{}^{\nu}{}_{C]}\delta\chi^C. \qquad (5.12)$$

VI. ARBITRARY FUNCTION THEOREM (NOETHER'S THEOREM)

Proof

An important conclusion may be drawn about the expression for S^ν given by Eq. (5.2) in cases in which the parameters p^A are completely arbitrary functions of the coordinates. The p^A that have this property are called Λ^A. Thus,

$$S^\nu = \sum_{N=0} S(N)^\nu{}_A{}^{\tau\cdots}\Lambda^A{}_{,\tau\cdots}, \qquad (6.1)$$

which is a special case of (5.2), satisfies the local conservation law (3.1) for all choices of the Λ^A. It is now shown that the S^ν of (6.1) can always be written in the form (3.2),[29] i.e., that it describes a strong law. In fact, the quantity $S^{[\nu\mu]}$ is explicitly displayed as[30]

$$S^{[\nu\mu]} = \sum_{N=0}\sum_{M=0}[(-1)^M(N+1)/(N+M+2)] \times [S(N+M+1)^\nu{}_A{}^{\mu\sigma\cdots\tau\cdots} - S(N+M+1)^\mu{}_A{}^{\nu\sigma\cdots\tau\cdots}]_{,\sigma\cdots}\Lambda^A{}_{,\tau\cdots}. \qquad (6.2)$$

The convention used throughout this section is that there are N indices $\tau\cdots$ and M indices $\sigma\cdots$.

The verification that (6.1) and (6.2) satisfy (3.2) is straightforward but requires some very careful bookkeeping. As a guide, the essential steps are now reproduced. The key to the proof lies in the realization that (6.1) can satisfy the conservation law (3.1) for all choices of the Λ^A if, and only if, the coefficient of each derivative of each Λ^A in (3.1) vanishes separately. This means that, for all N,

$$S(N-1)^{(\tau}{}_A{}^{\cdots)} + S(N)^\nu{}_A{}^{\tau\cdots}{}_{,\nu} = 0, \qquad (6.3)$$

[28] The only reference known to the present author is Heller (34).

[29] This was first proved by Noether (44).

[30] For the special case in which $S(N)^\nu{}_A{}^{\tau\cdots}$ vanishes for all $N>2$, the explicit formula for $S^{[\nu\mu]}$ has been given by Bergmann and Schiller (14). See also Anderson (1).

as may be seen by substituting (6.1) into (3.1). This relation is used in the following proof in the special form,

$$[1/(N+M+1)][S(N+M)^{\nu}{}_{A}{}^{\sigma\cdots\tau\cdots} +NS(N+M)^{(\tau}{}_{A}{}^{\cdots)\nu\sigma\cdots}+MS(N+M)^{(\sigma}{}_{A}{}^{\cdots)\nu\tau\cdots}] +S(N+M+1)^{\mu}{}_{A}{}^{\nu\sigma\cdots\tau\cdots}{}_{,\mu}=0. \quad (6.4a)$$

The proof now begins. By substituting (6.1) and (6.2) into (3.2), one finds that the relation to be verified is

$$\sum_{N=0} S(N)^{\nu}{}_{A}{}^{\tau\cdots}\Lambda^{A}{}_{,\tau\cdots}=\sum_{N=1}\sum_{M=0}[(-1)^{M}N/(N+M+1)] \times[S(N+M)^{\nu}{}_{A}{}^{\sigma\cdots\tau\cdots}-S(N+M)^{(\tau}{}_{A}{}^{\cdots)\nu\sigma\cdots}]_{,\sigma\cdots} \times\Lambda^{A}{}_{,\tau\cdots}-\sum_{N=0}\sum_{M=1}[(-1)^{M}(N+1)/(N+M+1)] \times[S(N+M)^{\nu}{}_{A}{}^{\sigma\cdots\tau\cdots}-S(N+M)^{(\sigma}{}_{A}{}^{\cdots)\nu\tau\cdots}]_{,\sigma\cdots} \times\Lambda^{A}{}_{,\tau\cdots}. \quad (6.4b)$$

It would be convenient to rewrite this in a form in which both double sums had the same range. Clearly, there is no problem in extending the range of the first double sum down to $N=0$, since the terms for $N=0$ all vanish. In the second double sum, $S(N+M)^{(\sigma}{}_{A}{}^{\cdots)\nu\tau\cdots}$ vanishes by definition when $M=0$; so to extend the range of that double sum down to $M=0$ means that $-\sum_{N=0}S(N)^{\nu}{}_{A}{}^{\tau\cdots}\Lambda^{A}{}_{,\tau\cdots}$ would be added to the right side of the equation. Providentially, this term already occurs on the left side of the equation with opposite sign, therefore, the relation to be proved may be rewritten as

$$\sum_{N=0}\sum_{M=0}[(-1)^{M}/(N+M+1)] \times[-S(N+M)^{\nu}{}_{A}{}^{\sigma\cdots\tau\cdots}-NS(N+M)^{(\tau}{}_{A}{}^{\cdots)\nu\sigma\cdots} +(N+1)S(N+M)^{(\sigma}{}_{A}{}^{\cdots)\nu\tau\cdots}]_{,\sigma\cdots}\Lambda^{A}{}_{,\tau\cdots}=0. \quad (6.4c)$$

Now, by differentiating (6.4a) with respect to $\sigma\cdots$, multiplying it by $(-1)^{M}\Lambda^{A}{}_{,\tau\cdots}$, summing over M and N, and adding the result to (6.4c), one obtains the following, which can be easily verified:

$$\sum_{N=0}\sum_{M=0}(-1)^{M}[S(N+M)^{(\sigma}{}_{A}{}^{\cdots)\nu\tau\cdots} +S(N+M+1)^{\mu}{}_{A}{}^{\sigma\cdots\nu\tau\cdots}{}_{,\mu}]_{,\sigma\cdots}\Lambda^{A}{}_{,\tau\cdots}=0. \quad (6.4d)$$

Application to General Relativity

In a generally covariant theory, unlike a Lorentz-covariant theory, the parameters describing infinitesimal coordinate and similarity transformations are entirely arbitrary functions. Hence, the above theorem is applicable. Corresponding to the expressions (5.3), one finds

$$S^{[\nu\mu]}=(1/\epsilon)(-g)^{\frac{1}{2}}(\xi^{\mu;\nu}-\xi^{\nu;\mu} +(F^{\nu\mu}A_{\rho}+\tfrac{1}{16}\bar{\psi}\{\gamma_{\rho},[\gamma^{\nu},\gamma^{\mu}]\}\psi)\xi^{\rho}-F^{\nu\mu}\Lambda). \quad (6.5)$$

One should *not over*estimate the significance of this precise expression. The quantity S^{ν} of (5.3) was not unique in the first place. It could have had any term of the form $S^{[\nu\mu]}{}_{,\mu}$ added to it. By judicious choice of this undetermined quantity, one could cause S^{ν} to vanish whenever the equations of motion are satisfied. The important fact is that the conservation laws following from general covariance are all strong laws.

It is interesting, however, that in the absence of matter fields ($\bar{\psi}$, ψ, $F^{\nu\mu}$, and A_{ρ} all vanish) Eq. (6.5) is the same as one suggested by Komar (37). The method by which (6.5) was derived tends to substantiate Komar's statement that his expression is closely connected with the possible infinitesimal coordinate transformations of the theory. The expression of Møller (43) is the special instance of Komar's expression in which ξ^{μ} is taken to be normal to the surface for which one is calculating a global law; this choice automatically gives the expression the properties mentioned at the end of Sec. III.

VII. SIGNIFICANCE OF CONSERVATION LAWS IN GENERALLY COVARIANT THEORIES

Relation Between the Various Conservation Laws

The last five sections have marshaled an array of facts about local conservation laws. The present section uses these facts to fulfill the aims outlined in the introduction. The development begins by reviewing certain of the points already made. First, all conservation laws may be derived by use of the symmetry method given in Sec. V; in particular, in the special examples considered here, the infinitesimal symmetry operations used in that method are well behaved when the equations of motion are satisfied. Second, the only infinitesimal symmetry operations a generally covariant theory *must* possess are infinitesimal coordinate transformations and, if spinors appear in the theory, infinitesimal similarity transformations, both of which are symmetry operations described by arbitrary functions. Any other infinitesimal symmetry operations would be characteristic only of particular theories. Third, all conservation laws derived in consequence of symmetries described by arbitrary functions are strong laws. These three points together show that *all conservation laws characteristic of generally covariant theories are strong laws.*

This conclusion illuminates the connection between the various conservation laws which have been proposed. They all represent different choices of the quantity $S^{[\nu\mu]}$ of Eq. (3.2). One could "rediscover" these laws and arbitrarily many more just by choosing suitable quantities $S^{[\nu\mu]}$. The difference between these various laws is no greater than between the unsymmetrized and symmetrized forms of the energy-momentum law in Lorentz-covariant theories. The unusual feature of the situation is that there also is no greater difference than this between any one of these laws and the trivial law consequent from the choice $S^{\nu}=0$. As a matter of fact, the choice $S^{\nu}=0$ has been seriously considered.[31] (An interesting interpretation of such a choice is as follows: The quantity $-2R_{\mu\nu}+g_{\mu\nu}R$ is identified as the stress-energy tensor of the gravitational field. According to the equations of motion, the sum of this quantity and the well-known stress-energy tensor of matter [see (2.12) and (8.1)] is zero; the gravitational field carries energy as fast one way as the matter fields carry it the other, so that the energy always vanishes point by point.) The importance of the various derivations used in obtaining the various proposed laws lies in the fact that they each rewrite the equations of motion in a different form; each form may reveal different mathematical consequences of the equations of motion.

The whole range of possible conservation laws in generally covariant theories is now exposed; they all result from suitable choices of $S^{[\nu\mu]}$ in (3.2). The only way to decide which of these laws is important is to inquire what one wishes to do with the law. This section examines possible uses of conservation laws one by one to see which choices of $S^{[\nu\mu]}$, if any, are appropriate for each use. It cannot be pretended that *all* possible kinds of uses of conservation laws in generally covariant theories are considered, since the only limit on the usefulness of something is placed by the ingenuity of those who wish to use it. Only those uses which have been widely mentioned are examined here.

First Integral

The basic use of conservation laws, for which they were originally conceived, is to provide first integrals of the equations of motion. If one can find, as a consequence of the equations of motion, that the time derivative of some function of the field variables vanishes, then it follows that that function of the field variables is a constant independent of the time, i.e., that it constitutes a first integral of the equations. This result represents a step toward solving the equations of motion, i.e., toward expressing the field variables at any time as a function of their values at an initial time. It is now shown that such a use of a strong law can be made only in special cases and even then only in a limited sense.

The local conservation law (3.1) does not directly state that the time derivative of some quantity vanishes. To obtain such an expression, one must go to the integral form (3.6). One then has an expression of the required form if, and only if, it is assumed that $-\oint_{\beta}S^{n}d\sigma_{on}$ vanishes in consequence of the boundary conditions imposed. These boundary conditions may be specified either at the space-like limits of ω (when it is an entire initial surface) or at some boundary β isolating the region covered by ω from the rest of the universe. Now suppose that a strong law is being considered, i.e., that (3.7) applies. Then the boundary condition must be imposed that $(d/dx^{o})\oint_{\beta}S^{[on]}d\sigma_{on}$ vanishes. The "first integral" that one obtains when this condition is imposed states that $\oint_{\beta}S^{[on]}d\sigma_{on}$ equals a constant. This so-called first integral is nothing more than a restatement of the boundary conditions; no information has been gained about the field variables whose changes are being investigated.

This result is not surprising if one considers a situation familiar in Lorentz-covariant theories. There one considers the first integrals obtained from two conservation laws differing by a strong law (as do the unsymmetrized and symmetrized energy laws) as essentially the same, since they differ only by boundary values.[32] Thus, the first integrals obtained from strong laws are all essentially the same as the trivial first integral, zero equals a constant.

All uses of conservation laws that are really uses of the first integrals obtained from them are also ruled out by this result. First, there is, for example, the "check-up service" cited in Sec. I: One judges whether a set of final data is compatible with a set of initial data by computing the values of the first integrals implied by the two sets of data to see whether they are equal. In the case in which the "first integrals" used are obtained from strong laws, all one does when applying this procedure is to verify that the two sets of data do in fact fulfil the imposed boundary conditions. Second, first integrals are also important in the ergodic hypothesis. Here, once again, the "first integrals" consequent from strong laws are of no value since they do not limit the regions of phase space accessible to the changing field variables. It would seem that this fact would have far-reaching consequences in any effort to form a really complete theory of general relativistic thermodynamics.[33]

At first glance it might seem that there must be a "catch" somewhere in the arguments of the last few paragraphs. They might seem to imply that the well-known integral expressing the conservation of charge is of no value. This integral is now examined in an effort to explain further the result which has been obtained. For definiteness, the theory described by the Lorentz-covariant Lagrangian L_D+L_E (2.10a, b) is considered. The quantity appearing in the law of local charge conservation is [refer to (5.3e, f) and (6.5)]

$$S^{\nu}=ie\bar{\psi}\gamma^{\nu}\psi \tag{7.1a}$$

$$=F^{\nu\mu}{}_{,\mu}. \tag{7.1b}$$

When S^{ν} is written in the form (7.1a), the first integral corresponding to S^{ν} looks like a result of some significance. However, when the form (7.1b) is used, the result is patently trivial. It seems that the manner in which S^{ν} is written is crucial. A ready explanation of this dilemma is that the form (7.1a) is misleading; the

[31] See Lorentz (40) and Souriau (53).

[32] See Landau and Lifschitz (39, p. 82).

[33] The treatment by Tolman (56) is not sufficiently complete for this matter to cause any concern.

law has merely been written in such a form as to conceal its triviality. This explanation is certainly correct, but the matter does go somewhat deeper.

First, the Lagrangian L_D alone possesses the law of conservation of charge in the form (7.1a) irrespective of the behavior of the electromagnetic field variables. This law corresponds to the infinitesimal gauge transformations for which Λ is a constant. These transformations constitute an infinitesimal symmetry property of L_D alone even when the vector potential is regarded as a fixed quantity. Now, since Λ must be a constant, Noether's theorem does not apply. In this case, (7.1a) is not a strong law, and the first integral obtained from it is not trivial. That is, a first integral that is trivial in a given complete theory (e.g., that described by L_E+L_D) is not necessarily trivial in a part of that theory (e.g., that is described by L_D). To illustrate what it means to say that a first integral ceases to be trivial when one shifts one's attention from a complete to a partial theory, let the "check-up service" application be considered. Let it be supposed that the initial and final sets of data that one is given are sets of data only about the partial theory; i.e., let it be supposed that the data consists of information only about $\bar{\psi}$ and ψ and not about A_μ. Then the conclusions one can draw from the law of charge conservation (7.1a) are of true significance; the first integral supplies information that would normally be supplied by the now unknown values of A_μ. Unfortunately, this partial theory approach does not seem to have an immediate analogy in the case of conservation laws consequent upon infinitesimal coordinate transformations. It seems that one must look for conservation laws possessed by L_M (2.10d) irrespective of the behavior of the metric. Such conservation laws would follow from a subset of the infinitesimal coordinate transformations which are symmetry properties of L_M even when the $g_{\mu\nu}$ are regarded as fixed. That is, one must find an infinitesimal coordinate transformation which leaves $g_{\mu\nu}$ unchanged, a transformation described by a ξ^μ satisfying

$$\xi_{\mu;\nu}+\xi_{\nu;\mu}=0. \tag{7.2}$$

But in general, no vector of this kind (no Killing vector) exists.[34] The partial theory has no general symmetries, no conservation laws. However, the analogy to electromagnetism just made may not be the only possible one. There may be other situations when all the quantities appearing in the equations of motion are not known. (This often happens in experimental physics.) Then the conservation laws may supply information which could normally be obtained directly from the equations of motion. This would have to be investigated in particular cases. Thus, the first integral corresponding to the law of charge conservation is useful for certain special kinds of problems. How the first integrals corresponding to other conservation laws can be used similarly is not obvious and requires special study in each case.

A second point about the concept of charge is that it is a physically measurable concept; the presence and quantity of charge is experimentally detectable on a local basis. In the case of a locally measurable quantity, it is of interest that the total amount is expressible as an integral over a bounding surface. This fact *indirectly* gives some importance to the first integral, for the boundary conditions then supply some information regarding measurements made within the boundary. One is thus led to the discussion of a second possible use of conservation laws.

Local Physical Significance

A local strong conservation law may supply useful information if the quantity S^ν appearing in it has a local physical significance (of its own, apart from its definition in terms of the field variables). Any facts about experimental quantities are likely to be of interest in some way or other. This raises the question of whether there are any quantities $S^\nu=S^{[\nu\mu]}{}_{,\mu}$ characteristic of generally covariant theories which have a local physical significance (other than charge, which has already been considered). It is enough to remark that no one has ever thought of one; further, it seems unlikely that any one ever will. As noted in Sec. III, the quantity S^ν must be a contravariant vector density of weight one if it and its integral are to be covariant concepts, and it is difficult to imagine how a noncovariant concept could be experimentally measurable locally. But the quantities S^ν which one expects to find as a consequence of general covariance (energy, momentum, and angular momentum densities) are not vector densities.

An exception to the conclusions of the last paragraph may perhaps lie in the suggestions of Pirani (47) that the conserved quantities of physical significance might be composed out of objects describing the field and objects describing the observers. Some of the quantities so formed could be contravariant vector densities of weight one. For example, the quantity ξ^μ in the expression of Komar (37) or in (6.5)[35] could characterize observers in some meaningful way. There are clearly elements of this approach in the expression of Møller (43) also, since the observers could be taken to define somehow the surfaces for which the energy is to be calculated. Further investigation will have to be made before definite conclusions can be reached as to the possibilities of this approach.

Schwarzschild Mass

The uses discussed so far are ones that one is likely to consider in relation to any conservation law. The use considered now is a rather peculiar and special one to which a particular law, the energy-momentum law,

[34] This suggests a topic which goes beyond the present investigation, namely, the problem of what symmetries L_M possesses for certain definite solutions for the $g_{\mu\nu}$.

[35] There is also a hint of a similar kind of expression in Lanczos (38).

is often put in general relativity [the theory described by L_G (2.10d)]; namely, it is used as a means of computing the Schwarzschild mass of a system. The exposition of this use is given here in more detail than is usual, so as to put the conclusions to be drawn on a firm basis. The procedure outlined is somewhat analogous to that for computing the total charge of a system by integrating the electric field over a distant surface.

Physical space is assumed to be asymptotically Lorentzian; this means that one can cover all of space with a coordinate system such that, in the limit as the three space-like coordinates approach (plus or minus) infinity, the metric tensor approaches its Lorentz value, $\bar{g}_{\mu\nu}$. Usually the following somewhat more stringent requirement is made:

$$g_{\mu\nu} \underset{r\to\infty}{=} \bar{g}_{\mu\nu}+\text{order}(1/r), \tag{7.3}$$

where $r=+(g_{mn}x^m x^n)^{\frac{1}{2}}$, Latin indices running only over spacelike coordinates. Attention is directed to space-like three-surfaces which are asymptotically hyperplanes, i.e., which become flat as r approaches infinity. A certain quantity $S^{[\nu\mu]}$ is selected, and the quantity $S(\omega)$ [see Eq. (3.4)] computed from it when ω is asymptotically a hyperplane is called the energy in ω. Three other quantities $S^{[\nu\mu]}$ are also selected; the $S(\omega)$ corresponding to them are the three components of momentum in ω. Collectively, these four quantities are symbolized $S_\rho{}^{[\nu\mu]}$, where $\rho=0$ corresponds to the energy and $\rho\neq 0$ to the momentum. The energy and momentum are required to transform as a vector under Lorentz transformations of the coordinates. A further requirement is that for the space described (for large r) by the Schwarzschild metric,

$$\begin{aligned} g_{oo}&=-(1-M/32\pi r)^2/(1+M/32\pi r)^2, \quad g_{om}=0, \\ g_{mn}&=\delta_{mn}(1+M/32\pi r)^4, \end{aligned} \tag{7.4}$$

there exist a set of asymptotically parallel space-like hyperplanes such that the energy computed for those hyperplanes is the Schwarzschild mass M and the momentum is zero. Thus, by applying the energy-momentum computation to non-Schwarzschild spaces, one has a way of generalizing the notation of Schwarzschild mass.

The problem of finding a suitable $S_\rho{}^{[\nu\mu]}$ will now be considered. The boundary surfaces β of the surfaces ω may be taken to be asymptotic spheres (surfaces r equals a constant):

$$S_\rho(\omega)=\bar{S}_\rho(\beta)=\oint_\beta S_\rho{}^{[om]}\eta_m r^2 d\Omega, \tag{7.5}$$

where ω is a surface of constant x^o, η_m is normal to β, $d\Omega$ is a differential solid angle, and r is approaching infinity. Clearly, it is the terms in $S_\rho{}^{[om]}$ of order $1/r^2$ (when averaged over angles) that are important. Now, since $S_o(\omega)$ is to be nonvanishing for the Schwarzschild metric, $S_\rho{}^{[\nu\mu]}$ must involve at least one term which does not vanish when the fields other than the metric (i.e., the matter fields) vanish at infinity. Attention is directed to this term, since a suitable $S_\rho{}^{[\nu\mu]}$ can be found if and only if this term can be suitably chosen. Since $S_\rho{}^{[\nu\mu]}$ must be a tensor density under Lorentz transformations, this term must be a function of $g^{\mu\nu}$, the various derivatives of $g^{\mu\nu}$, and the permutation symbol $\epsilon^{\kappa\lambda\mu\nu}$, these being the only objects available out of which to form tensors. Now the only quantities $S_\rho{}^{[\nu\mu]}$ that can be formed out of these objects and which are at infinity of order $1/r^2$ or greater for the Schwarzschild metric are

$$S_\rho{}^{[\nu\mu]}=a_\rho{}^{[\nu\mu]}{}_\sigma{}^{(\kappa\lambda)}\Gamma_{\kappa\lambda}{}^\sigma, \tag{7.6}$$

where $a_\rho{}^{[\nu\mu]}{}_\sigma{}^{(\kappa\lambda)}$ is a function of $g^{\mu\nu}$ and $\epsilon^{\kappa\lambda\mu\nu}$. Actually, there are other possibilities if one were willing to allow $S_\rho{}^{[\nu\mu]}$ to involve terms which are divided by Lorentz-scalar quantities that vanish as infinity is approached. These possibilities are rejected because the order to which the scalar vanishes would differ for different metrics; hence, S_ρ would probably diverge for some metrics and, even if it did not, would have a very questionable relation to the Schwarzschild mass. In any event, all the well-known suggestions for energy expressions employ an $S_\rho{}^{[\nu\mu]}$ of the form (7.6).

Because of the symmetries of the indices of $a_\rho{}^{[\nu\mu]}{}_\sigma{}^{(\kappa\lambda)}$ and $g^{\mu\nu}$ and $\epsilon^{\kappa\lambda\mu\nu}$, there are only seven distinct kinds of terms (aside from multiplication by arbitrary powers of the determinant g) which may comprise $a_\rho{}^{[\nu\mu]}{}_\sigma{}^{(\kappa\lambda)}$, namely, $\delta_\rho{}^{[\nu}\delta_\sigma{}^{\mu]}g^{\kappa\lambda}$, $\delta_\rho{}^{[\nu}g^{\mu](\kappa}\delta_\sigma{}^{\lambda)}$, $\delta_\rho{}^{(\kappa}g^{\lambda)[\nu}\delta_\sigma{}^{\mu]}$, $g_{\rho\alpha}g_{\sigma\beta}\epsilon^{\alpha\beta\nu\mu}g^{\kappa\lambda}$, $g_{\rho\alpha}\epsilon^{\alpha\nu\mu(\kappa}\delta_\sigma{}^{\lambda)}$, $g_{\sigma\beta}\epsilon^{\beta\nu\mu(\kappa}\delta_\rho{}^{\lambda)}$, and $g_{\sigma\alpha}g_{\rho\beta}(\epsilon^{\alpha\beta\nu(\kappa}g^{\lambda)\mu}-\epsilon^{\alpha\beta\mu(\kappa}g^{\lambda)\nu})$. Computation shows that only one linear combination of these terms is nonvanishing for the Schwarzschild metric and is unchanged by coordinate transformations of that metric which are asymptotic to the identity and preserve the conditions (7.3). For that linear combination, one has

$$\begin{aligned} S_\rho{}^{[\nu\mu]}=(-g)^{W/2}[&\delta_\rho{}^\nu(g^{\kappa\lambda}\Gamma_{\kappa\lambda}{}^\mu-g^{\mu\kappa}\Gamma_{\kappa\lambda}{}^\lambda)-g^{\nu\lambda}\Gamma_{\rho\lambda}{}^\mu \\ &-\delta_\rho{}^\mu(g^{\kappa\lambda}\Gamma_{\kappa\lambda}{}^\nu-g^{\nu\kappa}\Gamma_{\kappa\lambda}{}^\lambda)+g^{\mu\lambda}\Gamma_{\rho\lambda}{}^\nu], \end{aligned} \tag{7.7}$$

where the undetermined constant factor that could multiply the expression has been chosen so that $S_o(\omega)$ is indeed equal to M for the Schwarzschild metric. The argument given below to show that the energy and momentum calculated from (7.7) do indeed have the proper behavior under coordinate transformations indicates the nature of the arguments that show that (7.7) is the only suitable expression.

The quantities (7.7) correspond to the infinite number of mixed-index energy-momentum pseudotensors given by Goldberg (32). The special case $W=1$ corresponds to the original expression given by Einstein.[36] If the index ρ is raised, then the special case $W=2$ corresponds to the expression of Landau and Lifschitz (39). These facts shed further light on the relation between some of the proposed energy expressions.

[36] See Freud (29).

The coordinate transformation properties of the quantity (7.7) are now examined. It behaves as a tensor density of weight W except for certain additional terms arising from the Christoffel symbols, namely,

$$(-g)^{W/2}[\delta_\rho{}^\nu(g^{\mu\kappa}x^\lambda{}_{,\sigma}x^{\sigma'}{}_{,\kappa\lambda}-g^{\kappa\lambda}x^\mu{}_{,\sigma}x^{\sigma'}{}_{,\kappa\lambda})+g^{\nu\lambda}x^\mu{}_{,\sigma}x^{\sigma'}{}_{,\rho\lambda} -\delta_\rho{}^\mu(g^{\nu\kappa}x^\lambda{}_{,\sigma}x^{\sigma'}{}_{,\kappa\lambda}-g^{\kappa\lambda}x^\nu{}_{,\sigma}x^{\sigma'}{}_{,\kappa\lambda}) -g^{\mu\lambda}x^\nu{}_{,\sigma}x^{\sigma'}{}_{,\rho\lambda}], \quad (7.8)$$

where the derivatives of $x^{\alpha'}$ are with respect to x^β and conversely. These terms affect the value of the energy-momentum only if they contribute to the integral (7.5); therefore, interest settles on the case $\nu=0$, $\mu=m$ and on terms of order $1/r$ to $1/r^2$. Now derivatives tangent to the surface β (i.e., with respect to the coordinates other than x^o or x^m) always lower the order of the object differentiated by one order of r. Since the first derivatives of the new coordinates with respect to the old are, except for constant terms, of order $1/r$ or less [so as to preserve (7.3)], the second derivatives in (7.8) which involve at least one differentiation tangent to β are of order $1/r^2$ or less. Hence, the terms involving such derivatives do not contribute to (7.5) as may be seen by integrating by parts with respect to angles. For, to order $1/r^2$, those derivatives only multiply the Lorentz values of $g^{\mu\nu}$, which are constants. Hence, the only terms which can contribute to (7.5) are the terms involving $x^{\sigma'}{}_{,oo}$, $x^{\sigma'}{}_{,om}$, $x^{\sigma'}{}_{,mm}$. Direct calculation shows that such terms do not appear in (7.8) when $\nu=0$, $\mu=m$.

Hence the $S_\rho{}^{[\nu\mu]}$ given by (7.7) may, for the application considered here, be considered a tensor. There has been no need to impose coordinate conditions in order to establish this fact [except for the condition (7.3), which is not only a coordinate condition but also a limitation on the kinds of spaces to which (7.5) will be applied]. In fact, coordinate conditions would be of no help in the present problem. For the only objects available for imposing coordinate conditions are the same objects (i.e., the components of the metric) whose changes are to be limited by the coordinate conditions. To limit the changes in, say, $\Gamma_{\kappa\lambda}{}^\sigma$, the coordinate condition that would be imposed would state that $\Gamma_{\kappa\lambda}{}^\sigma$ itself equaled some given value. Thus, to be effective the coordinate conditions would have to fix at preselected values the very objects one wished to use to indicate the Schwarzschild mass.

Two cases will now be distinguished: the case in which $S_\rho{}^{[om]}\eta_m$ involves no terms of order $1/r$ and the case in which it does involve such terms. In the first case (which holds for the Schwarzschild metric), the fact that $S_\rho{}^{[\nu\mu]}$ of (7.7) effectively transforms as a tensor density means that under transformations asymptotic to the identity, $S_\rho{}^{[om]}\eta_m$ is unchanged to order $1/r^2$. Hence, in this case the energy-momentum has the properties required of it. This case includes as a special case the situation considered by Einstein (23)[37] in which all Christoffel symbols are of order $1/r^2$. For that situation, energy is conserved (the system does not radiate), and one can use that fact to prove that energy and momentum behave properly under coordinate transformations. But the present case is more general and includes, for example, the metrics considered by Trautman (57)[38] for which the Christoffel symbols involve terms of order $1/r$ and for which energy is not conserved.

In the case in which $S_\rho{}^{[om]}\eta_m$ is of order $1/r$, it must be required that the terms of order $1/r$ vanish when integrated over angles in order that the energy have a finite value. Even when this requirement is fulfilled, the situation is unsatisfactory, for coordinate transformations asymptotic to the identity can affect the terms of order $1/r^2$ and thus change the energy value. The fact that $S_\rho{}^{[om]}\eta_m d\Omega$ is effectively a vector is not sufficient to assure a unique energy value because of the noncovariant nature of the integral of a vector. Thus, the expression (7.5), using an $S_\rho{}^{[\nu\mu]}$ defined by (7.7), may be used to define the Schwarzschild mass of a system provided, first, that the conditions (7.3) are fulfilled and, second, that $S_\rho{}^{[om]}\eta_m$ is of an order less than $1/r$ for the hypersurfaces for which the calculation is to be performed. To be able to perform the calculation of the Schwarzschild mass on any hypersurface, $S_\rho{}^{[\nu\mu]}$ itself must be of an order less than $1/r$. This more stringent requirement implies that, unless material fields are nonvanishing at infinity, there is no radiation (i.e., that $S_\rho(\omega)$ is conserved). This is now shown.

Direct calculation from (7.7) shows that if $S_\rho{}^{[\nu\mu]}$ is of an order less than $1/r$, then, to order $1/r$, $g_{\mu\nu,\lambda} = g_{(\mu\nu,\lambda)}$. To see whether $S_\rho(\omega)$ is conserved, the time derivative of (7.5) must be taken. This requires $S_\rho{}^{[om]}{}_{,o}$ to be computed to order $1/r^2$. But to this order, $S_\rho{}^{[om]}{}_{,o}$ equals $S_\rho{}^{[\nu m]}{}_{,\nu}$, for the derivative normal to β vanishes due to the skew-symmetry while the derivatives tangent to β is of an order less than $1/r^2$. In differentiating $S_\rho{}^{[\nu m]}$, terms are obtained involving products of two first derivatives of the metric; these are of an order less than $1/r^2$ due to the above condition on $g_{\mu\nu,\lambda}$. Thus, the only terms of interest are those involving second derivatives of the metric. Re-expressing these derivatives with the use of the definitions of the Ricci tensor and the scalar curvature (2.9b, c), one may write

$$S_\rho{}^{[om]}{}_{,o}=(-g)^{W/2}[2(R_\rho{}^m-\tfrac{1}{2}\delta_\rho{}^mR) +2g^{m\nu}g^{\kappa\lambda}g_{\sigma\tau}(\Gamma_{\rho\nu}{}^\sigma\Gamma_{\kappa\lambda}{}^\tau-\Gamma_{\rho\kappa}{}^\sigma\Gamma_{\nu\lambda}{}^\tau) +\delta_\rho{}^mg^{\alpha\beta}g^{\kappa\lambda}g_{\sigma\tau}(\Gamma_{\alpha\kappa}{}^\sigma\Gamma_{\beta\lambda}{}^\tau-\Gamma_{\alpha\beta}{}^\sigma\Gamma_{\kappa\lambda}{}^\tau)] \quad (7.9)$$

to order $1/r^2$. The terms involving products of two Christoffel symbols are all of an order less than $1/r^2$ due to the condition on $g_{\mu\nu,\lambda}$. Therefore $S_\rho{}^{[om]}{}_{,o}$ vanishes to order $1/r^2$ if there are no matter fields at infinity, since $R_\rho{}^m-\frac{1}{2}\delta_\rho{}^mR$ then vanishes in consequence of the equations of motion. Radiation can occur only if there are

[37] A similar discussion is made by Landau and Lifschitz (39, p. 320).

[38] Trautman proves a result similar to the one given here, but the present author feels that his proof lacks rigor.

matter fields at infinity. However, when there are matter fields at infinity, $S_\rho{}^{[\nu\mu]}$ need not be of the form (7.6), and further analysis is needed. These results also apply if the index ρ is raised.

Concerning the expression of Møller (43), even though his expression is unchanged by transformations of the coordinates within the initial surfaces, it does not fulfill the requirements which are demanded here. Coordinate transformations involving the time coordinate may be made which do change the energy value. In fact, in his paper Møller shows that the energy calculated for any hypersurface is zero if the metric is chosen (as it always may be, even for the Schwarzschild space) such that $g_{mo}=0$, $g_{oo}=-1$. The properties that Møller demands of his expression are incompatible with the properties required here. The present author finds this fact interesting and surprising.

A set of expressions (7.7) suitable for defining the Schwarzschild mass, even in spaces which are not Schwarzschildian, has been found. The question of the physical significance of this generalized mass is a topic for further investigation. Such an investigation (which will ask the question, of what *use* is the calculated mass) will perhaps be able to decide what is the value W should assume in (7.7) and whether or not the index ρ should be written in the raised position. In any event, the limitations on the kinds of spaces in which the calculation may be made must always be kept in mind.

Mathematical Uses

The uses of conservation laws considered up until now have been uses of the numerical values of the conserved quantities. It is now time to turn to the other major class of uses, namely, those in which the mathematical form in which the laws are expressed is the thing of importance. It is not possible to even attempt to catalog such uses. On the contrary, it is the author's belief that the possibilities inherent in such uses have not been fully examined. In order to display some possibilities, Sec. VIII presents three new uses of this kind. In the interest of completeness, two well-known uses, adequately discussed elsewhere, are now mentioned briefly.

One use is in the problem of motion in general relativity, i.e., the problem of inferring the motion of the sources of the gravitational field from a knowledge of the equations of motion of the gravitational field itself. This problem was first satisfactorily handled by Einstein, Infeld, and Hoffman (25, 26, 27).[39] More recent work[40] has shown the intimate connection between the way in which the equations of motion are rewritten in the Einstein-Infeld-Hoffman treatment and the way they are rewritten in certain conservation laws.

Another use, rather indirect, of conservation laws is in the Hamiltonian form of the equations of motion. This is a mathematical use since the functional dependence of the Hamiltonian on coordinates and momenta is important. Hamiltonian forms of general relativity are discussed by Dirac (18) and Bergmann *et al.* (2, 10, 12, 13, 14). Some remarks on this subject are also made in Sec. VIII.

[39] See also Scheidegger (49).

[40] See Goldberg (30, 31).

Definitions

As to which law should be called the energy law, the momentum law, or the angular momentum law, the author prefers to let the reader decide. The only guide one has in the assignment of a name to a quantity is analogy with the ways the name has been used previously. The analogy in this case would seem to come from Lorentz-covariant theory. But the discussion of this section shows that the analogy is poorer than one might have expected. The author finds himself unable to make any analogy which satisfies him completely, but the reader may feel differently. It should only be kept in mind that if one defines the energy in one way [as, say, being derived from one of the expressions (7.7) or as a particularly suggestive expression used in the problem of motion], it does not then follow that the expression has any of the properties of energy familiar from Lorentz theory.

VIII. THREE MATHEMATICAL USES OF CONSERVATION LAWS

Stress-Energy of the Dirac Field

This section presents three mathematical uses of conservation laws, i.e., uses in which the conservation laws are employed to display in an especially simple manner mathematical relationships between the field variables. First, the problem of computing the stress-energy tensor of the Dirac is considered. Here is meant the symmetrized stress-energy tensor, the tensor which is the source of the gravitational field. [The theory discussed here is described by (2.10d).] It is defined by

$$T_{\mu\nu}=2(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R). \tag{8.1}$$

This expression is actually a set of equations of motion, namely, those which follow from varying $g^{\mu\nu}$ [see (2.13)]. As explained in Sec. II, because of the nonindependence of $g^{\mu\nu}$ and γ^μ, $T_{\mu\nu}$ is somewhat difficult to calculate directly when L_M involves spinors, as it does when the Dirac field is present. It is now shown how the calculation of $T_{\mu\nu}$ for the Dirac field may be shortened by use of a fact learned from the study of conservation laws.

The desired result follows immediately from Eq. (6.3) with $N=1$. As a guide, a few steps in the calculation are indicated. The quantities that must be known are $S(0)^\nu{}_\rho$ and $S(1)^\nu{}_\rho{}^\sigma$; these have already been obtained in (5.3b, c). (The case in which $L_M=L_E+L_D$ is being considered here.) It would be convenient to simplify them by using the equations of motion which follow from varying $\bar{\psi}$ and ψ, since these are easily obtained directly. The result is

$$S(0)^{\nu}{}_{\rho}=(-g)^{\frac{1}{2}}(R\delta_{\rho}{}^{\nu}-\tfrac{1}{4}F^{\lambda\mu}F_{\lambda\mu}\delta_{\rho}{}^{\nu}-g^{\lambda\mu}\Gamma_{\lambda\mu}{}^{\nu}{}_{,\rho}+g^{\lambda\nu}\Gamma_{\lambda\mu}{}^{\mu}{}_{,\rho} -F^{\sigma\nu}A_{\sigma,\rho}+\tfrac{1}{2}\bar{\psi}\gamma^{\nu}\psi_{|\rho}-\tfrac{1}{2}\bar{\psi}_{|\rho}\gamma^{\nu}\psi+ie\bar{\psi}\gamma^{\nu}\psi A_{\rho} -\tfrac{1}{16}\bar{\psi}\{\gamma^{\nu},[\gamma_{\mu},\gamma^{\lambda}]\}\psi\Gamma_{\lambda\rho}{}^{\mu}), \quad (8.2a)$$

$$S(1)^{\sigma}{}_{\rho}{}^{\nu}=(-g)^{\frac{1}{2}}(g^{\lambda\mu}\Gamma_{\lambda\mu}{}^{\nu}\delta_{\rho}{}^{\sigma}-2g^{\mu\nu}\Gamma_{\mu\rho}{}^{\sigma}+g^{\nu\sigma}\Gamma_{\rho\mu}{}^{\mu}+F^{\sigma\nu}A_{\rho} +\tfrac{1}{16}\bar{\psi}\{\gamma^{\nu},[\gamma_{\rho},\gamma^{\sigma}]\}\psi). \quad (8.2b)$$

In the first of these expressions, $\gamma_{\lambda,\rho}$ has been rewritten by use of the defining equation for Γ_{ρ}, (2.6). If these expressions are substituted into (6.3) with $N=1$, there results (after the factor $(-g)^{\frac{1}{2}}$ has been divided out)

$$-2(R_{\rho}{}^{\nu}-\tfrac{1}{2}R\delta_{\rho}{}^{\nu})+(F^{\nu\mu}F_{\rho\mu}-\tfrac{1}{4}\delta_{\rho}{}^{\nu}F_{\lambda\mu}F^{\lambda\mu}) +\tfrac{1}{2}(\bar{\psi}\gamma^{\nu}\psi_{|\rho}-\bar{\psi}_{|\rho}\gamma^{\nu}\psi+\tfrac{1}{8}(\bar{\psi}\{\gamma^{\nu},[\gamma_{\rho},\gamma^{\mu}]\}\psi)_{|\mu})=0. \quad (8.3)$$

This expression has been simplified with the use of the equation of motion obtained by varying A_{μ}. In this expression, the stress-energy tensor of the electromagnetic field is easily identifiable as the second term; therefore, the last term must be the contribution to the stress-energy by the Dirac field. By employing once again the equations of motion obtained by varying $\bar{\psi}$ and ψ, this may be simplified to give[41]

$$T_{\mu\nu}=\tfrac{1}{4}(\bar{\psi}\gamma_{\nu}\psi_{|\mu}-\bar{\psi}_{|\mu}\gamma_{\nu}\psi+\bar{\psi}\gamma_{\mu}\psi_{|\nu}-\bar{\psi}_{|\nu}\gamma_{\mu}\psi). \quad (8.4)$$

Relation Between Gauges and Constraints

A second mathematical use of conservation laws is now considered. The use of the "edge-on" view of Sec. III raises the interesting problem of what constitutes a sufficient set of field variables which, when measured on an initial surface, enable one to completely predict the future. There is an immediate answer: the Hamiltonian variables, since Hamilton's equations involve no higher that first derivatives of these quantities. There are, however, two features to be noted, namely, the phenomena of initial constraints and of gauge. For convenience, the initial surfaces are here taken as surfaces for which x^o equals a constant.

Some of Hamilton's equations express relations between the χ^A which do not involve derivatives with respect to x^o. For example, one of Hamilton's equations for the electromagnetic field is $F^{o\mu}{}_{,\mu}\equiv F^{om}{}_{,m}=0$. Such a relation among the χ^A is called an *initial constraint.* There are not only initial constraints which follow didirectly from Hamilton's equations but further ones obtained by differentiating with respect to x^o the constraints so obtained; constraints so derived are called secondary. The existence of constraints shows that the specification of all the Hamiltonian variables on an initial surface is sufficient but not necessary for the prediction of the future. However, to get rid of this redundancy, one usually must resort to specifying quantities which are not field variables in the sense used here, since they are not functions of a single point. For example, for the electromagnetic field, nonredundant quantities to replace F^{om} are the "transverse" components of F^{om}, a notion not definable at a point.

Hamiltonian variables are not only not necessary for the required specification but in a certain sense are not sufficient either. The derivatives of some field variables with respect to x^o just do not occur in Hamilton's equations; thus, there is no way to discover the future behavior of those variables. There is clearly nothing that can be done to correct this circumstance; it must be accepted that the future is to a certain extent unpredictable. Specification of the Hamiltonian variables enables the future to be predicted insofar as possible. Thus, there are certain changes described by arbitrary parameters which may be performed upon the field variables in the future without violating the equations of motion; such a change is called a *gauge* transformation in analogy with that famous example, the electromagnetic gauge. Other examples of gauges are coordinate and similarity transformations in generally covariant theories. Since all effects of physical significance should be predictable, two field configurations differing by a gauge transformation are assumed to be of the same physical significance.

That there must be some relationship between the existence of a gauge and the existence of initial constraints can be seen in a rough way: For every one of the $^*F$ Hamiltonian variables which does not have a derivative of it with respect to x^o appearing in Hamilton's equations, there must be one of the $^*F$ Hamilton's equations involving no derivatives at all with respect to x^o.[42] This type of counting procedure does not, however, show how one might use one's knowledge of the existence of a gauge to discover initial constraints. The following discussion shows how a conservation law enables one to do this. More precisely, a method is given for deriving, from the existence of a gauge, a set of initial constraints on arbitrary variations in the Hamiltonian variables which conform to the equations of motion.

The first part of the derivation of this method is done in two ways. The first way shows that the expression obtained is a suitable rewriting of Eq. (6.3). This way very definitely does not constitute a "transparent" derivation. Therefore, a second mode of derivation is given, a mode which, although it obscures the close connection of the result with (6.3), makes the result far easier to "see."

Let (6.3) be considered. By differentiating this equation, multiplying it by suitable factors, and then summing, one obtains

$$\sum_{N=0}[(-1)^{N}(N+M)!/M!N!][S(N+M-1)^{(\sigma_{A}\cdots\tau\cdots)} +S(N+M)^{\nu}{}_{A}{}^{\sigma\cdots\tau\cdots}{}_{,\nu}]_{,\tau\cdots}=0. \quad (8.5a)$$

In this section, the convention is that there are N

[41] This result is the same as that obtained by a different method by Brill and Wheeler (15).

[42] This argument is presented more rigorously by Bergmann and Schiller (14).

indices $\tau\cdots$; hence, there are M indices $\sigma\cdots$ in this last equation and the next. Equation (8.5a) may be immediately rewritten as

$$\sum_{N=0}[(-1)^N(N+M-1)!/(M-1)!N!] \times S(N+M-1)^{(\sigma}{}_A{}^{\cdots)\tau\cdots}{}_{,\tau\cdots}=0. \quad (8.5b)$$

This equation is specialized to the case in which the indices $\sigma\cdots$ are all equal to the same value, namely o. Then, if the derivatives with respect to x^o are separately noted, Eq. (8.5b) becomes

$$\sum_{N=0}\sum_{L=M}[(-1)^{L+M+N}(L+N-1)!/(L-M)!(M-1)!N!] \times S(N+L-1)^o{}_A{}^{\cdots t\cdots}{}_{,t\cdots o\cdots}=0. \quad (8.5c)$$

Here Latin indices (e.g., t) are summed only over values not equal to o; there are L-M derivatives with respect to x^o on each term, and the index o occurs L times in $S(N+L-1)^o{}_A{}^{\cdots t\cdots}$. Note that o is a fixed index and therefore not summed. If (8.5c) is multiplied by $(M-1)!/(M-K-1)!K!$ and summed over M (for $M-1\geq K\geq 0$), the result is, after the summations are rearranged into an equivalent expression,

$$\sum_{N=0}\sum_{L=K+1}\sum_{M=K+1}^{L}[(-1)^{L+M+N}(L+N-1)!/ (L-M)!(M-K-1)!N!K!] \times S(N+L-1)^o{}_A{}^{\cdots t\cdots}{}_{,t\cdots o\cdots}=0. \quad (8.5d)$$

The sum over M vanishes unless $L=K+1$,[43] in which case it consists of a single term. In consequence, the sum over L also consists of just one term. So, it follows that, for each $K\geq 0$,

$$\sum_{N=0}[(-1)^N(N+K)!/K!N!] \times S(N+K)^o{}_A{}^{\cdots t\cdots}{}_{,t\cdots}=0, \quad (8.6)$$

which is the expression which has been sought.

This last relation is now derived in another, less straightforward, but far more transparent way. The first step is to integrate (6.1) over a d-volume bounded by two x^o equals constant surfaces and a cylindrical surface joining them. On the cylindrical surface and one of the x^o equals constant surfaces the arbitrary parameters Λ^A together with a sufficient number of their derivatives are assumed to vanish. By converting the volume integral into a surface integral by means of Gauss' theorem, one obtains

$$\sum_{N=0}\int S(N)^o{}_A{}^{\tau\cdots}\Lambda^A{}_{,\tau\cdots}d\sigma_o=0 \quad (8.7a)$$

on the surface on which the Λ^A do not vanish. Derivatives with respect to the $d-1$ coordinates other than x^o may now be transferred from Λ^A to $S(N)^o{}_A{}^{\tau\cdots}$ by use of integration by parts, since the Λ^A and their derivatives vanish on the boundaries of the surface. The result is

$$\sum_{N=0}[(-1)^N(N+K)!/K!N!] \times\int S(N+K)^o{}_A{}^{\cdots t\cdots}{}_{,t\cdots}\Lambda^A{}_{,o\cdots}d\sigma_o=0, \quad (8.7b)$$

where Λ^A is differentiated K times with respect to x^o. Since each $\Lambda^A{}_{,o\cdots}$ may be arbitrarily chosen all over the surface, the coefficient of each must vanish, from which fact follows (8.6).

The significance of (8.6) is this: When $S(N)^\nu{}_A{}^{\tau\cdots}$ is given by (5.12), then (6.1) and (6.3) refer directly to gauge transformations of the Hamiltonian variables; hence (8.6) is a direct consequence of the existence of a gauge. But, when $S(N)^\nu{}_A{}^{\tau\cdots}$ is given by (5.12), (8.6) is a constraint on the $\delta\chi^C$, since it involves no derivatives of the $\delta\chi^C$ with respect to x^o. Thus, the existence of a gauge has led directly to the existence of constraints on variations in the Hamiltonian variables which conform to the equations of motion, namely,

$$2\sum_{N=0}[(-1)^N(N+K)!/K!N!] \times[\chi(N+K)^B{}_A{}^{o\cdots t\cdots}\mathfrak{A}_{[B}{}^o{}_{C]}\delta\chi^C]_{,t\cdots}=0. \quad (8.8)$$

From these one can infer constraints on the Hamiltonian variables themselves. In general, each gauge freedom has more than one initial constraint corresponding to it; in fact, there is one initial constraint for each derivative order (including the zeroth) to which a gauge parameter appears in (5.1), though some of these constraints may be trivial.

As illustrations of the results that may be obtained from the relation just derived, consider the Lorentz-covariant Maxwell-Dirac field and the pure gravitational field. The easiest way to apply the relation is to set $\nu=0$ in (5.9b) and (5.10b) and then transpose derivatives (other than with respect to x^o) as though one were integrating by parts until no derivatives (other than with respect to x^o) appear on the parameters Λ and ξ^ρ; the coefficients of these undifferentiated parameters are the required initial constraints. For the Maxwell-Dirac field one obtains

$$\delta[F^{mo}{}_{,m}+ie\bar{\psi}\gamma^o\psi]=0, \quad (8.9)$$

which is the coefficient of Λ. For the gravitational field, on the other hand, there results

$$\delta[2\mathfrak{g}^{\lambda o}\mathfrak{G}_{\lambda r}{}^o+\mathfrak{g}^{oo}{}_{,r}]=0, \quad (8.10a)$$

$$\delta[-\mathfrak{g}^{\lambda\mu}\mathfrak{G}_{\lambda\mu}{}^o+2\mathfrak{g}^{\lambda o}\mathfrak{G}_{\lambda o}{}^o-\mathfrak{g}^{om}{}_{,m}]=0, \quad (8.10b)$$

$$\delta[\mathfrak{g}^{\lambda\mu}\mathfrak{G}_{\lambda\mu}{}^o{}_{,r}-(\mathfrak{g}^{\lambda m}\mathfrak{G}_{\lambda r}{}^o)_{,m}-\mathfrak{g}^{om}{}_{,mr}]=0, \quad (8.10c)$$

[43] This may be seen as follows: Except for a factor independent of M, (8.5d) equals

$$\sum_{M=K+1}^{L}(-1)^{M-K-1}(L-K-1)!/(M-K-1)!(L-M)! \equiv(1-1)^{L-K-1},$$

which vanishes if $L>K+1$.

$$\delta[-2(\mathfrak{g}^{\lambda m}\mathfrak{G}_{\lambda o}{}^{o})_{,m}+\mathfrak{g}^{mn}{}_{,mn}]+(\mathfrak{G}_{\lambda\nu}{}^{m}\delta\mathfrak{g}^{\lambda\nu})_{,m} +\mathfrak{G}_{\lambda\mu}{}^{o}{}_{,o}\delta\mathfrak{g}^{\lambda\mu}-\mathfrak{g}^{\lambda\mu}{}_{,o}\delta\mathfrak{G}_{\lambda\mu}{}^{o}=0, \quad (8.10\text{d})$$

which are the coefficients of $\xi^{r}{}_{,o}$, $\xi^{o}{}_{,o}$, ξ^{r}, and ξ^{o}, respectively.

It is of interest to consider whether every constraint of the $\delta\chi^C$ can be written in the form (8.8) with coefficients $\chi(N+K)^B{}_A{}^{o\cdots t\cdots}$ which describe a gauge transformation (as is the case for the examples just considered). The answer is no.[44] Careful consideration of just how gauges and constraints arise from the equations of motion shows that the $\chi(N+K)^B{}_A{}^{o\cdots t\cdots}$ corresponding to a constraint *do* describe a gauge if they themselves satisfy the constraints. Examination of this point does not make use of conservation laws and therefore is not discussed here.[45]

It may be wondered whether the relations derived above have any more than curiosity value. One might feel, for example, that, while it is rather interesting that the constraints (8.10) are intimately associated with gauge invariances, it is not particularly useful to express the constraints of the pure gravitational field in the form (8.10). It is hoped that the following discussion of a third mathematical use of conservation laws, which applies to above ideas to the Schwinger formalism, dispels such feelings.

Treatment of Gauge in the Schwinger Formalism

In order to set up the quantum theory of a physical system, two basic items must be determined: first, the equations of motion for the Heisenberg operators; second, the commutation (or anticommutation) relations for these operators on an initial surface. Once this has been done, "all" that remain are the mathematical difficulties encountered when solving the equations of motion in particular cases. The Schwinger formalism (51, 52)[46] is a prescription for arriving at these two basic items, based upon the supposition that one already knows the Lagrangian of the theory. An unfortunate feature of the formalism, however, is that it does not seem to be immediately applicable to theories possessing gauge freedoms; when so applied, certain *ad hoc* operations must usually be performed in order to obtain proper results (3). The following discussion presents a definite method for dealing with such situations, a method which makes use of conservation laws.

Certain features of quantum theory are not related to the basic objective of the present discussion but necessarily intrude themselves and greatly complicate the issue; there is, in particular, the factor-ordering problem. In order to avoid such difficulties, the Schwinger formalism is here described as a classical theory. It then becomes a prescription for finding equations of motion and, instead of commutators, Poisson brackets. By this approach, the essential features of the technique for treating gauge are brought out; there then remain only the usual problems of converting from a classical to a quantum theory, a subject outside the scope of this paper.

One does, however, pay a price when one makes the Schwinger formalism into a classical theory. By so doing one makes it a highly artificial formalism; certain essential arguments made in establishing the quantum-mechanical formalism have no classical analog. Therefore, no attempt is made here to justify or interpret the formalism. Rather, the two procedures, one for arriving at equations of motion, one for arriving at commutators, which are derived in the quantum-mechanical formalism, are taken over (in their classical analogs) as postulates.

As shown in Sec. II, any theory derived from a Lagrangian may be derived from a Schwinger Lagrangian [see (2.19)]:

$$^*L=\chi^B\mathfrak{A}_{[B}{}^{\nu}{}_{C]}\chi^C{}_{,\nu}-H(\chi^A). \quad (8.11)$$

In order to use the Schwinger formalism, one must know *a priori* the χ^A and the $^*L$ appropriate to the theory at hand. One can then write, upon varying $^*L$ in the manner of (2.11),

$$M_A\delta\Phi^A=(2\chi^B{}_{,\nu}\mathfrak{A}_{[B}{}^{\nu}{}_{C]}+\partial H/\partial\chi^C)\delta\chi^C, \quad (8.12)$$

$$\Delta^\nu=\chi^B\mathfrak{A}_{[B}{}^{\nu}{}_{C]}\delta\chi^C. \quad (8.13)$$

The expression for Δ^ν is unique, since it is required that no derivatives of the χ^A or $\delta\chi^A$ appear in it. The first postulate of the classical Schwinger formalism is that the equations of motion follow from the condition $M_A\delta\Phi^A=0$, as usual. This postulate is unaffected by the presence of a gauge, and no more need be said of it.

The Poisson bracket (which is here symbolized with angle brackets $\langle\,\rangle$) is introduced into the formalism as an undefined binary operation restricted by the requirements that it is skew-symmetric in its arguments, that it vanishes whenever either argument is independent of the χ^A, that it is linear and distributive if either argument is expressed as a sum of products, and that it satisfies the Jacobi identity. These properties are so similar to the properties of the derivative that it is clear that the constraints (8.8) [and similarly (8.9) and (8.10)] hold if $\delta\chi^C$ is replaced by the Poisson bracket of χ^C with any other variable; this already shows that (8.9) and (8.10) have some practical value. An important use of the Poisson bracket operation is in carrying out an *infinitesimal canonical transformation* of the χ^A. Under such a transformation, the changes in the χ^A are

$$\delta\chi^A=\langle\chi^A, G(\omega)\rangle, \quad (8.14)$$

where $G(\omega)$ is a function of the χ^A called the *generator* of the transformation.

[44] For example, the Lagrangian, $L=pq_{,o}+rp+sq$, in a one-dimensional physical space (coordinate x^o) has primary constraints $p=0$ and $q=0$ and secondary constraints $r=0$ and $s=0$. Clearly, it possesses no gauges.

[45] This point is covered in a very different formalism by Dirac (17).

[46] Several of Schwinger's remarks suggest, but do not discuss in detail, the results obtained here.

The "edge-on" view is adopted. A canonical transformation is, by its nature, a transformation on a given initial surface; this accounts for the notation $G(\omega)$. In the present instance only *local* canonical transformations are of interest, i.e., transformations for which $G(\omega) = \int_\omega G(x^m)d\sigma_o$. The second postulate of the classical Schwinger formalism is that there exists an infinitesimal canonical transformation on any initial surface ω such that

$$\langle \chi^{A'}, \delta\chi^B \rangle = 0 \tag{8.15a}$$

for all A and B and any two points on ω (the prime on $\chi^{A'}$ is to indicate that it is, in general, at a point different from $\delta\chi^B$) and such that

$$G(\omega) = 2\int_\omega \Delta^o d\sigma_o \equiv 2\int \chi^B \mathfrak{A}_{[B^oC]}\delta\chi^C d\sigma_o. \tag{8.15b}$$

This second postulate enables one to calculate the Poisson brackets of the χ^A with one another all over ω. For, from (8.14) and (8.15), it follows that

$$\delta\chi^{A'} = 2\int_\omega \langle \chi^{A'}, \chi^B \rangle \mathfrak{A}_{[B^oC]}\delta\chi^C d\sigma_o. \tag{8.16}$$

Since the matrix $\mathfrak{A}_{[B^oC]}$ is in general singular, this directly determines only some of the $\langle \chi^{A'}, \chi^B \rangle$. However, with the use of the equations of motion, the others may be found. This is best illustrated by an example.

By applying the relation (8.16) to the neutral scalar meson [see (2.20c)], one finds

$$\delta\phi' = \int (\langle \phi', p^o \rangle \delta\phi - \langle \phi', \phi \rangle \delta p^o) d\sigma_o, \tag{8.17a}$$

$$\delta p^{o'} = \int (\langle p^{o'}, p^o \rangle \delta\phi - \langle p^{o'}, \phi \rangle \delta p^o) d\sigma_o, \tag{8.17b}$$

from which it follows that

$$\langle \phi', \phi \rangle = \langle p^{o'}, p^o \rangle = 0, \quad \langle \phi', p^o \rangle = \delta(x'-x). \tag{8.17c}$$

But nothing has been found about Poisson brackets involving p^n. However, with the use of the equations of motion and the derivatives of (8.17c), they may be obtained as

$$\langle p^{n'}, p^m \rangle = \langle p^{n'}, \phi \rangle = 0,$$
$$\langle p^{n'}, p^o \rangle = -(\partial/\partial x^{n'})\delta(x'-x). \tag{8.17d}$$

The skew-symmetry of the $\mathfrak{A}_{[B^oC]}$ assures the skew-symmetry of the Poisson brackets.

The theory just considered as an example does not possess a gauge. If one tries to carry out a procedure similar to the one just used on a theory, such as electromagnetic theory, that possesses a gauge, two difficulties appear: first, one is not able to find all Poisson brackets (e.g., those involving A_o in electromagnetic theory), and, second, some of the Poisson brackets obtained are incompatible with the equations of motion (e.g., $\langle F^{om}{}_{,m}, A_n \rangle$ does not vanish, as it should). The reader may verify these statements regarding the electromagnetic field; that field is discussed in some detail after the statement of the modified formalism for use in theories with a gauge (which reduces to the usual formalism in theories without a gauge).

A few remarks are now made to motivate this modification. In a theory without a gauge, specification of the χ^A over an initial surface completely determines the χ^A over all physical space. This is not so in a theory with a gauge; the χ^A are known all over space only "up to a gauge." It is therefore reasonable to postulate that sets of χ^A differing by a gauge are of the same physical significance; the apparatus of the theory should not be able to "know" the difference between χ^A that differ by a gauge. Hence in (8.14), the right and left sides cannot be exactly equated; they are only equal "up to a gauge." Therefore the new second postulate of the classical Schwinger formalism may be expressed as the replacement of (8.16) by

$$\delta\chi^{A'} \doteqdot 2\int \langle \chi^{A'}, \chi^B \rangle \mathfrak{A}_{[B^oC]}\delta\chi^C d\sigma_o, \tag{8.18}$$

where $\doteqdot$ means that the expressions on the left and right may be made equal by a suitable gauge transformation of one of them.

It is now shown what this postulate means for electromagnetic theory. One has [see (2.20a)]

$$\delta A_n' \doteqdot \int (\langle A_n', F^{mo} \rangle \delta A_m - \langle A_n', A_m \rangle \delta F^{mo}) d\sigma_o, \tag{8.19a}$$

$$\delta F^{no'} \doteqdot \int (\langle F^{no'}, F^{mo} \rangle \delta A_m - \langle F^{no'}, A_m \rangle \delta F^{mo}) d\sigma_o. \tag{8.19b}$$

From these follow

$$\langle A_n', A_m \rangle = (\partial/\partial x^{n'})\Lambda_m(x,x') - (\partial/\partial x^m)\Lambda_n(x',x) \tag{8.19c}$$

$$\langle F^{no'}, F^{mo} \rangle = 0, \tag{8.19d}$$

$$\langle A_n', F^{mo} \rangle = \delta_n{}^m \delta(x'-x) + (\partial/\partial x^{n'})\Lambda_1{}^m(x,x'). \tag{8.19e}$$

The Λ_m and $\Lambda_1{}^m$ are arbitrary functions. They arise from two causes: first, the uncertainty implied by the notation $\doteqdot$, second, the fact that $\delta F^{mo}{}_{,m} = 0$, which implies that if a gradient is added to a term multiplying δF^{mo}, the values of the integrals (8.19a, b) are unchanged as may be seen by integrating by parts. (All fields fall off "sufficiently fast" at infinity.) The fact that these two causes lead to the same kinds of arbitrariness in the Poisson brackets is a consequence of the relation between gauges and constraints described previously, a relation derived from consideration of a conservation law. By use of the constraint equations of motion one finds

$$\langle F_{nk}', F_{ml} \rangle = 0, \tag{8.19f}$$

$$\langle F_{nk}{}',A_m\rangle = -(\partial/\partial x^m)(\partial/\partial x^{n'})\Lambda_k(x',x) + (\partial/\partial x^m)(\partial/\partial x^{k'})\Lambda_n(x',x), \quad (8.19g)$$

$$\langle F_{nk}{}',F^{mo}\rangle = \delta_k{}^m(\partial/\partial x^{n'})\delta(x'-x) - \delta_n{}^m(\partial/\partial x^{k'})\delta(x'-x). \quad (8.19h)$$

The Poisson brackets involving A_o cannot be found. This is as should be expected, since A_o may be anything "up to a gauge" on an initial surface. Hence, the Poisson brackets involving A_o may each be equated to an arbitrary function. There are some limitations on the arbitrary functions due to the constraints; in fact, the presence of these arbitrary functions assures that the constraints may be satisfied. In particular, in (8.19e), since $F^{om}{}_{,m}=0$,

$$(\partial/\partial x^{n'})[-\delta(x'-x)+(\partial/\partial x^m)\Lambda_1{}^m(x,x')]=0. \quad (8.19i)$$

One $\Lambda_1{}^m$ satisfying this is

$$\Lambda_1{}^m = -(1/4\pi)(\partial/\partial x^m)(1/|x'-x|).$$

Also, the time derivatives of some arbitrary functions are related to other arbitrary functions when the equations of motion are solved. The arbitrary functions do not appear in the expression for Poisson brackets of gauge invariant quantities (F_{nk} and F^{mo}) and hence do not affect the physical predictions of the theory. In practice, however, one usually fixes the arbitrary functions, thereby "choosing a gauge."

The consistency which the modified formalism has displayed in the case of the electromagnetic field is now shown in greater generality. Two points have to be made. First, that the arbitrariness in the $\langle\chi^{A'},\chi^B\rangle$ due to the meaning of $\doteqdot$ in (8.18) is the same as the arbitrariness due to the presence on constraints on the $\delta\chi^C$. Second, it must be shown that constraints on the $\langle\chi^{A'},\chi^B\rangle$ can be satisfied. In regard to the first point, the symbol $\doteqdot$ implies that $\langle\chi^{A'},\chi^B\rangle$ is determined only up to a gauge transformation of the $\chi^{A'}$. Consistency requires that it also be determined only up to a gauge transformation of the χ^B, i.e., that the right-hand side of (8.18) must remain unchanged when χ^B is gauge transformed. The proof of this hinges on (8.8), which was found from a consideration of conservation laws. The difference between the values of the right-hand side of (8.18) for two choices of χ^B may be written

$$2\left\langle\chi^{A'},\int\int \eth\chi^B \mathfrak{A}_{[B}{}^o{}_{C]}\delta\chi^C d\sigma_o\right\rangle, \quad (8.20)$$

where $\eth\chi^B$ is an infinitesimal gauge transformation and is integrated between the two values of χ^B. By expressing $\eth\chi^B$ with the use of (5.11), integrating by parts, and then applying (8.8), it is verified that (8.20) vanishes. In the examples considered here, all arbitrariness in the $\langle\chi^{A'},\chi^B\rangle$ due to the constraints on the $\delta\chi^C$ is of the form of a gauge transformation on the χ^B; hence, the first point is established. As has been mentioned, however, there is, for some theories, further arbitrariness in the χ^B due to the constraints on the $\delta\chi^C$ than follows from the existence of gauges. But it has also been mentioned that such arbitrariness vanishes when the constraints are required to be satisfied. The first point is thus proven in general.

In regard to the second point, the arbitrariness in $\langle\chi^{A'},\chi^B\rangle$ arising from the constraints on the $\delta\chi^C$ is of just such a nature to assure that there is sufficient freedom for the constraints on $\chi^{A'}$ to be satisfied. Because of the skew-symmetry of $\langle\chi^{A'},\chi^B\rangle$ established by the first point, this assures that the constraints on χ^B can be satisfied also.

The procedure presented here is closely related to procedures developed by Dirac (17, 18) and Bergmann *et al.* (2, 10, 12, 13, 14). In fact, it very likely produces physical predictions equivalent to those procedures. The relationship to those procedures has not yet been explicated, however. The exact relationship is not immediately obvious because of the great difference in viewpoint between those procedures and the present one. Dirac and Bergmann place arbitrary functions in the Hamiltonian, while the present procedure does not discuss the Hamiltonian and puts the arbitrary functions in the Poisson brackets.

The application of the procedure given here to the pure gravitational field is not as tractable from a mathematical standpoint as was the application to the electromagnetic field. For the electromagnetic field, the procedure yielded a few Poisson brackets directly; these were then used to find other Poisson brackets; the constraint equations were then applied; and finally, the time development of the brackets was obtained. (This last step has not been performed here.) For the gravitational field, this step-by-step procedure is not possible; one does not obtain any Poisson brackets directly but rather certain combinations of them involving arbitrary functions. Because of these difficulties, the Poisson brackets have not yet been entirely worked out for the gravitational field. This section concludes by making a few observations about the nature of the gravitational field in the light of the present formalism.

An immediate result of the formalism is the $\langle\mathfrak{g}^{\alpha\beta'},\mathfrak{g}^{\mu\nu}\rangle=0$ "up to a gauge" on the surface x^o equals constant. The phrase "up to a gauge" is important; it implies that $\langle\mathfrak{g}^{\alpha\beta'},\mathfrak{g}^{\mu\nu}\rangle=0$, not for any two points on the surface for which x^o equals constant but rather for any two points on the surface described by the constancy of that function of the gauge-transformed coordinates which equals x^o in the original coordinates. The formalism is "trying to tell us" that the coordinates describing the initial surface are unimportant. It has led to the conclusion that the Poisson bracket of the metric tensor at one point on the initial surface of topological points with the metric tensor at another point on that surface vanishes.

When one considers another result of the formalism, namely, that $\langle\mathfrak{G}_{\alpha\beta}{}^{o'},\mathfrak{g}^{\mu\nu}\rangle=\frac{1}{2}(\delta_\alpha{}^\mu\delta_\beta{}^\nu+\delta_\alpha{}^\nu\delta_\beta{}^\mu)\delta(x'-x)$ "up to a gauge," the mathematical difficulties become clear. A gauge transformation of $\langle \mathfrak{G}_{\alpha\beta}{}^{o\prime}, \mathfrak{g}^{\mu\nu} \rangle$ involves Poisson brackets of the form $\langle \mathfrak{G}_{\alpha\beta}{}^{k\prime}, \mathfrak{g}^{\mu\nu} \rangle$, which must be found by applying constraint equations to $\langle \mathfrak{G}_{\alpha\beta}{}^{o\prime}, \mathfrak{g}^{\mu\nu} \rangle$ and $\langle \mathfrak{g}^{\alpha\beta\prime}, \mathfrak{g}^{\mu\nu} \rangle$. Thus, one must solve simultaneously a set of equations in order to find $\langle \mathfrak{G}_{\alpha\beta}{}^{\kappa\prime}, \mathfrak{g}^{\mu\nu} \rangle$. It is hoped that these calculations can be carried out in the near future.

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