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Elementary Particles and Symmetry Principles*

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"···*science is always most completely assimilated when it is in the nascent state.*" J. C. Maxwell

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* An early version of this paper was delivered by invitation to the biennial conference of the Swedish National Committee for Physics, Upsala, June 4, 1957. The text has been considerably revised in the light of later information and deliberations (see Acknowledgments). Originally, it was to a great extent directed toward readers wishing to become acquainted with the main features underlying the recent explosion of interest in symmetry properties of elementary particles. The treatment was therefore generally descriptive. Later it was decided to insert a more deductive account of group concepts and some of their connections with relativistic quantum theory (Secs. II and III): A review of the fundamentals of Lorentz group representation theory is needed to give a theoretical classification of elementary particles and to elucidate the vexing questions of physical vs conventional significance of relative parities. In the Appendix the problem of defining *centroidal position* in a general field theory is discussed. Commuting position operators appearing in the work of various writers are here obtained in a general way.

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I. SURVEY AND SUMMARY

A. Particles—Their Production and Decay

WITHIN the last few years our knowledge of elementary particles has increased greatly. There are indications that we are approaching a leveling off in the discovery of new particles, and also some understanding of their behavior. In the first chart (Fig. 1), and in Table I, the mass levels of the various elementary particles (of nonzero rest mass) known today and their decay modes are represented. The particles may be grouped into three main categories in order of increasing mass: (1) light fermions or *leptons* which comprise the electron and muon; (2) the middleweight bosons or *mesons* which comprise the mass triplet of pions at

TABLE I. Properties and decay modes of the nonzero rest mass elementary particles.[a]

Particle name, symbol, charge	Spin	3rd component of isospin T_3	Strangeness $S=2(q-T_3)-N$ $q\equiv$charge $N\equiv$baryon number	Hypercharge (isoparity index) $U=S+N$	Mass in[b] electron masses	Lifetime in seconds τ	Decay modes (all except the Σ^0 and π^0 decays follow the weak interaction rule: $\lvert\Delta S\rvert=1$ (or $\lvert\Delta T_3\rvert=\frac{1}{2}$))	Experimental branching ratios	Theoretical branching ratios from the assumed rule: $\lvert\Delta T\rvert=\frac{1}{2}$
Xi Ξ^- (Ξ^0)	$\frac{1}{2}$	$-\frac{1}{2}$ $\frac{1}{2}$	-2	-1	2586	$(5<\tau<200)\times10^{-10}$	$\Lambda+\pi^-$ $(\Lambda+\pi^0)^c$		
Sigma Σ^-		-1			2341	$1.72\pm0.15\times10^{-10}$	$n+\pi^-$		
Σ^+	$\frac{1}{2}$	1	-1	0	2328	$0.84\pm0.1\ \times10^{-10}$	$p+\pi^0$ $n+\pi^+$	0.5 ±0.1 0.5 ±0.1	$\tau_{\Sigma^-}/\tau_{\Sigma^+}\approx10$
Σ^0		0			2327	$\simeq\ 10^{-12}$	$\Lambda+\gamma$		
Lambda Λ^0	$\frac{1}{2}$	0	-1	0	2181	$2.6\ \pm0.2\ \times10^{-10}$	$p+\pi^-$ $n+\pi^0$ $p+\mu^-+\bar{\nu}$, $p+e^-+\bar{\nu}$	0.63 ±0.03 0.37 ±0.03 $\lesssim10^{-3}$	$\frac{2}{3}$ $\frac{1}{3}$ $\boxed{\gtrsim10^{-2}\text{ on UFI}^d}$
Nucleon N^0 (n)	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	1838.65	$1.11\ \ \times10^3$	$p+e^-+\bar{\nu}$		
N^+ (p)		$\frac{1}{2}$			1836.12	∞			
Kayon K^+	0	$\frac{1}{2}$	1	1	966	$1.22\pm0.02\times10^{-8}$	$\mu^++\nu$ $\pi^++\pi^0$ $e^++\nu+\pi^0$ $\mu^++\nu+\pi^0$ $\pi^++\pi^-+\pi^+$ $\pi^++2\pi^0$	0.58 ±0.02 0.25 ±0.02 0.051 ±0.008 0.039 ±0.005 0.062 ±0.003 0.0215±0.03 $\div=2.9$	4/1.3=3.1[e]
$K^0\begin{cases}K_1\\K_2\end{cases}$		$-\frac{1}{2}$			965	$0.99\pm0.07\times10^{-10}$ $0.9\ \pm0.3\ \times10^{-7}$	$\begin{cases}\pi^++\pi^-\\\pi^0+\pi^0\end{cases}$ $\begin{cases}\pi^\pm+e^\mp+\nu,\ \pi^\pm+\mu^\mp+\nu\\\pi^++\pi^-+\pi^0\end{cases}$	0.42 ±0.05 (0.7 or 0.11$\pm$0.03) 0.51 ±0.07 $1/\tau_{3\pi}<1.5\times10^6$	$\tau_{K^+}/\tau_{K_1}\approx500^e$ $\boxed{0.14^e}$ $\boxed{1/\tau_{3\pi}=2.4\times10^{6e}}$
Pion π^-, π^+	0	-1, 1	0	0	273	$2.6\ \ \ \times10^{-8}$	$\mu^-+\bar{\nu}$, $e^-+\nu$; $\mu^++\bar{\nu}$, $e^++\nu$		
π^0		0			264	$(<4)\ \times10^{-16}$	$e^-+\nu$, $\gamma+\gamma$, $(\gamma+e^++e^-)$ $(2e^++2e^-)$		
Muon μ^-	$\frac{1}{2}$				206.8	$2.2\ \ \ \times10^{-6}$	$e^-+\nu+\bar{\nu}$		
Electron e^-	$\frac{1}{2}$				1	∞			

[a] The zero rest mass quanta which appear in the electromagnetic and weak interactions of these particles are the photon γ and the neutrino (antineutrino), $\nu(\bar{\nu})$, respectively.

[b] A more exact table of the experimentally observed masses can be found in W. H. Barkas and A. H. Rosenfeld, UCRL 8030, November, 1957. In the absence of any theory of masses of elementary particles, we have thought it sufficient to give rounded off values.

[c] The published experimental evidence for Ξ^0 is so far very meager (expectedly so). It consists principally of a picture taken by the Pic du Midi group (1958) in their double cloud chamber experiment. This picture shows two V-shaped tracks of which the apex of the upper one is a possible origin for the lower one suggesting a genetic relationship between the two events according to the cascade decay:

$$\Xi^0\rightarrow\Lambda+\pi^0$$
$$\pi^0\rightarrow\gamma+e^++e^-\ \text{(upper V track: a "Dalitz pair")}$$
$$\Lambda\rightarrow p^++\pi^-\ \text{(lower V track).}$$

[d] UFI represents "Universal Fermi Interaction."

[e] Some of these values involve considering an admixture of $\lvert\Delta T\rvert=3/2$. For a detailed discussion see R. Dalitz, Repts. Progr. Phys. **20**, 163 (1957); Gatlinburg Conference on Weak Interactions, October, 1958. The three outstanding discrepancies with the $\lvert\Delta T\rvert=\frac{1}{2}$ rule have been indicated in our table by putting a box around the theoretical predictions.

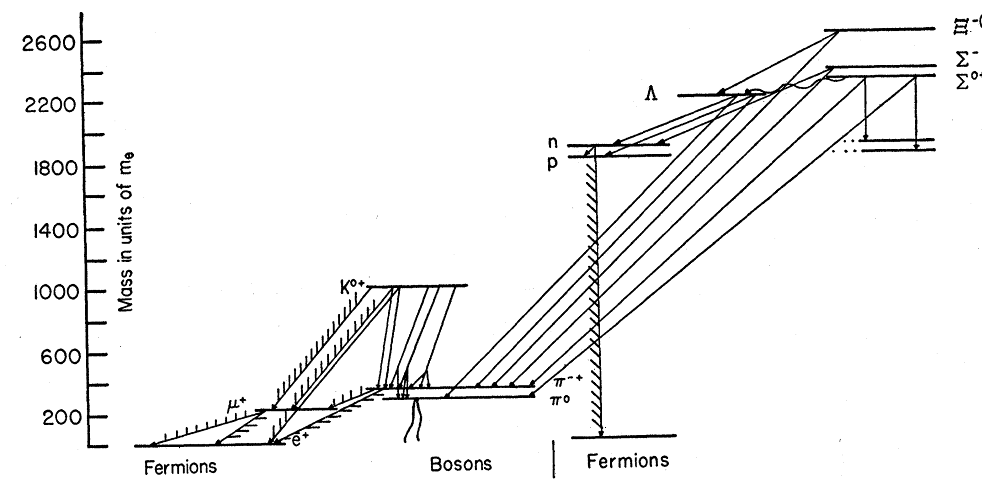

FIG. 1. System of decays of the elementary particles. Only "particles" are indicated except for the "antiparticles" e^+ and μ^+ and (in spite of the breakdown of charge-conjugation invariance in weak interactions) it is believed on the basis of the *CPT* theorem that there occur in nature exact charge-conjugate reactions of all those indicated, with antiparticles replacing particles. All established decay modes of the indicated "particles" are shown except those of K^0; these, however, parallel the six decay modes of K^+ except for the absence of the $\mu+\nu$ mode. "Radiationless" decays (i.e., without photon or neutrino emission) are indicated by straight lines joining decaying particle to daughter particle(s). The only three types of photon-emitting decays which certainly occur,

$$\pi^0 \rightarrow \gamma+\gamma, \quad \pi^0 \rightarrow \gamma+e^+ + e^-, \quad \Sigma^0 \rightarrow \Lambda^0+\gamma,$$

are indicated by wavy arrows from parent to daughter particle. The neutrino-emitting decays are indicated by a one-sided feathered arrow from parent to daughter particle(s).

about 270 electron masses, and the doublet of kayons (K^+ and K^0) at about 970 electron masses; (3) the heavy fermions or *baryons* ranging from the neutron-proton doublet up through the singlet Λ particles, the Σ triplet, and the doublet comprising Ξ^- and Ξ^0 (the cascade particles).

How do we encounter all these particles? The manner of occurrence of the older known particles, including the muon and pion, is by now generally familiar. The new "strange" ones are the kayons and the *hyperons* (baryons heavier than nucleons). Both were first observed in cosmic rays but are now produced copiously in the laboratory. Although kayons were clearly observed at least as early as 1947 (Rochester and Butler, 1948)[1] in the same year that the pions' existence was established experimentally, it was not until recently that their complicated behavior began to become clear. Hyperons are produced together with kayons in reactions in which high-energy pions, gammas, nucleons, collide with nucleons. The reactions, some of which are listed in detail in Table II, are of the general form

pion+nucleon gives *hyperon+kayon(s)*
gamma+nucleon gives *hyperon+kayon(s)*
nucleon+nucleon gives *hyperon+nucleon+kayon(s)*.

As might be expected, these reactions also occur in "partially transposed" form so that we may also have

antikayon+nucleon gives *hyperon+pion(s)*.

[1] References are given at the end of this article in chronological order by year.

These reactions are fast (high transition probability), like the reactions of pion exchange which characterize the nuclear forces. To visualize the speed of these reactions we consider as a unit of time the interval required for a particle moving nearly with the speed of light to travel a distance of one nucleon "diameter," a distance which we define for present purposes as very nearly the Compton wavelength of the nucleon. Such a unit of time, of magnitude about 0.7×10^{-24} sec we call a *flash*.[2] Then the times for the "strong interactions,"

TABLE II. Some production reactions for the kayons and hyperons ($N=p$ or n).

	Calculated threshold in Gev (lab) (on nucleon at rest)
$\pi+p \rightarrow \Lambda+K+\pi$	
$\rightarrow \Sigma+K+\pi$	
$\rightarrow \Lambda+K$	0.76
$\rightarrow \Sigma+K$	0.90
$\rightarrow N+K+\bar{K}$	1.36
$\rightarrow \Xi+2K$	2.23
$p+n \rightarrow \Lambda+K+N$	1.57
$p+p \rightarrow \Sigma+K+N$	1.80
$p+n \rightarrow K+\bar{K}+2N$	2.50
$p+p \rightarrow \Xi+2K+N$	3.74

[2] This unit of time goes with the natural units, $\hbar=c=1$, if we measure the energy in kMev (Gev or Bev). Since one kMev is very close to 1.6×10^{-3} erg, the corresponding unit of time—defined to be "one flash"—is very closely 0.66×10^{-24} sec. To make the relations nearly exact, we must introduce as length unit the "modified compton" $=0.943\times$Compton wavelength of a nucleon. In summary: We can achieve harmony between the natural unit system, in which $\hbar=c=1$, and the practical Mev system by

by which hyperons are formed, are of the order of one flash. It is found, however, that, once formed, all the hyperons, with exception of the very short-lived Σ^0, decay to a lighter baryon plus a pion in a time roughly of the order of 10^{-10} sec. These, and the decays of the kayons, we call *weak decays*. The basic facts, (a) the very rapid production of hyperons in association with at least one kayon, and (b) the relatively very slow decay of the hyperon or kayon once formed, constitute the basis for one of the most striking discoveries in physics of the last five years.[3] This is the existence of a new property of elementary particles, *strangeness* (or *isoparity*—see Sec. V), which is conserved in all fast interactions, i.e., in all processes except weak decays. The essence of the new concept is that it is the necessity to violate strangeness conservation which inhibits the decay process and makes it slow. It is worthwhile to emphasize how *very slow* are the weak decays of these newly discovered particles compared with their production. They are produced in a flash and they decay in a time of the order of 10^{14} flashes. For comparison, we note that by recent estimates the age of the universe in seconds is 10^{17}. Or, to make another comparison: If the scale of the constants of nature were changed so that the time of a strong interaction was one second, instead of a few flashes, a hyperon and kayon formed in one second would travel through free space about a million years before they each decayed! It is only in these and the likewise relatively rare events of "β decay" that there occur also parity violations.[2]

In the middleweight meson region, besides the familiar $\pi \rightarrow \mu$ decays, there occur the complicated modes of decay of the kayons which have led to such interesting developments of ideas about parity within the last two years. In Fig. 1 we have indicated the six well-established modes of decay of the charged kayon (see Table I). There are also a large number of modes for the neutral kayon which appears in two forms having different lifetimes. Except for one of the two subtypes of the neutral particle (see Sec. IV), which can decay a little faster, the lifetimes of all kayons are found to be 10^{-8} sec. In addition, there are conjugate modes of decay for the antiparticles of the particles indicated. It is hard to observe these, for, by reason of their high production energy threshold, few antihyperons are found.

B. Theoretical Problems of Existence and Behavior of Elementary Particles

After this descriptive summary, we turn to the principal problems of understanding the existence and behavior of the elementary particles. A theory of existence is hard to come by, but we may expect that a first step would be a rational classification. This too would be a matter of stages. With our present philosophy of nature, in which we see all physical phenomena occurring in a given background of space-time, it is natural to try a first classification of elementary particles in terms of their symmetry properties in space-time. Indeed, the tendency at the present time is to define and classify elementary particles on the basis of the general and precise concept of an *elementary system* (cf. Newton and Wigner, 1949). An elementary system is defined as *a physical system for which every possible state is described by a set of functions transforming under space-time transformations according to one of the symmetry types of the extended Poincaré group.* ("Symmetry type" is an informal equivalent of "indecomposable-irreducible representation" or "rep." The "extended Poincaré group" is the full Lorentz group with space and time translations. Both concepts are discussed in detail in Secs. II and IV.)

A first answer to the question, "What is an elementary particle?" would then be that an elementary particle is a physical system which satisfies the following conditions:

(1) It is an elementary system.

(2) It is useful to consider it as simple and not as composite.

The heuristic form of the second condition is evident. Applied liberally, it leads one to the Orwellian observation imputed to Salam: "All particles are elementary, but some particles are more elementary than others." We do not here discuss how the second condition might be made more explicit, but instead refer to Newton and Wigner's illustrative discussion of the reasons for considering a pion to be an elementary particle, and a hydrogen atom in the ground state not to be one, though both are elementary systems. A classification of the theoretically possible elementary particles upon the basis of the classification of elementary systems is given later in Table V.

We turn now to the problem of understanding the behavior of elementary particles as summarized earlier and in Tables I and II. The initial problems for theory fall into two main categories: We wish to understand (a) *selection rules*, and (b) *lifetimes* or *branching ratios*, and *cross sections*. In other words, we would like to understand first why certain processes (decays and reactions) occur at all and others do not. Second, of those processes which do occur, we would like to understand why some occur more rapidly and others more slowly; alternatively, why some processes are more frequent and others rarer. Since, apart from intensity and phase-space factors, the rate of a microphysical process is proportional to the squared matrix element of an *interaction* operator (Sec. I.C), we can rephrase all

measuring distance in modified comptons, time in flashes, and energy (and mass and momentum) in kMev.

[3] That there exist parity violations in some basic natural processes constitutes another striking discovery of the last half-decade. What the two discoveries have in common is that in each case a conservation principle which holds strictly for stronger interactions is found to be violated in "weak interactions."

these questions about rates in terms of magnitudes of interactions.

The discussion is directed principally to the primary problem of selection rules. As a result of the very strong evidence now for a universal Fermi interaction (Sec. V), some quantitative account of lifetimes also can be given; and, in some cases, as indicated in Table I, estimates of branching ratios may be made by isospin and phase-space considerations. These matters are not discussed here.

The goal is to attain a "complete selection-rule scheme," i.e., a theoretical scheme with the following two properties:

(1) *No interaction which is observed is completely forbidden by the scheme.*

(2) *All interactions which remain unobserved after being carefully sought after are accounted for as forbidden* (at least in low order).

Assuming the existence of such a complete scheme means that one assumes the applicability of Gell-Mann's (1956) neototalitarian "principle of compulsory strong interactions" which may be stated in the form: Anything that is not forbidden is compulsory.

The terms "symmetry" and "invariance" are used interchangeably in the literature to indicate the meaning: "the property of not changing under one or more distinguishable operations." An (X) *invariance* or *symmetry principle* holds for a system if we can make a statement of the type: *All basic aspects*[4] *of the system are invariant under* (X). Various examples are those where one can insert for (X): "translation," "rotation," "reflection"—of space and/or time coordinates; "permutation of particles"; "charge conjugation of fields"; etc. (see Table III and Sec. III). Any (X) is called a *symmetry operation.*

We can expect that a complete selection-rule scheme will result from knowing all the symmetry principles which hold for systems of interacting elementary particles. Why? An interaction is forbidden if the corresponding matrix element would not be invariant under all the symmetry operations of the system, for as a consequence of the underlying symmetry the matrix element should be unchanged. Specifically, suppose that the symmetry types (behavior under transformations) of the interaction operator and of the initial and final states in the matrix element are such that, under a symmetry operation, the matrix element would be changed. Such an interaction is forbidden. It is exactly in this way, in the older problems of absorption and emission of energy, with particle number conserved, that symmetry or invariance principles proved of such value for exploring the microphysical world. A somewhat parallel application of symmetry principles can be made to the middlesized world of classical physics (cf. Curie, 1894; Melvin, 1949, 1956). To the older of the symmetry principles, which were common to the two worlds, there recently have been added some interesting new ones, which are specifically microphysical and which govern processes in which elementary particles are not conserved.

A very important alternative way of looking at selection rules, one which is much beloved of physicists because it lends itself easily to intuitive formulations, is in terms of *conservation principles.* When we say that a system obeys a particular conservation principle we instinctively think of an entity associated with the system, which has the "substancelike" property of keeping its magnitude constant in time. The bearing on selection rules is that the system may evolve only into states associated with the same value of the conserved entity as it had initially.

That there should exist the two alternative ways of looking at selection rules follows from a deepseated fact which we discuss in detail in the following. Just like the two faces of a coin which can never be separated from each other, so with every invariance principle of the usual type there is associated inseparably a conservation principle. (This is not the case with respect to time reversal invariance; see Sec. III.) In Table III we have listed on the left all of the familiar invariance principles and some not so familiar ones of recent vintage. On the right we have indicated, in so far as they are known, the corresponding conserved quantities or "constants of motion."

C. Survey of Theoretical Background and Connection between Symmetry Principles and Constants of Motion

The association between the invariance and conservation points of view has its prototype in classical Hamiltonian dynamics; this is thoroughly discussed by Hill (1949). Here we take up the question in the context of microphysical phenomena. What is a "constant of motion" in quantum mechanics, and why is there such an entity associated with every (usual) invariance principle? Details of the answers are given in Sec. III. In the present section we summarize the postulational background in quantum theory, first in the approximation where particle number is conserved, and then in the more exact quantum theory of fields. The basic terminology is that found in Dirac's book (1958).

It appears to be a fact that no matter what precautions are taken, the making of certain observations upon a microphysical system disturbs it so that it is impossible to predict certain other subsequent observations in individual detail (allowing, however, statistical prediction). Quantum theory translates this elementary fact, that observation disturbs microphysical systems

[4] A precise definition of what we mean by "all basic aspects" of a microphysical system is given in footnote 6, Sec. I.C. For the moment we use it in a loosely defined descriptive sense to mean the fundamental laws and building-block quantities upon which the theory is constructed.

quasi-unpredictably, into a mathematical structure involving an ∞-dimensional vector space: In this space each vector represents a state of the system, and mutually disturbing observations on states are related to noncommuting operators on the vectors.

Specifically, for any physical system, every measurable dynamical variable is represented by a linear operator $\mathbf{F}$ which is called an *observable*. The result λ of every measurement of the dynamical variable is an eigenvalue of $\mathbf{F}$, i.e., $\mathbf{F}-\lambda\mathbf{I}$ has no inverse. ($\mathbf{I}$ is the identity operator.) If the eigenvalue is not degenerate, the eigenvector or *eigenket* $|\lambda\rangle$ defined by $\mathbf{F}|\lambda\rangle=\lambda|\lambda\rangle$ is determined only up to an arbitrary complex number factor c, and the mathematical entity $c|\lambda\rangle$ is called an *eigenstate*. If a given eigenvalue of a given operator is degenerate it does not specify a corresponding eigenstate uniquely; to specify a unique eigenstate and thereby the condition of the system, it is necessary to use a *complete set of commuting observables* $\mathbf{Q}_a$, $\mathbf{Q}_b$, $\cdots$. With a definite set of eigenvalues λ_a, λ_b, $\cdots$, one from each observable, there is then associated to the system a definite simultaneous eigenstate $c|\lambda_a, \lambda_b, \cdots\rangle$. (Again the eigenvector is not completely determined even with unitary normalization; c remains an arbitrary phase factor.) This state describes the physical behavior of the system in all admissible detail, yielding for example the statistics of all possible measurements of dynamical variables of the system. Specifically, let the system be continually reprepared in the state $|\lambda\rangle$ and repeated measurements be made of any chosen dynamical variable G. Then the magnitudes of the projections of $|\lambda\rangle$ along the eigenstates of $\mathbf{G}$ (i.e., unitary scalar products of $|\lambda\rangle$ with these eigenstates) yield the amplitudes for the probabilities with which the corresponding eigenvalues of $\mathbf{G}$ will appear. The requirement that the eigenvalues always be real restricts the observables to be Hermitian operators in the space of all possible states of the system. This state space is a *Hilbert space* since (1) every linear combination of states is a state (*principle of superposition*); (2) the unitary scalar product of any pair of states is defined (*probability interpretation*); (3) every state is a linear combination of the eigenstates of the commuting observables (*completeness condition*). (In some physical problems where unitary scalar products do not converge, either the concept of a Hilbert space must be extended by relaxing postulates (2) and (3), or the problem must be approximated by one for a system in a "box.")

The Hilbert space of the theory acquires specific structure when it is assumed that for many systems (those which have classical analogs) there is a close analogy with the canonical coordinate formulation of classical dynamics, commutators replacing Poisson brackets: Every observable connected with the system is a function of two fundamental conjugate sets of observables, coordinates $\mathfrak{q}_j$ and momenta $\mathfrak{p}_j$, which at any particular time satisfy the *commutation relations*

$$\mathfrak{q}_j\mathfrak{p}_k-\mathfrak{p}_k\mathfrak{q}_j=i\delta_{jk}\mathbf{I}.$$

For systems without a classical analog, canonical coordinates and momenta as a basis for building all observables may not exist. (It seems, however, that for any *closed system* with total spin zero, regarded as a whole, canonical momenta and coordinates do exist; see Appendix.)

In all cases, whether with a classical analog or not, the Hilbert space is given specific structure if the following is assumed: There exists a basic set of not necessarily Hermitian operators, in terms of which all others may be constructed, which form a *canonical* set. Such a set consists of two conjugate subsets—one of "$\mathfrak{a}$'s," one of "$\mathfrak{b}$'s"—equal in number, which satisfy either the canonical *commutation relations* (Bose-Einstein case)

$$\begin{aligned} &\mathfrak{a}_j\mathfrak{a}_k-\mathfrak{a}_k\mathfrak{a}_j=0 \quad \mathfrak{a}_j\mathfrak{a}_k^\dagger-\mathfrak{a}_k^\dagger\mathfrak{a}_j=\delta_{jk}\\ &\mathfrak{a}_j\mathfrak{b}_k-\mathfrak{b}_k\mathfrak{a}_j=0 \quad \mathfrak{a}_j\mathfrak{b}_k^\dagger-\mathfrak{b}_k^\dagger\mathfrak{a}_j=0\\ &\mathfrak{b}_j\mathfrak{b}_k-\mathfrak{b}_k\mathfrak{b}_j=0 \quad \mathfrak{b}_j\mathfrak{b}_k^\dagger-\mathfrak{b}_k^\dagger\mathfrak{b}_j=\delta_{jk}\\ &\qquad (j, k=1, 2, \cdots) \end{aligned} \tag{1}$$

or the canonical *anticommutation relations* (Fermi-Dirac case)

$$\begin{aligned} &\mathfrak{a}_j\mathfrak{a}_k+\mathfrak{a}_k\mathfrak{a}_j=0 \quad \mathfrak{a}_j\mathfrak{a}_k^\dagger+\mathfrak{a}_k^\dagger\mathfrak{a}_j=\delta_{jk}\\ &\mathfrak{a}_j\mathfrak{b}_k+\mathfrak{b}_k\mathfrak{a}_j=0 \quad \mathfrak{a}_j\mathfrak{b}_k^\dagger+\mathfrak{b}_k^\dagger\mathfrak{a}_j=0\\ &\mathfrak{b}_j\mathfrak{b}_k+\mathfrak{b}_k\mathfrak{b}_j=0 \quad \mathfrak{b}_j\mathfrak{b}_k^\dagger+\mathfrak{b}_k^\dagger\mathfrak{b}_j=\delta_{jk}\\ &\qquad (j, k=1, 2, \cdots). \end{aligned} \tag{2}$$

In the consistent development of the theory the $\mathfrak{a}$'s are to be interpreted as particle destruction operators and the $\mathfrak{b}$'s as antiparticle destruction operators of a quantum field. The Hermitian conjugate quantities are the corresponding creation operators. In the case of neutral self-charge conjugate fields (see Table III), $\mathfrak{a}^\dagger=\mathfrak{b}$, $\mathfrak{b}^\dagger=\mathfrak{a}$, and the last two lines in each set of relations are redundant.

In any case, either that of Bose-Einstein or that of Fermi-Dirac fields, observables obey commutation relations. This follows in the only questionable case—the anticommuting case of Fermi-Dirac—from the fact that all observables are bilinear combinations of "$\mathfrak{a}$'s" and "$\mathfrak{b}$'s" (cf. Appendix 3; Fermi, 1951).

A very important theorem of uniqueness of the operators $\mathfrak{a}_j$ and $\mathfrak{b}_j$ holds if the following two assumptions are made (see Sec. III and the bibliographic references to Becker and Leibfried 1946, 1948, and Wightman and Schweber, 1955):

(i) There exists in the Hilbert space a *vacuum state* (or *vacuum* ket) $|0\rangle$ such that

$$\mathfrak{a}|0\rangle=0 \quad \mathfrak{b}|0\rangle=0,$$

where the zero on the right stands for the zero ket, i.e., such that any operator on it gives the zero ket again.

(ii) There exists among the observables in the Hilbert space a total *particle number operator*

$$\mathbf{N}=\sum_{j=1}^{\infty} \mathfrak{a}_j^\dagger \mathfrak{a}_j$$

such that

$$\mathbf{N}|0\rangle=0.$$

The uniqueness theorem then states:

Provided assumptions (i) and (ii) are satisfied, an irreducible[5] *set of operators satisfying the canonical relations (1) or (2) is determined uniquely up to a similarity transformation.*

More specifically, if there are two irreducible sets of operators $(\mathfrak{a}_j',\mathfrak{b}_j')$, and $(\mathfrak{a}_j,\mathfrak{b}_j)$ which obey the relations (1) or (2), then an operator $\mathbf{U}$ exists such that

$$\mathfrak{a}_j'=\mathbf{U}\mathfrak{a}_j\mathbf{U}^{-1} \quad \mathfrak{b}_j'=\mathbf{U}\mathfrak{b}_j\mathbf{U}^{-1}. \tag{3}$$

The following two properties also follow: (a) The operator $\mathbf{U}$ is defined uniquely up to a factor. (b) If $\mathbf{U}$ transforms Hermitian operators into Hermitian operators and a pair of Hermitian adjoint operators into a pair of Hermitian adjoint operators, then $\mathbf{U}$ can be chosen as a unitary operator.

It is then clear that any operator $\mathbf{A}'$ constructed from the a_j' and b_j' in the same way as an operator $\mathbf{A}$ is constructed from the a_j and b_j satisfies the relation $\mathbf{A}'=\mathbf{U}\mathbf{A}\mathbf{U}^{-1}$. In particular, we have $\mathbf{N}'=\mathbf{U}\mathbf{N}\mathbf{U}^{-1}$.

Between disturbing observations a microphysical system develops causally. To formulate the theory of the development in time of physical processes, it is perhaps simplest to use the *Heisenberg picture* since it is the one in which we can deal with relations between operators directly without necessarily referring to matrix elements. Also, it is closest formally to classical Hamiltonian dynamics. The Heisenberg picture is defined by the property that the eigenvectors of the complete set of commuting observables at *one moment of time* (the "fixed time") are chosen to be the basic "coordinate unit vectors" in Hilbert space, the same at *all times* (these other times may be called "displaced times"). This means that all movement and oscillation is associated with operators rather than with eigenvectors. The generalization of this picture to other transformations (frame displacements) besides time displacements is indicated in the following by the adjective phrase "Heisenberg type." One should compare with the "Schrödinger type" picture where, in contrast to the "objective" emphasis of the Heisenberg type picture, two states are called the "*same*" when they are "subjectively" the same, i.e., when one appears in the displaced frame exactly the same as the other appears in the fixed frame.

We now consider several Hilbert-space operators each of which exists by virtue of some space-time symmetry principle, i.e., by the requirement of invariance of the *basic aspects*[6] of a system under some redescription[6a] of the space-time coordinates. Particularly important examples of such Hilbert-space operators are provided by considering the theory of *closed systems*. By definition, such systems are those in which the basic aspects depend only on the *relative* space-time positions, relative space orientations, and relative velocities of different parts of the system with respect to each other. In other words, the basic aspects do not depend on the absolute location of the origin, or absolute orientation or absolute velocity of the frame of reference in space-time. Such systems, exhibiting *full relativistic invariance*, are obtained in practice when no "external fields" act.

For simplicity it is perhaps best to describe the consequences of this broad invariance in three stages, first with respect to space-time displacements, then space rotations, and finally, at a somewhat later point, with respect to pure Lorentz transformations.

The fact that one may reset the zeros of time and position without changing the basic aspects (invariance under time-displacement and space-displacement redescriptions), together with the uniqueness theorem, implies (Sec. III) that there exists a unitary "evolution-displacement" operator $\mathbf{U}_{t\mathbf{r}}$ which relates observables at two different space-time positions in accord with

$$\mathbf{Q}(t,\mathbf{r})=\mathbf{U}_{t\mathbf{r}}\mathbf{Q}(0,0)\mathbf{U}_{t\mathbf{r}}^{-1},$$

where for simplicity we have taken one of the positions to be the origin. Because the possible time and space displacements can be taken continuously increasing from the identity, there are defined four Hermitian operators $\mathbf{H}$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ such that

$$\mathbf{U}_{t\mathbf{r}}=\exp[i(t\mathbf{H}-x_1\mathbf{P}_1-x_2\mathbf{P}_2-x_3\mathbf{P}_3)].$$

[5] This concept is discussed in detail in Sec. II.

[6] The descriptive phrase "the basic aspects" should be interpreted for microphysical systems to mean explicitly:

(1) commutation or anticommutation relations of the basic $\mathfrak{a}$'s and $\mathfrak{b}$'s out of which all observables are built,

(2) definition of the vacuum state and of the number operators. Invariance of these two aspects is necessary for making the basic—and all other—operators well defined (*uniqueness theorem*). With operators transforming according to Eq. (3) and with eigenvectors unchanging—"Heisenberg type" transformation induced by space-time redescription—observable consequences of the well-defined operators are essentially the same in the old and new descriptions, i.e., observable matrix elements are invariant.

If instead of the Heisenberg type the "Schrödinger type" induced transformations—operators unchanging, eigenvectors transforming—are considered, we should interpret "basic aspects" to mean rather the complete set of transition probabilities between any two states. This condition, which again guarantees covariance of observable consequences, implies (Wigner 1931, 1959) that the Schrödinger type operator $\mathbf{U}$ which gives

$$|\rangle'=\mathbf{U}|\rangle$$

is either unitary or antiunitary (see Sec. II.E).

[6a] We use the term "redescription" to emphasize the purely *passive* nature of the transformation, i.e., the same aspects are being described at the same point but in the transformed language of the changed frame. One can hardly emphasize too much that *it is because of the uniqueness theorem* that this passive redescription in coordinate space induces an active transformation in Hilbert space, i.e., the unitary operator which relates observables (and state vectors) at one physical point to those at another (see Sec. III).

The operator $\mathbf{H}$, which relates an observable $\mathbf{Q}$ at one moment of time to $\mathbf{Q}$ at the next in accord with

$$\mathbf{Q}(t+\delta t, \mathbf{r})-\mathbf{Q}(t,\mathbf{r})\equiv\delta\mathbf{Q}=i[\mathbf{H},\mathbf{Q}]\delta t,$$

is the Hilbert-space representative of infinitesimal time displacement. It is called the *Hamiltonian*, and the commutator expression $i[\mathbf{H},\mathbf{Q}]$ is called the "implicit time rate of change" of $\mathbf{Q}$. In systems with a classical analogue involving a Q and an H, the commutator is the equivalent of the Poisson bracket expression for the rate of change of Q,

$$\frac{dQ}{dt}=\frac{\partial Q}{\partial q_j}\frac{dq_j}{dt}+\frac{\partial Q}{\partial p_j}\frac{dp_j}{dt}=\frac{\partial Q}{\partial q_j}\frac{\partial H}{\partial p_j}-\frac{\partial Q}{\partial p_j}\frac{\partial H}{\partial q_j}.$$

Similarly, the operator $\mathbf{P}_1$, which is called the "x_1 component of linear momentum," relates an observable $\mathbf{Q}$ at one position in space to $\mathbf{Q}$ at a slightly translated position in the x_1 direction, in accord with

$$\delta\mathbf{Q}/\delta x_1=-i[\mathbf{P}_1,\mathbf{Q}].$$

There are two other similar expressions in which $\mathbf{P}_2$ and $\mathbf{P}_3$ play the same role with respect to x_2 and x_3.

Independence of over-all space orientation is also included in the concept of a closed system, i.e., the basic aspects are invariant under redescription from a space-rotated set of axes. In a similar way to the foregoing, it follows that there exist three Hermitian operators $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ which are called the "angular momentum" operators and which relate any scalar observable $\mathbf{Q}$ at one position to $\mathbf{Q}$ at a slightly rotated position in accord with

$$\left(x_2\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_2}\right)\mathbf{Q}=i[\mathbf{M}_1,\mathbf{Q}].$$

There are similar expressions with 1, 2, and 3 cyclically permuted. A somewhat more complicated expression appears when $\mathbf{Q}$ is not a scalar but a general spinor (or tensor, in special cases). This is derived in Sec. III.

Invariance under space-time displacements and space rotations are only special examples of symmetry operations which systems may display. It is established in Sec. III upon the basis of the uniqueness theorem that *associated with every symmetry principle there exists a unitary operator* $\mathbf{U}$ *in Hilbert space relating state-vectors and observables at two different "physical points."* We call this unitary operator, or rather its "Hermitian generator" $\mathbf{F}$ defined by

$$\mathbf{U}\equiv\exp(i\epsilon\mathbf{F}),$$

the *invariance operator* associated with the principle. In cases where the parameter ϵ can be made infinitesimal ($\equiv\delta\epsilon$), $\mathbf{F}$ may be interpreted as an operator relating observables $\mathbf{Q}$ at neighboring physical points in accord with

$$\delta\mathbf{Q}/\delta\epsilon=i[\mathbf{F},\mathbf{Q}].$$

As in other instances in physics, once quantities have been introduced in particular cases in which their meaning is exceptionally simple, their definitions can be extended to allow them to be recognized in other more general situations. So with the invariance operators, they may be considered also in cases where the corresponding invariances no longer hold, and still the operators may be defined as "displacement operators," etc. In such cases it is generally found that the operator evolves in time. Furthermore, it may have an explicit as well as an implicit time dependence. The *law of motion*, which may be taken as a fundamental postulate, now states that *the total rate of change with time of any operator* $\mathbf{Q}$ *equals the sum of the implicit and explicit rates of change with time*,

$$d\mathbf{Q}/dt=i[\mathbf{H},\mathbf{Q}]+\partial\mathbf{Q}/\partial t.$$

By definition, a "conserved quantity" or "constant of motion" $\mathbf{Q}$ is an operator for which $d\mathbf{Q}/dt=0$. (This means that, amongst other matrix elements, the average value of $\mathbf{Q}$ is constant during the evolution of *any* state of the system; or, more particularly, if we put the system in an eigenstate of $\mathbf{Q}$, it stays in that state, and the eigenvalue is a "good quantum number.") Applying the fundamental law of motion we now see that in a closed system every one of the components of the total linear momentum $\mathbf{P}_k$ and angular momentum $\mathbf{M}_k$ ($k=1, 2, 3$) is a constant of motion. The argument is as follows: The existence of these operators follows from the invariance of all processes in closed systems under infinitesimal translation and rotation redescriptions. These redescriptions are not time dependent and commute with the time displacement redescription. Consequently, the corresponding invariance operators are not explicitly time dependent and also commute with the Hamiltonian. This leads to the conditions

$$d\mathbf{P}_k/dt=0 \quad d\mathbf{M}_k/dt=0,$$

which express the familiar principles of conservation of linear and angular momentum.

Quite generally we can say that *any invariance operator* $\mathbf{F}$ *which is the Hilbert-space representative of a not explicitly time dependent space-time redescription is itself not explicitly time dependent and commutes with the Hamiltonian. Any such* $\mathbf{F}$ *is a constant in time.*

The commutation of $\mathbf{H}$ with the invariance operator $\mathbf{F}$ and the absence of explicit time dependence in $\mathbf{F}$ go together. Both conditions fail together if $\mathbf{F}$ is one of the "centroidal moment" operators $\mathfrak{M}_k$ ($k=1, 2, 3$). (Just as the angular momentum operators $\mathbf{M}_k$ appear as the Hilbert-space representatives of infinitesimal space rotations, so the centroidal moment operators $\mathfrak{M}_k$ appear as the Hilbert-space representatives of infinitesimal space-time rotations—or pure Lorentz transformations in the three space directions x_1, x_2, x_3; see Sec. II.) Here $\mathbf{H}$ mixes with other components of the energy-momentum under transformation by any $\mathfrak{M}_k$, and

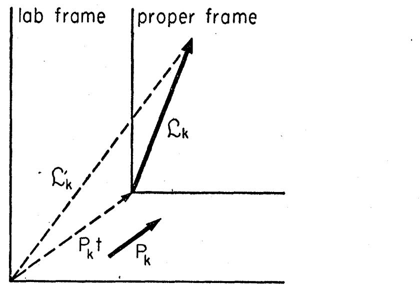

FIG. 2. Interpretation of the constancy of *proper orbital centroidal moment*, $\mathfrak{L}_k=\mathfrak{L}_k'-P_kt$ $(k=1, 2, 3)$, of a system. The *lab frame orbital centroidal moment* $\mathfrak{L}_k'$ is the integral over all space of the first moment of the energy density T_{00}: $\mathfrak{L}_k'=\int x_kT_{00}d^3x$. The energy distribution as a whole is moving to the right corresponding to a total linear momentum P_k. Therefore the lab centroidal moment has an implicit time rate of change $d\mathfrak{L}_k'/dt=i[H,\mathfrak{L}_k']=P_k$. This just cancels the explicit time rate of change of $\mathfrak{L}_k$ so that $d\mathfrak{L}_k/dt=0$. The case of a zero spin system, where the conserved total moment is the orbital moment alone, has been represented.

therefore $\mathbf{H}$ does not commute with $\mathfrak{M}_k$. However, $\mathfrak{M}_k$ is still a constant of motion because its explicit time dependence just cancels its implicit time dependence. Specifically, the operators $\mathfrak{M}_k=\mathfrak{M}_k'-\mathbf{P}_kt$ ($\mathfrak{M}_k'$, $\mathbf{P}_k$ time independent) satisfy the nonvanishing commutator relation $[\mathfrak{M}_k,\mathbf{H}]=i\mathbf{P}_k$. The $\mathfrak{M}_k$ are nevertheless constants of motion because

$$d\mathfrak{M}_k/dt=i[\mathbf{H},\mathfrak{M}_k]+\partial\mathfrak{M}_k/\partial t=\mathbf{P}_k-\mathbf{P}_k=0.$$

Figure 2 shows the simple intuitive meaning of the constancy of the operator vector $\mathfrak{M}$ when its average value is taken. It is simply the first moment of the system in a frame at rest with respect to the center of mass. The vector $\mathfrak{M}'$ which carries the implicit time dependence is the first moment in the "lab frame" with respect to which the system is moving with momentum P_k. Just as the constants P_k indicate the "*velocity*" of the centroid of a system, so the constants $\mathfrak{M}_k$ indicate its "*position*" (see Appendix).

D. Discrete Space-Time Symmetries. Other Symmetries and Conservations

In the upper part of Table III are listed thê familiar space-time invariance principles which give rise to the conservation of energy, momentum, angular momentum, and centroidal moment. The selection rules following from these principles together with the principle of conservation of charge indicated near the bottom of the table are indispensable for discriminating between the different types of particles and determining their masses and spins. On the left side of the table is given an intuitive interpretation of the various principles in terms of a comparison between transformed records of natural processes taken under various conditions, and natural processes observed directly in nature.

The constants of motion $\mathbf{P}_a$ $(a=0, 1, 2, 3$ corresponding to H, $P_k)$, $\mathbf{M}_k$, and $\mathfrak{M}_k$ for a closed system are operator representatives of generators of a *continuous* group of symmetry redescriptions. Besides, there also occurs an additional constant of motion connected with another space-time symmetry—in this case *discrete.* This is the parity operator $\mathbf{P}$, Hilbert-space representative of the redescription reversing the handedness of the space-frame (i.e., the signs of the three space coordinates).[7] This unitary Hilbert-space operator relates observables at $\mathbf{x}$ and at $-\mathbf{x}$. For example, for a system of particles with positions $\mathfrak{q}_j$, momenta $\mathfrak{p}_j$, and spins $\boldsymbol{\sigma}_j$, we have

$$\mathbf{P}\mathfrak{q}_j\mathbf{P}^{-1}=-\mathfrak{q}_j,\quad \mathbf{P}\mathfrak{p}_j\mathbf{P}^{-1}=-\mathfrak{p}_j,\quad \mathbf{P}\boldsymbol{\sigma}_j\mathbf{P}^{-1}=\boldsymbol{\sigma}_j.$$

We cannot define an infinitesimal Hermitian generator for such a discrete operator. However, $\mathbf{P}$ itself is Hermitian, for in addition to having the property

$$\mathbf{P}^\dagger\mathbf{P}=\mathbf{P}\mathbf{P}^\dagger=\mathbf{I},$$

it can be chosen so that its square equals the identity (Sec. III)

$$\mathbf{P}^2=\mathbf{I}.$$

Thus $\mathbf{P}=\mathbf{P}^\dagger$ is a Hermitian operator as well as a unitary one. It is itself therefore an observable. If $\mathbf{H}$ is even under reflection, $\mathbf{P}$ commutes with it and is a constant of motion.[7]

Another discrete space-time symmetry, invariance under reversal of the sense of time, plays an important role in the theory of many physical properties. But here the fact that a redescription reversing the time variable t induces a sign change in the canonical commutation relations in Hilbert space means that we must use an *antiunitary* rather than a unitary operator to represent time reversal. This is more fully discussed in footnote a of Table III and in Sec. II.E.

The terms in Sec. A of Table III, with the one exception of charge-conjugation (or particle-antiparticle conjugation), refer to invariance under transformations of space-time kind only. Those in Sec. C refer in "geometrical" language to invariances under transformations in isospin space—a space which used to be regarded by physicists merely as a formal device but which of late is being taken much more physically. We note with special emphasis the abortive attempt, but actual gap, in interpretation of the first of the two conservation principles indicated in Sec. B of Table III. At the present time we lack any geometrical interpretation, in either ordinary or isospin space, of the very important and apparently universally valid "baryon conservation principle," which may be stated as follows: *A baryon always appears or disappears in a pair with an antibaryon.* It is the baryon conservation principle—forbidding the solo appearance of antinucleons—which prevents the cancerous infection of matter by antimatter, from which the material universe might vanish

[7] It is now clear that this operator is not quite a constant of motion with the same generality as the $\mathbf{P}_a$'s, $\mathbf{M}_k$'s, and $\mathfrak{M}_k$'s. It fails in closed weakly interacting systems, where the Hamiltonian is not even under reflection.

TABLE III. Symmetry principles which are expected to hold generally for the fundamental laws of nature (first column). Associated redescriptions of dependent variables or fields appearing in these laws. These are the formal operations on the fields with respect to their coordinate space dependence and are given in differential forms in those cases where they occur, i.e., when the redescription operations are continuously connected with the identity (second column). Corresponding conserved quantities or Hermitian operators in quantum theory; in each case the conserved quantity is the Hilbert space representative of the generator of the symmetry redescription of the field (third column).

Each redescription of field quantities (dependent variables) is so defined that together with that of the coordinates it keeps all fundamental equations invariant—and therefore yields new solutions from old ones. The redescriptions have been given only in their spin 0 and spin $\frac{1}{2}$ representatives. The spin $\frac{1}{2}$ representatives of the extended symmetry operations, time-reversal T, space-inversion P, and charge-conjugation C, have been simplified by employing a unitary-real (Majorana) form for the Dirac theory gamma matrices, $\gamma_0, \gamma_1, \gamma_2, \gamma_3$. Aside from phase factors, and the appropriate change in the coordinate arguments, $t \to -t$ or $r \to -r$, these redescription operations for spin $\frac{1}{2}$ fields are as follows: The time-reversal redescription T consists of complex conjugating all c numbers k and multiplying field operators by $\gamma_1\gamma_2\gamma_3$. The charge-conjugation redescription C consists merely of Hermitian conjugation but not spinor transposition of the field operators. The spinor-parity redescription P consists of multiplying by $i\gamma_0 \equiv \gamma_4$. (The last is true not only in the Majorana but in every other gamma matrix basis also.)

Symmetry principle Given an actual process (the "control" process), one can construct from it another possible process by the transformation:	Redescription of spin 0 fields Φ and spin $\frac{1}{2}$ fields Ψ_a associated with the coordinate redescription R in accord with $\Phi(x) \to \Phi'(\xi) = \Phi(R^{-1}\xi)$, $\Psi^a(x) \to \Psi^{a\prime}(\xi) = S^{ab}\Psi^b(R^{-1}\xi)$ Spin 0 fields Φ		Spin $\frac{1}{2}$ fields Ψ^a	Conserved quantity	
A. Space-time symmetries					
Time shift		$\left.\begin{matrix}\Phi\\ \Psi\end{matrix}\right\} \to \left(1+i\epsilon\frac{\partial}{\partial t}\right)\left\{\begin{matrix}\Phi\\ \Psi\end{matrix}\right.$		Energy	$\mathbf{P}_0 \equiv \mathbf{H}$
Position shift		$\left.\begin{matrix}\Phi\\ \Psi\end{matrix}\right\} \to (1-i\epsilon\nabla)\left\{\begin{matrix}\Phi\\ \Psi\end{matrix}\right.$		Linear momentum	$\mathbf{P}$
Orientation shift	$\Phi \to [1-i\delta\boldsymbol{\varphi}\cdot\mathbf{r}\times\boldsymbol{\nabla}]\Phi$		$\Psi \to \{1+\delta\boldsymbol{\varphi}\cdot[\frac{1}{2}\boldsymbol{\sigma}-i\mathbf{r}\times\boldsymbol{\nabla}]\}\Psi$	Angular momentum	$\mathbf{M}$
Frame-velocity shift	$\Phi \to [1+i\epsilon_{0k}(x_k\partial_0+x_0\partial_k)]\Phi$		$\Psi \to \{1+\epsilon_{0k}[\frac{1}{2}\gamma_0\gamma_k+i(x_k\partial_0+x_0\partial_k)]\}\Psi$	Centroidal moment	$\mathfrak{M}$
Time reversal (e.g., a movie of the control process is run backward)[a]	$\Phi(t) \to \eta_T\Phi(-t)$ $\Phi^*(t) \to -\eta_T{}^*\Phi^*(-t)$	$k \to k^*$	$\Psi(t) \to \lambda_T\gamma_1\gamma_2\gamma_3\Psi(-t)$ $\Psi^*(t) \to \lambda_T{}^*\gamma_1\gamma_2\gamma_3\Psi^*(-t)$		$\mathbf{T}$
Space inversion (e.g., the control process is filmed through a mirror or pinhole camera)	$\Phi(\mathbf{r}) \to \eta_p\Phi(-\mathbf{r})$ $\Phi^*(\mathbf{r}) \to \eta_p{}^*\Phi^*(-\mathbf{r})$		$\Psi(\mathbf{r}) \to \lambda_p\gamma_4\Psi(-\mathbf{r})$ $\Psi^*(\mathbf{r}) \to \lambda_p{}^*\gamma_4\Psi^*(-\mathbf{r})$	Coparity	$\mathbf{P}$
Charge conjugation (particles and antiparticles are interchanged)	$\Phi(x) \to \eta_c\Phi^*(x)$ $\Phi^*(x) \to \eta_c{}^*\Phi(x)$		$\Psi(x) \to \lambda_c\Psi^*(x)$ $\Psi^*(x) \to \lambda_c{}^*\Psi(x)$		$\mathbf{C}$
B. Gauge transformation symmetries for baryons.			$\Psi(l) \to \exp(il\varphi)\Psi(l)$ ($l=1, -1$ for lepton, antilepton)	Lepton number	$\boldsymbol{l}$
			$\Psi(B) \to \exp(iB\theta)\Psi(B)$ ($B=1, -1$ for baryon, antibaryon)	Baryon number	$\mathbf{B}$

TABLE III—*Continued.*

Symmetry principle Given an actual process (the "control" process), one can construct from it another possible process by the transformation:	Redescription of spin 0 fields Φ and spin $\frac{1}{2}$ fields Ψ_a associated with the coordinate redescription R in accord with $\Phi(x)\to\Phi'(\xi)=\Phi(R^{-1}\xi)$, $\Psi^a(x)\to\Psi^{a\prime}(\xi)=S^{ab}\Psi^b(R^{-1}\xi)$ Spin 0 fields Φ	Spin $\frac{1}{2}$ fields Ψ^a	Conserved quantity
C. Isospace symmetries			
Charge gauge transformation	$\Phi\to\exp[iq\alpha(x)]\Phi$	$\Psi\to\exp[iq\alpha(x)]\Psi$ ($q=1, -1$ for positive, negative charge)	Charge q
Equatorial plane reflection in isospace		$\Psi\to\exp[iq\pi]\Psi$ isospinors: $\Psi=\begin{pmatrix}p\\n\end{pmatrix}$, $q=\begin{pmatrix}1 & \\ & 0\end{pmatrix}$ $\Psi=\begin{pmatrix}\Xi^0\\\Xi^-\end{pmatrix}$, $q=\begin{pmatrix}0 & \\ & -1\end{pmatrix}$	Special case of charge conservation
Inversion and cylindrical rotations in isospace (neglecting weak interactions)		$\Psi\to\exp[iU\pi/2]\Psi$, $\exp[iT_3\alpha]\Psi$ isospinors: $\Psi=\begin{pmatrix}p & \\ & n\end{pmatrix}$, $U=\begin{pmatrix}1 & \\ & 1\end{pmatrix}$, $T_3=\begin{pmatrix}\frac{1}{2} & \\ & -\frac{1}{2}\end{pmatrix}$	Hypercharge $\mathbf{U}=\mathbf{B}+\mathbf{S}$ 3rd component of isospin $\mathbf{T}_3$
All rotary reflections in isospace (neglecting electromagnetic and weak interactions)		$\Psi\to\exp[iU\pi/2]\Psi$, $\exp[i\boldsymbol{\varphi}\cdot\mathbf{T}]\Psi$ isospinors: $T_1=\begin{pmatrix} & \frac{1}{2}\\ \frac{1}{2} & \end{pmatrix}$, $T_2=\begin{pmatrix} & -\frac{1}{2}\\ \frac{1}{2} & \end{pmatrix}$, $T_3=\begin{pmatrix}\frac{1}{2} & \\ & -\frac{1}{2}\end{pmatrix}$	Isospin $\mathbf{T}$

[a] Consider first one-particle quantum mechanics. Let $\mathfrak{r}$ and $\mathfrak{p}$ be conjugate position and momentum observables of the particle. Then any time-reversal operator, when considered as making a transformation in Hilbert space, ought to satisfy the intuitively necessary relations,

$$\mathsf{T}\mathfrak{r}\mathsf{T}^{-1}=\mathfrak{r}\qquad \mathsf{T}\mathfrak{p}\mathsf{T}^{-1}=-\mathfrak{p}.$$

Because any unitary operator would maintain the basic commutation relation

$$\mathfrak{r}\mathfrak{p}-\mathfrak{p}\mathfrak{r}=i\hbar$$

and the T operator does not, it cannot be unitary. In order to insure invariance of Schrödinger's equation, transition probabilities, and positiveness of positive energy states, T has to be taken as "antilinear" and "antiunitary" (Wigner, 1932), i.e., whenever it operates on a wave function it complex-conjugates it as well as carrying out a unitary operation upon it. (In this respect T is characteristically different from C and P which are linear.)

With a scalar particle, this amounts to complex conjugation of the wave function in position space, i.e., in the representation in which $\mathfrak{r}$ is diagonal. With a spin $\frac{1}{2}$ particle, since the spin ($\Sigma_1=i\gamma_2\gamma_3$, *et cycl*) reverses sign with time-reversal, we must have

$$\mathsf{U}\Sigma^*\mathsf{U}^{-1}=-\mathsf{U}\Sigma\mathsf{U}^{-1}=-\Sigma,$$

where U is the unitary part of T. The only such matrix U which commutes with the products of γ_1, γ_2, γ_3, two at a time, is $\gamma_1\gamma_2\gamma_3$.

So far for one-particle quantum mechanics. In quantum field theory the unitary part of the T redescription remains the same, but with regard to complex conjugation, some subtle variations in possibilities arise. It may help to mention that some authors try to express the peculiar character of T by complex (not Hermitian!) conjugation of the field operators. Still others (following Schwinger, 1951) obtain the same result as complex conjugation by reading all operator relations from right to left instead of from left to right; in its effect on matrix elements, this is equivalent in most cases to complex conjugation because all physical observables are represented by Hermitian operators. By contrast, and consistent with the basic mathematical definitions of antiunitarity, Lüders (1952, 1954, 1957) complex conjugates, instead of the field operators, all c numbers k. We have followed him in this.

into radiation through progressive annihilation. More subtly, this conservation principle manifests itself in characteristic differences in behavior between, for instance, an electron-proton system and an electron-positron system even in the midst of "life" of the latter. The effect of the great mass difference between proton and positron is unimportant in this respect; it is easily allowed for by simply shifting the centroidal position to the midpoint, and halving the reduced mass. The much larger magnetic moment of the positron is more important and introduces considerable fine and hyperfine differences. These are not, however, qualitative differences. The really qualitative differences—the portents of mutual destruction—are exhibited in the positron-electron system by certain characteristic electrodynamic effects which are absent in the proton-electron system. These are the effects formally represented by extra "annihilation terms." They appear in the energy-level system of the positronium atom (particularly for S states where the wave function for zero distance between electron and positron does not vanish). They appear also in the expression for "Bhabha scattering" of positrons from electrons.

Later (Sec. IV) we discuss the unsuccessful attempts to find a symmetry principle associated with baryon conservation as well as the newly clarified situation with respect to "lepton conservation" (Sec. V).

II. GROUPS AND REPRESENTATIONS

A. Meaning of the Concepts of Transformation and Invariance

To understand the content of all the conservation laws it is necessary first to spell out carefully what invariance means. Even classically the analysis of a symmetry situation may confuse us by its chameleon-like aspect in which we are required to differentiate among samenesses. Further, in modern physics, we must go from the intuitive picture, suitable for macrophysical experience, to a more abstract conception adequate to interpret microphysical phenomena. Formally, the first step involves "group theory" and the second "quantum field theory." For elementary-particle physics, neither discipline can be avoided. Once the foundations have been laid, however, in simple situations, one can minimize the formal details of the two disciplines and use special or approximate versions of their methods.

First, what do we mean by a "transformation?" Because various operations (for instance, reflection) cannot be carried out directly on a physical system but only on the coordinate frame, we limit ourselves to the "passive" interpretation of transformation. We mean the change in description which the system undergoes when the coordinate frame is changed while the system remains fixed. (It is convenient to imagine this "fixture" to be with respect to a "protoframe" which is never altered. Any point fixed in the protoframe is a "physical point.") Such a passive transformation is nothing but a *redescription* (Sec. I.C).

For concreteness we consider a "scalar" field φ in a two-dimensional space, i.e., a function which associates a value to each physical point. (The value can be a number or an operator.) Let a rotation be made; we interpret this to mean the redescription which comes about when the space-frame is turned rigidly and the field (i.e., the value of φ at each point) is regarded as fixed. The redescription changes the space coordinate numbers (x_1,x_2) of a given physical point to new values

$$x_1'=R_{11}x_1+R_{12}x_2 \quad x_2'=R_{21}x_1+R_{22}x_2,$$

which we write in matrix-vector notation

$$\mathbf{x}'=\mathbf{R}\mathbf{x};$$

since we are talking about rotations, $\mathbf{R}$ satisfies the orthogonal-matrix condition $\mathbf{R}^{-1}=\mathbf{R}^T$. Instead of the old description, $\varphi(x)$, the field is given in the turned frame by a new function φ' of the new coordinate-numbers $\mathbf{x}'$. The definition of the field as scalar requires that at the same physical point, i.e., $\mathbf{x}'=\mathbf{R}\mathbf{x}$, $\varphi'(\mathbf{x}')$ equals $\varphi(\mathbf{x})$ (see Fig. 3). The symbol R may be interpreted to mean an operator which replaces $\mathbf{x}'$ by $\mathbf{x}$ in whatever follows. It is still better to interpret it to mean an operator which converts an old function to a new one, which is then evaluated for the new coordinates $\mathbf{x}'$. In summary: *By definition, if φ is a scalar, the new function $R\varphi$ evaluated at $\mathbf{x}'$ equals the original function evaluated at $\mathbf{x}$,*

$$(R\varphi)(\mathbf{x}')\equiv R\varphi(\mathbf{x}')\equiv \varphi'(\mathbf{x}')=\varphi(\mathbf{x}).$$

These ideas can be generalized to apply to any operation which may be more abstract than rotations, but which have meaning when applied to the argument of the field function. We keep the notation "x" for this argument and the name "frame" for any standard organization of its values, and continue to use the name "physical point" for a given value of x in the particular fundamental frame we call the "protoframe."

For a more general definition of a *scalar field*, consider a redescription which changes the argument (or coordinate numbers) x_j of a given physical point to new values

$$x_i'=A_i(x_j),$$

which we write symbolically as

$$x'=Ax.$$

It is not assumed that A is a linear operator but merely that it defines the x' in terms of the x. Now we say that a field function $\varphi(x)$ is a "scalar under the redescription A" when it goes over into a new function (of the new coordinates) which, *at each physical point*, has the same value as the old function:

$$\varphi(x)\rightarrow\varphi'(x')=\varphi(x).$$

Assuming that the redescription may be inverted, i.e.,

A^{-1} exists so that

$$x = A^{-1}x',$$

the condition for a scalar is equivalent to saying that we can get $\varphi'(x')$ by taking the old functional form and substituting for its argument the redescription by the new coordinate numbers:

$$\varphi'(x') = \varphi(A^{-1}x').$$

Examples of redescription which are physically important and which are discussed in the following are: inhomogeneous and homogeneous Lorentz transformations of the space-time coordinates; the parity transformation P consisting of reversing the handedness of the space-frame (i.e., the signs of the three space coordinates); the formal time-reversal G consisting of reversing the sign of the time coordinate; physical time-reversal or "motion-reversal" T; and charge conjugation (matter-antimatter interchange) C. An additional nongeometrical example of a redescription is given by a permutation operation upon the labels of a collection of particles.

We have introduced the idea of a function which is scalar under a given redescription. An even more important idea than "scalarity" for physics is that of *invariance*. A function $\varphi(x)$ is said to be invariant under a redescription A when its redescription not only has the same value at the same physical point, but also is expressed in the prime coordinates in exactly the same form as $\varphi(x)$; i.e., we not only have

$$R\varphi(x') \equiv \varphi'(x') = \varphi(x), \quad \textit{same value},$$

but also

$$\varphi'(x') = \varphi(x'), \quad \textit{same form},$$

so that we have for a function which is *scalar and invariant*

$$\varphi'(x') = \varphi(R^{-1}x') = \varphi(x'). \tag{4}$$

If the function is *scalar but not invariant*, we have instead

$$\varphi'(x') = \varphi(R^{-1}x') \neq \varphi(x'). \tag{4'}$$

For illustration let us consider a simple example which is also instructive for the later discussion of the Lorentz group: If the function

$$\varphi(x_1,x_2) = x_1^2 - x_2^2$$

is required to be only a scalar under the redescription

$$x_1 = ax_1' + bx_2' \quad x_2 = cx_1' + dx_2'$$

or

$$\mathbf{x} = \mathbf{R}\mathbf{x}',$$

we have

$$\varphi'(x_1',x_2') = (ax_1' + bx_2')^2 - (cx_1' + dx_2')^2.$$

If, besides being a scalar, the function $\varphi(x)$ is to be invariant, this imposes a restriction on the redescription,

$$a^2 - c^2 = d^2 - b^2 = 1 \quad ab - cd = 0.$$

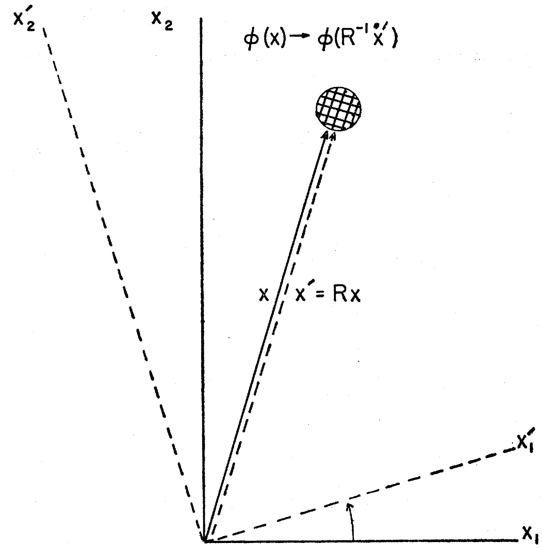

FIG. 3. Redescription of a scalar field which results from rotating the coordinate frame.

With the help of the further requirement that the inverse of $\mathbf{R}$ exists and has the same property of leaving φ invariant, we find that

$\det\mathbf{R} = \pm 1$ and

$$\mathbf{R} = \begin{pmatrix} a & b \\ \pm b & \pm a \end{pmatrix}, \quad \mathbf{R}^{-1} = \begin{pmatrix} a & \mp b \\ -b & \pm a \end{pmatrix}. \tag{5}$$

Note. This "two-dimensional transformation" is *not* an orthogonal transformation satisfying $\mathbf{R}^{-1} = \mathbf{R}^T$, but rather what can be called a "pseudo-orthogonal" transformation satisfying

$$\mathbf{R}^{-1} = \boldsymbol{\sigma}_3 \mathbf{R}^T \boldsymbol{\sigma}_3 \quad \boldsymbol{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The proper transformation $\mathbf{R}_+$, with $\det\mathbf{R} = +1$, is, however, symmetric: $\mathbf{R}_+ = \mathbf{R}_+^T$. (This symmetry, or Hermiticity, means that it can represent a real physical quantity. In fact, when the transformation is interpreted as a Lorentz transformation, the eigenvalues of $\mathbf{R}_+$ give the velocity.)

This example shows that, unlike the situation with scalarity where one can *define* a given function to be scalar under any redescription, here we have to do with a more demanding and fruitful idea: *We ask for the complete set of redescriptions under which a given function is invariant.* Such a complete set necessarily has the properties of a mathematical group: *closure* under the associative property of carrying out two redescriptions successively, *identity inclusion*, and *inverse inclusion.*

One can abstract in a one-to-one way from the set of redescriptions to a corresponding group of abstract operators $r_1, r_2, \cdots$ defined only by their mutual multiplicative relations. The whole abstract-group

TABLE IV. Linear groups and subgroups of n-row matrices with complex number elements, and some connections with physics (Det≡determinant).

Notation	Name	Type of matrices	Dimension (number of real parameters)	Type of matrices of the infinitesimal algebra
$\mathbf{G}_n$	Full linear (general linear), related to permutation groups; theory of complex spectra	Nonsingular	$2n^2$	Arbitrary
$\mathbf{G}_n'$	Real linear	Nonsingular real	n^2	Real
$\mathfrak{g}_n$	Unimodular (special linear), $n=2$: two-sheeted covering group of proper homogeneous Lorentz group	Det$=1$	$2(n^2-1)$	Trace$=0$
$\mathfrak{g}_n'$	Real unimodular	Real, Det$=1$	n^2-1	Real, trace$=0$
$\mathbf{U}_n$	Unitary, admissible transformation groups of the Hilbert spaces of quantum theory (antiunitary transformations are admitted for time reversal)	Unitary ($\xi_1\xi_1^*+\cdots+\xi_n\xi_n^*$ invariant)	n^2	Anti-Hermitian ($a_{ik}+a_{ki}^*=0$)
$\mathfrak{u}_n$	Unimodular unitary, $n=2$: two-sheeted covering group of three-dimensional rotation group	Unitary, Det$=1$	n^2-1	Anti-Hermitian, trace$=0$
$\mathbf{R}_n$	Orthogonal, $n=3$: rotations proper and improper	Orthogonal, real ($\xi_1^2+\cdots+\xi_n^2$ invariant)	$\frac{n(n-1)}{2}$	Antisymmetric, real ($a_{ik}+a_{ki}=0$)
$\mathfrak{r}_n$	Rotation group (proper orthogonal), $n=3$: rotations proper; $n=3, 4$: excess degeneracy in hydrogen atom, harmonic oscillator and rigid rotator problems; $n=4$: Global symmetry theories of elementary particles	Orthogonal, real, Det$=1$	$\frac{n(n-1)}{2}$	Antisymmetric, real

structure theory then follows, i.e., the theory of subgroups, invariant or "normal" subgroups and factor groups, unique dissection of a group into classes, homomorphisms, etc. Some wide categories of these groups, their connections with physics, and some of their general characteristics are listed in Table IV. With the additional development of the theory of representations of groups by systems of matrices, we are led to a rather deep insight and valuable methods for classification and solution of concrete problems.

In later sections a sketch is given of some features of continuous group theory as it bears on elementary particle physics. First, however, we introduce the inhomogeneous Lorentz group; it is an important example in itself and further, it illustrates the bearing of invariance groups on physical systems in general. After discussing the structure of the Lorentz group we make some remarks about the meaning of "irreducible" and "indecomposable" representations of groups, and then apply some of the resulting ideas to examples.

B. Lorentz Group

The inhomogeneous Lorentz group consists of all those redescriptions under which the space-time interval between two space-time points (x_0,x_1,x_2,x_3) and (y_0,y_1,y_2,y_3) is invariant:

$$(x_1'-y_1')^2+(x_2'-y_2')^2+(x_3'-y_3')^2-(x_0'-y_0')^2 \\ =(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2-(x_0-y_0)^2. \quad (6)$$

Here x_0 or y_0 is the time coordinate t of the point. This equation may be put in the standard form for an invariant interval in a flat space,

$$\sum g^{ij}(x_i'-y_i')(x_j'-y_j')=\sum g^{ij}(x_i-y_i)(x_j-y_j),$$

by introducing a metric tensor $g^{ij}=(g^{ij})'$. The (Latin index) metric tensor for measuring interval in space-time which we use is

$$-g^{00}=g^{11}=g^{22}=g^{33}=-g_{00}=g_{11}=g_{22}=g_{33}=1,$$

constituting a diagonal matrix $\mathbf{G}$. Alternatively, we may introduce $x_4=x^4=ix_0=-ix^0$, which goes with the (Greek index) metric tensor

$$g^{\mu\nu}=g_{\mu\nu}=\delta_{\mu\nu} \quad (\mu, \nu=1, 2, 3, 4).$$

We state without proof a number of consequences, the proofs of which can be worked out easily, or found in references. With a slight generalization of notation these results are equally true for the inhomogeneous rotation group in n dimensions and with any metric.

Every inhomogeneous transformation L of a space-time vector $\mathbf{x}$ may be decomposed into a homogeneous transformation represented by a matrix $\mathbf{\Lambda}$ and a translation $\mathbf{a}$,

$$\mathbf{x}'=L(a,\Lambda)\mathbf{a}=\mathbf{a}+\mathbf{\Lambda x},$$

which may be written in matrix form

$$\begin{bmatrix}1\\x_1'\\\vdots\\x_4'\end{bmatrix}=\begin{bmatrix}1&0&0\\a_1&\Lambda_{11}&\Lambda_{14}\\\vdots&\vdots&\vdots\\a_4&\Lambda_{41}&\Lambda_{44}\end{bmatrix}\begin{bmatrix}1\\x_1\\\vdots\\x_4\end{bmatrix}$$

or

$$\begin{pmatrix}1\\\mathbf{x}'\end{pmatrix}=\begin{pmatrix}1&0\\\mathbf{a}&\mathbf{\Lambda}\end{pmatrix}\begin{pmatrix}1\\\mathbf{x}\end{pmatrix}.$$

Here, if the Greek-index metric is used, the matrices $\mathbf{\Lambda}$ satisfy the orthogonality condition; $\mathbf{\Lambda}^{-1}=\mathbf{\Lambda}^T$ derived from the defining condition that the scalar product of two 4-vectors is invariant. (Note that the matrices are complex and orthogonal, and therefore not unitary!) If the real metric $\mathbf{G}$ is used, the homogeneous transformation matrices—now real—satisfy the pseudo-orthogonality condition

$$\mathbf{\Lambda}^{-1}=\mathbf{G\Lambda}^T\mathbf{G}. \quad (7)$$

This is derived again from the condition that the scalar product of two space-time vectors is invariant, i.e.,

$$(\mathbf{\Lambda x})\mathbf{G}(\mathbf{\Lambda y})=\mathbf{xGy}$$
$$\mathbf{x\Lambda}^T\mathbf{G\Lambda y}=\mathbf{xGy} \quad \text{or} \quad \mathbf{\Lambda}^T\mathbf{G\Lambda}=\mathbf{G}. \tag{7'}$$

The product and inverse of transformations are given by

$$\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{a} & \mathbf{\Lambda}\end{pmatrix}\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{b} & \mathbf{M}\end{pmatrix}=\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{a}+\mathbf{\Lambda b} & \mathbf{\Lambda M}\end{pmatrix},$$
$$\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{a} & \mathbf{\Lambda}\end{pmatrix}^{-1}=\begin{pmatrix}1 & \mathbf{0}\\ -\mathbf{\Lambda}^{-1}\mathbf{a} & \mathbf{\Lambda}^{-1}\end{pmatrix}.$$

Both subsets, that of all homogeneous transformations, $\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & \mathbf{\Lambda}\end{pmatrix}$, and of pure translations $\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{a} & \mathbf{I}\end{pmatrix}$, form subgroups. The latter is Abelian and is an invariant subgroup, i.e.,

$$\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{a} & \mathbf{\Lambda}\end{pmatrix}\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{b} & \mathbf{I}\end{pmatrix}\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{a} & \mathbf{\Lambda}\end{pmatrix}^{-1}=\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{\Lambda b} & \mathbf{I}\end{pmatrix}.$$

The fact that inhomogeneous rotation groups have a nontrivial Abelian invariant subgroup, and are therefore not "semisimple" (see footnote 15), has important consequences for the representation theory (see the following). It is evident from the equation

$$\det(\mathbf{\Lambda}^{-1}\mathbf{\Lambda})=(\det\mathbf{G})^2\det(\mathbf{\Lambda}^T\mathbf{\Lambda})=(\det\mathbf{\Lambda})^2=1$$

that the homogeneous group breaks into two disconnected parts, the proper subgroup with all $\mathbf{\Lambda}$ such that

$$\det\mathbf{\Lambda}=+1 \quad \{\Lambda_+\},$$

and its improper coset consisting of all $\mathbf{\Lambda}_-$ such that

$$\det\mathbf{\Lambda}_-=-1 \quad \{\Lambda_-\}.$$

A further decomposition of the homogeneous group occurs if we remember from the physics that the $\mathbf{\Lambda}$'s must be real. Equating the matrix element in the first row, first column on both sides of Eq. (7′), we find

$$\sum_{i=1}^{3}(\Lambda_0{}^i)^2-(\Lambda_0{}^0)^2=-1$$

or

$$(\Lambda_0{}^0)^2=1+\sum_{i=1}^{3}(\Lambda_0{}^i)^2\geq 1.$$

We have, therefore, two possibilities: Either

$$\Lambda_0{}^0\geq 1,$$

when $\mathbf{\Lambda}$ is said to be in the set $\{\Lambda^\uparrow\}$ of future-preserving (or "orthochronous") transformations; or

$$\Lambda_0{}^0\leq -1,$$

when $\mathbf{\Lambda}$ is said to be in the set $\{\Lambda^\downarrow\}$ of time-reversing transformations.

It is easy to verify (see, for example, Murnaghan, 1938) that $\{\Lambda^\uparrow\}$ is an invariant subgroup of $\{\Lambda\}$, and that the intersection, $\{\Lambda_+{}^\uparrow\}$ of $\{\Lambda^\uparrow\}$ and $\{\Lambda_+\}$ is an invariant subgroup of both $\{\Lambda\}$ and $\{\Lambda_+\}$. Further, each of the four separate pieces into which $\{\Lambda\}$ breaks up is *connected* within itself; i.e., each $\mathbf{\Lambda}$ in a piece may be connected to a standard $\mathbf{\Lambda}$ within the piece by a continuous sequence of $\mathbf{\Lambda}$'s lying entirely in the piece. The scheme is as follows:

(1) $\{\Lambda_+{}^\uparrow\}$: every $\mathbf{\Lambda}$ connected with $\mathbf{I}$ (identity)

(2) $\{\Lambda_+{}^\downarrow\}$: every $\mathbf{\Lambda}$ connected with $-\mathbf{I}\equiv\mathbf{I}_{st}$ (strong reflection)

(3) $\{\Lambda_-{}^\uparrow\}$: every $\mathbf{\Lambda}$ connected with $-\mathbf{G}\equiv\mathbf{I}_s$ (space reflection or parity)

(4) $\{\Lambda_-{}^\downarrow\}$: every $\mathbf{\Lambda}$ connected with $\mathbf{G}\equiv\mathbf{I}_t$ (formal time-reversal).

The redescription matrix expressing the parity operation is designated $\mathbf{P}$, as well as $\mathbf{I}_s\equiv-\mathbf{G}$; when considered abstractly it is represented by P. Likewise, the formal time-reversal redescription may be represented abstractly by G.

The inhomogeneous future-preserving proper Lorentz group $\{L_+{}^\uparrow\}$ consists of all transformations of the form

$$\mathbf{x}'=\mathbf{a}+\mathbf{\Lambda}_+{}^\uparrow\mathbf{x}.$$

For short, and roughly following Wigner (1957), we call this group the "Poincaré group." When it is extended to include all transformations of the sets (1), (2), (3), and (4), we call it the "extended Poincaré group."

C. Decomposability of Matrices to Blockdiagonal Form. Reducibility to Semi-blockdiagonal Form

We now introduce briefly the subject of representations of groups by systems of matrices. Some elementary formal definitions are omitted as these can be found in many references. Also omitted is any detailed discussion of the properties of sets of basis vectors or functions transforming according to matrices of a representation. This topic, which is of great practical importance for applications in nonrelativistic quantum theory (cf. Wigner 1931; Melvin 1956), is largely dispensable for our purposes. We are concerned with only the elementary principles of the relativistic theory, and even with respect to that theory we adopt the point of view which stresses operators rather than eigenvectors (Heisenberg picture). A brief introduction to the concept of functions transforming according to a representation is given at the end of Sec. II.D. This is adequate for the following. Confining ourselves then to the matrix representations themselves, we first discuss the meaning of "irreducible" and "indecomposable" representations which play such a basic role as building blocks. Let there be a representation consisting of matrices $\mathbf{R}_1, \mathbf{R}_2, \cdots$ corresponding to the group of operators $r_1, r_2, \cdots$, i.e., to each r_i in the group there corresponds a matrix $\mathbf{R}_i$ with the basic representation property that the image of a product is the product of the images, i.e., if $r_ir_j=r_k$, then $\mathbf{R}_i\mathbf{R}_j=\mathbf{R}_k$. If we think of the vector space on which the matrices act, it suggests itself that it would be a simplification to separate out any subspaces which go into themselves alone under all the **R**'s and deal with each such subspace separately. Algebraically, this idea of "reducing" the representation means trying, by a single similarity transformation

$$\mathbf{M}(\mathbf{R})\mathbf{M}^{-1} \quad (\mathbf{M} \text{ the same matrix for all } \mathbf{R})$$

in the representation space, to put all **R**'s simultaneously into a simplified form which exhibits a submatrix structure. How far can we go in this simplification?

Even in the simplest nontrivial instance, that of a single matrix of order $n\times n$ $(n\geq 2)$, there is a distinction between the diagonalizable case and the nondiagonalizable case. The first case occurs when the eigenvector set is complete, i.e., when there are n linearly independent eigenvectors, and the second when the eigenvector set is incomplete, numbering less than n. (This way of stating the condition for diagonalizability, so natural for a physicist, is in line with the diagonalizability conditions on Hermitian matrices which quantum theory imposes in the ∞-dimensional Hilbert space of state vectors.) It is easy to prove that completeness of the eigenvector set is a necessary and sufficient condition for diagonalizability[8]; *sufficient* because the eigenvectors themselves form the columns of the transformation matrix which accomplishes the diagonalization, *necessary* because, given that the matrix is diagonal in a certain coordinate system, the basis vectors of that coordinate system give the complete set of eigenvectors.

Thus, for example, the best that can be done with the matrix

$$\begin{pmatrix} a+b & b \\ -b & a-b \end{pmatrix}$$

which has only one eigenvector, is to reduce it to the semidiagonal form

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

More generally, the best that can be done with a single matrix having an incomplete eigenvector system is to put it in the "Jordan normal form,"

$$\left[\begin{array}{cccc:cccc:c}
a & 1 & & 0 & & & & & \\
 & \ddots & \ddots & & & & 0 & & \\
 & & \ddots & 1 & & & & & \\
0 & & & a & & & & & \\ \hdashline
 & & & & b & 1 & & 0 & \\
 & & 0 & & & \ddots & \ddots & & \\
 & & & & & & \ddots & 1 & \\
 & & & & 0 & & & b & \\ \hdashline
 & & & & & & & & \ddots
\end{array}\right].$$

The over-all structure of this, as we see, is block-diagonal, but within each block the structure is only semidiagonal.

When we come to consider the possibility of simplifying an entire *system of matrices* simultaneously by a single equivalence transformation, we find that some such systems are "decomposable" to the blockdiagonal form

$$\left[\begin{array}{c:c:c}
\begin{matrix} a_{11}\cdots a_{1m} \\ \cdots \\ a_{m1}\cdots a_{mm} \end{matrix} & 0 & \\ \hdashline
0 & \begin{matrix} b_{11}\cdots b_{1n} \\ \cdots \\ b_{n1}\cdots b_{nn} \end{matrix} & 0 \\ \hdashline
 & 0 & \ddots
\end{array}\right],
\quad
\left[\begin{array}{c:c:c}
\begin{matrix} a_{11}'\cdots a_{1m}' \\ \cdots \\ a_{m1}'\cdots a_{mm}' \end{matrix} & 0 & \\ \hdashline
0 & \begin{matrix} b_{11}'\cdots b_{1n}' \\ \cdots \\ b_{n1}'\cdots b_{nn}' \end{matrix} & 0 \\ \hdashline
 & 0 & \ddots
\end{array}\right],\cdots$$

whereas other sets can at most be simplified to the semiblockdiagonal form

$$\left[\begin{array}{c:c:c}
\begin{matrix} a_{11}\cdots a_{1m} \\ \cdots \\ a_{m1}\cdots a_{mm} \end{matrix} & \begin{matrix} a_{1,m+1}\cdots \\ \cdots \\ a_{m,m+1}\cdots \end{matrix} & \begin{matrix} \cdots \\ \\ \cdots \end{matrix} \\ \hdashline
0 & \begin{matrix} b_{11}\cdots b_{1n} \\ \cdots \\ b_{n1}\cdots b_{nn} \end{matrix} & \begin{matrix} \cdots \\ \\ \cdots \end{matrix} \\ \hdashline
 & 0 & \ddots
\end{array}\right],$$

[8] In the usual mathematical literature on finite matrices the condition is stated rather in the form that all the elementary divisors of the matrix must be simple. Since the characteristic polynomial is the product of all the elementary divisors, this means there must be exactly n of them. Finally, since there is a one-to-one correspondence between elementary divisors and eigenvectors (also not usually stated), we see how the standard criterion for diagonalizability is related to the one we have formulated.

$$\left[\begin{array}{c|c|c} \begin{matrix} a_{11}'\cdots a_{1m}' \\ \cdots \\ a_{m1}'\cdots a_{mm}' \end{matrix} & \begin{matrix} a_{1,m+1}'\cdots \\ \cdots \\ a_{m,m+1}'\cdots \end{matrix} & \begin{matrix} \cdots \\ \\ \cdots \end{matrix} \\ \hline 0 & \begin{matrix} b_{11}'\cdots b_{1n}' \\ \cdots \\ b_{n1}'\cdots b_{nn}' \end{matrix} & \begin{matrix} \cdots \\ \\ \cdots \end{matrix} \\ \cline{2-3} & 0 & \ddots \end{array}\right], \cdots$$

with only the entries below the diagonal blocks all definitely vanishing. A system that simplifies to only the latter type is called "indecomposable" but, if it can truly be brought to a nontrivial semiblockdiagonal form, we still call it "reducible" for the following good reason. Suppose the semiblockdiagonalization of the original total representation has been carried as far as possible. Each of the matrices carries a vector lying entirely in the uppermost m-dimensional subspace into the same subspace. (We call this subspace the "top invariant subspace" of the total representation space.) Furthermore, the top block along the diagonal of $\mathbf{R}_i$ is a matrix which by itself also represents r_i in a representation consisting solely of the set of top blocks in all the $\mathbf{R}$'s; for *these top blocks are the sole effective parts of the entire matrices in carrying the top invariant subspace into itself*. Since, by hypothesis, the reduction cannot be carried further, the top block representation is an "irreducible representation." If the above-diagonal blocks do not vanish in all the $\mathbf{R}$'s of the total representation, the total representation does not decompose into parts; in other words, vectors lying outside the top invariant subspace also get mixed into this subspace. Such a representation is *indecomposable*, or "not fully reducible" though it is reducible. (The name reducible is used even when the system is looked at in the original basis, in which the semiblockdiagonal structure is not yet in evidence.)

A simple example of such a reducible but indecomposable system of matrices, already in semiblockdiagonal form, is

$$\left[\begin{array}{c|c} 1 & \theta_1 \\ \hline 0 & 1 \end{array}\right], \qquad \left[\begin{array}{c|c} 1 & \theta_2 \\ \hline 0 & 1 \end{array}\right].$$

This system is a representation of the Abelian group of translations along a line

$$\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_2)\mathbf{R}(\theta_1) = \mathbf{R}(\theta_1+\theta_2).$$

Before the simplifying equivalence transformation was applied, the general matrix of the system may have looked like

$$\begin{bmatrix} 1+\theta_i\mathrm{sc} & \theta_i\mathrm{c}^2 \\ -\theta_i\mathrm{s}^2 & 1-\theta_i\mathrm{sc} \end{bmatrix}\cdots$$

$$\mathrm{s}\equiv\sin\alpha,\ \mathrm{c}\equiv\cos\alpha,\ \alpha \text{ arbitrary}.$$

It is not fundamental that the zeros are placed "downstairs" in the usual representation of an indecomposable reducible system. They could equally well have been placed upstairs. The significance of this interchange is merely that it is then the subspace complementary to the top subspace that is invariant.

Further, it is of interest to remark that not only the top block on the diagonal gives an irreducible representation of the group, but each of the other diagonal blocks also gives an irreducible representation since these also have the basic representation property that the image of a product is the product of the images. These other blocks are, however, no longer necessarily associated with invariant subspaces; there is such an invariant subspace only if all blocks lying vertically above the given diagonal block vanish along with those below. Thus we see that the one-to-one correspondence between invariant subspaces and irreducible representations which exists for decomposable—fully reducible—representations is lost when we deal with indecomposable reducible representations.

We may sum up the rather subtle relations of decomposability and reducibility in an imaginative figure. Suppose one could say that each complete living organism provides in itself a mapping in some sense of a cosmic entity (analogous to a group). Then one might say, leaning on the figure: A decomposable representation is like a mitotic ameba, whereas an indecomposable but reducible representation is like a young-bearing animal. An irreducible representation (which is *a fortiori* indecomposable) is like an animal which bears no young within it.

D. Conditions for Indecomposability and Irreducibility

Very important are Schur's lemma (Schur, 1905), which provides necessary conditions for irreducibility (sufficient for reducibility) and Schur's theorem (Schur, 1928), which provides a necessary and sufficient condition for indecomposability.

Schur's lemma. (*a*) *Given two irreducible representations* $\mathbf{A}(s)$, $\mathbf{B}(s)$ *of a group* $\{s\}$ *and a matrix* $\mathbf{P}$ *such that* $\mathbf{P}$ *intertwines* $\mathbf{A}$ *and* $\mathbf{B}$, *i.e.*,

$$\mathbf{B}(s)\mathbf{P} = \mathbf{P}\mathbf{A}(s) \quad (\textit{every } s),$$

then either $\det\mathbf{P}\neq 0$ or $\mathbf{P}\equiv 0$.

(*b*) *If* $\mathbf{B}(s) = \mathbf{A}(s)$, $\mathbf{P} = \lambda\mathbf{I}$.

Part (a) of the lemma states that either $\mathbf{P}$ is a square nonsingular matrix and the two representations are equivalent, or else $\mathbf{P}$ is identically zero. In other words, any matrix $\mathbf{P}$ which intertwines two irreducible representations is "square or nothing."

Part (b) of the lemma gives more information on a special case: Any matrix which intertwines an irreducible representation with itself is a multiple of the unit matrix. In other words, a representation can be irreducible only if every commutor[9] is a multiple of the identity.

We do not reproduce here the easily available proof of Schur's lemma, but remark that the arguments which establish Schur's lemma for finite dimensional representations can be carried over to infinite dimensional representations (cf. Bargmann, 1947).

Schur's theorem. A representation $\mathbf{A}(s)$ *is indecomposable if and only if every commutor of* $\mathbf{A}(s)$ *has all its eigenvalues equal. Such a commutor is, in general, of the form*

$$\begin{bmatrix} \lambda & \theta_{12} & \theta_{13} & \cdots & \\ 0 & \lambda & \theta_{23} & \cdots & \\ \cdot & 0 & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 0 & 0 & \cdot & 0 & \lambda \end{bmatrix}.$$

The meaning of representations in defining symmetry types among states of physical systems was discussed in a previous paper (Melvin, 1956) and applications involving the geometrical symmetry groups were also given. Since in all such groups and even wider categories (see the following) indecomposability and irreducibility go together, the two were not distinguished. The term "rep" was coined to refer to these indecomposable-irreducible representations. When one comes to consider other groups such as the inhomogeneous rotation and Lorentz groups, it becomes important to make the distinction since in these cases indecomposable but still reducible representations occur along with more "normal" ones. Here, too, an abbreviation is useful and it seems best to continue to use the term "rep" to refer to just those normal cases where the two properties of indecomposability and irreducibility go together. It is the rep cases which play a dominant role in physics (see next section).

Nevertheless, the question may come up: When the two are distinguishable, which is the more relevant, indecomposability or irreducibility? We specify a physical system by a set of operators, some of them Hermitian and some of them not necessarily Hermitian (e.g., some finite symmetry operators). The observables must be represented by Hermitian matrices and in this case there is no difference between reducibility and decomposability. Operators which are not Hermitian or unitary, or more generally "normal" (i.e., Hermitian and anti-Hermitian parts commuting), can be reducible while indecomposable. Examples of such non-Hermitian nonnormal operators occur commonly among spinor transformations which correspond to space-time transformations on the coordinates (i.e., finite-dimensional representations of the Lorentz group, including the vector representation—"spin 1"—case itself). Reflection suggests that in such cases the important thing is the indecomposability property; it is this property which permits us to describe the system as an *independent* system. If a representation were decomposable it would refer to a composite of several independent systems. On the other hand, if a representation is indecomposable but reducible, the contained irreducible parts—the "young" within—cannot be considered to be completely independent.

As a simple illustration of these ideas we consider the "two-dimensional Lorentz group" of matrices which we discussed in the foregoing, Eq. (5). It is easily verified that the proper subgroup ($\det\mathbf{R}=+1$) has among its commutors matrices which are more general than the identity, i.e., matrices of the form

$$\begin{pmatrix} w & x \\ x & w \end{pmatrix}.$$

Thus, the system of proper matrices is reducible and therefore decomposable (being Hermitian). On the other hand, the system of improper matrices ($\det\mathbf{R}=-1$) has no commutors other than multiples of the identity. Thus it is indecomposable (Schur's theorem). We do not yet know whether it is also irreducible. Schur's lemma is not sufficient to establish irreducibility; it merely shows that the necessary condition for irreducibility is satisfied in this case. In fact, for the choice of parameters, $b=-a$, $\mathbf{R}$ takes the form

$$\begin{pmatrix} a & -a \\ a & -a \end{pmatrix}$$

which upon being unitarily transformed by

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

takes the form

$$\begin{pmatrix} 0 & -2a \\ 0 & 0 \end{pmatrix},$$

the typical indecomposable-reduced form. Thus the two-dimensional Lorentz group has such representations, and likewise the four-dimensional Lorentz group. We note again that the possibility of such representations has come up only with the consideration of the reflection (e.g., time reversal) transformations. In fact the *proper* homogeneous Lorentz group can be proved to be "simple," i.e., to have no nontrivial invariant subgroup and therefore, by a theorem to be discussed in the next section, all reducible representations are also decomposable.

That this is far from a trivial result appears when we realize that, as already indicated by the two-dimensional case, none of the finite-dimensional representations of $\{\Lambda\}$ are unitary,[10] and therefore do not partake of the

[9] We call a matrix which commutes with every member of a system of matrices a "commutor" of the system. It seems best to avoid the term "commutator" used, for instance, in Weyl (1946) because it conflicts with different usages of that term in abstract group theory and in quantum theory.

[10] Wigner (1939) and earlier work of Majorana (1932), Dirac (1936), and Proca (1936). Wigner's methods are global whereas the earlier work is based on the study of infinitesimal generators (see Sec. II.F).

automatic equivalence of reducibility and decomposability which characterizes unitary representations. Nevertheless, by the result just mentioned, they are reps. The equivalence of reducibility and decomposability in any unitary representation follows from the fact that every matrix represents an element R^{-1} inverse to some other element R in the group, and for any unitary representation $\mathbf{R}^{-1}=\mathbf{R}^{\dagger}$, so that if all the matrices have zeros "downstairs" they also have zeros "upstairs."

We can introduce the idea of such a finite-dimensional rep of any group of coordinate redescriptions $\{\mathbf{R}_1=\mathbf{I}, \mathbf{R}_2, \cdots \mathbf{R}_\alpha \cdots\}$ also in terms of the transformation behavior of a basic set of functions. This might be described as a generalization from the concept of an invariant function to that of a "covariant set of functions": We take a function $\varphi^1(\xi)$ which is scalar but not invariant [see Eq. (4')], i.e., it transforms under the αth member of the group to a function $\varphi^\alpha(\xi)$, in general different from $\varphi^1(\xi)$ (except in the case of the identity redescription $\mathbf{R}_1=\mathbf{I}$):

$$\begin{aligned} R_2\varphi^1(\xi) &\equiv \varphi^2(\xi)=\varphi^1(\mathbf{R}_2^{-1}\xi) \qquad \xi=\mathbf{R}x \\ &\vdots \\ R_\nu\varphi^1(\xi) &\equiv \varphi^\nu(\xi)=\varphi^1(\mathbf{R}_\nu^{-1}\xi). \end{aligned}$$

By operating with all the elements of the group, we in general get a set of functions of which it may happen that only a finite subset are linearly independent. Let these be n in number:

$$\varphi^1(\xi), \quad \varphi^2(\xi), \cdots \varphi^n(\xi).$$

We then have

$$R\varphi^\alpha(\xi)=\sum_{\beta=1}^{n} S^{\beta\alpha}(\mathbf{R})\varphi^\beta(\xi). \tag{8}$$

It is now proved that the matrices $\mathbf{S}(\mathbf{R})$ constitute a representation of the group, i.e.,

$$\mathbf{S}(\mathbf{R}')\mathbf{S}(\mathbf{R})=\mathbf{S}(\mathbf{R}'\mathbf{R}).$$

By definition, letting $\mathbf{I}$ represent the identity of the group, we have

$$I\varphi^\alpha(\xi)=\varphi^\alpha(\xi)=\varphi^1(\mathbf{R}_\alpha^{-1}\xi)=\sum_\beta S^{\beta\alpha}(\mathbf{I})\varphi^\beta,$$

and for a general element $\mathbf{R}$ of the group we have

$$R\varphi^\alpha(\xi)=\varphi^1(\mathbf{R}^{-1}\mathbf{R}_\alpha^{-1}\xi)=\sum_\beta S^{\beta\alpha}(\mathbf{R})\varphi^\beta(\xi),$$

which brings out the mode of generating the representation from the original simple function φ^1. Taking another element $\mathbf{R}'$ of the group and operating, we have

$$R'\varphi^\beta=\sum_\kappa S^{\kappa\beta}(\mathbf{R}')\varphi^\kappa,$$

and

$$R'R\varphi^\alpha=\sum_\beta S^{\beta\alpha}(\mathbf{R})R'\varphi^\beta=\sum_{\beta,\kappa} S^{\kappa\beta}(\mathbf{R}')S^{\beta\alpha}(\mathbf{R})\varphi^\kappa.$$

But $\mathbf{R}'\mathbf{R}$ operating on φ^α gives an expansion in φ^κ with coefficients $S^{\kappa\alpha}(\mathbf{R}'\mathbf{R})$. Thus we have

$$S^{\kappa\beta}(\mathbf{R}')S^{\beta\alpha}(\mathbf{R})=S^{\kappa\alpha}(\mathbf{R}'\mathbf{R}),$$

and the complete set of matrices $\{\mathbf{S}\}$, one corresponding to each $\mathbf{R}$, forms in the n-dimensional space of spinor components $\varphi^\beta(\xi)$ a representation of the group of transformations $\{\mathbf{R}\}$. If this representation is indecomposable irreducible, it is called a "rep," and the associated spinor a "rep spinor." This spinor provides a *basis* for the rep.

It is worthwhile to emphasize the point already indicated in our proof, i.e., that to generate the whole representation *one need know how to express, in terms of a basic set, only the transform of the original function* φ^1 *under all operations of the group.* To illustrate the whole procedure, and the last point in particular, we consider a simple example. Let the group $\{\mathbf{R}\}$ be the two-dimensional rotation group in the plane and let $\varphi_1\equiv\cos\theta$ where θ is the polar angle. Then we have

$$\begin{aligned} R_\alpha \cos\theta \equiv \varphi^\alpha(\theta) &= \cos(R_\alpha^{-1}\theta) \\ &= \cos(\theta+\alpha)=\cos\alpha\cos\theta-\sin\alpha\sin\theta. \end{aligned}$$

The second basis function needed for a complete set here is $R_{\frac{3}{2}\pi}\cos\theta\equiv\varphi^2(\theta)=\sin\theta$. We then have for a general rotation

$$\begin{aligned} R_\alpha\varphi^2 &= \varphi^1(R_\alpha^{-1}R_{\frac{3}{2}\pi}^{-1}\theta)=\cos(\theta+\tfrac{3}{2}\pi+\alpha) \\ &= \cos(\alpha+\tfrac{3}{2}\pi)\cos\theta-\sin(\alpha+\tfrac{3}{2}\pi)\sin\theta \\ &= \sin\alpha\cos\theta+\cos\alpha\sin\theta, \end{aligned}$$

which are the usual redescriptions which the coordinates $x=\cos\theta$ and $y=\sin\theta$ undergo when the frame is rotated through an angle α.

E. Physical Motivation. Decomposability of Representations of Groups

Groups may now be surveyed from the point of view of finding which of their indecomposable representations are also irreducible, since such reps are of great significance for physics. Fortunately the problem of finding the reps is simplified greatly when we concentrate on representations in state-vector space. The postulational basis of quantum theory compels us (Wigner, 1931, 1959) to limit ourselves in state-vector space to unitary representations (or antiunitary representations—see the following), and for these indecomposability means irreducibility. In other types of representations these two properties need not concur. Such representations may appear when groups which are not compact or "semisimple" [11] are being represented since noncompact groups have nonunitary representations, and, if the group is also not semisimple, indecomposability need not imply irreducibility.

We can be more specific: In accord with the super-

[11] See footnote 14 and the following discussion.

position principle, probability interpretation, and completeness condition, the states of a quantum-mechanical system are representable in a Hilbert space in which a *unitary* scalar product is defined (Sec. I.C). Each state of the physical system corresponds uniquely to a unit-length *ray*, i.e., a unit vector with an arbitrary phase factor. The arbitrariness in the phase factor comes about in that the quantities which are physically observable are not the state rays $\varphi, \psi, \cdots$ themselves but rather the absolute squares of their scalar products, $|(\varphi,\psi)|^2$. This square gives the probability that, the system being in state ψ, an experiment to see whether it is also in state φ, gives the result "yes"—the so-called "transition probability."

Suppose that we are dealing with a microphysical system whose basic aspects are invariant under a certain group of coordinate redescriptions. We interpret the term "basic aspects" to mean the set of all transition probabilities between states. The states themselves are not invariant ("Schrödinger type" interpretation), but corresponding to each redescription the new states are given by a certain operator acting on the old states. Wigner (1931, 1959) proved that the invariance of the transition probability leads to the following mathematical property of such mapping operators in state-vector space:

The operator can be either one of two kinds. Either (1) it is *linear* and *unitary*, i.e.,

$$\mathbf{U}(\lambda\varphi+\mu\psi)=\lambda\mathbf{U}\varphi+\mu\mathbf{U}\psi, \quad (\mathbf{U}\varphi,\mathbf{U}\psi)=(\varphi,\psi);$$

or (2) it is *antilinear*

$$\mathbf{A}(\lambda\varphi+\mu\psi)=\lambda^*\mathbf{A}\varphi+\mu^*\mathbf{A}\psi$$

norm-preserving and has an inverse, and is therefore *antiunitary*, i.e.,

$$(\mathbf{A}\varphi,\mathbf{A}\psi)=(\varphi,\psi)^*=(\psi,\varphi). \tag{9}$$

(Included in the definitions is the requirement that all conditions must hold for any numbers λ, μ and any functions φ, ψ.)

It is immediately established that the product of any two antiunitary operators $\mathbf{A}$ and $\mathbf{B}$ is unitary:

$$(\mathbf{AB}\varphi,\mathbf{AB}\psi)=(\mathbf{B}\psi,\mathbf{B}\varphi)=(\varphi,\psi).$$

In particular, this holds for the square of any antiunitary operator. An especially important subclass occurs when this square is a scalar, i.e., a multiple of the identity. For this case there applies the following.[12] *Scalar square theorem. If* $\mathbf{V}$ *is an antiunitary operator whose square is a numerical multiple* ω *of the identity, then* $\omega=\pm1$.

[12] I wish to thank L. Michel for a discussion of this theorem of which proofs are found in Michel and Wightman's notes and in Wigner's book (1959). The simple proof given here is based on discussions with P. Erdös and R. Herndon.

Proof. (1) The absolute magnitude of ω is 1. This results immediately from the fact that $\mathbf{V}^2$ is unitary:

$$(\varphi,\psi)=(\mathbf{V}^2\varphi,\mathbf{V}^2\psi)=(\omega\varphi,\omega\psi)=\omega\omega^*(\varphi,\psi), \quad \therefore\ \omega\omega^*=1.$$

$$(2)\ \omega\mathbf{V}\varphi=\mathbf{VVV}\varphi=\mathbf{V}\omega\varphi=\omega^*\mathbf{V}\varphi, \quad \therefore\ \omega/\omega^*=1.$$

Physical arguments were given by Wigner (1932) to the effect that if a given type of invariance operation is in one instance represented unitarily (antiunitarily), it should be represented unitarily (antiunitarily) in all representations. Furthermore, all Hilbert-space operators representing redescriptions lying in any connected piece of the full Lorentz group must have the same unitary or antiunitary nature (see, for instance, Bargmann, 1954). Wigner found, assuming positive energy states were carried into positive energy states, that of all the invariance operations, only those connected with time reversal should be represented antiunitarily, all the others unitarily (see Table III). For pure time reversal $\mathbf{T}$ (but not for the product of time reversal with operators representing other symmetry operations!) the "scalar square theorem" applies since the result of reversing the time twice must give back the same state—hence, the same ket up to a factor. Thus, we have

$$\mathbf{T}^2=\pm\mathbf{I}, \tag{10}$$

which serves to classify physical systems into two sharply separated classes according as their states are even or odd under $\mathbf{T}^2$ (see Sec. IV and Table V).

The involutional property of $\mathbf{T}$ which limits $\mathbf{T}^2$ to ±1 facilitates the development of a theory of *co-representations*. This natural extension of representation theory to groups containing an involutional antiunitary operator is thoroughly discussed by Wigner (1959). This is the general case, because any antiunitary operator equals $\mathbf{UK}$ where $\mathbf{U}$ is unitary and $\mathbf{K}$ antiunitary, with $\mathbf{K}^2=1$ (*conjugation*).

The two results, that basically we must operate with unitary representations in the state-vector space, and that there are no finite-dimensional unitary representations of the Lorentz group, taken together mean that the state-vector space must be an ∞-dimensional Hilbert space. This provides an affirmative answer to the query of van der Waerden[13] as to whether there is an infinity of eigenfunctions associated with every Hamiltonian.

We return now to representations in general, not limited to those in state-vector space. For all finite groups, and for many infinite groups of physical interest—specifically for all compact[14] groups or semi-

[13] B. L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (Verlag Julius Springer, Berlin, 1932; Edwards Brothers, Inc., Ann Arbor, Michigan, 1944), p. 5.

[14] A "closed" or "compact" group is defined as a continuous group where every infinite set of group elements contains at least one limit element ("point of accumulation") in the group. It is remarkable that from this rather general basis there follow a number of powerful results, i.e., there exists an invariant density function such that integration with it over the space of group parameters gives a finite result (the group "volume"), and this integral is invariant under a shift in the arrangement of the group

simple groups[15]—reducibility means decomposability. For the compact groups, *all* representations can be similarity transformed to a unitary form, i.e., such that every r_i in the group is represented by a unitary matrix $\mathbf{U}_i$ (with $\mathbf{U}_i^\dagger = \mathbf{U}_i^{-1}$). Reducibility and decomposability mean the same thing, i.e., upon reduction to the semi-blockdiagonal form all elements above the diagonal blocks also vanish. For infinite-dimensional unitary representations in general, there is no guarantee that they decompose into a discrete sum of reps, but instead, a "direct integral" over a continuous infinity of reps may be needed (Murray and von Neumann, 1936; Mackey 1952). For all compact groups, however, Wigner (1939) has proved on the basis of the Peter-Weyl theorem[16] that all reducible representations, finite or infinite, are also decomposable into a sum of finite-dimensional reps. For connected semisimple groups, one has the theorem (Harish-Chandra 1953, 1954) that all representations are "of type I," i.e., they decompose into a discrete sum of (not necessarily finite!) reps.

Groups which are more general then compact or semisimple ones occur in physics as is obvious from the elementary examples of the homogeneous Lorentz group and the rotation-translation group. These have among their indecomposable representations some which are reducible, and we have looked at some of them in the foregoing. Their general theory is complicated. If however, we limit ourselves to unitary representations a great deal can be said. (The antiunitary ones can be dealt with by an extension of the theory; see Wigner, 1959.) Only the infinite ones present any problem. Here Murray and von Neumann have shown that these infinite unitary representations are always composed out of a discrete sum of reps and "continuous" representations involving a "direct integral" over reps. For specific groups, even though they are not compact or semisimple, the situation may simplify further. Thus, for example, Wigner (1939) has shown that the Poincaré group has no "continuous" representations: *all unitary representations of the Poincaré group can be decomposed into a discrete sum of reps.*

Finally, there is the question of the arbitrary phase factor and the possibility of fixing it consistently by a convention, a question which is very important for physics since it underlies analyses of "physical equivalence" of similar particles, and "relative parity" of dissimilar particles, etc.

The unitary (or antiunitary) operators which represent the Lorentz group are defined uniquely only up to a phase factor; multiplication of a state vector by an arbitrary phase factor makes no physical difference. The operators form therefore a "representation up to a factor" or a "ray representation." One must consider whether, by a proper selection of elements, one can consistently reduce to a smaller degree this physically irrelevant arbitrariness in the phase of the representation matrices. Following upon earlier work of Silberstein (1924) and Weyl (1928) on the homogeneous Lorentz group, Wigner (1939) proved that by proper selection of elements all the unitary ray representations of the Poincaré group can be consistently replaced by two-valued representations, i.e., representations up to a $\pm$ sign. This irremovable two-valuedness marks the occurrence of spinors in quantum theory. A similar argument in classical theory, with its purely tensorial objects, leads to single-valued representations rather than representations up to a sign.

Ambiguities of phase are associated with physical state vectors in their behavior with respect to other invariance groups besides the Lorentz groups, and thus it is both interesting and useful that Bargmann (1954) has generalized Wigner's considerations to other continuous groups.

F. Lie Algebra of a Continuous Group. Generators of the Poincaré Group

A group is said to be a *connected continuous* group if it consists of a single one or higher-dimensional continuous infinity of elements. For many connected continuous groups the structure may be studied by considering only a finite number of "generators" obtained from the group elements differing infinitesimally from the identity. The generators are such that out of them the entire group may be constructed by iteration. Such continuous groups, whose elements are determined by a finite number of parameters, are called Lie groups, and the algebra which the set of generators satisfies is called the *Lie algebra* of the group. It too has a rich theory of matrix representations and, in many groups of interest in physics, there is a one-to-one correspondence with representations of the group. It is often much easier to study the representation theory for the Lie algebra, and infer therefrom the representation theory of the group, than to study the group directly. It should be remarked, however, that Wigner obtained his more general and complete account of the Lorentz groups by direct "global" methods and not through the infinitesimal method. The latter was used earlier by Majorana (1932).

Let there be given a set of elements in which multi-

---

elements by multiplication from the left or right. Further, every continuous function can be integrated over the group. The category of compact groups includes: the n-dimensional unitary group and all of its subgroups, including the real-orthogonal and the rotation groups. It does not include, for example, the two-dimensional unimodular group g_2 which is in 2-to-1 correspondence with the proper homogeneous Lorentz group (see Table IV).

[15] A "semisimple" group $\{G\}$ is one which has no Abelian invariant subgroup $\{F\}$ besides the identity. This means that there is no nontrivial $\{F\}$ which commutes with all the elements of $\{G\}$, i.e., such that $\{F\}$ contains all conjugates, GFG^{-1}.

[16] Included in the Peter-Weyl (1927) theorem are the results that the compact groups have a denumerable infinity of reps, and functions belonging to these provide a complete orthogonal basis for expansion of any function. A generalized version of the orthogonality theorem was given by Wigner (1939) for the unitary reps of an arbitrary group. Altogether, these results provide the most general basis for the expansion methods used in physics. [See Melvin (1956) for illustrations and applications.]

plication by scalars from a field is defined. If, in addition, the set is closed under two composition operations, "addition" and "multiplication," which are distributively related, an algebra is defined. The "sum" of two elements in a Lie algebra is to be taken in an obvious formal sense (so that the sum maps into the usual sum of matrices in the matrix representations of the algebra). The "product" of two elements, on the other hand, has to be defined as the commutator of the two elements. (This product is nonassociative!) It is only if this is done that we get the desired closure property, i.e., the "product" of any two elements is always a linear combination of a set of basis elements. The expressions of the commutators of the abstract generators as linear combinations of generators are called the "commutation relations of the Lie algebra." [17]

The Poincaré group $\{L_+{}^{\uparrow}\}$ is probably the most important example for physics. The group of rotations in n-dimensional space depends on $n(n-1)/2$ parameters. The translations provide another n parameters. Thus, in the four-dimensional Poincaré group $\{L_+{}^{\uparrow}\}$, there are 10 parameters. The corresponding 10 generators, the kinematical significance of the associated group elements, and the names of the dynamical representatives of the generators in the ∞-dimensional Hermitian representations in Hilbert space [see Sec. III and Table III and, for a formal derivation, Pauli (1956)] are

Generators of the Poincaré group	*Kinematical significance*	*Hermitian dynamical representative*
$P_a \sim (P_0,\mathbf{P})\,(a=0,1,2,3)$	translations	energy momentum $\mathbf{P}_a$
$J_{ij}=-J_{ji}=M_k$ $(i,j,k=1,2,3$ *et cycl*$)$	pure space rotations	angular momentum $\mathbf{M}_k$
$J_{k0}=-J_{0k}=\mathfrak{M}_k$	pure Lorentz transformations	centroidal moment $\mathfrak{M}_k$

[For physical reasons, we choose $\mathfrak{M}$ equal $-N$ of Pauli (1956).]

The commutation relations of the Lie algebra of the generators (written not with full mathematical symmetry, but in a form easier to interpret kinematically and dynamically—see Appendix) are

$$
\begin{aligned}
&[M_1,M_2]=iM_3 \quad \textit{et cycl} && \text{(a)}\\
&[M_1,\mathfrak{M}_1]=0\ \ [M_1,\mathfrak{M}_2]=[\mathfrak{M}_1,M_2]=i\mathfrak{M}_3 \quad \textit{et cycl} && \text{(b)}\\
&[M_1,P_1]=0\ \ [M_1,P_2]=[P_1,M_2]=iP_3 \quad \textit{et cycl} && \text{(c)}\\
&[\mathfrak{M}_1,\mathfrak{M}_2]=-iM_3 \quad \textit{et cycl} && \text{(d)}\\
&[P_j,P_k]=0\ \ [\mathfrak{M}_j,P_k]=iP_0\delta_{jk} \quad (j,k=1,2,3) && \text{(e)}\\
&[P_k,P_0]=[M_k,P_0]=0\ \ [P_0,\mathfrak{M}_k]=-iP_k. && \text{(f)}
\end{aligned}
\tag{11}
$$

Looking over the commutation relations one is led to consider the possibility of separating out from M_j and $\mathfrak{M}_j$ a part which is responsible for the noncommutation with P_k, and a residual part which commutes with P_k. To avoid complicated questions of uniqueness we do this not with the abstract generators but in a concrete representation basis in which the P's are simultaneously diagonal. (Since they commute this is possible.) In this basis $\partial^{P_a}\equiv\partial/\partial P_a$ is formally defined, and it is easily verified by direct differentiations that the separation is represented by the following splitting:

$$
\begin{aligned}
M_k &\equiv J_{mn}=i(P_n\partial^{P_m}-P_m\partial^{P_n})+S_k\equiv L_k+S_k\\
\mathfrak{M}_k &\equiv J_{k0}=i(P_0\partial^{P_k}+P_k\partial^{P_0})+\mathcal{S}_k\equiv \mathcal{L}_k+\mathcal{S}_k\\
&k,\,m,\,n=cycl\ (123),
\end{aligned}
$$

where we call L the *orbital part* and S the *spin part* of M, and similarly we call $\mathfrak{L}$ and $\mathfrak{S}\equiv(\mathcal{S}_1\mathcal{S}_2\mathcal{S}_3)$ the orbital and spin parts of $\boldsymbol{\mathfrak{M}}\equiv(\mathfrak{M}_1\mathfrak{M}_2\mathfrak{M}_3)$. Every orbital component commutes with every spin component.

In any representation of the J's, not only the one with the diagonal momentum basis which we have just used, the decomposition into orbital and spin parts takes the form of a sum of two "matrices." These may be of discrete or continuous type—or a direct product of the two types. For example, in the Hilbert space of one-particle quantum mechanics with the basis in which the space coordinates x_k are diagonal, J_{ab} has a mixed discrete-continuous representation. The discrete labeling of any matrix element is given by the discrete indices $\alpha\beta$, and the "continuous matrix" aspect is expressed by the linear operator form. In accord with $i\partial^{P_k}\to x_k$, $-i\partial^{P_0}\to x_0\equiv t$, $P_k\to -i\partial_k$ $P_0\to i\partial_0$ $(\partial_a\equiv\partial/\partial x_a)$, we may write

$$
\begin{aligned}
M_k{}^{\alpha\beta} &= (x_mP_n-x_nP_m)\delta^{\alpha\beta}+S_k{}^{\alpha\beta}\\
&= i(x_n\partial_m-x_m\partial_n)\delta^{\alpha\beta}+S_k{}^{\alpha\beta}\\
\mathfrak{M}_k{}^{\alpha\beta} &= (x_kP_0-P_kt)\delta^{\alpha\beta}+\mathcal{S}_k{}^{\alpha\beta}\\
&= i(x_k\partial_0+x_0\partial_k)\delta^{\alpha\beta}+\mathcal{S}_k{}^{\alpha\beta}.
\end{aligned}
\tag{12}
$$

The first or "orbital part" governs the infinitesimal change in each component of the spinor field due to the transformation. The second or "spin part" $S^{\alpha\beta}$ is the spinor representation matrix which governs the mixing of components by the given rotation.

Initially the operators are to be considered abstractly merely as generators of invariance operations in an abstract group-theoretic sense. The commutation relations are bilinear in the P and J components and can be interpreted as saying that, regarded as a passive object undergoing transformation, P_a behaves like a vector and J_{ab} behaves like a tensor in space-time: Under infinitesimal "rotations" and "displacements," such as are generated by J's and P's regarded actively, they transform in the appropriate simple linear way. More specifically, the first relations describe the fact that the

[17] One should clearly distinguish between these generator commutation relations and the additional ones that come up in the quantum theory of fields, where commutators of the field operators with each other and with the generators of the invariance groups of the theory also make their appearance (see Sec. III).

three-component generator of pure rotations transforms like a vector in the three-dimensional spacial subspace of space-time. The next two sets of relations describe the fact that $\mathfrak{M}$ and $\mathbf{P}$ also transform like a vector under purely spacial rotations. The fourth set of relations expresses another interesting fact, i.e., that the commutator of two infinitesimal Lorentz transformations $\mathfrak{M}$ and $\mathfrak{M}'$, is a pure infinitesimal rotation. In particular, the commutator of $\mathfrak{L}$ and $\mathfrak{L}'$ is a pure $\mathbf{L}$.[18] Finally the last two relations describe the invariance of space displacements under space displacement, the mixing of time and space displacements under Lorentz transformation, and the invariance of time displacement under space displacement and space rotation.

While the system of P, M, and $\mathfrak{M}$ operators has no finite dimensional unitary or Hermitian reps, it does have various infinite dimensional ones. As we have already indicated, these infinite dimensional Hermitian representatives $\mathbf{P}$, $\mathbf{M}$, and $\mathfrak{M}$ can be identified as the dynamical observables of total linear momentum, angular momentum, and centroidal moment, respectively, for the various possible cases of relativistically invariant systems (see Sec. III and Appendix). More detailed dynamical interpretations of the commutation relations become possible if one takes as the system a *field*. Then linear momentum, orbital angular momentum, and centroidal moment are all expressible in terms of integrals of a stress-energy-momentum density tensor, and it follows that the $\mathfrak{M}_k$ can be written

$$\mathfrak{M}_k=\mathfrak{M}_k'-\mathbf{P}_k t \quad (k=1, 2, 3),$$

where $\mathfrak{M}_k'$ may be interpreted as the components of the centroidal moment in the observer's frame. Interpretations of the commutation relations on the basis of this connection between the $\mathfrak{M}$ and $\mathbf{P}$ operators are discussed in the Appendix. The set of Eqs. (11f) for example, state, in accord with the fundamental law of motion, that $\mathbf{P}, \mathbf{M}$, and $\mathfrak{M}$ are all constants of motion.

We now define the 4-pseudovector $W_a \sim (W_0, \mathbf{W})$ by

$$W^a=\tfrac{1}{2}\epsilon^{bcda}J_{bc}P_d \quad (\text{indices}=0, 1, 2, 3),$$

where ϵ^{abcd} is the completely antisymmetric tensor of rank four ($\epsilon^{1230}=1$). The four operators $W_0=W^0$, $W_k=-W^k$ $(k=1, 2, 3)$, can be expressed in terms of the basic generators in space-vector notation as follows:

$$\mathbf{W}=\mathbf{M}P_0-\mathfrak{M}\times\mathbf{P}=\mathbf{S}P_0-\mathfrak{S}\times\mathbf{P}\sim(-W^1, -W^2, -W^3)$$

$$W_0=\mathbf{M}\cdot\mathbf{P}=\mathbf{S}\cdot\mathbf{P}.$$

As we see, because of the structure of the orbital parts, only the spin parts $\mathbf{S}$ and $\mathfrak{S}$ contribute to W_a. The space-vector part $\mathbf{W}$ may be thought of as an "inner angular momentum" multiplied by a factor with the dimensions of energy. Because of the role it plays in applications (Bouchiat and Michel, 1958) one may also call W_a the "polarization 4 vector." It is easily verified that the polarization 4 vector is Lorentz orthogonal to P_a (is therefore a spacelike vector), commutes with P_a, and has the same commutation relations with J's as P_a. Finally, its different components do not commute with each other, and we find

$$[W_1,W_2]=i(W_3P_0-W_0P_3)\ \ \textit{et cycl} \qquad [\mathbf{W},W_0]=-i\mathbf{W}\times\mathbf{P}.$$

With the help of these relations and the earlier relations, Eq. (11), we find that the operators,

$$P^2=-P_aP^a=P_0^2-\mathbf{P}^2$$

$$\begin{aligned}W^2=-W_aW^a&=-W_0^2+\mathbf{W}^2\\&=P_0^2\mathbf{S}^2+P_0(\mathbf{P}\times\mathfrak{S}\cdot\mathbf{S}+\mathbf{S}\cdot\mathbf{P}\times\mathfrak{S})\\&\qquad+(\mathbf{P}\times\mathfrak{S})^2-(\mathbf{P}\cdot\mathbf{S})^2,\end{aligned}$$

commute with all the others. Therefore, by Schur's lemma, in each rep each is represented by a matrix which is a multiple of the identity $\mathbf{I}$. These multipliers, or eigenvalues, can be used to characterize the rep. In the dynamically significant ∞-dimensional Hermitian reps, the eigenvalue of $\mathbf{P}^2$ is to be interpreted as the square of the rest-mass magnitude m associated with the system, and the eigenvalue of $\mathbf{W}^2$ is m^2 times the spin squared. This can be seen by considering $\mathbf{P}^2$ and $\mathbf{W}^2$ in the *rest system* which is defined by the property that in it the only nonzero component of $\mathbf{P}$ is $\mathbf{P}_0$ (with eigenvalue m). Thus, in all the Hilbert-space representations,

$$\mathbf{P}^2=m^2\mathbf{I} \quad \mathbf{W}^2=m^2\mathbf{S}^2=m^2s(s+1)\mathbf{I},$$

and we are led to the same characterization of the reps as Wigner (1939) has obtained by global methods. (See Table V.) Explicit constructions of the Lorentz group rep matrices in a basis in which $\mathbf{M}^2$, $\mathfrak{M}^2$, and M_3 are diagonal are given by Pauli (1956). There one finds also a concise mathematical discussion of the reps of the inhomogeneous group (Poincaré group).

III. INVARIANCE AND FIELD EQUATIONS

The earlier discussion of space-time transformations was general and related to invariance of the space-time interval and of transition probabilities. We now consider the concept of invariance in the more comprehensive context of "quantum dynamics" which in its most complete form, accounting for the appearance and disappearance of particles, is *quantum field theory*.

We have discussed, with many ramifications, the idea of an "invariant function." Now in order to consider the analogous but more elaborate idea of an "invariant set of equations" or an "invariant theory," it is necessary to make a generalization. In Sec. II.A we considered functions which were scalar and invariant; in Sec. II.D we generalized our considerations to functions which were scalar but not invariant and which,

[18] It is noteworthy that this relation can be extended to finite transformations Λ, and Λ'. Here the "commutator" must be taken in the finite operator sense—i.e., as that operator which applied to $\Lambda'\Lambda$ takes it into $\Lambda\Lambda'$. [This operator, $(\Lambda'\Lambda)^{-1}\Lambda\Lambda'$, becomes the identity plus the usual commutator of infinitesimal generators in the limit of small transformations.] It can be proved that this finite-sense commutator of two finite Lorentz transformations is a pure finite rotation. [See, for example, Silberstein (1924), p. 167.]

TABLE V. Possible kinds of elementary particles and their behavior under space and time inversions—based on Wigner's analysis of the unitary reps of the inhomogeneous Lorentz group and on Michel-Wightman (unpublished). In the spinless case, in Type I $\phi(p)$ is still a one-component function of the momentum $\mathbf{p}$, but already Types II, III, and IV have to be represented by two-component functions. A similar doubling of dimensions for Types II, III, and IV with respect to Type I occurs with all spinors of higher rank. The operators $V(I_x)$ represent the inversion operators. The symbol K used in defining them represents the antiunitary operator conjugation. The operators $V(I_x)$ act on the one- or many-component functions $\phi(\mathbf{p})$, and give the result of the inversion in accord with

$$[U(0,I_x)\phi](\mathbf{p})=V(I_x)\phi(I_x\mathbf{p}),\quad I_s\mathbf{p}=-\mathbf{p}\quad I_t\mathbf{p}=-\mathbf{p}\quad I_{st}\mathbf{p}=\mathbf{p}.$$

To get these operators for the higher rank spinors we always start with the same basic two-dimensional matrices, and insert in place of the 1's the spinor or tensor matrices of Type I (direct product). The types have been assigned so as to make the square of the operators of a given type for integer spin the negatives of those for half-odd integer spin. Arbitrary phase factors have been left out of the definitions. Their values in no way affect the classification of particles into types.

Symbol	Range of mass m	Range of spin s	Intrinsic parity		$[V(I_t)]^2$	$[V(I_{st})]^2$	$V(I_s)$	$V(I_t)$	$V(I_{st})$	Occurrence in nature
				Massive						
${}_mU_0{}^{\epsilon}$	$m>0$	$s=0$	$\epsilon=+1(-1)$ scalar (pseudoscalar)	Type I	1	1	1	K	K	Mesons
				Type II	$\mathbf{1}$	$-\mathbf{1}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}K$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}K$	?
				Type III	$-\mathbf{1}$	$\mathbf{1}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}K$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}K$	?
				Type IV	$-\mathbf{1}$	$-\mathbf{1}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}K$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}K$	?
${}_mU_{\frac{1}{2}}{}^{\epsilon}$	$m>0$	$s=\frac{1}{2}$		Type I	-1	-1	γ^0	$\gamma^0\gamma^5K$	γ^5K	Leptons Baryons
				Type II	$-\mathbf{1}$	$\mathbf{1}$	$(\,)\otimes\gamma^0$	$(\,)\otimes\gamma^0\gamma^5K$	$(\,)\otimes\gamma^5K$	?
				Type III	$\mathbf{1}$	$-\mathbf{1}$	$(\,)\otimes\gamma^0$	$(\,)\otimes\gamma^0\gamma^5K$	$(\,)\otimes\gamma^5K$	?
				Type IV	$\mathbf{1}$	$\mathbf{1}$	$(\,)\otimes\gamma^0$	$(\,)\otimes\gamma^0\gamma^5K$	$(\,)\otimes\gamma^5K$	?
$\vdots$										$\vdots$
${}_mU_s{}^{\epsilon}$	$m>0$	$2s=\lvert\text{integer}\rvert$								?
$\vdots$										
				Massless						
${}_0U_0{}^{\epsilon}$	$m=0$	$s=0$	$\epsilon=\pm1$							?
${}_0U_{\frac{1}{2}}$	$m=0$	$s=\frac{1}{2}$		Type I	-1	-1	γ^0	$\gamma^0\gamma^5K$	γ^5K	Neutrino
				$\vdots$						?
${}_0U_1$	$m=0$	$s=1$		Type I	1	1	$-G$	GK	$-K$	Photon
$\vdots$				$\vdots$						?
${}_0U^{\epsilon}{}_{\Xi\,0}U^{\epsilon}{}_{\Xi'}$	$m=0$	all $s>0$								?

together with a set of covariant partners, served as basis functions for reps of groups. These functions, while they formed components of "rep spinors" were still "coordinate-scalars," i.e., under the coordinate redescription $\boldsymbol{\xi}=\mathbf{R}\mathbf{x}$ we had for each component $\varphi(x)$,

$$R\varphi(\xi)\equiv\varphi'(\xi)=\varphi(x).$$

We now consider functions which are not coordinate scalars but *vectors* or *tensors*, or more generally *spinors*, with respect to a change of frame. For these we have

$$R\psi^{\alpha}(\xi)=\psi^{\alpha'}(\xi)=R^{\alpha\beta}\psi^{\beta}(x)$$

or

$$\psi^{\alpha'}(\xi)=R^{\alpha\beta}\psi^{\beta}(R^{-1}\xi). \tag{13}$$

An important limiting case of Eq. (13) occurs when the transformation $R^{\alpha\beta}$ is a member of a family which is continuously connected with the identity transformation. Then, setting $-\delta\boldsymbol{\xi}\equiv\mathbf{R}^{-1}\boldsymbol{\xi}-\boldsymbol{\xi}$, we have in the neighborhood of the identity,

$$R^{\alpha\beta}=\delta^{\alpha\beta}+\delta\epsilon S^{\alpha\beta},$$

where $\delta\epsilon$ is an infinitesimal parameter corresponding to $\delta\boldsymbol{\xi}$ in order of smallness. We can then write[19]

$$R^{\alpha\beta}\psi^{\beta}(\mathbf{R}^{-1}\xi)=(\delta^{\alpha\beta}+\delta\epsilon S^{\alpha\beta})\psi^{\beta}(\xi-\delta\xi)$$
$$=(\delta^{\alpha\beta}+\delta\epsilon S^{\alpha\beta})[(\psi^{\beta}(\xi)-\delta\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\psi^{\beta})].$$

For the total infinitesimal change in any component of ψ by any such transformation infinitesimally near the identity, we have then

$$\delta\psi^{\alpha}\equiv R^{\alpha\beta}\psi^{\beta}(\xi-\delta\xi)-\psi^{\alpha}(\xi)$$
$$=-\delta\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\psi^{\alpha}(\xi)+\delta\epsilon S^{\alpha\beta}\psi^{\beta}(\xi). \tag{13'}$$

The first term on the far right-hand side gives the change in the component ψ^{α} due to the change in coordinates alone, the last term gives the change in ψ^{α} due to the mixing in of other components. Well-known special cases of these infinitesimal transformations are (1) translation by $\delta\mathbf{a}$ in three-dimensional space: $\delta\boldsymbol{\xi}=\delta\mathbf{a}$, (2) rotation by $\delta\boldsymbol{\varphi}$ in three-dimensional space: $\delta\boldsymbol{\xi}=\delta\boldsymbol{\varphi}\times\boldsymbol{\xi}$. In case (2) we refer to the first term, involving $\delta\boldsymbol{\varphi}\times\boldsymbol{\xi}\cdot\boldsymbol{\nabla}=\delta\boldsymbol{\varphi}\cdot\boldsymbol{\xi}\times\boldsymbol{\nabla}$, in Eq. (13′) as the "orbital part" of the infinitesimal rotation. We call the second part, involving $S^{\alpha\beta}\psi^{\beta}$, the "spin part" of the infinitesimal rotation.

[19] The symbol ∇ represents a generalized gradient operator.

An expression quite similar to that for case (2) occurs when we consider rotations in a four-dimensional space. In fact, the resemblance of the operators acting on ψ^β in Eq. (13′) to the infinitesimal generators of space-time rotations [Eq. (12)] is obvious upon inspection.

A set of equations governing physical systems is said to be invariant under a redescription R if, when we substitute for all independent variables and functions of the variables their redescriptions, *numerical constants remaining unchanged*, we find that the set takes exactly the same form as it had originally. Invariance, defined in this way, is physically important because the new variables form as "equally good" a set as the old from a physical point of view. This has the following consequence: The redescribed solution of the original equations is just as good a solution of the original equations as the old solution. Thus, we obtain *a new solution of the original equations with the same numerical constants.* This solution is generally different from the original, though in special cases it may be proportional to it.

These considerations, to the effect that every invariance principle yields new solutions from old, hold equally for any classical system of equations. But in quantum dynamics, where the field variables are operators in a Hilbert space, something additional comes out of the existence of this invariance principle. This is an explicit connection, independent of position, which must exist between the field variable at one "physical point" and at another "physical point" which may be finitely separated from it, *even in cases where the transformation connecting the two points does not belong to a set of transformations continuously connected with the identity.*

Specifically, suppose that the basic abstract Hilbert-space operators, out of which the theory is built, are expressed as functions of coordinates $\mathbf{A}^\alpha(\mathbf{x})$. If all the governing equations are invariant under some coordinate transformation or redescription $\xi\equiv\mathbf{R}\mathbf{x}$, then the operators must undergo some associated linear redescription, e.g.,

$$\mathbf{A}^{\alpha\prime}(\xi)=R^{\alpha\beta}\mathbf{A}^\beta(\mathbf{x})=R^{\alpha\beta}\mathbf{A}^\beta(\mathbf{R}^{-1}\xi).$$

But here there appears the basic *physical* principle that the new coordinates—and therefore the new solutions—are just as admissible physically as the old ones. The new solutions provide an equally admissible representation of the basic abstract operators. This is the invariance or "relativity" principle governing the theory, and it has a very important consequence in quantum dynamics.

Let the basic set of operators, in terms of which all others may be constructed, form a canonical commuting or anticommuting set [Eqs. (1) and (2), Sec. I.C]. Every Hilbert-space representation of this set decomposes into a sum of continuous and discrete representations. [Continuous representations were introduced by Murray and v. Neumann (1936)—see Sec. II.E. Their formal definition is discussed concisely by Wightman and Schweber (1955). Any representation which is not continuous is discrete.] A very important *uniqueness theorem* may be proved concerning the discrete representations of canonical systems: *Every discrete representation by bounded operators is made up of ("decomposes" into) reps. Of all these reps there is a unique one for which a vacuum state and a number operator exist.*[20] In other words, in any rep the new set of operators obtained by the redescription, as described previously, is equivalent to the old. In particular, a given pair of corresponding old and new operators $A(\xi)$ and $A'(\xi)$ may be regarded as two representatives in Hilbert space of the same abstract operator, one being a unitary transform of the other in accord with

$$\mathbf{A}'(\xi)=\mathbf{U}\mathbf{A}(\xi)\mathbf{U}^{-1}\ [\mathbf{U}\equiv\mathbf{U}(R)].$$

The transformed quantities—actually functions of the transformed coordinates—are here related to the original quantities evaluated at the *same coordinate numbers. Thus the operator* $\mathbf{U}$ *gives a relation between the field quantities at two different physical "points."* Describing the situation at the two points in the same (for instance, new coordinate) langugage, we have, using the redescription equations, Eq. (13),

$$\mathbf{U}\mathbf{A}^\alpha(\xi)\mathbf{U}^{-1}=R^{\alpha\beta}\mathbf{A}^\beta(\mathbf{R}^{-1}\xi). \tag{14}$$

Equation (14) may be called the "finite interval field equations."

If the redescription $\mathbf{R}$ belongs to a set continuously connected with the identity then, correspondingly, $R^{\alpha\beta}$ is continuously connected with the identity and, in the limit of transformations infinitesimally near the identity, the quantities

$$\delta\mathbf{A}^\alpha=R^{\alpha\beta}\mathbf{A}^\beta(R^{-1}\xi)-\mathbf{A}^{\alpha\prime}(\xi)$$

are infinitesimally small. Under these conditions $\mathbf{U}$ is a member of a set of unitary operators continuously connected with the identity. By "Stone's theorem" in the theory of Hilbert space, a Hermitian operator $\mathbf{G}$, the *generator* of $\mathbf{U}$, is then uniquely defined by the relation

$$\mathbf{U}=\exp(-i\epsilon\mathbf{G}),$$

where ϵ is the parameter of the set. (This is analogous to an elementary fact about ordinary numbers: a unit-magnitude complex number, continuously connected with 1, uniquely determines a real number, its phase.)

[20] The uniqueness theorem for a *finite* set of operators obeying canonical anticommutation relations, (i.e., describing Fermi-Dirac particles) was proved by Jordan and Wigner (1928). The corresponding proof for a finite set obeying commutation relations (i.e., describing Bose-Einstein particles) is included in a general theorem of v. Neumann (1931) on representations of canonical operators. As already discussed in Sec. I.C, Wightman and Schweber (1955) showed that the uniqueness theorem for both commuting and anticommuting cases can be extended to an *infinite* set of operators, such as occurs in any continuous field theory, only if it is assumed that there exists for the given field a no-particle ("vacuum") state and an operator of the total number of particles.

We then have the *differential field equations* which relate the values of state of field quantities at two neighboring points

$$\delta\mathbf{A}^{\alpha}/\delta\epsilon \equiv -(\delta\xi/\delta\epsilon)\cdot\nabla\mathbf{A}^{\alpha}+S^{\alpha\beta}\mathbf{A}^{\beta} = i(\mathbf{G}\mathbf{A}^{\alpha}-\mathbf{A}^{\alpha}\mathbf{G})\equiv i[\mathbf{G},\mathbf{A}^{\alpha}]. \quad (15)$$

Every system of natural processes is invariant under the redescription connected with the resetting of the zero of time ("time displacement"). This possibility of displacement of "epoch" t is continuous. The corresponding differential field equation which relates the values of a quantity at two neighboring moments involves the Hermitian representative of the generator P_0, and we call this $\mathbf{H}$, the Hamiltonian. The differential field equations then take the form of the equations of motion,

$$\delta\mathbf{A}^{\alpha}/\delta t = i[\mathbf{H},\mathbf{A}^{\alpha}], \quad (16)$$

which as we see are consequences of the invariance under time displacement. The fact that the operators $\mathbf{Q}$ corresponding to the other invariance operations of the Poincaré group either commute with $\mathbf{H}$ or in any case give $d\mathbf{Q}/dt=0$, leads to the theory of the continuous-transformation constants of the motion (Sec. I.C and Appendix).

If the redescription $\mathbf{R}$ does not belong to a set continuously connected with the identity, as for instance is the case with the space reflection operation, we still have a "finite interval field equation." For example, in the space reflection case the corresponding Hilbert-space operator is called the parity $\mathbf{P}$, and it satisfies

$$\mathbf{P}\mathbf{A}^{\alpha}(\mathbf{r},t)\mathbf{P}^{-1}=P^{\alpha\beta}\mathbf{A}^{\beta}(-\mathbf{r},t). \quad (17)$$

Interpreting $\mathbf{P}$ as a "Schrödinger type" operator on kets we have in this case, just as with time reversal (Sec. II.D), that reversing space coordinates twice must give back the same state—hence the same ket up to a factor. Thus, we can write

$$\mathbf{P}^2=\lambda_P{}^2\mathbf{I} \quad \mathbf{P}=\lambda_P\mathbf{P}' \quad (\mathbf{P}'^2=\mathbf{I}).$$

However, P is not antiunitary, and the "scalar square theorem" does not apply. Therefore $\lambda_P{}^2$ is not compellingly restricted to any particular values. (If normalization is required, $\lambda_P{}^2$, and therefore λ_P, must be of absolute magnitude 1.) It can be shown, however, that λ_P can be chosen consistently to be $+1$. Alternatively, by interpreting $\mathbf{P}$ in "Heisenberg-type" manner one arrives at the same conclusion, establishing on the way that $\mathbf{P}\mathbf{P}^{\dagger}=\mathbf{I}$. Thus, P can be chosen as a Hermitian operator with square unity (cf. Lüders, 1955). Further discussion of possible restrictions on the freedom of choice of λ_P under conditions of interaction will be found in Sec. IV. We have allowed for possible phase factors different from $+1$ in Table III.

Classically, connections of type analogous to Eq. (17) do not exist when the two points cannot be joined by a "continuous motion," i.e., by the unfolding of a contact transformation, and this difference between the classical and quantum-mechanical situations is at the basis of the oft-quoted statement that "parity is not a classical concept." In the form quoted it is misleading, because oddness and evenness under reflection of eigenfunctions of a classical continuum problem is certainly a good classical concept. But for *classical particle mechanics* in contrast to continuum mechanics, there is no corresponding general "constant of motion" as there is in *quantum particle mechanics*.

Because it does not preserve the canonical relations, the case of time reversal is peculiar among the discrete symmetry operations. (They are all peculiar, but in different ways!) The analysis is indicated briefly in Table III. We describe the situation here from a somewhat different point of view. The redescription of formal time reversal, $G: f(t)\to f(-t)$, and that of infinitesimal time displacement ∂_t do not commute but anticommute as is evident from the equations

$$\partial_t[Gf(t)]=\partial_t f(-t)=-f'(-t)$$
$$G\partial_t f(t)=Gf'(t)=f'(-t).$$

The redescription which has the Hamiltonian as a representative is, however, not ∂_t but $i\partial_t$. We can have a time-reversing redescription which commutes with $i\partial_t$ if we combine formal time-reversal G with a complex conjugation operation K on all c numbers following:

$$T\equiv GK \quad Ti\partial_t - i\partial_t T=0.$$

Correspondingly, any Hilbert-space representative $\mathbf{T}\equiv\mathbf{G}\mathbf{K}$ of T commutes with $\mathbf{H}$. This also means that a time-reversed operator $\mathbf{Q}_T$, defined by $\mathbf{T}\mathbf{Q}\mathbf{T}^{-1}$, obeys the standard equations of motion, since

$$\mathbf{T}(d\mathbf{Q}/dt)\mathbf{T}^{-1}=\mathbf{T}(i[\mathbf{H},\mathbf{Q}])\mathbf{T}^{-1}$$

becomes

$$-d\mathbf{Q}_T/dt=-i[\mathbf{H},\mathbf{Q}\mathbf{T}^{T}].$$

The minus sign on the left stems from the G part of T and that on the right from the K part! The fact that the canonical relations are not preserved—$\mathbf{T}$ is not unitary—does not affect the physical validity of the time-reversed solution. The physical results of quantum theory are unchanged if in all fundamental equations i is replaced by $-i$.

A quantum field theory of elementary particles consists of: (a) *Canonical commutation or anticommutation relations;* these are of the type of Eqs. (1) or (2). (b) *Field equations;* these may be interpreted as of the type of Eqs. (15) and (16), with an explicit form for the Hamiltonian $\mathbf{H}$ and other generators $\mathbf{G}$. The field equations and commutation relations may (at least in theory) be considered for cases where particles of only one kind are present (free fields) or for cases where two or more kinds of interacting particles are present (interacting fields). Once the theory has been formulated, invariance principles are of great value for drawing practical inferences, especially with regard to selection rules (Sec. II.B). The way in which such invariance principles are developed in quantum field theory may be described as an intellectual exploration in two stages. In the first stage we *define* invariance operations for free fields; in the second stage we *test* the invariance operations for interacting fields. As we see already from our derivation of the equations of motion on the basis of invariance under the Poincaré group, part of this exploration may be carried out in a more deductive spirit: We can first find all possible forms of free field equations of motion by the requirement that they shall conform to a rep of an assumed invariance group. Under the same requirement for coupled field equations, the possible invariant forms of coupling can be found.

The apparent over-all isotropy and homogeneity of space-time apart from gravitation gives physicists strong reason to start out with field equations which are invariant under the Poincaré group, and a rather exhaustive list of such possible systems for free fields has been obtained (Bargmann and Wigner, 1948). The usefulness of assuming Poincaré group invariance for free fields and for the couplings between them is tested in practice by comparing inferences drawn from this invariance with empirical observations. So far the Poincaré group has survived every observational test in which gravitational phenomena are unimportant (cf all experiments which verify the theory of special relativity, including microphysical cases, e.g., dilation of the lifetime of mesons at high velocities, energy-momentum relationship, etc.). With sufficiently detailed experimental study circumscribing the invariances, even the specific form of the invariant coupling can be determined (see Sec. V).

It appears further, that by suitable definitions a number of other invariance operations besides those of the Poincaré group can be set up for the free field system. In almost every useful case thus far, these operations can be regarded as "induced" by certain relatively simple redescriptions of the space-time coordinates ("space-reflection," "time-reversal," $\cdots$), or of the physical operators ("charge conjugation") in terms of which the system of relations was formulated. This "induction" provides a quasi-intuitive motivation for the definition of the invariance operation, but one should not be confused about its heuristic nature. The essential point is that the first stage is merely formal—we define a redescription of the dependent variables in the free field system such that, together with the redescription of the independent variables, we achieve invariance. Only in the second stage, upon "switching on" the interaction, does the real physical test of these further invariances come.

Examples which are of fundamental interest in elementary particle physics are the following systems of equations for free fields in space-time[21]:

Spin 0 case: $(\Box^2-m^2)\Phi(x)=0$

$$\begin{gathered}\text{Hermitian conjugate equations}\\ [\Phi^*(x),\Phi(y)]=-i\Delta(x-y)\\ [\Phi(x),\Phi(y)]=0.\end{gathered}\tag{18}$$

Spin $\frac{1}{2}$ case: $(\gamma_n\partial^n+m)\mathbf{\Psi}(x)=0$

$$\begin{gathered}\text{Hermitian conjugate equations}\\ \{\mathbf{\Psi}_a(x),\bar{\mathbf{\Psi}}_b(y)\}=i(\gamma_n\partial^n-m)_{ab}\Delta(x-y),\\ \bar{\mathbf{\Psi}}(x)=-i\mathbf{\Psi}^\dagger\gamma_0\\ \{\mathbf{\Psi}_a(x),\mathbf{\Psi}_b(y)\}=0\quad(\{A,B\}\equiv AB+BA),\end{gathered}\tag{19}$$

where the γ's are 4×4 Dirac matrices satisfying

$$\gamma_k\gamma_n+\gamma_n\gamma_k=2g_{kn}.$$

In particular, the γ's may be chosen real and unitary (representation of Majorana, 1937), and therefore with γ_1, γ_2, γ_3 symmetric and γ_0 antisymmetric,

$$\gamma_k^\dagger=\gamma_k^T=\gamma_k^{-1}=g_{kn}\gamma_n,$$

e.g.,

$$\gamma_0=\begin{pmatrix}0&-\sigma_1\\ \sigma_1&0\end{pmatrix}\quad \gamma_1=\begin{pmatrix}\mathbf{I}&0\\ 0&-\mathbf{I}\end{pmatrix}$$

$$\gamma_2=\begin{pmatrix}0&-i\sigma_2\\ i\sigma_2&0\end{pmatrix}\quad \gamma_3=\begin{pmatrix}0&-\mathbf{I}\\ -\mathbf{I}&0\end{pmatrix}.$$

Spin 1 case: $(\Box^2-m^2)\mathbf{\Phi}_a(x)=0$

$$\begin{gathered}\text{Hermitian conjugate equations}\\ \partial^n\mathbf{\Phi}_n=0\\ [\mathbf{\Phi}_a(x),\mathbf{\Phi}_b^*(y)]=i[g_{ab}-(1/m^2)\partial_a\partial_b]\Delta(x-y),\\ [\mathbf{\Phi}_a(x),\mathbf{\Phi}_b(y)]=0.\end{gathered}\tag{20}$$

System (18) is invariant under all transformations of the Poincaré group provided Φ is a function belonging to the ${}_mU_0$ rep, i.e., when represented in momentum space Φ transforms under $U(a,\Lambda)$ according to

$$[\mathbf{U}(a,\Lambda)\Phi](p)=e^{ip\cdot a}\Phi(\Lambda^{-1}p),\tag{21}$$

which is the spin 0 case of Eq. (13) rewritten for the Poincaré group and in momentum space (see Table V). Similarly, system (19) is invariant provided $\mathbf{\Psi}$ is a column matrix of four functions belonging to the ${}_mU_{\frac{1}{2}}$ rep, i.e., transforming according to the spin $\frac{1}{2}$ case of Eq. (13). In the same way, system (20) is invariant provided the $\mathbf{\Phi}_a$ are four functions belonging to the ${}_mU_1$ rep, i.e., transforming according to the spin 1 case of Eq. (13).

When these systems are interpreted only as free field systems, each is also invariant under the operations of space reflection P, of charge conjugation C, and of time reversal T, respectively, as defined in Table III. The important question for physics, as emphasized in

[21] With the Latin-index metric, we have

$$\partial_a\equiv\partial/\partial x^a\quad \partial^a\equiv g^{ab}\partial_b\quad \Box^2\equiv\partial^a\partial_a.$$

The singular function Δ is defined by

$$\Delta(x)\equiv-\frac{1}{(2\pi)^3}\int e^{ik\cdot x}\frac{\sin\omega x^0}{\omega}d^3k\quad \omega\equiv+(k^2+m^2)^{\frac{1}{2}}$$

and has the symmetry properties

$$\Delta(\mathbf{r},t)=\Delta(-\mathbf{r},t)=-\Delta(\mathbf{r},-t)=-\Delta(-\mathbf{r},-t).$$

the foregoing, is whether these invariances survive the actual interactions observed in nature.[22]

IV. *CPT* EXTENSIONS OF THE POINCARÉ GROUP. BARYON CONSERVATION AND PHYSICAL INEQUIVALENCE

The question of the persistence of the extended invariances is best examined in the "interaction picture" in which the field operators continue to obey the free field equations and commutation relations. However, with interaction, we have to consider also the equation of motion of the state vector $|\rangle$ whose evolution is governed by the Schrödinger equation,

$$i\partial_t|\rangle = H(t)|\rangle,$$

where $H(t)$ is the interaction Hamiltonian, assumed to be Hermitian. The invariance behavior of this interaction Hamiltonian is at question.

Examination of this question in a broad context has recently brought about a considerable change in our conceptions of invariance under the three operations C, P, and T. As is well known now, Lüders (1954, 1957) proved an important invariance theorem, to which the ideas of Schwinger (1951, 1953), Zumino (1957), and Pauli (1955) have contributed, both before and after Lüders formulation. The "*CPT* theorem" makes it very likely that the product of the three operations C, P, and T, taken together as one composite operation, constitutes a universally valid invariance operation. That is to say, presenting the matter somewhat picturesquely: within the frame of proper-local field theories,[23] *the processes seen directly in matter look exactly like those in a film taken of processes in antimatter—taken through a mirror—and run off backwards.*

We naturally ask: What is the situation with respect to extended symmetry operations in the actual interacting fields which occur in nature? Because these are much more restricted in form than those contemplated in the general theorem, we might expect more than just total *CPT* invariance—perhaps invariance under partial groupings of the C, P, and T operations separately.

As the result of intensive analysis initiated by Lee and Yang (1956) in connection with the puzzle of discrepant modes of decay of the kayons, the situation at the present time may be described as follows: Until recently, on the basis of experiments upon strong interactions and electromagnetic interactions, it appeared that the true extended space-time symmetry group for all fundamental processes in nature was a direct product of the Poincaré group by the group of eight elements consisting of the identity, C, P, and T and their products CP, CT, PT, and CPT. As a consequence of experiments with weak interactions (Wu, 1957 and many others), we are now sure that this extended space-time group is not universally as large as this. It may consist of only the direct product of the Poincaré group by the two-element group (identity, CPT). On the basis of present experiments, however, *it is rather likely that the correct extended space-time group, with which all fundamental interaction Hamiltonians commute, is the intermediate one consisting of the direct product of the Poincaré group by four elements: Identity, T, CP, and CPT.* With the intensive experimentation now going on we may expect to know more of the true situation before the end of another year. (See Telegdi, 1959; Clark, Robson, and Nathans, 1959.)

It is evident that if the four-element hypothesis about the extended space-time group should be confirmed, it will be desirable to have a single name for the composite operation of space inversion and charge conjugation. The name "combined inversion" has been used. One might consider as an alternative name for this operation, as well as for the associated quantum number, the term *coparity*, and the symbol Π. Coparity invariance may be described as a general "pseudoscalarity" of all matter: *Under inversion, all particle properties (charge, baryon number, etc.) go into antiparticle properties.*

The coparity can be sharp only for a system of zero charge. This is because, like the charge conjugation, it only anticommutes with the charge operator though it does (under the 4-element hypothesis) always commute with the Hamiltonian. For those cases of zero charge, where also coparity is sharp, we can obtain useful selection rules. Some examples are $\pi^0\pi^0$, $\pi^+\pi^-$, and definite orbital states of systems of $N\bar{N}$ and pions of total charge zero.

In other cases of zero charge the coparity is not sharp, but the state of the system is a superposition of eigenstates of coparity. These states may differ in their physical properties. An interesting example is the case of a beam which when freshly prepared consists of pure K^0's (strangeness $+1$). After a time the residual

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[22] For emphasis, we quote a prophetic remark by Dirac (1949). Writing of space reflections and of time reversals, he remarked:

"I do not believe there is any need for physical laws to be invariant under these reflections, although all the exact laws of nature so far known do have this invariance. The restricted principle of relativity arose from the requirement that the laws of nature should be independent of the position and velocity of the observer, and any change the observer may make in his position and velocity, taking his coordinate system with him, will lead to a transformation (1) of a kind that can be built up from infinitesimal transformations and cannot involve a reflection. Thus it appears that restricted relativity will be satisfied by the requirement that physical laws shall be invariant under infinitesimal transformations of the coordinate system of the type (1)."

Even though this skepticism of the scope of restricted relativity has been brilliantly vindicated, yet the situation is subtle, as witness the *CPT* theorem.

[23] By a "proper-local field theory" is meant one in which *the fields are represented by covariant boson or fermion operators,* and the *interactions between fields are represented by proper-invariant local operators.* More explicitly, the following two groups of properties are satisfied: (i) The basic fields on which the theory is built transform in space-timelike tensors or spinors and, in their role as operators which augment or deplete numbers of quanta present, they satisfy the usual commutation and anticommutation relations. (ii) The interactions between the fields are described by terms in a properly symmetrized Lagrangian (or Hamiltonian) which are invariant under the proper Lorentz group and are "local," i.e., they consist of a linear combination of products of the fields and finite-order derivatives of the fields.

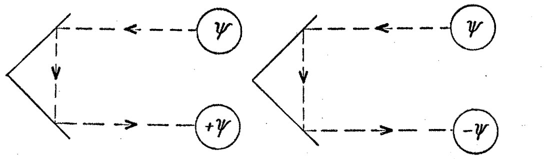

FIG. 4. Picturization of Yang and Tiomno's attempt to distinguish two kinds of spinors by their behavior under double reflection.

beam contains $\bar{K}^0$'s (strangeness -1). In effect the state of the fresh beam may be regarded as consisting of a superposition of coparity eigenstates

$$|K^0\rangle = (1/\sqrt{2})(|K_1\rangle + |K_2\rangle),$$

where the even and odd coparity eigenstates, $|K_1\rangle$ and $|K_2\rangle$, are defined by the equations

$$|K_1\rangle = (1/\sqrt{2})(|K^0\rangle + \mathbf{\Pi}|K^0\rangle)$$
$$|K_2\rangle = (1/\sqrt{2})(|K^0\rangle - \mathbf{\Pi}|K^0\rangle) \quad (\mathbf{\Pi}|K^0\rangle = \pm|\bar{K}^0\rangle).$$

The component K_1 can decay into two pions, but K_2 cannot (because, as pions obey Bose statistics, a neutral two-pion system in any definite orbital state can be in only an even eigenstate of $\mathbf{\Pi}$). Consequently K_2, which decays into three particles, does so a thousand times more slowly than K_1 and soon dominates. Thus, while the fresh beam had sharp strangeness and unsharp coparity, the residual beam has unsharp strangeness and sharp coparity:

fresh beam, K_0's: sharp $\mathbf{S}$, unsharp $\mathbf{\Pi}$;

residual beam, K_2's: unsharp $\mathbf{S}$, sharp $\mathbf{\Pi}$.

The emergence of opposite strangeness in the beam shows up strikingly in that the aged beam can make hyperons upon collision with nucleons whereas the original fresh beam could not (Gell-Mann and Pais, 1955; Pais-Piccioni effect, 1955).

We turn now to the question of the origin of the baryon conservation principle. Because this has so far not been elucidated, it is instructive to examine the history with some care. As stated before, with every conservation principle there is associated an invariance principle. The question is to find it and give it expression in a geometrically or physically interesting form, i.e., one which is not just a formal restatement of the conservation principle itself. It was suggested by Yang and Tiomno (1950) that the invariance principle associated with baryon conservation is the principle of space inversion invariance, and in the attempt to make it work they assigned definite and different type parities to the spinor state functions of leptons and nucleons. We have tried to make their assignments picturesque by "absolutizing" them in Fig. 4. It is of course a swindle to suggest, as we have, that it is possible to *observe* the effect of two successive mirrorings in restoring the field operator of a lepton to itself and of a baryon to minus itself, an observation which would lead us to assign the former to "spinor parity kind" $+1$ and the latter to "spinor parity kind" -1 (Fig. 4 and Table VI).[24] But one seemed to infer selection-rule consequences of such gedanken-behavior, which in several cases amounted to baryon conservation. In the old way of thinking, which required parity conservation, this followed because any field operator of the -1 spinor kind necessarily goes into $i \equiv (-1)^{\frac{1}{2}}$ times itself upon inversion and the corresponding charge-conjugate field goes into minus i times itself upon inversion. Thus for baryonlike spinors, fields and conjugate fields have opposite inversion behavior whereas, for leptonlike spinors, fields and conjugate fields have the same behavior under inversion. Thus, for example, the interaction leading a neutron to β^+ decay into an antiproton would necessarily vanish since it would not be inversion invariant. Such an argument to develop the baryon conservation principle, which resembles that originally given by Yang and Tiomno, is no longer strong enough to stand up under weak interactions in view of the possibility of parity violation in weak interactions. But it, or any coparity reedition of it, presents difficulties in principle anyway for two reasons, one special and one general.

TABLE VI. Yang and Tiomno's attempt to distinguish two kinds and four type of spinors, with respect to behavior under space-inversion. In the first kind, particle fields and antiparticle fields are of opposite type. In the second kind, particle fields and antiparticle fields are of the same type.

Kind I (Double reflection ~ -1)			Kind II (Double reflection $\sim +1$)		
Types:	A	B	Types:	C	D
Phase factor λ_p	i	$-i$		1	-1
Particle field:	A	B		C	D
Antiparticle field:	B	A		C	D

Special reason. Certain four-fermion interactions, such as two neutrons going over spontaneously into two antineutrons, cannot be ruled out by any real or imaginary type parity argument, since $(\pm i)^4 = 1$. This was recognized by Yang and Tiomno in their original paper.

General reason. There are certain far-reaching objections in principle to taking seriously the assignment of specific relative parity types to different kinds of fermions, assuming that they are separated by exact conservation laws.

The possibility of $\pm i$ types, as well as ± 1 types, of spinor parity was first suggested by Racah (1937). The question of whether it is physically meaningful to distinguish four types (Yang, Tiomno, 1950) or only two (Caianiello, 1952; Eriksson, 1953) has been much disputed. The most general view appears to be that of the "Three W's" (Wick, Wightman, and Wigner, 1952;

[24] Actually, the assignments also could be taken to be the reverse of these which were given by Yang and Tiomno. This makes no real difference because only relative parities are in question (see Table VII).

TABLE VII. Demonstration of arbitrariness of *absolute* parities of fermions.

We are allowed to make a transformation $\lambda_p \to i^F \lambda_p$ in the parity phase factor λ_p of all particle and antiparticle fields, where F is the fermion number ($+1$ for fermions, -1 for antifermions). This is admissible because in any interaction Lagrangian there appear just as many antifermion operators as fermion operators (fermion conservation). Under this transformation the parity types of Table VI transform as follows:

Particle field		Antiparticle field	
$A \to D$	$C \to A$	$A \leftarrow D$	$C \leftarrow A$
$B \to C$	$D \to B$	$B \leftarrow C$	$D \leftarrow B$

A more general transformation which is equally admissible is $\lambda_p \to e^{i\alpha F}\lambda_p$, where α is arbitrary.

If baryons and leptons are conserved separately, such (special or general) transformations may be applied to the parity phase factors of either arbitrarily. It is clear that under these circumstances the *relative* parities of baryon to lepton fields are also arbitrary.

Wigner, 1956; Mathews, 1957), which may be stated as follows: Since any attempt to measure relative parities can yield only the three results "same," "opposite," or "undeterminable," the only way in which there can be different parity kinds, e.g., "real" and "imaginary," is when the result is "undeterminable." Thus, the existence of different parity kinds goes with the existence of two classes of fermions separated from each other by a *superselection rule*, i.e., not only is spontaneous transition between the two classes impossible, but, further, it is impossible to induce transitions by any measurable operator. But this is equivalent to the existence of an ironclad conservation principle preventing one class of fermions from ever transforming into the other. From the "Three W's" point of view, the relative phase factor, after inversion, between the two spinor kinds is entirely arbitrary (see Table VII).

Though it has deepened our conception of parity, and is now generally accepted, there is one qualification which should be made in the WWW point of view. That concerns the possibility, in thought at least, of comparing the parity of two identical fermions with a boson. For suppose that the original Majorana neutrino theory were correct (which it apparently is not), then processes in which the two exactly like neutrinos annihilated to give a boson could occur. Though by the well-known superselection rule separating fermion states from boson states (WWW, 1952) the relative parity of the individual neutrino is unobservable, one would nevertheless *decide to call* the parity of the neutrino relative to the boson imaginary (see the following). Yang and Tiomno's "minimum-generality" point of view in the matter of relative parity types would then be partly vindicated (but not the attempt to establish baryon conservation, since the counter arguments given in the foregoing would still apply). The fact that these things do not happen—conservation of leptons—makes acceptable the WWW point of view that any arbitrary relative phase under P may be assigned with equal reason to boson and lepton fields. However, a further matter of formal convenience comes in here. If one wishes C and P to commute in their action on fermion fields, one should assign all fermion fields the i-type parity (Lüders, 1955). For the Majorana neutrino theory this assignment seems to be compelling (see Racah, 1937; Wightman and Schwever, 1955; and the argument in the following—next paragraph but one). This suggests that i-type parity for spinors may be a good choice to make in general. It also fits in naturally with the way we conveniently describe inversion properties of spinors in three-dimensional space (see Sec. V).

It should be remarked that the foregoing considerations by no means affect the well-known theoretical inference that a fermion-antifermion system in an S state has odd parity (Yang, 1950). It is important here to distinguish clearly between the parity behavior of *fields* and the parity behavior of *states* of the field. The parity factor of a fermion field and its antifermion field are subject only to the requirement that their product be $+1$ (conservation of fermions) and are otherwise arbitrary as is demonstrated in Table VII. On the other hand, the product of the intrinsic parities of a one-fermion state and the corresponding one-antifermion state is always -1 no matter what type the field is. In the final analysis this is due to the fact that the parity redescription does not commute with all other redescriptions of the Poincaré group (e.g., translation). Therefore, P cannot be a multiple of the identity in any rep (except the trivial scalar rep). Thus, for example, the matrix γ_4 appears as a factor in the parity transformation of spin $\frac{1}{2}$ particles (Table III). The structure of γ_4 is such that it controls the signs of the phase factors in front of particle and antiparticle states, so that they come out $\lambda_P{}^*$ and $-\lambda_P$, respectively. This can be verified most easily in the split (Dirac-Pauli) representation

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} (k=1,2,3) \quad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

by studying the expansion of each field operator in particle absorption and antiparticle emission operators.

The theoretical inference that a fermion-antifermion system in an S state has odd parity (behaving like a pseudoscalar particle) has been well-verified experimentally in the ^1S annihilation of positronium through the property that the two photons come off with opposite polarization (Wu and Shaknov, 1950; Siegbahn, 1957).

We note that a system consisting of two identical Majorana neutrinos in an S state would be a special case of a fermion-antifermion system—therefore with a parity -1. In the interest of maintaining continuity of concepts, it is then hardly escapable, as has been indicated previously, that a one-neutrino state should as assigned the i-type parity. (Admittedly, we cannot directly test this assignment empirically—if the fundamental superselection rule holds!)

To sum up, it does not seem that the explanation of baryon conservation can be found along the lines of a familiar space-time symmetry, nor of a discrete symmetry group of any kind. An interesting possible approach has been opened up by Ferretti (1957 and private communication) who, upon generalizing some earlier work of Lee and Yang (1955), has analyzed the possibility of consistently introducing a baryon "charge-current density" and has inferred that this makes it necessary to introduce a neutrino field as well as a meson field coupled to the nucleon field.

In general, it is of basic importance for elementary particle physics to be able to decide the question: When are two seemingly different particle theories *physically equivalent?* This, for example, is much of the issue between the "i-type parity" position taken by Yang and Tiomno and the "arbitrary-parity" position taken by Wick, Wightman, and Wigner with respect to the relative parity of baryons and leptons.

The problem is that of finding clear formal criteria to decide when two relativistic elementary particle theories are physically equivalent. As already emphasized, every such theory, besides being characterized by a Poincaré group rep made up of redescriptions in coordinate space, is also associated with a Poincaré group rep—up to a factor—in state-vector space; in other words, the Hilbert space spanned by the state vectors of the elementary particle described by the theory also serves as a representation space for an ∞-dimensional unitary rep of the Poincaré group (Sec. III).

The question of physical equivalence has been discussed extensively by Michel and Wightman in their unpublished notes, where they also refer to unpublished work of Bargmann. For a first gross classification they consider the theories to be represented adequately by their associated Hilbert space reps up to a factor, and they set up criteria for "physical equivalence up to a factor" of the latter: "Two representations up to a factor, $\mathbf{U}$ and $\mathbf{U}'$, are physically equivalent if there is a one-to-one correspondence, $\mathbf{W}$, between their states $\psi\leftrightarrow\psi'$ which (1) preserves transition probabilities:

$$|(\psi_1,\psi_2)|^2=|(\psi_1',\psi_2')|^2,$$

and (2) is invariant under Lorentz transformation: If $\psi\leftrightarrow\psi'$, then $\mathbf{U}(a,\Lambda)\psi\leftrightarrow\mathbf{U}'(a,\Lambda)\psi'$." By applying the fundamental theorem of Wigner, it follows that $\mathbf{W}$ can be chosen as a unitary or antiunitary operator, and the condition of physical equivalence reduces to that for mathematical equivalence,

$$\mathbf{U}'=\mathbf{W}\mathbf{U}\,\mathbf{W}^{-1},$$

with W unitary or antiunitary.

To refine the criterion of equivalence further, one must look at *all* the observables $\mathbf{Q}$. If the correspondence $\mathbf{W}$ between states extends to a correspondence between observables, so that for every $\mathbf{Q}$ we have a $\mathbf{Q}'$ such that

$$(\psi,\mathbf{Q}\psi)=(\psi',\mathbf{Q}'\psi'),$$

then the theories are completely equivalent.

The possible physically inequivalent kinds of elementary particles (systems), based on Wigner's analysis of the inhomogeneous Lorentz group, are described in Table V. In making this classification, including the distinctions due to differing behaviors under the space-time inversions, essential use is made of the scalar square theorem on antiunitary operators (Sec. II.E) according to which

$$\mathbf{T}^2=+1 \quad\text{or}\quad \mathbf{T}^2=-1.$$

This result is important because classification of a system according to its $\mathbf{T}^2$ eigenvalue (see Table V) leads not to a pseudo but to a real characterization, independent of an arbitrary change of phase in $\mathbf{T}$. To see this we set $\mathbf{T}'=\mathbf{T}e^{i\alpha}$, whereupon we find

$$(\mathbf{T}')^2=\mathbf{T}e^{i\alpha}\mathbf{T}e^{i\alpha}=\mathbf{T}e^{i\alpha}e^{-i\alpha}\mathbf{T}=\mathbf{T}^2.$$

The arbitrariness of the phase factor under the parity operation P extends to particles of higher half-integral spin. It appears possible to prove, on the basis of the quoted criteria for physical inequivalence that the only physically inequivalent spin j particle fields are the four types listed in Table V, separated only by their behavior under time-reversing transformations, [Bargmann (unpublished) cited by Michel and Wightman (unpublished)].

V. OTHER CONSERVATIONS AND SYMMETRIES

We come now to those symmetries of natural processes which, while they do not appear to be directly related to space-time symmetries, nevertheless lend themselves to an elegant geometrical mode of description.[25] These are the *isospace* symmetries. The role of the concept of *isospin* in nuclear physics resembles strongly the role of the concept of spin itself in atomic physics. More exactly, a component of the isospin which represents the total electric charge plays a role paralleling that of the z component of spin. We recall that, although the direct spin interaction of two particles in atomic physics is so small that it may be neglected in first approximation, there is nevertheless a great qualitative importance to the presence of spin in that, with it *the number of states allowed by the exclusion principle is increased*. The different new allowed states are labeled by the different values associated with them of the spin projection along some axis (e.g., the z axis) which is preferred in a spin determining experiment.

To see the analogous role of isospin in the realm of charge independence of nuclear interactions, compare a system of four neutrons with a system of two neutrons and two protons. Here, as with spin in the atomic case,

[25] This is only true up to a point. Certain complications which appear when one tries to include charge conjugation show that the geometrical metaphor has its limits.

the presence of charge serves merely to increase the number of allowed states. This leads to the idea of interpreting the charge q of a nucleon as the component T_3 along a "3 axis" of a spinlike property T. To get a neat "charge-multiplet formula" it is necessary to add $\frac{1}{2}$ to T_3 in the expression for q. The concept of isospin is then successfully extended to the pions in which case, instead of having an *isospin doublet* as with the nucleons, we have an *isospin triplet*. But here the relation $q=T_3$ holds. It is nice then to notice that the same formal relation as in the doublet case can be maintained between charge and 3 component of isospin by putting, instead of the 1 in the numerator of the $\frac{1}{2}$, the baryon number B, with the result that the charge-multiplet formula, for both doublet and triplet, reads[26]

$$q=T_3+B/2.$$

Gell-Mann (1953, 1954, 1956) and Nishijima (1953, 1955) independently realized that one could successfully embrace the new unfamiliar particles, and account for the simultaneous properties of associated production (strong interactions) and slow decay (weak interactions) by generalizing this charge-multiplet formula still further. One had only to introduce in the numerator of the fraction, in addition to the baryon number, the *strangeness number* S: $q=T_3+(B+S)/2$. While zero for all the familiar particles, S assumes integral values for the new particles. What its inventors had in mind was that while strangeness is conserved in strong and electromagnetic interactions, its nonconservation is an inhibition which allows a process to go only by weak decay.

By means of this restrictively conserved quantity, strangeness, it has been possible to give a successful account of the decay behavior and reaction behavior of the elementary particles indicated in our first chart and Table I. First, the strong associated production is interpreted as a strangeness conserving reaction, i.e., strange particles are produced from pions and protons (of strangeness zero) only in pairs of opposite strangeness. Once produced, each particle is off alone with its strangeness. If the different types of particles have the values of S indicated in Table I, no lone strange particle[27] can decay (subject to conservation of q and B) except by changing its strangeness by at least one unit. This takes a long time; as we have seen, a time of the order of 10^{14} flashes.

[26] This formula also describes antiparticles, for which q, T_3, and B simultaneously have reversed signs. More generally, the same formula works for a system of particles with charge Z and mass number B. For example, we observe a nuclear state which is the same (except for clearly understandable differences due to secondary electromagnetic effects) in $2T+1$—and not more—different isobaric nuclei. Then we assign to this nuclear state an isospin value T, and the different isobars are distinguished by the value of

$$T_3=Z-B/2.$$

[27] Except Σ^0, which actually does go into $\Lambda^0+\gamma$ very fast in accord with the condition $\Delta S=0$.

With the recognition of a new conserved quantity in physics, the problem arises of finding a corresponding invariance principle. Various attempts in this direction have been made. Of these one of the more appealing appears to be that of D'Espagnat and Prentki (1955, 1956), Racah (1956), and Murai (1956). In this interpretation the strangeness number S plus baryon number B is an *isoparity index or hypercharge* U, namely, it is the exponent in that power of i which appears in front of the field function of a particle when an inversion in isospace is carried out. The charge-multiplet formula becomes

$$q=T_3+U/2. \tag{22}$$

Since B is always conserved, and $U=B+S$, Gell-Mann's rule of conservation of strangeness in strong and electromagnetic interactions becomes a rule of "conservation of hypercharge." Whereas the strangeness number is found to take on various values ranging from 0 to ± 2, the hypercharge ranges only over the values 0, ± 1. The value 0 applies to *isobosons* (T_3 integral), and ± 1 to *isofermions* (T_3 half-integral), as is evident from the integral value of the charge. The strangeness, and likewise the hypercharge of an antiparticle is the negative of that for a particle.

In writing simple numbers for q, T_3, and U, we have essentially been dealing with eigenvalues, or with operators in isospace in diagonal form. We now introduce the operators explicitly. We can write the isoparity operation itself,

$$\mathfrak{p}=\exp(i\mathbf{U}\pi/2).$$

Here the Hermitian operator $\mathbf{U}$ may be considered as the "generator" of a continuous transformation which for the argument $\pi/2$ is equivalent to the isoparity operation. A corresponding interpretation also can be given to the other physical quantities by considering them as generators of geometrical operations in isospace. Just as the z component of ordinary angular momentum generates rotations about the z axis of ordinary space so the 3 component of isospin appears as the generator of a turn $\mathbf{A}(\alpha)=\exp(i\mathbf{T}_3\alpha)$, through any angle α about the 3 axis in isospace. Equation (22) may now be interpreted as expressing the charge operator in the form of a "generator" of a half-turn and inversion operation,

$$\mathfrak{p}\mathbf{A}(\pi)=\mathbf{A}(\pi)\mathfrak{p}=\exp(i\mathfrak{q}\pi).$$

In other words, *the charge operator is the "generator" of a reflection through the equatorial plane at right angles to the* 3 *axis*. The inversion $\mathfrak{p}$, the rotations $\mathbf{A}(\alpha)$—and all other rotations in isospace[28]—are separately good symmetry operations for strong interactions. For electromagnetic interactions only $\mathfrak{p}$ and $\mathbf{A}(\alpha)$ remain. While finally, for weak interactions, only $\mathfrak{q}\sim\mathbf{A}(\pi)\mathfrak{p}$ remains as a good invariance operator. Thus, the progressive re-

[28] This full rotation symmetry is the geometrical expression of *charge independence*.

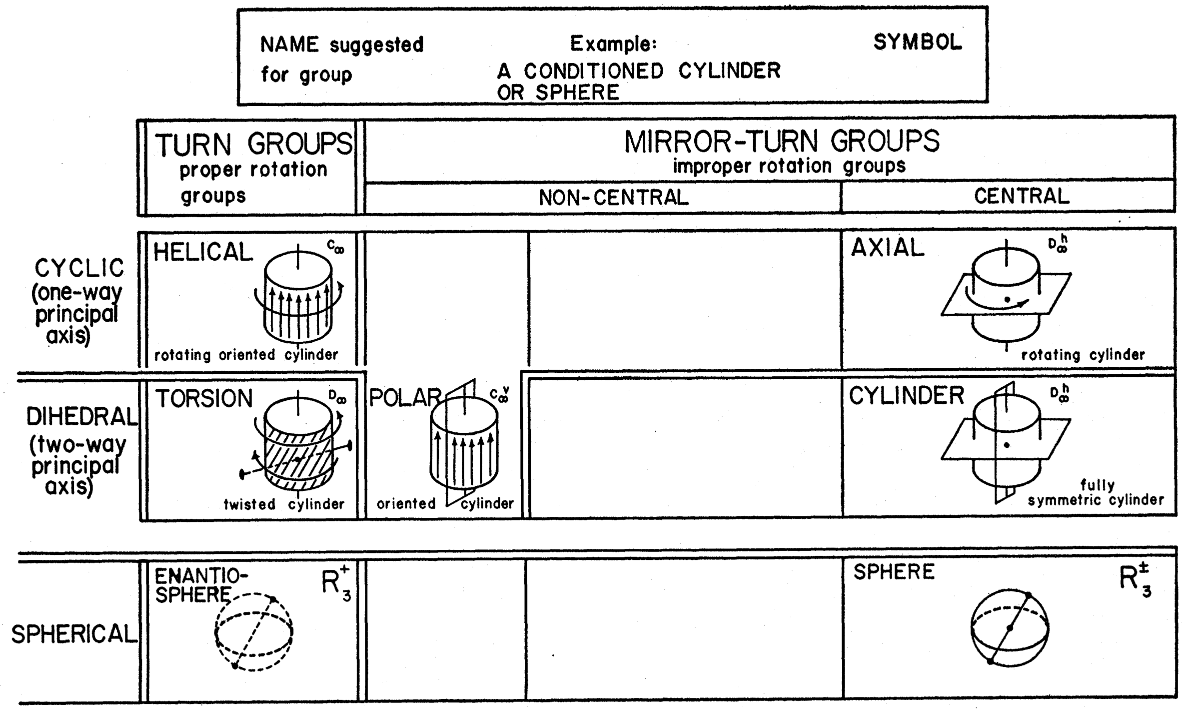

FIG. 5. The possible three-dimensional rotation and rotary-inversion groups which have one or more infinity-fold axes.

duction in symmetry which accompanies the passage from strong interactions—to electromagnetic interactions—to weak interactions may be pictured as a reduction from the full rotary-inversion group $R_3^{\pm}$—to the rotating cylinder group C_∞^h—to the simple mirror group in isospace. The nature of the first two groups and their place among all the possible continuous groups in a three-dimensional space is indicated in Fig. 5.

VI. RECENT HISTORY OF THE UNIVERSAL FERMI INTERACTION

Through recent experiments it has become very probable that weak interactions are of a universal four-fermion type. Such an interaction is called "universal" in the sense that it underlies all, and is the same for all parity or strangeness nonconserving processes, always with the same coupling constant.

It appears that such a universal four-fermion interaction as a basis for all weak processes finally may be established. This appealing idea, first suggested in a discussion by Klein (1948), would endow the theory of weak interactions with a basic simplicity analogous to that given to the theory of electrical phenomena when Faraday (1831) showed that all "forms of electricity" are the same. Out of Maxwell's formulations, via the peregrination into space-time, it emerged later that the primary interaction between electricity and the electromagnetic field is universally of the "vector" form $\alpha J\cdot A$ where α is the dimensionless electrical coupling constant $e^2/\hbar c$, J is the four vector of charge-current density, and A is the four vector of electromagnetic potential. In the present conception, all other electromagnetic interactions (such as those involving magnetic moments) are derivative from the primary vector interaction.

In line with the original approach of Fermi (1934) when he first formulated a theory of beta decay, and by generalized analogy with the electromagnetic interaction, one assumes that the weak interaction is one in which any charged (fermion-antifermion) pair is coupled to any other charged (fermion-antifermion) pair. If certain selection rules—specifically that of associated neutrino production described in the following—as well as the two principles, conservation of baryons and conservation of leptons, are always to hold, then such interactions can involve only a baryon-antibaryon pair and a lepton-antilepton pair with one particle in each pair charged; i.e., interactions occur only between two pairs of the general forms $(\bar{B}B^0)$, $(\bar{B}^0B)$, $(\bar{L}L^0)$, $(\bar{L}^0L)$. (For μ decay we should replace the baryon-antibaryon pair by a meson-antineutrino pair.) Thus, for example, one of the possible forms of the interaction might be

$$\text{general form: } (\bar{B}^0B)(\bar{L}^0L); \quad \text{special case: } (\bar{n}p)(\bar{\nu}e).$$

The pair of fields in the first bracket has nonvanishing matrix elements representing the disappearance of a (neutral antibaryon-charged baryon) pair, and similarly, the second bracket may represent appearance of a (neutral lepton-charged antilepton). The way in which these interpretations refer to the characteristic weak interaction of β^+ decay is evident in the "special case" form given on the right.

Since the only neutral lepton is the neutrino, it is clear that one can immediately establish on these general grounds what has sometimes been called the "principle of associated neutrino production" and which may be stated as follows: *No single lepton of any one kind, namely, an electron by itself or a muon by itself, can ever appear in weak interactions unless accompanied by a single antineutrino.*

We turn now to the experimental indications for the specific form of the interaction. For some time the indirect evidence had been accumulating, and then the direct evidence began coming in (Herrmannsfeldt *et al.* 1957, 1959; Goldhaber *et al.* 1958) that the universal form underlying all weak interactions is entirely *vector-axial.* This characterization refers to the relativistic tensor quantities which appear in the interaction. These are the quantities whose scalar products (or pseudoscalar products since only proper Lorentz invariance—but not parity—is conserved) represent the interaction between the one fermion-antifermion pair (a,b) and the other (c,d). Except for coupling constants, these interaction terms are then the products of the vectors, $\bar{\psi}_a\gamma_\lambda\psi_b$ or $\bar{\psi}_c\gamma_\lambda\psi_d$, and the axial vectors, $\psi_a\gamma_5\gamma_\lambda\psi_b$ or $\bar{\psi}_c\gamma_5\gamma_\lambda\psi_d$, associated with each pair. To establish this "pure V-A coupling" experimentally as the universal form of the weak interaction, it was necessary to demolish the validity of such an apparently well-established result as that of the 1955 He^6 recoil experiment which seemed to imply that the interaction in the He^6 Gamow-Teller type transition is pure tensor (T) (i.e., involving the scalar or pseudoscalar product of terms of the form $\bar{\psi}_a\gamma_\mu\gamma_\nu\psi_b$). Reasons for criticism of the experiment were found by the experts, and with the destructive thought concentrated on it, it soon crumbled.

In addition to the removal of the He^6 difficulty, the situation in other respects also cleared up considerably from that of the summer of 1957 when "for every experiment there was an antiexperiment." The two outstanding types of parity-insensitive experiments ("classical" experiments), i.e., energy distribution and recoil experiments, for the most part began to agree well with pure V-A interaction, with the squared coupling-constant ratio estimated at

$$g_A^2/g_V^2 = 1.42 \pm 0.08$$

(Gerhart, 1958; Winther and Kofoed-Hansen, 1958; Sosnovskij *et al.* 1958. There is some discrepancy with the ratio evaluated from ft values; see Kistner and Rustad, 1958). The two outstanding types of parity-sensitive experiments have been those on (1) polarization of decay β's; (2a) β asymmetry from polarized nuclei; and (2b) β-γ (circular polarization) angular correlation. Both of these measure the expectation value of a pseudoscalar product $\mathbf{j}\cdot\mathbf{P}_e$ of angular momentum $\mathbf{j}$ and linear momentum $\mathbf{P}_e$ of the β particle. From the results, inferences were drawn concerning the neutrino helicity, i.e., its spin linear-momentum correlation (or "handedness"). These inferences had to be drawn via assumptions about the form of coupling, which governs the correlation between neutrino momentum and electron momentum. It was very satisfying that, to these experiments, a new type was added which measured the helicity of the neutrino rather more directly and therefore, in conjunction with the results of the polarization experiments, served to determine the form of coupling independently of the recoil experiments. This new type was an e^- capture and subsequent γ-ray measurement experiment, which was suggested independently by Page (1958), and Goldhaber *et al.* (1958), and carried out by Goldhaber *et al.* (1958). In that experiment successful use was made of the fact that the emitted neutrino gives the daughter nucleus a twist and a kick which then appears in the direction of circular polarization and in the sign of Doppler shift of the subsequently emitted γ. The results on the $Eu^{152m}\xrightarrow{EC} Sm^{152}$ transition showed unequivocally that the neutrino emitted was left-helical. That, combined with the result of all polarization experiments showing weak-interaction emitted β^+'s to be right-polarized, led directly to the conceptually important result: *In the Europium transition the basic Gamow-Teller interaction is axial and not tensor.*

Besides the He^6 difficulty, which was finally completely removed, perhaps the most serious difficulty which remained for the pure V-A coupling was the fact that pion-electron neutrino $(\pi e\nu)$ decay was not observed,[29] while pion-muon neutrino $(\pi\mu\nu)$ decay was observed. Because the removal of this discrepancy by the discovery of $\pi e\nu$ decay in just the right amount was of considerable importance [see Gatlinburg Conference reports, Revs. Modern Phys. **31**, 782, (1959)], we give a brief history of the analysis of the connection between the type of coupling and the ratio ρ of the $\pi e\nu$ to the $\pi\mu\nu$ decay mode.

Ruderman and Finkelstein (1949) assumed that decays of this type take place via a virtual nucleon-antinucleon pair and therefore involve the four-fermion interaction as the slow step. (Nowadays we would recognize this also as the parity-nonconserving step.) They found that of the five possible types of four-fermion interaction, the scalar (S), vector (V), and tensor (T) types are completely forbidden. But the pseudoscalar (P) and axial vector (A) are allowed. This may be seen to come from the fact that the virtual pair arises out of the pion—a pseudoscalar entity, for which

[29] Early investigations set an experimental upper limit to the ratio ρ of $\pi^+\rightarrow e^+ + \nu$ to $\pi^+\rightarrow\mu^+ + \nu$ less than 5×10^{-6}. More recent work suggested that the possible value of ρ was less than 10^{-5}. The most recent experimental work, in which the decay is observed, confirms the theoretically expected ratio 1.3×10^{-4} (see the following) very closely (Fazzini *et al.* 1958; Impeduglia *et al.* 1958; Anderson *et al.* 1959).

the only linear couplings to the nucleon-antinucleon pair are of pseudoscalar or axial type. Determinate values for ρ were obtained under somewhat restrictive assumptions by Ruderman and Finkelstein for P and A coupling, respectively. The value for P was not satisfactory, being much too high to accord with observation, thus ruling out any appreciable admixture of P in the interaction. The general result for the A type of interaction was much better. Including the density-of-states factor which we may compute with present data to be 5.4, the A interaction by their method gives ρ the value

$$\rho = 5.4(m_e/m_\mu)^2 = 1.3\times 10^{-4}.$$

This result, that in the A interaction the matrix element for the $\pi e\nu$ mode is reduced by the factor m_e/m_μ compared to that for the $\pi\mu\nu$ mode, was also obtained by Miyazawa and Oehme (1955) without any detailed assumptions about the intermediate state. The result followed from a general theorem which they proved to the following effect. Under certain generally satisfied conditions the matrix element for the decay of spin-zero mesons of mass m into two leptons of mass m_1 and m_2, respectively, contains only terms proportional to m_1/m and m_2/m. The conditions under which the general theorem holds are satisfied whether the meson-lepton interaction is direct or occurs via nucleon pairs or other intermediate fields, provided only that the lepton coupling is vector or axial vector (odd number of γ's). It is assumed only that the pion has spin zero and that the muon, electron, and neutrino have spin one-half. Of course, if the further restriction is made that the pion is pseudoscalar, decay by vector coupling becomes completely forbidden, and the theorem is applied to the axial coupling channel.

While the experimental evidence was still confused, theories of pure V-A interaction were considered by Feynman and Gell-Mann (1958), Sudarshan and Marshak (1958), and Sakurai (1958) following the general pattern of "two-component theory" which had been initiated independently by Landau (1957), Lee and Yang (1957), and Salam (1957). [An early anticipation of these ideas, before parity violation was known, was discussed by Stech and Jensen (1955).] Of these we sketch only the early development of ideas of the first-named authors. Feynman and Gell-Mann started with the idea that all Fermi particles should be represented by two-component spinors obtained by projecting the usual four-component forms with the projection operator $\frac{1}{2}(1+\gamma_5)$, and that all weak interactions between such particles should be represented by direct couplings between these spinors. They showed that the only possible interaction then is the pure V-A coupling in equal strengths in the first naive approach. The exclusive use of the projection operator $\frac{1}{2}(1+\gamma_5)$ with all fermions means that the phase difference between the V and the A amplitudes is 180°; thus it is a "V minus A coupling." Had the *a priori* equally admissible assumption been made that the other projection operator $\frac{1}{2}(1-\gamma_5)$ be used with one of the fermion pairs, the coupling would have been "V plus A." (This corresponds to interchange of particles and antiparticles in one of the pairs.) Experiment (Telegdi and others, 1959) appears to confirm the "V minus A" coupling which, because of the reality of the relative phase factor, also satisfies time-reversal invariance. The physical essence of the $\frac{1}{2}(1+\gamma_5)$ projection is that, *in all weak interactions, left-polarizations for particles and right-polarization for antiparticles is favored, this favoring tendency becoming absolute as the velocity of particle (or antiparticle) approaches the velocity of light.* Thus the V-A theory automatically comprises two-component neutrinos (specifically, purely left-helical neutrinos and right-helical antineutrinos) as well as conservation of leptons.

With the V-A theory, the lifetime of the muon is obtained to within the experimental error of 2%. Since the magnitude of the coupling constant used is the one obtained from the O^{14} decay, where nucleons and their fields of virtual pions are involved, and it is being applied to a system where ostensibly there are no virtual pion effects, the question comes up why the agreement is so good. This led to some interesting suggestions as to why the universal coupling constant (or at least the vector part of it) should not be subject to renormalization due to virtual mesons—a situation which resembles that of electrodynamics. Since the time they were first made, there have been further interesting consequences of these suggestions (see Gatlinburg Conference reports). We confine ourselves to remarking that, by extending the universality to couplings involving a Λ or Σ fermion, a qualitative account is also obtained of the weak decays of the strange particles with their parity nonconservation.[30] For example, a K^+ can go virtually into an anti-Λ and proton by strong coupling. By the weak decay $(\bar{\Lambda}p)(\bar{p}n)$ a virtual $\bar{p}$ and n can then form. On annihilating, these give two or three pions.

We cannot leave the subject of a universal V-A type weak interaction without mentioning a third test, first of the two-component consequence of the theory and then of the specific form of the interaction. The shape of the energy spectrum of the decay electron in the $\mu\rightarrow e+\nu+\bar{\nu}$ process, upon the basis of a direct four-fermion theory, involves a single parameter, the Michel shape-parameter whose magnitude indicates how slowly the spectrum falls off at high energies. For any two-component theory with maximum parity nonconservation (the older ST interaction also could be given this form) the value of this parameter is $(\rho_M)_{\text{theoret}}=0.75$. Recent experimental values are Rosenson (1958): ρ_M

[30] *Note added in proof.* At the present time it appears that the coupling constant for weak interactions involving a strange particle is about one-tenth as large as that for the weak interactions of the nonstrange particles. How the difference is to be interpreted is not known, but one is struck by the fact that a similar ratio is encountered for the strong interactions involving strange particles (kayons) and nonstrange particles (pions).

$=0.67\pm0.05$; Plano and LeCourtois (1959): 0.79 ± 0.03. From the range of the reported experimental values it seems as though this type of measurement too may conclude in agreement with a two-component interaction. Similarly, the shape of the angular distribution of the emitted electron provides a test of the two-component nature of the interaction. The scale for this shape is provided by the value of the asymmetry parameter for the maximum energy. This maximum asymmetry parameter ξ should have the value 1 for pure V-A interaction. Both the two-component shape and the V-A value $\xi=1$ are well confirmed by recent experiments [Plano and LeCourtois (1959); Bardon, Berley, and Lederman (1959)].

It appears then that we are on the verge of a universal and rather simple conception of weak interactions. Since at least one new conservation principle (lepton conservation) is involved, as well as the breakdown of two other conservations (parity and strangeness), one expects remarkable new invariance properties to be somehow involved. It is likely that when these new invariance properties are unraveled we will have gone a long way toward understanding the nature of weak interactions.

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VIII. APPENDIX. PHYSICAL INTERPRETATION OF COMMUTATION RELATIONS OF THE SPACE-TIME DISPLACEMENT GENERATORS. DEFINABILITY OF CENTROIDAL POSITION

In Sec. II.F we discussed the commutation relations of the generators of the Poincaré group in their original abstract setting. We now interpret the Hilbert-space representatives $(\mathbf{P}_0,\mathbf{P}_k)$, $(\mathbf{M},\mathfrak{M}_k)$ of these generators to have certain specific dynamical meanings. We see that the commutation relations correspond to *physical relations which are to be expected on familiar grounds* if the quantities $\mathbf{P}_0$, $\mathbf{P}_k$, $\mathbf{M}_k$, and $\mathfrak{M}_k$ are interpreted as *total energy*, *linear momentum*, *angular momentum*, and *centroidal moment* of a dynamical system. In connection with the last concept, we are led to a definition of an operator defining the *centroidal position* of a system which, for a one-particle system, coincides with the position of the particle. (Position in this sense should be distinguished clearly from the space-time coordinates which are used as parameters in quantum field theory and which are to be associated with measurements on the observing apparatus rather than on the observed system.)

The reference to "familiar grounds" in the preceding paragraph means that it is by a "correspondence principle" type of argument—from classical physics, or elementary quantum theory—that we first become confident that it is wise to define the Hilbert-space representatives of the generators as basic dynamical variables. Once this confidence is established, as often happens in similar situations in physics, we drop the relatively concrete approach of the past and introduce the new quantities in an autonomous way. This means *introduction by definition*, and may be confusing if one does not know the historical motivation.

First we reproduce the commutation relations in a somewhat modified form:

Interpreted either as

$$\left.\begin{matrix}\mathbf{M}=\mathbf{L},\ \mathfrak{M}=\mathfrak{L}\\ \text{or as}\\ \mathbf{M}=\mathbf{S},\ \mathfrak{M}=\mathfrak{S}\end{matrix}\right\}\quad \begin{aligned} &[\mathbf{M}_1,\mathbf{M}_2]=i\mathbf{M}_3 \quad \textit{et cycl} && \text{(a)}\\ &[\mathbf{M}_1,\mathfrak{M}_1]=0 \quad [\mathbf{M}_1,\mathfrak{M}_2]=[\mathfrak{M}_1,\mathbf{M}_2]=i\mathfrak{M}_3 \quad \textit{et cycl} && \text{(b)}\\ &[\mathfrak{M}_1,\mathfrak{M}_2]=-i\mathbf{M}_3 \quad \textit{et cycl} && \text{(c)}\end{aligned}$$

$$\begin{aligned} &[\mathbf{L}_1,\mathbf{P}_1]=0 \quad [\mathbf{L}_1,\mathbf{P}_2]=[\mathbf{P}_1,\mathbf{L}_2]=i\mathbf{P}_3 \quad \textit{et cycl} && \text{(d)}\\ &[\mathbf{P}_j,\mathbf{P}_k]=[\mathbf{S}_j,\mathbf{P}_k]=[\mathfrak{S}_j,\mathbf{P}_k]=0 \qquad [\mathfrak{L}_j,\mathbf{P}_k]=i\mathbf{P}_0\delta_{jk} && \text{(e)}\\ &[\mathbf{P}_k,\mathbf{P}_0]=[\mathbf{L}_k+\mathbf{S}_k,\ \mathbf{P}_0]=0 \qquad [\mathfrak{M}_k,\mathbf{P}_0]=i\mathbf{P}_k && \text{(f)}\end{aligned} \tag{I}$$

$$(j,k=1,2,3).$$

TABLE VIII. Table of commutators of quantities constructable with generators of the Poincaré group. We start with the vector $\mathbf{P}_a \sim (\mathbf{P}_0, \mathbf{P})$ and the bivector or antisymmetric tensor $\mathbf{J}_{ab} \sim (\mathbf{M}, \mathfrak{M})$. The space vectors $\mathbf{P}$, $\mathbf{M}$, $\mathfrak{M}$ are polar-odd, axial-odd, and polar-even, respectively, where the adjective following the hyphen in each case refers to behavior under time reversal. The "outer product," $\mathbf{J}_{bc}\mathbf{P}_d + \mathbf{J}_{cd}\mathbf{P}_b + \mathbf{J}_{db}\mathbf{P}_c \equiv \mathbf{W}_{bcd} = \mathbf{W}^a$ ($a, b, c, d = cycl$ of 0123), defines the trivector or pseudovector $\mathbf{W}^a \sim (\mathbf{W}_0, -\mathbf{W})$ of which $\mathbf{W}$ is axial-odd. The "inner product" $\mathbf{W}_{abc}\mathbf{P}^c$ defines the bivector $\mathbf{W} \wedge \mathbf{P} \sim (\mathbf{W}_0\mathbf{P} - \mathbf{W}\mathbf{P}_0, \mathbf{W}\times\mathbf{P})$, of which $\mathbf{W}_0\mathbf{P} - \mathbf{W}\mathbf{P}_0$ is axial-odd and $\mathbf{W}\times\mathbf{P}$ is polar-even. Other quantities are expressible in terms of these, provided suitable scalars are adjoined. For example, the "inner product" $\mathbf{J}_{ab}\mathbf{P}^b$ gives the vector $\mathbf{U} \sim (\mathbf{U}_0, \mathbf{U})$, where $\mathbf{U}_0 = -\mathfrak{M}\cdot\mathbf{P}$, $\mathbf{U} \equiv -\mathfrak{M}\mathbf{P}_0 - \mathbf{M}\times\mathbf{P} = -[m^2\mathfrak{M} + \mathbf{W}\times\mathbf{P} + (\mathfrak{M}\cdot\mathbf{P})\mathbf{P}]/\mathbf{P}_0$. Thus, the polar-even vector $\mathbf{U}$ is expressible in terms of $\mathfrak{M}$, $\mathbf{W}\times\mathbf{P}$, and $\mathbf{P}$, provided the scalar-odd coefficient $\mathfrak{M}\cdot\mathbf{P} = -\mathbf{U}_0$ is allowed. The commutator $\mathbf{U}_k U_m$ equals $im^2 M_n$.

	$\mathbf{P}_0$	$\mathbf{P}_m$	$\mathbf{M}_m$	$\mathfrak{M}_m$	$\mathbf{W}_0$
$\mathbf{P}_0$	0	0	0	$-i\mathbf{P}_m$	0
$\mathbf{P}_k$	0	0	$i\mathbf{P}_n$	$-i\mathbf{P}_0\delta_{km}$	0
$\mathbf{M}_k$	0	$i\mathbf{P}_n$	$i\mathbf{M}_n$	$i\mathfrak{M}_n$	0
$\mathfrak{M}_k$	$i\mathbf{P}_k$	$i\mathbf{P}_0\delta_{km}$	$i\mathfrak{M}_n$	$-i\mathbf{M}_n$	$i\mathbf{W}_k$
$\mathbf{W}_0$	0	0	0	$-i\mathbf{W}_m$	0
$\mathbf{W}_k$	0	0	$i\mathbf{W}_n$	$-i\mathbf{W}_0\delta_{km}$	$-i(\mathbf{W}\times\mathbf{P})_k$
$(\mathbf{W}_0\mathbf{P}-\mathbf{W}\mathbf{P}_0)_k$	0	0	$i(\mathbf{W}_0\mathbf{P}_n-\mathbf{W}_n\mathbf{P}_0)$	$i(\mathbf{W}\times\mathbf{P})_n$	$i(\mathbf{W}\times\mathbf{P})_k\mathbf{P}_0$
$(\mathbf{W}\times\mathbf{P})_k$	0	0	$i(\mathbf{W}\times\mathbf{P})_n$	$-i(\mathbf{W}_0\mathbf{P}_n-\mathbf{W}_n\mathbf{P}_0)$	$-i(\mathbf{W}_0\mathbf{P}_0\mathbf{P}_k-\mathbf{W}_k\mathbf{P}^2)$
$\mathbf{U}_0=-\mathfrak{M}\cdot\mathbf{P}$	$-i\mathbf{P}^2$	$-i\mathbf{P}_0\mathbf{P}_m$	$-i(\mathfrak{M}\times\mathbf{P})_m$	$-i\mathbf{U}_m$	$-i\mathbf{W}_0\mathbf{P}_0$
$\mathbf{U}_k$	$-i\mathbf{P}_k\mathbf{P}_0$	$-i(m^2\delta_{km}+\mathbf{P}_k\mathbf{P}_m)$	$i[\mathbf{U}_n\mathbf{P}_0-(\mathfrak{M}\times\mathbf{P})_m\mathbf{P}_k]/\mathbf{P}_0$	$-i\mathbf{U}_0\delta_{km}$	$-i\mathbf{W}_k\mathbf{P}_0$
$(\mathbf{U}_0\mathbf{P}-\mathbf{U}\mathbf{P}_0)_k/m^2$	$i\mathbf{P}_k$	$i\mathbf{P}_0\delta_{km}$	$i[\mathfrak{M}_n+(\mathbf{W}\times\mathbf{P})_n/m^2]$	$-i[\mathbf{M}_n+(\mathbf{W}_0\mathbf{P}_n-\mathbf{W}_n\mathbf{P}_0)/m^2]$	$-i(\mathbf{W}_0\mathbf{P}_k-\mathbf{W}_k\mathbf{P}_0)\mathbf{P}_0/m$

	$\mathbf{W}_m$	$(\mathbf{W}_0\mathbf{P}-\mathbf{W}\mathbf{P}_0)_m$	$(\mathbf{W}\times\mathbf{P})_m$	$\mathbf{U}_0=-\mathfrak{M}\cdot\mathbf{P}$
$\mathbf{P}_0$	0	0	0	$i\mathbf{P}^2$
$\mathbf{P}_k$	0	0	0	$i\mathbf{P}_0\mathbf{P}_k$
$\mathbf{M}_k$	$i\mathbf{W}_n$	$i(\mathbf{W}_0\mathbf{P}_n-\mathbf{W}_n\mathbf{P}_0)$	$i(\mathbf{W}\times\mathbf{P})_n$	$i(\mathfrak{M}\times\mathbf{P})_k$
$\mathfrak{M}_k$	$i\mathbf{W}_0\delta_{km}$	$i(\mathbf{W}\times\mathbf{P})_n$	$-i(\mathbf{W}_0\mathbf{P}_n-\mathbf{W}_n\mathbf{P}_0)$	$i\mathbf{U}_k$
$\mathbf{W}_0$	$i(\mathbf{W}\times\mathbf{P})_m$	$-i(\mathbf{W}\times\mathbf{P})_m\mathbf{P}_0$	$i(\mathbf{W}_0\mathbf{P}_0\mathbf{P}_m-\mathbf{W}_m\mathbf{P}^2)$	$i\mathbf{W}_0\mathbf{P}_0$
$\mathbf{W}_k$	$-i(\mathbf{W}_0\mathbf{P}_n-\mathbf{W}_n\mathbf{P}_0)$	$-i[m^2\mathbf{W}_n\delta_{km}+\mathbf{P}_k(\mathbf{W}\times\mathbf{P})_m]$	$i[m^2\mathbf{W}_0\delta_{km}+\mathbf{P}_k(\mathbf{W}_0\mathbf{P}_m-\mathbf{W}_m\mathbf{P}_0)]$	$i\mathbf{W}_0\mathbf{P}_k$
$(\mathbf{W}_0\mathbf{P}-\mathbf{W}\mathbf{P}_0)_k$	$i[-m^2\mathbf{W}_n\delta_{km}+(\mathbf{W}\times\mathbf{P})_k\mathbf{P}_m]$	$im^2\mathbf{P}_0\mathbf{W}_n$	$im^2[\mathbf{P}_k\mathbf{W}_m-\mathbf{W}_0\mathbf{P}_0\delta_{km}]$	$i(\mathbf{W}_0\mathbf{P}_0\mathbf{P}_k-\mathbf{W}_k\mathbf{P}^2)$
$(\mathbf{W}\times\mathbf{P})_k$	$-i[m^2\mathbf{W}_0\delta_{km}+(\mathbf{W}_0\mathbf{P}_k-\mathbf{W}_k\mathbf{P}_0)\mathbf{P}_m]$	$-im^2[\mathbf{W}_k\mathbf{P}_m-\mathbf{W}_0\mathbf{P}_0\delta_{km}]$	$im^2\mathbf{W}_0\mathbf{P}_n$	$i(\mathbf{W}\times\mathbf{P})_k\mathbf{P}_0$
$\mathbf{U}_0=-\mathfrak{M}\cdot\mathbf{P}$	$-i\mathbf{W}_0\mathbf{P}_m$	$-i(\mathbf{W}_0\mathbf{P}_0\mathbf{P}_m-\mathbf{W}_m\mathbf{P}^2)$	$-i(\mathbf{W}\times\mathbf{P})_m\mathbf{P}_0$	0
	$-i\mathbf{W}_k\mathbf{P}_m$	$-i[m^2\mathbf{W}_0\delta_{km}+\mathbf{P}_k(\mathbf{W}_0\mathbf{P}-\mathbf{W}\mathbf{P}_0)_m]$	$i[\mathbf{P}_k(\mathbf{W}\times\mathbf{P})_m-m^2\mathbf{W}_n\delta_{km}]$	$im^2\mathfrak{M}_k$
	$-i(\mathbf{W}_0\mathbf{P}_k-\mathbf{W}_k\mathbf{P}_0)\mathbf{P}_m/m^2$	$i(\mathbf{W}_0\mathbf{P}_0\delta_{km}-\mathbf{W}_m\mathbf{P}_k)$	$i\mathbf{W}_n\mathbf{P}_0$	$i[\mathbf{U}_k+(\mathbf{W}\times\mathbf{P})_k\mathbf{P}_0/m^2]$

In the first three sets of relations $\mathbf{M}$ is to be interpreted either as $\mathbf{L}$ or $\mathbf{S}$ provided we at the same time interpret $\mathfrak{M}$ as $\mathfrak{L}$ or $\mathfrak{S}$ correspondingly. Otherwise we use $\mathbf{M}$ and $\mathfrak{M}$ in the more inclusive sense of $\mathbf{M}=\mathbf{L}+\mathbf{S}$ and $\mathfrak{M}=\mathfrak{L}+\mathfrak{S}$. In the remaining equations we have separated out the spin and orbital parts explicitly.

A more detailed list of commutators, including those for certain interesting quantities constructed from the polarization pseudovector $\mathbf{W}_a$, is given in Table VIII. It is worthwhile also to remark that a general formula may be established for the commutator of $\mathfrak{M}_k$ with any function of the energy $\mathbf{P}_0$ which is expressible as a power series in $\mathbf{P}_0$, i.e.,

$$[\mathfrak{M}_k, f'(\mathbf{P}_0)] = if(\mathbf{P}_0)\mathbf{P}_k. \qquad \text{(I')}$$

This is established by induction. First, it is easily verified that it holds not only for $f(\mathbf{P}_0)=\mathbf{P}_0$, Eq. (If), but also for $f(\mathbf{P}_0)=\mathbf{P}_0^{-1}$. Then it is established that if it holds for $\mathbf{P}_0^n$ it holds also for $\mathbf{P}_0^{n+1}$ and $\mathbf{P}_0^{n-1}$.

The first two commutation relations in Eq. (If) express the total constancy in time of $\mathbf{P}_k$ and $\mathbf{M}_k$ (neither of which has any explicit time dependence). For the physical interpretation of the last relation we suppose only that we are dealing with a system describable in the most general terms of a field theory. By definition, the dynamical quantities which we have listed are then constructable from integrals involving the energy-momentum components of the stress-energy-momentum density tensor $\mathbf{T}_{ik}$.[31] With $\mathbf{T}_{00}$ representing the energy density, and $\mathbf{T}_{0k}$ the k component of the momentum density, the constructions are

$$\textit{energy:} \qquad \mathbf{H} \equiv \mathbf{P}_0 = \int dv\,\mathbf{T}_{00}$$

[31] The tensor $\mathbf{T}_{ik}$ itself may be constructed out of field operators on the basis of an invariant Lagrangian, following the prescriptions of the "canonical formalism." We do not, however, need this explicit construction for the following purposes. For simplicity, and to compare with the case illustrated in Fig. 2, *we limit ourselves in the following two paragraphs to a system with spin zero* (the canonical stress-energy-momentum tensor $\mathbf{T}_{ik}$ is symmetric); the centroidal moment, as well as the angular momentum, is then a constant of the motion.

linear momentum: $$\mathbf{P}_k=\int dv\mathbf{T}_{0k}$$

orbital angular momentum: $$\mathbf{L}_k=\int dv(x_i\mathbf{T}_{0j}-x_j\mathbf{T}_{0i})\cdot$$

orbital centroidal moment: $$\mathfrak{L}_k=\int dv(x_k\mathbf{T}_{00}-x_0\mathbf{T}_{0k}),$$

where the integration is over the total simultaneous three-dimensional space v at time $x_0\equiv t$ in a given Lorentz frame.

The reason for calling the last quantity "orbital centroidal moment," is apparent if we first discuss the integral of the first term in the expression for $\mathfrak{L}_k$. This quantity,

$$\mathfrak{L}_k'=\int dv\; x_k\mathbf{T}_{00}, \tag{II}$$

is the first moment, with respect to a "lab" coordinate frame, of the energy density $\mathbf{T}_{00}$. A limited definition of the "position of the centroid" of this energy distribution is possible even in quantum mechanics (see the following), and therefore we use the term "lab orbital centroidal moment" for $\mathfrak{L}'$. We may now write

$$\mathfrak{L}_k=\mathfrak{L}_k'-\mathbf{P}_k t. \tag{II$'$}$$

It is clear that the significance of $\mathfrak{L}$ is that it is the "proper orbital centroidal moment," i.e., if there exists a proper Lorentz frame—one in which the total momentum $\mathbf{P}_k$ vanishes—then $\mathfrak{L}_k$ are the components in that frame of the first moment of the energy distribution. An immediate interpretation of the last commutation relation in Eq. (If) is that it expresses the net constancy in time of $\mathfrak{L}_k$ in accord with the law of motion, i.e.,

$$d\mathfrak{L}_k/dt=i[\mathbf{H},\mathfrak{L}_k]+\partial\mathfrak{L}_k/\partial t=\mathbf{P}_k-\mathbf{P}_k=0.$$

The intuitive meaning of this result has already been indicated in Sec. I.C and Fig. 2.

The statement that $\mathfrak{L}_k$ is constant in time is of course just another way of stating that the motion of the system as a whole is uniform.[32] Writing the last commutation relation in Eq. (If) explicitly, and using $[\mathbf{P}_k,\mathbf{P}_0]=0$, we get the result

$$\mathbf{H}\mathfrak{L}_k'-\mathfrak{L}_k'\mathbf{H}=-i\mathbf{P}_k. \tag{If$'$}$$

But by the basic law of motion this gives

$$d\mathfrak{L}_k'/dt=\mathbf{P}_k, \tag{If$''$}$$

which can be described as the field-theoretic version of the relation of velocity to momentum in a one-particle theory,

$$E\dot{x}_k=P_k$$

where E is the total energy.

We leave the interpretation of the commutation relations Eqs. (Ia), (Ib), and (Id) simply at this—that they express the mutual interference of measurement (and consequent nonsimultaneous sharpness) of different components of angular momentum and centroidal moment with each other and with linear momentum.

More freshly interesting interpretations, which bear on the concept of centroidal position may be read out of the relations (Ie) and (If). Before discussing these interpretations we consider briefly the question of what quantities we can add, while maintaining *full space-time covariance*, to $\mathbf{J}\equiv(\mathbf{M},\mathfrak{M})$ without changing any of the commutation relations. This question, which is of interest in itself, will be given a more detailed investigation elsewhere. Here we limit ourselves to the main features.

To this purpose we investigate the two bivectors which may be constructed by taking the "outer product" of the four vector $(\mathbf{P}_0,\mathbf{P})$ with the vectors $(\mathbf{W}_0,\mathbf{W})$ and $(\mathbf{U}_0,\mathbf{U})$, respectively (see Table VIII). These bivectors are

$$\mathbf{A}\equiv(-\mathbf{U}\times\mathbf{P},\,\mathbf{U}_0\mathbf{P}-\mathbf{U}\mathbf{P}_0)\quad \mathbf{B}\equiv(\mathbf{W}_0\mathbf{P}-\mathbf{W}\mathbf{P}_0,\,\mathbf{W}\times\mathbf{P}).$$

These bivectors are not, however, independent of each other and of $\mathbf{J}$. It can be verified that the three are related by the identity

$$\mathbf{A}\equiv\mathbf{B}+m^2\mathbf{J}.$$

Thus, it is sufficient to consider the possibility of adding to $\mathbf{J}$ a term solely in $\mathbf{B}$. It is easily verified that the only linear combination $a\mathbf{J}+b\mathbf{B}$ which preserves all the commutation relations involving the $\mathbf{M}$ part of $\mathbf{J}$ is the one for which $a=1$, $b=0$.

This attempt having failed, we are then led to consider a generalization which does not have full space-time covariance, i.e., one involving the $\mathfrak{M}$ part of $\mathbf{J}$ alone. Naturally, we try to keep the covariance which holds within any given Lorentz frame. We call any three-component operator $\mathfrak{R}$ a *complete moment operator* provided it satisfies the following three postulates:

(A) *Like* $\mathfrak{L}$ *it is a vector under space rotations* [i.e., it satisfies Eq. (Ib)] *which is odd under space reflection* (polar) *and even under time reversal.*

(B) *It also satisfies equations like* Eqs. (Ie) *and* (If). *We write these as follows:*

$$[\mathfrak{R}_j,\mathbf{P}_k]=i\mathbf{P}_0\delta_{jk} \tag{IIIa}$$

$$[\mathfrak{R}_k,\mathbf{P}_0]=i\mathbf{P}_k\quad d\mathfrak{R}_k/dt=0 \tag{IIIb}$$

[32] The "system as a whole" refers to some kind of centroid, defined for instance by Eq. (VI) in the following. Equation (If″) signifies that all such centroids, defined in differently moving lab frames, are at rest in the proper frame.

(C) *It satisfies* Eq. (Ic), *with the ordinary angular momentum* $\mathbf{M}$ *on the right-hand side.*

Postulate (A) requires that any moment operator be of the same dimensions and vectorial nature as $\mathfrak{L}$ or $\mathfrak{L}'$. These have the structure of an energy times a distance and are therefore odd under space reflection and even under time reversal, in contrast with angular momentum which behaves oppositely.

In postulate (B), Eq. (IIIa) guarantees that the quantities $\mathfrak{R}_j$ "divided" by $\mathbf{P}_0$ (see Eq. (VII) in the following) satisfy the canonical commutation relations of position and momentum. Equation (IIIb) states the constancy in motion of $\mathfrak{R}$—and implies that the explicit time dependence of $\mathfrak{R}$ is the same as that of $\mathfrak{L}$, i.e., $-\mathbf{P}t$.

We do not discuss postulate (C) here, but merely remark that if an operator satisfies only postulates (A) and (B), we omit the adjective "complete," denoting it by the term *moment operator*.

We work within the framework of quantities constructable from the generators of the Poincaré group (see Table VIII). The only independent spacial vectors, commuting with all $\mathbf{P}_a$ and polar with respect to space reflections, which are available are scalar coefficients times $\mathbf{P}$ and $\mathbf{W}\times\mathbf{P}$, or a pseudoscalar coefficient times $\mathbf{W}$. (Other possibilities, such as the vectors $\mathbf{W}P_0-W_0\mathbf{P}$ and $(\mathbf{W}\times\mathbf{P})\times\mathbf{W}$, are linear combinations of $\mathbf{P}$ and $\mathbf{W}$. Forms such as $\mathbf{M}\times\mathbf{P}$ or $\mathbf{U}\equiv-\mathfrak{M}P_0-\mathbf{M}\times\mathbf{P}$—see Table VIII—are linear combinations of $\mathbf{W}\times\mathbf{P}$, $\mathbf{P}$, and $\mathfrak{M}$, provided we allow for the presence of the spacial scalar $\mathfrak{M}\cdot\mathbf{P}=-\mathbf{U}_0$ in front of $\mathbf{P}$. Actually, we see from the last column of Table VIII that the presence of such a $\mathbf{U}_0$ would spoil Eqs. (III). Alternatively, from the requirement—postulate (B)—that the explicit time dependence of $\mathfrak{R}_k$ be $-\mathbf{P}_kt$, we infer that no other form involving $\mathbf{P}_k$ occurs.)

We see then that to satisfy postulates (A) and (B) the category of moment operators is limited to those which can be obtained by adding to $\mathfrak{M}$ operators of the form

$$\alpha(m,\mathbf{W}^2,\mathbf{E};t)\mathbf{W}\times\mathbf{P}\quad\gamma(\mathbf{W}_0;m,\mathbf{W}^2,\mathbf{E};t)\mathbf{W},$$

where the coefficients α and γ are functions of the indicated arguments. The function α is scalar (even) under space reflection and even under time reversal, while γ is pseudoscalar (odd) under space reflection and odd under time reversal. To indicate the pseudoscalarity of γ we have included among its arguments the pseudoscalar quantity $\mathbf{W}_0=\mathbf{W}\cdot\mathbf{P}/\mathbf{P}_0=\mathbf{M}\cdot\mathbf{P}$, of which γ must be an odd function.

For simplicity we limit the discussion in the following to the possibility of adding the term $\alpha\mathbf{W}\times\mathbf{P}$, leaving for elsewhere consideration of the term in $\mathbf{W}$.[33] It is then found, with the help of Table VIII, that imposing postulate (C) leads to the following condition on the function α:

$$\mathbf{W}(\alpha'\mathbf{P}^2+2\alpha P_0)=\mathbf{P}W_0(\alpha'P_0+2\alpha-\alpha^2m^2),\qquad\text{(IV)}$$

where α' is the derivative of α with respect to E. The solution of this equation, when $\mathbf{W}$ is not parallel to $\mathbf{P}$, is easily found since right and left sides separately must be zero (nonparallelism of $\mathbf{W}$ and $\mathbf{P}$ means the "mass" m is not zero, for it can be shown that $m=0$ implies $W_a=\lambda P_a$ where $\lambda=W_0/P_0$ is the longitudinal polarization or helicity—spin in this case). The two equations are compatible and yield the unique solution

$$\alpha=-2/\mathbf{P}^2=-2/(E^2-m^2).$$

In the case of $m=0$, the equation is automatically satisfied without any restriction on α. But of course the additive term $\mathbf{W}\times\mathbf{P}$ then vanishes.

We now relinquish the requirement that postulate (C) be satisfied and return to the consideration of moment operators in general in order to be able ultimately to define a position operator with commuting components. We rewrite Eqs. (IIIa,b) as follows:

$$\mathfrak{R}_j\mathbf{P}_k-\mathbf{P}_k\mathfrak{R}_j=i\delta_{jk}\mathbf{E}\quad\mathfrak{R}_k\mathbf{E}-\mathbf{E}\mathfrak{R}_k=i\mathbf{P}_k.\qquad\text{(III}'\text{a,b)}$$

The symbol $\mathbf{E}$ has been written instead of $\mathbf{P}_0$ for the total energy for the following reason: *we want the total energy operator to have an inverse*. This can be assured if it is positive definite *per se*, or if we limit ourselves to the domain of Hilbert space in which it is positive definite (or, alternatively, negative definite). This means that great care must be exercised in applying the following considerations to processes involving the vacuum state. The energy operator with this limitation we call $\mathbf{E}$. As $\mathbf{E}$ is positive definite it has an inverse $\mathbf{E}^{-1}$ and this is Hermitian like $\mathbf{E}$. Since $\mathbf{E}$ commutes with all the $\mathbf{P}_k$, its inverse does likewise and we may define the *velocity operator*

$$\mathbf{V}_k=\mathbf{P}_k\mathbf{E}^{-1}=\mathbf{E}^{-1}\mathbf{P}_k.\qquad\text{(V)}$$

It is easily verified that the $\mathbf{V}_k$ satisfy commutation relations like those which the $\mathbf{P}_k$ satisfy except for the following [use Eq. (I′)]:

$$[\mathfrak{M}_j,\mathbf{V}_k]=i(\delta_{jk}-\mathbf{V}_j\mathbf{V}_k).\qquad\text{(V}'\text{)}$$

To make a comparison with familiar ideas we introduce the "centroidal position operator" vector $\mathbf{X}_k$ by the following definition:

$$\mathbf{X}_k=\tfrac{1}{2}(\mathfrak{R}_k\mathbf{E}^{-1}+\mathbf{E}^{-1}\mathfrak{R}_k).\qquad\text{(VI)}$$

It is evident from the form of Eq. (II) that in the special case $\mathfrak{R}=\mathfrak{L}'$, this symmetrized, and therefore Hermitian, operator is an approach to defining a "center-of-mass" (or rather "center-of-energy") of the

[33] It is not difficult to show that inclusion of the term in $\gamma\mathbf{W}$, as well as consideration of a possible dependence of α on $\mathbf{W}_0{}^2$ does not alter the final solution essentially.

system, and in this case we could call $\mathbf{X}$ the "orbital position" of the centroid.

With the help of Eq. (IIIb) we have

$$\begin{aligned}\mathbf{E}\mathbf{X}_k&=\tfrac{1}{2}(\mathbf{E}\mathfrak{R}_k\mathbf{E}^{-1}+\mathfrak{R}_k)=\mathfrak{R}_k-\tfrac{1}{2}i\mathbf{V}_k\\ \mathbf{X}_k\mathbf{E}&=\tfrac{1}{2}(\mathfrak{R}_k+\mathbf{E}^{-1}\mathfrak{R}_k\mathbf{E})=\mathfrak{R}_k+\tfrac{1}{2}i\mathbf{V}_k,\end{aligned}\qquad \text{(VI}'\text{a,b)}$$

from which we may infer the relations,

$$d\mathbf{X}/dt=i(\mathbf{E}\mathbf{X}_k-\mathbf{X}_k\mathbf{E})=\mathbf{V}_k,\qquad \tfrac{1}{2}(\mathbf{E}\mathbf{X}_k+\mathbf{X}_k\mathbf{E})=\mathfrak{R}_k,\qquad \text{(VI}''\text{a,b)}$$

which show again the consistency of the definitions (V) and (VI) of the velocity and "position" operators.

By using Eqs. (VI′) and (IIIa), we have

$$\mathbf{X}_i\mathbf{P}_k-\mathbf{P}_k\mathbf{X}_i=[\mathbf{E}^{-1}\mathfrak{R}_j,\mathbf{P}_k]=i\delta_{jk},\qquad \text{(VII)}$$

since the velocity components all commute with each other. Thus the commutation relations between a centroidal position operator and the total momentum of the field are of the canonical form of the position-momentum commutation laws in one-particle mechanics.

Equations (Ie) show also that in field theory, just as in one-particle mechanics, the components of $\mathbf{P}$ commute with each other. It is not true, however, that the components of a centroidal position operator always commute with each other. By letting the commutator of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be represented by $-i\mathbf{R}_3$ (in the special case where $\mathfrak{R}=\mathfrak{M}$ we have $\mathbf{R}_3=\mathbf{M}_3$), we find, when Eqs. (VI′) are substituted, that

$$[\mathbf{E}\mathbf{X}_1,\mathbf{E}\mathbf{X}_2]=(\mathbf{X}_1\mathbf{E},\mathbf{X}_2\mathbf{E}]=-i\mathbf{R}_3\ \textit{et cycl},\qquad \text{(VIII)}$$

where β' is the derivative of β with respect to $\mathbf{E}$, and where we have used Eq. (V′). With the help of the expression for the commutator of $\mathbf{E}$ and $\mathbf{X}$, Eq. (VI″a), this may be written

$$\begin{aligned}\mathbf{E}\mathbf{E}[\mathbf{X}_1,\mathbf{X}_2]&=-i[\mathbf{R}_3-(\mathbf{X}_1\mathbf{P}_2-\mathbf{X}_2\mathbf{P}_1)]\\ &=-i[\mathbf{R}_3-\mathbf{E}^{-1}(\mathfrak{R}_1\mathbf{P}_2-\mathfrak{R}_2\mathbf{P}_1)].\end{aligned}\qquad \text{(VIII}'\text{)}$$

For the particular choice of the moment operator, $\mathfrak{R}=\mathfrak{M}$, the bracket on the right-hand side of Eq. (VIII′) equals

$$\mathbf{M}-\mathbf{E}^{-1}\mathfrak{M}\times\mathbf{P}=\mathbf{S}-\mathbf{E}^{-1}\mathfrak{S}'\times\mathbf{P}=\mathbf{E}^{-1}\mathbf{W},$$

where $\mathbf{W}$ is the "inner angular momentum," or the space part of the polarization 4-vector, defined in Sec. II.F. Thus, the components of the position operator $\mathbf{X}$ defined by Eq. (VI) for the moment $\mathfrak{M}$ are not in general commuting. They commute, and can serve as a simultaneous set of observables for a system with rest mass if the spin is zero. (From the vanishing of W^2 and of $\mathbf{S}$, the vanishing of $\mathfrak{S}\times\mathbf{P}$ and therefore of $\mathbf{W}$ follows.) For the mass zero case $\mathbf{W}$, which is then proportional to $\mathbf{S}$, can vanish *only* if the spin is zero. We can, however, choose a modified moment $\mathfrak{R}=\mathfrak{M}+\mathfrak{R}'$ so as to make the right side of Eq. (VIII′) vanish for all cases of nonzero mass and spin. We write $\mathfrak{R}'$ in the general form $\beta\mathbf{W}\times\mathbf{P}$ where β is a function of $\mathbf{E}$, m, etc., to be determined. After a systematic calculation with the help of Table VIII, the condition for the vanishing of the right-hand side of Eq. (VIII′) becomes

$$\begin{aligned}\mathbf{W}[\beta'\mathbf{P}^2+\beta E^{-1}(E^2+m^2)-E^{-1}]\\ =\mathbf{P}W_0[\beta'E+\beta-\beta^2m^2].\end{aligned}\qquad \text{(IX)}$$

For $\mathbf{W}$ parallel to $\mathbf{P}$ the only admissible solution is $\mathbf{W}_0=0$ (zero spin), β arbitrary. For $\mathbf{W}$ not parallel to $\mathbf{P}$ ($m\neq 0$), each side must vanish separately. The equations are compatible and we find that there are two solutions,

$$\beta_+=1/m(E+m)\quad \beta_-=-[1/m(E-m)],\qquad \text{(IX}'\text{)}$$

and it is noted that $\beta_++\beta_-=+\alpha$, where α was the coefficient of $\mathbf{W}\times\mathbf{P}$ in the complete moment operator defined by Eq. (IV). The commuting position operators corresponding to the solutions (IX′) are

$$\begin{aligned}\mathbf{Q}_+&=\mathbf{X}+(m\mathbf{E})^{-1}(\mathbf{E}+m)^{-1}\mathbf{W}\times\mathbf{P}\\ \mathbf{Q}_-&=\mathbf{X}-(m\mathbf{E})^{-1}(\mathbf{E}-m)^{-1}\mathbf{W}\times\mathbf{P}.\end{aligned}\qquad \text{(X)}$$

The commuting position operator $\mathbf{Q}_+$ was found by Pryce (1935, 1948) in a different manner. Such an operator permits a "localized particle interpretation." Newton and Wigner (1949), making a "Schrödinger type" analysis instead of the "Heisenberg type" analysis employed by Pryce and by us here, showed on the basis of certain postulates for localizability of an elementary system that the localized position operator $\mathbf{Q}_+$ (for positive energy states) is unique. They gave explicit forms for the operator and its eigenfunctions in all cases where their analysis showed that it exists, i.e., for massive systems with any spin and massless systems with spin 0 and $\frac{1}{2}$. Full bibliographic references to earlier and later papers on this subject are given by Wightman and Schweber (1955).

IX. GLOSSARY OF NOTATIONS

(Within each category in order of appearance)

(a) Scalars (Italic type):

λ, c, λ_a, δ_{jk}, x_j, t, ϵ, φ, R_{ij}, $g_{\mu\nu}$, Λ_{ij}, a, b, a_{ij}, b_{ij}, θ_i, α_1, θ_{ij}, $\varphi_i(\xi)$, y, ψ, μ, ω, q, T_3, B, ρ, ρ_M, ξ.

Complex conjugates: λ^*, μ^*, $\lambda_P{}^*$.

(b) Vectors and Matrices (boldface):

$\mathbf{N}$, $\mathbf{P}$, $\mathbf{r}$, $\mathbf{x}$, $\mathbf{R}$, $\boldsymbol{\Lambda}$, $\mathbf{a}$, $\mathbf{G}$, $\mathbf{b}$, $\mathbf{P}$, $\boldsymbol{\sigma}$, $\mathbf{I}_s$, $\boldsymbol{\Lambda}_+{}^\dagger$, $\mathbf{M}$, $\mathbf{A}(s)$, $\mathbf{B}(s)$, $\mathbf{I}$, $S^{\alpha\beta}(\mathbf{R})$, $\mathbf{U}_i$, $\mathbf{W}$, $\boldsymbol{\xi}$.

Transposed matrices: $\mathbf{R}^T$, $\boldsymbol{\lambda}^T$.

Hermitian conjugates: $\mathbf{U}_k{}^\dagger$.

(c) Operators (corresponding Roman and script letters are used for space-space and space-time parts of six-vectors):

$$A,\ P,\ G,\ T,\ C,\ r_i,\ L(a,\Lambda),\ M_i,\ P_i,\ \mathcal{M}_k,\ S,\ \mathcal{S},\ P,\ \nabla,\ \Box,\ \boldsymbol{\gamma}_n,\ \boldsymbol{\pi}.$$

(d) Operators in Hilbert space (sans serif M_k, German $\mathfrak{M}_k$, or last letters of Greek alphabet Φ; corresponding sans serif and German letters are used for space-space and space-time parts of six-vectors):

$$\mathsf{F},\ \mathsf{I},\ \mathsf{Q},\ \mathfrak{p},\ \mathfrak{q},\ \mathfrak{a},\ \mathfrak{b},\ \mathsf{N},\ \mathsf{U},\ \mathsf{A},\ \mathsf{H},\ \mathsf{P}_k,\ \mathsf{M}_k,\ \mathfrak{M}_k,\ \mathsf{P},\ \mathsf{V},\ \mathsf{B},\ \mathsf{T},\ \mathfrak{N}_k,\ \mathfrak{L}_k,\ \mathfrak{S}_k,\ \mathfrak{S},\ \mathsf{G},\ \mathsf{K},\ \Phi,\ \boldsymbol{\Psi}_a,\ \boldsymbol{\Phi}_a,\ \mathsf{W},\ \pi.$$

Hermitian conjugates: $\mathfrak{a}_k^\dagger$, $\mathfrak{b}_k^\dagger$, $\mathsf{P}^\dagger$, $\Psi^\dagger$.

(e) Groups of matrices or operators:

$$\{\Lambda_+\},\ \{\Lambda^\uparrow\},\ \{\Lambda_+^\uparrow\},\ \{\Lambda\},\ \{L_+^\uparrow\},\ \{G\},\ \{F\}.$$

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