Also

$$\mathbf{Z}_\alpha{}^\dagger=\left[\begin{array}{cccc|cccc} 1&0&0&0&&&& \\ 0&0&1&0&&&& \\ \hline &&&&1&0&0&0 \\ &&&&0&0&1&0 \\ \hline &&&&&&& \\ &&&&&&& \end{array}\right],$$

$$\mathbf{D}_\alpha{}^\dagger=\left[\begin{array}{cccc|cccc} 0&1&0&0&&&& \\ 0&0&0&1&&&& \\ \hline &&&&0&1&0&0 \\ &&&&0&0&0&1 \\ \hline &&&&&&& \\ &&&&&&& \end{array}\right],$$

so that

$$\mathbf{D}_\alpha{}^\dagger+\mathbf{G}_r{}^\dagger\mathbf{Z}_\alpha{}^\dagger=\left[\begin{array}{cccc|cccc} c_0&1&0&0&&&& \\ c_1&0&c_0&1&&&& \\ \hline c_q&0&0&0&c_0&1&0&0 \\ 0&0&c_q&0&c_1&0&c_0&1 \\ \hline c_s&0&0&0&&&& \\ 0&0&c_s&0&&&& \\ \hline &&&&c_s&0&0&0 \\ &&&&0&0&c_s&0 \end{array}\right].$$

Modifications for side, edge, and corner effects can be determined in the manner of Secs. 5 and 6. The problem of obtaining the largest eigenvalue of matrices with this type of structure is under study.

CONCLUSION

We have shown that (1) a very general type of problem in statistical mechanics can be reduced to an associative combinatorial problem, and (2) this problem can again be reduced to the determination of the set of eigenvalues of a certain matrix.

A variety of problems already known to be soluble are easily handled by the automatic application of this technique, and at least one problem is removed from the unsolved category. We believe that the systematic exploitation of this method will provide many interesting new results.

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Linearized Theory of Plasma Oscillations*

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I. Hydrodynamic Treatment

INTRODUCTION

BASICALLY there are two ways of dealing with plasma problems: a microscopic gas-kinetic treatment using the Boltzmann equation together with Maxwell's equations of electrodynamics; or a macroscopic, hydrodynamic treatment using Euler's equation of motion together with the Maxwell equations. For various mathematical reasons it seems to be impossible to investigate the different general modes of oscillation using gas-kinetic methods without serious physical restrictions. To avoid excessively complicated mathematics in using the kinetic approach, it is necessary to make assumptions of such kind that it is more reasonable to use the hydrodynamic equations. For this reason we deal here only with the hydrodynamic equations together with Maxwell's equations. Questions concerning the range of validity of our treatment are deferred to Sec. II. The hydrodynamic treatment is always justified when there is a *stationary* distribution of velocities in the plasma which is not disturbed "essentially" by the collective oscillations.

Although one succeeds in this way in simplifying the procedure a great deal, the treatment of the unabridged hydrodynamic equations [except in a few cases such as the work of R. W. Larenz (1955)†] is further simplified. For this purpose, one supposes the plasma to be uniform and of infinite extent, and the oscillations to be small sinusoidal perturbations. These concepts are not very close to reality, and are unable to explain complicated processes such as the origin of cosmic radio-frequency radiation. Still, the linearized theory succeeds well in explaining the ionospheric observations, so one may hope that at least some idea is obtained of how and where to begin a later nonlinear approach.

All investigations of the linearized theory, until now, dealt with special cases—Langmuir oscillations, ionospheric theory, Alfvén's magnetohydrodynamics, etc.—

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† References are given in alphabetical order in Bibliography.

hence, it would be interesting to describe the whole field of applications by using a uniform treatment.

We start by deriving the basic equations as completely as possible, and then try to clear up the relation between different approaches previously published. Then we discuss the possible basic modes of plasma oscillations; the influence of translatory motions, i.e., the influence of a moving reference frame (here we find some serious errors in the literature); and the case of a nonzero constant magnetic field. We then obtain a connection with Alfvén's magnetohydrodynamics.

1. Basic Equations

The basic equations may be written in such a general form that all special cases to be dealt with later are included; however, we do not include gravitational forces, macroscopic electrical fields, and macroscopic gradients of density or temperature. The plasma is assumed to be of infinite extent and macroscopically neutral. It consists, on the average, of N_0 electrons per cm^3, and of the same number of singly ionized ions. There are to be no neutral particles; their influence upon the plasma characteristics is discussed elsewhere, e.g., A. Schlueter (1951).

All equations are referred to a coordinate system in which the plasma, as a whole, may have a constant but otherwise unrestricted velocity. In this system the electrons and the ions in a given volume element have the actual velocities $\mathbf{V}_e$ and $\mathbf{V}_i$, respectively; $\mathbf{V}_e$ and $\mathbf{V}_i$ therefore may include constant average velocities which, in general, are supposed to have the same direction and absolute value for both constituents. We assume the validity of the Galilean transformation for all coordinate transformations; relativistic effects are considered only when specially noted.

For completeness we introduce a damping term in our basic equations which is proportional to the velocity itself. In general we do not consider dissipative effects, but try to estimate their importance in Sec. I.9; in that section we also discuss different alternative formulations. The constant of proportionality α in the damping term contains the scattering cross section for electron-ion and electron-electron collisions which can be calculated only by kinetic theory; for this reason we do not consider the derivation of this constant.

In general, we allow for compressibility and specialize only at times (e.g., in the case of magnetohydrodynamics) to an incompressible gas, but we deal always with a scalar pressure. We do not define immediately the connection between the partial pressures P_e and P_i of electrons and ions and the actual electron density N_e and ion density N_i; therefore, the parameters T_e and T_i which are identified later with electron and ion temperatures may be any function of N_e and N_i.

Let $\mathbf{E}$ be the alternating electrical field, and $\mathbf{H}$ the magnetic field; the latter may include a time-independent term. The universal constants are the elementary charge (electron charge: $-e$); the electron and ion masses m_e and m_i; the Boltzmann constant K; and the velocity of light c.

The hydrodynamic equations of motion for the electrons and ions are

$$\frac{\partial}{\partial t}\mathbf{V}_e+(\mathbf{V}_e\cdot\nabla)\mathbf{V}_e=-\frac{e}{m_e}\left[\mathbf{E}+\frac{1}{c}(\mathbf{V}_e\times\mathbf{H})\right]-\frac{1}{N_em_e}\nabla P_e-\frac{N_i}{m_e}\alpha(\mathbf{V}_e-\mathbf{V}_i), \quad (1)$$

$$\frac{\partial}{\partial t}\mathbf{V}_i+(\mathbf{V}_i\cdot\nabla)\mathbf{V}_i=+\frac{e}{m_i}\left[\mathbf{E}+\frac{1}{c}(\mathbf{V}_i\times\mathbf{H})\right]-\frac{1}{m_iN_i}\nabla P_i+\frac{N_e}{m_i}\alpha(\mathbf{V}_e-\mathbf{V}_i). \quad (2)$$

These equations include a damping term proportional to the velocity differences; the expression $N_{e,i}\alpha/m_{e,i}$ is the mean collision frequency. We take

$$P_e=N_eKT_e(N_e,N_i), \quad (3)$$

$$P_i=N_iKT_i(N_e,N_i). \quad (4)$$

The equations of continuity are

$$\nabla\cdot(N_e\mathbf{V}_e)+\frac{\partial}{\partial t}N_e=0, \quad (5)$$

$$\nabla\cdot(N_i\mathbf{V}_i)+\frac{\partial}{\partial t}N_i=0. \quad (6)$$

The Maxwell equations become

$$c\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{H}, \quad (7)$$

$$c\nabla\times\mathbf{H}=4\pi\mathbf{j}+\frac{\partial}{\partial t}\mathbf{E}, \quad (8)$$

$$\mathbf{j}=e(N_i\mathbf{V}_i-N_e\mathbf{V}_e), \quad (9)$$

$$\nabla\cdot\mathbf{E}=4\pi e(N_i-N_e). \quad (10)$$

Equation (10) is identical with one component of the Maxwell equations but is easier to handle.

E. P. Gross (1951) has used, instead of the Maxwell equations, the equation

$$\nabla\times\nabla\times E=-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}-\frac{4\pi}{c}\frac{\partial\mathbf{j}}{\partial t}. \quad (11)$$

Here the magnetic field is eliminated. The use of Eq. (11) instead of Eqs. (7) and (8) makes no difference in the results.

In the case that

$$\nabla\times\mathbf{E}=0, \tag{12}$$

we may use a scalar potential which is introduced by the condition

$$\nabla\psi=-\mathbf{E}, \tag{13}$$

and write, instead of Eq. (10), the Poisson equation

$$\nabla^2\psi=-4\pi e(N_i-N_e). \tag{14}$$

This too makes no difference; J. R. Pierce (1948) has used this equation.

Finally, Larenz (1955) introduced, for the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$, the well-known expression for a vector potential which seems to be more convenient for his type of calculations.

2. Linearization

We now divide the variables $\mathbf{V}_e$, $\mathbf{V}_i$, N_e, N_i, and $\mathbf{H}$ into a constant average value and a *small* perturbation. As assumed previously $\mathbf{E}$ does not have a constant part:

$$N_e=N_0+n_e,\quad N_i=N_0+n_i \tag{15}$$

$$\mathbf{V}_e=\mathbf{v}_e^0+\mathbf{v}_e,\quad \mathbf{V}_i=\mathbf{v}_i^0+\mathbf{v}_i \tag{16}$$

$$\mathbf{H}=\mathbf{H}^0+\mathbf{h}. \tag{17}$$

By introducing these expressions into Eqs. (1) and (2), and neglecting products of perturbation terms, which determine the linearization, the equations of motion become

$$\frac{\partial}{\partial t}\mathbf{v}_e+(\mathbf{v}_e^0\cdot\nabla)\mathbf{v}_e+\frac{e}{m_e}\left[\mathbf{E}+\frac{1}{c}\{(\mathbf{v}_e^0\times\mathbf{H}^0)+(\mathbf{v}_e^0\times\mathbf{h})+(\mathbf{v}_e\times\mathbf{H}^0)\}\right]+\frac{1}{m_eN_0}\nabla P_e+\frac{\alpha}{m_e}\{N_i(\mathbf{v}_e^0-\mathbf{v}_i^0)+N_0(\mathbf{v}_e-\mathbf{v}_i)\}=0, \tag{18}$$

$$\frac{\partial}{\partial t}\mathbf{v}_i+(\mathbf{v}_i^0\cdot\nabla)\mathbf{v}_i-\frac{e}{m_i}\left[\mathbf{E}+\frac{1}{c}\{(\mathbf{v}_i^0\times\mathbf{H}^0)+(\mathbf{v}_i^0\times\mathbf{h})+(\mathbf{v}_i^0\times\mathbf{H}^0)\}\right]+\frac{1}{m_iN_0}\nabla P_i-\frac{\alpha}{m_i}\{N_e(\mathbf{v}_e^0-\mathbf{v}_i^0)+N_0(\mathbf{v}_e-\mathbf{v}_i)\}=0. \tag{19}$$

Special consideration is needed for Eqs. (3) and (4) which determine the connection between pressure and density. This requires that we give a physical meaning to the parameters T_e and T_i.

(a) The most obvious way is to assume isothermal conditions

$$T_e=T_i=T=\text{const}. \tag{20}$$

With Newton's sound velocities,

$$u_e^2=KT/m_e,\quad u_i^2=KT/m_i, \tag{21}$$

we obtain

$$\nabla P_c=m_cu_e^2\nabla n_e,\quad \nabla P_i=m_iu_i^2\nabla n_i. \tag{22}$$

Under the assumption of isothermal sound waves, the energy in the wave would be dissipated in distances less than a wavelength. A gas with many degrees of freedom, e.g., polyatomic organic compounds, most closely satisfies this condition. This means that our treatment is certainly only a first approach to the actual conditions in a plasma.

(b) Another obvious assumption would be that of separate adiabatic conditions for both plasma constituents:

$$T_e=T_0\left(\frac{N_e}{N_0}\right)^{\gamma-1},\quad T_i=T_0\left(\frac{N_i}{N_0}\right)^{\gamma-1}, \tag{23}$$

with

$$T_0=\text{const} \tag{24}$$

denoting the temperature. We then obtain

$$\nabla P_e=\frac{KT_0}{N_0^{\gamma-1}}\nabla(N_0+n_e)^\gamma=KT_0N_0\nabla\left(1+\frac{n_e}{N_0}\right)^\gamma \approx KT_0N_0\nabla\left(1+\gamma\frac{n_e}{N_0}\right)=\gamma KT_0\nabla n_e, \tag{25}$$

$$\nabla P_i\approx\gamma KT_0\nabla n_i. \tag{26}$$

For this assumption to be valid it is necessary that the interaction between the different plasma constituents be negligible.

(c) Finally, one may assume that both constituents *combined* behave adiabatically, at least as long as the plasma is completely ionized; otherwise, one would have to take into account the Saha equation.

We then obtain, in the case of $\gamma=5/3$,

$$T_e=T_i=T_0\frac{(N_e+N_i)^{\frac{2}{3}}}{(2N_0)^{\frac{2}{3}}} \tag{27}$$

and

$$P_e=KT_0N_e\left(\frac{N_e+N_i}{2N_0}\right)^{\frac{2}{3}} =KT_0(N_0+n_e)\left(\frac{2N_0+n_e+n_i}{2N_0}\right)^{\frac{2}{3}} \approx KT_0(N_0+n_e)\left(1+\frac{2}{3}\frac{n_e+n_i}{2N_0}\right), \tag{28}$$

or

$$\nabla P_e\approx\tfrac{4}{3}KT_0\nabla n_e+\tfrac{1}{3}KT_0\nabla n_i, \tag{29}$$

$$\nabla P_i\approx\tfrac{1}{3}KT_0\nabla n_e+\tfrac{4}{3}KT_0\nabla n_i. \tag{30}$$

Thus, the various assumptions do not make much difference as long as one considers a linearized theory, even when neglect of the ionic oscillations compared with the electronic ones is not allowed. The differences in the results using any of the foregoing assumptions is merely a number of the magnitude one. In Sec. I.8 we show that this statement still is valid in the case of the two-term expressions (29) and (30). In the following, the actual numerical coefficient is absorbed by using an appropriate sound velocity.

Finally, we obtain from the remaining equations (5) to (10), as a consequence of linearization,

$$N_0\nabla\cdot\mathbf{v}_e+(\mathbf{v}_e^0\cdot\nabla n_e)+\frac{\partial}{\partial t}n_e=0, \tag{31}$$

$$N_0\nabla\cdot\mathbf{v}_i+(\mathbf{v}_i^0\cdot\nabla n_i)+\frac{\partial}{\partial t}n_i=0, \tag{32}$$

$$c\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{h}, \tag{33}$$

$$c\nabla\times\mathbf{h}=4\pi\mathbf{j}+\frac{\partial}{\partial t}\mathbf{E}, \tag{34}$$

$$\mathbf{j}=eN_0(\mathbf{v}_i-\mathbf{v}_e)+e(\mathbf{v}_i^0n_i-\mathbf{v}_e^0n_e)+eN_0(\mathbf{v}_i^0-\mathbf{v}_e^0), \tag{35}$$

$$\nabla\cdot\mathbf{E}=4\pi e(n_i-n_e). \tag{36}$$

3. Derivation of a Dispersion Equation

The next step consists in inserting for the disturbances $\mathbf{v}_e$, $\mathbf{v}_i$, n_e, n_i, $\mathbf{E}$, and $\mathbf{H}$, an expression of the form

$$e^{i\omega t+ikx}\quad\text{with}\quad\frac{\partial}{\partial y}=\frac{\partial}{\partial z}=0 \tag{37}$$

into the Eqs. (18), (19), (22), and (31) to (36). In general, the result is a homogeneous linear system of equations. The condition of solubility—the vanishing of the determinant—yields the dispersion equation as an additional connection between the frequency and the wave number k.

However, we do not deal with the system (31)–(36) in its most general form, but make successive simplifications and calculate the dispersion equation for each one. In this way we obtain a notion of the fundamental types of plasma oscillations and understand how they are connected with each other under different sets of physical conditions. We thereby hope to obtain a complete picture in this somewhat less complicated manner.

PLASMAS WITHOUT CONSTANT MAGNETIC FIELD

4. Electron Waves. Plasma at Rest in the Coordinate System. Constant Density

For the most primitive model of a plasma we assume:

(a) The plasma as a whole is at rest, i.e., the observer does not move through the plasma as a whole: $\mathbf{v}_e^0=\mathbf{v}_i^0=0$.

(b) There is no constant magnetic field: $\mathbf{H}^0=0$.

(c) We neglect the pressure gradient in the equations of motion as compared with the effect of the alternating electric field, i.e., we assume a constant density. This means that the equation of continuity becomes

$$\nabla\cdot\mathbf{v}_e=0 \tag{38}$$

(condition of incompressibility), and, at least for all *longitudinal* oscillations, $k=0$. Thus, the plasma oscillates—longitudinally—as a whole over the infinite space; for the sake of avoiding difficulties with surface charges at infinity, we may consider the plasma in the shape of an infinite torus.

(d) The plasma does not absorb: $\alpha=0$.

(e) The oscillations of the ions are neglected when compared with the electronic ones: $m_i=\infty$.

With these assumptions we retain as fundamental equations (we drop the subscript e whenever doing so cannot cause confusion),

$$\frac{\partial}{\partial t}\mathbf{v}+\frac{e}{m}\mathbf{E}=0, \tag{39}$$

$$c\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{h}, \tag{40}$$

$$c\nabla\times\mathbf{h}=4\pi\mathbf{j}+\frac{\partial}{\partial t}\mathbf{E}, \tag{41}$$

$$\mathbf{j}=-eN_0\mathbf{v}. \tag{42}$$

With the expression (37) for plane harmonic waves propagating along the x axis, the system (39)–(42) degenerates into three separate groups.

The first group contains the variables v_x, E_x, h_x (with $h_x=0$), i.e., it describes a longitudinal wave of the "acoustic" type which, under condition (c) however, does not propagate. The "dispersion equation" is

$$\omega^2=\frac{4\pi e^2}{m}N_0\equiv\omega_e^2, \tag{43}$$

and additionally

$$k=0, \tag{44}$$

corresponding to Eq. (38).

This special case of "restricted plasma oscillations" which we discuss in more detail in Sec. I.6 has been described by Langmuir, Tonks, and others; an extensive review of the older literature is given by R. Rompe and M. Steenbeck (1939).

The remaining equations form the second and third groups containing the variables v_y, E_y, h_z and v_z, E_z, h_y. These are purely transverse oscillations which both lead—due to the assumption of *zero* magnetic field—to the dispersion equation

$$\omega^2-\omega_e^2-k^2c^2=0. \tag{45}$$

In deriving Eq. (45) we did not use the assumption of incompressibility, and in consequence, Eq. (45) is valid for all transverse oscillations with $k \neq 0$ or $=0$. Finally, the superposition of the two kinds of oscillations which correspond to Eq. (45) determine the type of polarization which is, in general, elliptical.

From Eq. (45) we have for phase velocity,

$$v_{\text{ph}}^2 = \omega^2/k^2 = c^2\omega^2/(\omega^2 - \omega_e^2), \tag{46}$$

and for the group velocity (which is the velocity of the energy flow),

$$g^2 = (d\omega/dk)^2 = c^2[1 - (\omega_e^2/\omega^2)] = c^2(c^2/v_{\text{ph}}^2). \tag{47}$$

Equation (47) makes sense only if the range of frequencies and phase velocities is not too large.

We seek for the physical meaning of Eq. (45). This discussion is of basic interest and is—mutatis mutandis—the model for several similar cases.

$\omega = \omega_e$ means $k=0$, i.e., a synchronous oscillation of the field vectors over the whole space with frequency ω_e. The phase velocity is infinite, the group velocity 0; there is no energy transport.

$\omega > \omega_e$ yields

$$k^2 = (1/c^2)|\omega^2 - \omega_e^2|, \quad v_{\text{ph}}^2 > c^2, \tag{48}$$

and—of course—a group velocity $<c$.

The energy flow is derived from the Poynting vector

$$\mathbf{S} = (c/4\pi)(\mathbf{E}\times\mathbf{h}). \tag{49}$$

By using Eq. (40), we have

$$h_y = (kc/\omega)E_z, \quad h_z = -(kc/\omega)E_y. \tag{50}$$

The x component of energy flow becomes

$$S_x = \frac{c}{4\pi}[E_y h_z - E_z h_y] = \frac{kc^2}{4\pi\omega}[E_y^2 + E_z^2]$$

$$= -\frac{g}{4\pi}[E_y^2 + E_z^2]. \tag{51}$$

The y and z components of the Poynting vector are zero because the space dependence of each product term (say, E_x and h_z) is different and gives a vanishing space average. We obtain, in general, an energy current in the direction of wave propagation—the negative x axis for the sign in Eq. (37)—which disappears in the resonance case $\omega = \omega_e$ and $g=0$.

When $\omega < \omega_e$, v_{ph}, g, and k become (purely) imaginary and the wave is now evanescent, well known from optics:

$$e^{\pm|k|x}e^{i\omega t}, \quad |k|^2 = \omega^2/|v_{\text{ph}}|^2. \tag{52}$$

Because k is imaginary, the wave includes only a periodicity in time and oscillates synchronously over the volume in question. The imaginary group velocity mathematically has absolute values even larger than c, while the corresponding Poynting vector still has a nonzero component only in the direction of wave propagation. The nonzero component disappears when averaged over the time because

$$S_x \sim \sin\omega t \cdot \cos\omega t. \tag{53}$$

The negative sign in the exponent of Eq. (52) is excluded, as in the case of total reflection, by the principle of conservation of energy (considering disturbances propagating in the negative x direction). The reverse case, with a positive sign in the exponent of Eq. (52), describes an exponentially decreasing wave amplitude as we go in the negative x direction; this means reflection with no loss of energy. A full description of this situation, in analogy to the optical case of total reflection, is impossible in the frame of our equations because we have not allowed for a macroscopic gradient which would be the essential feature for calculating the intensity of reflected waves.

Since we are dealing here with only self-excited oscillations, the rapid extinction of these fields for $\omega < \omega_e$ (with reasonable values of ω_e) means that they cannot be observed.

Finally, we may not call the evanescent wave (52) a "damped" one because there are no dissipative terms at all in the fundamental equations (39)–(42).

5. Allowance for Ionic Motion

Before discussing in detail the electronic oscillations, we want to make sure that our previous conclusions are not to be altered significantly by taking into account the ion oscillations. For this purpose we have to use, instead of Eq. (39), the expressions,

$$\frac{\partial}{\partial t}\mathbf{v}_e + \frac{e}{m_e}\mathbf{E} = 0, \tag{54}$$

$$\frac{\partial}{\partial t}\mathbf{v}_i - \frac{e}{m_i}\mathbf{E} = 0, \tag{55}$$

and, instead of Eq. (42),

$$\mathbf{j} = -eN_0(\mathbf{v}_e - \mathbf{v}_i). \tag{56}$$

The whole system consists again of three groups. The first one describes, with $(v_e)_x$, $(v_i)_x$, and E_x, the longitudinal oscillations:

$$\omega^2 = 4\pi e^2 N_0\left(\frac{1}{m_e} + \frac{1}{m_i}\right) = \omega_e^2 + \omega_i^2. \tag{57}$$

The second and third groups—$(v_e)_y$, $(v_i)_y$, E_y, h_z and $(v_e)_z$, $(v_i)_z$, E_z, h_y, respectively—yield the same dispersion equation for transverse waves:

$$\omega^2 - (\omega_e^2 + \omega_i^2) - k^2c^2 = 0. \tag{58}$$

By comparing Eqs. (43) and (45) with (57) and (58), we understand that there are *no* new wave types, and that only the critical frequency is somewhat higher; therefore, we have to multiply ω_e with

$$[1+(\omega_i^2/\omega_e^2)]^{\frac{1}{2}}=[1+(m_e/m_i)]^{\frac{1}{2}}. \tag{59}$$

This changes the plasma frequency in the case of hydrogen ions by about 3×10^{-4}.

6. Longitudinal "Restricted Plasma Waves"

We now discuss the longitudinal oscillations described by Eqs. (39)–(42) and determined by the dispersion equation (43). They have considerable practical interest and are the so-called "plasma waves." We call them "restricted plasma waves" to make clear that they correspond to only a very small range of possible oscillations in a plasma.

Physically speaking, they are high-frequency alternating currents with the plasma frequency ω_e. By neglecting the pressure gradients compared with the electric fields, we should have $k=0$. Thus, when using our equations for the more general case of $k\neq0$, this procedure is only an approximation. We must be sure that all changes in space are going on so slowly that we are allowed to neglect the accompanying pressure gradients as compared with the alternating electric fields.‡ This approximation is closely connected to the work of Pierce (1948) and A. V. Haeff (1948), who attempt to explain the operation of electron wave tubes. Besides the objection, which is discussed in Sec. I.6(a), their implicit use of a Maxwellian distribution is open to grave doubt; for further details, see Sec. II.

We must take into account the influence of motion of the reference frame on the observed wave form, i.e., the influence of a motion of the plasma as a whole. We expect that in the case $k\neq0$ we shall find some sort of Doppler effect.

(a) *Moving Coordinate System*

All our considerations, until now, took into account only a coordinate system which was at rest in the gas as a whole, or plasmas with components having no constant velocity in the coordinate system: $\mathbf{v}_e^0=\mathbf{v}_i^0=0$. We now drop this restriction but still allow only for motions *in* the direction of propagation (x axis). The case of motions transverse to the wave propagation is somewhat more complicated and is discussed in Sec. I.13. The reason for this separation is explained there. Also in Sec. I.13, a discussion is given of the influence of motions *in* the direction of wave propagation upon the transverse oscillations. Here we confine ourselves to the longitudinal oscillations.

For treating ions and electrons separately, we have to take into account the ionic oscillations. We start with two equations of motion—analogous to Eqs. (54) and (55)—

$$\frac{\partial}{\partial t}\mathbf{v}_e+(\mathbf{v}_e^0\cdot\nabla)\mathbf{v}_e+\frac{e}{m_e}\mathbf{E}=0, \tag{60}$$

$$\frac{\partial}{\partial t}\mathbf{v}_i+(\mathbf{v}_i^0\cdot\nabla)\mathbf{v}_i-\frac{e}{m_i}\mathbf{E}=0, \tag{61}$$

and, instead of Eq. (56), we write

$$\mathbf{j}=eN_0(\mathbf{v}_i-\mathbf{v}_e)+e(\mathbf{v}_i^0n_i-\mathbf{v}_e^0n_e). \tag{62}$$

The two equations of continuity are

$$N_0\nabla\cdot\mathbf{v}_e+(\mathbf{v}_e^0\cdot\nabla n_e)+\frac{\partial}{\partial t}n_e=0, \tag{63}$$

$$N_0\nabla\cdot\mathbf{v}_i+(v_i^0\cdot\nabla n_i)+\frac{\partial}{\partial t}n_i=0. \tag{64}$$

For longitudinal oscillations, we need not use the complete Maxwell equations but only the one relation,

$$\nabla\cdot\mathbf{E}-4\pi e(n_i-n_e)=0. \tag{65}$$

As a matter of fact, the preceding equations are incomplete. There is a gross convection term,

$$\mathbf{j}_0=+eN_0(\mathbf{v}_i^0-\mathbf{v}_e^0), \tag{66}$$

to be added to Eq. (62); compare Eq. (35). This term [Eq. (66)] is finite in the case of *different* constant velocities of ions and electrons and may even determine completely the numerical results. Moreover, as a consequence of Maxwell's equation

$$c\nabla\times\mathbf{H}=4\pi\mathbf{j}+\frac{\partial}{\partial t}\mathbf{E}, \tag{67}$$

there exists a macroscopic magnetic field $\mathbf{H}^*$ due to the constant convection current (66). A constant macroscopic displacement current $\partial\mathbf{E}/\partial t$ could arise only in the case of a plasma of finite extent in the x direction, resulting in a charge separation increasing linearly with time. By using an infinite torus-shaped plasma as a model, we avoid this difficulty. However, this type of treatment is not possible in the case of the magnetic field $\mathbf{H}^*$ which then should appear in the equation of motion through the force components

$$(\mathbf{v}_e\times\mathbf{H}^*)+(\mathbf{v}_e^0\times\mathbf{H}^*)\neq0. \tag{68}$$

We believe that a correct treatment of this problem should start from a plasma model with finite extent in the direction perpendicular to the electric current. Only then is it possible to determine the shape in space of $\mathbf{H}^*$ which, according to the well-known results of electrodynamics, *increases* in the *interior* of a conductor, assumed of circular shape, linearly with the distance o- the axis, but *decreases outside* the conductor proporf tionally to the distance. It is quite obvious that we

‡ In Sec. I.7, it is shown that this approximation, generally, is a poor one.

may not use a harmonic expression for the space-time periodicity of a variable in such a structure. This means that one cannot consider the treatment of a plasma in which the electrons as a whole are moving with respect to the ions.

Pierce (1948) has discussed this case: in his coordinate system the ions are at rest, but the electrons have a mean velocity $\mathbf{u}_0$. In his terminology the periodic part of the velocity component in the direction of wave propagation is written $\mathbf{v}_e-\mathbf{u}_0$. His equations are almost the same as ours, but Pierce uses the Poisson equation instead of the Maxwell equations and Eq. (63). In this set of equations the current (66) never appears, and Pierce neglects completely the expression (68) in the equation of motion.

Haeff (1949), in dealing with a similar situation of electron plasmas streaming across each other, makes the same error. The mistake in this case is not so obvious because he connects current and density through the principle of conservation of charge. Arising from the continuity equations, the convection current (66) cancels out by differentiation with respect to x. There is an additional objection against the procedure because use is made, essentially, of the electric space-charge density while the corresponding mass density and pressure gradients are neglected in the equations of motion.

We conclude that this manner of treatment of streaming plasmas is not correct.

J. H. Piddington (1956) criticized the interpretation of the derived dispersion equation by Pierce and Haeff for a quite different reason.§ In the following résumé we restrict ourselves to the situation Pierce (1948) has considered, but remark that the considerations are valid as well for the calculations of Haeff (1949).

Pierce derived from his incomplete equations the dispersion relation

$$\omega^2=\omega_i^2+\frac{\omega_e^2}{[1+(v_0/v_{\text{ph}})]^2}. \tag{69}$$

(He used for the phase velocity $v_{\text{ph}}=\omega/k$ and the constant velocity v_0 the symbols $\omega/j\Gamma$ and u_0, respectively.)

Starting from here, Pierce remarks that for frequencies $\omega<\omega_i$, the phase velocity and thus the wave number k (in his notation $j\Gamma$ with $j=\sqrt{-1}$) becomes imaginary. The wave then is the evanescent type discussed in Sec. I.4.

From here we immediately see the error: Pierce considers the exponentially increasing wave, assuming that this type of wave is observed in electron-wave tubes, without discussing the impossibility of an energy supply. The criticism of Piddington is directed against this point of view.

§ In an answer Pierce and Walker (1956) merely pointed out that there are oscillations observed in experiment; we think that this alone does not prove the correctness of the theoretical analysis.

Furthermore, as pointed out at the beginning of this section, strictly speaking the wave number k must be zero everywhere as a result of the neglect of the pressure gradients in the equation of motion.

(b) *Joint Motions of Electrons and Ions*

When we do *not* have different constant velocities for electrons and ions, i.e., when $v_e^0=v_0$, our Eqs. (60)–(64) are complete.

In this reference frame the dispersion equation for longitudinal oscillations is

$$(\omega+v_0k)^2-\omega_i^2-\omega_e^2=0, \tag{70}$$

while "waves"

$$\omega-v_0k=0 \tag{71}$$

are omitted; Eq. (71) describes periodical disturbances which are fixed in the gas as a whole and do not propagate. They always appear mathematically when use is made of a moving coordinate system, as pointed out by Piddington (1956) in connection with a paper by V. A. Bailey (1948).

The transition to a moving coordinate system does not change the physical properties of the gas. As a new feature, only Eq. (70) contains a Doppler effect; we therefore learn that a moving observer measures a frequency which depends on the wave number.

The situation becomes clear when use is made of the phase velocity,

$$v_{\text{ph}}=-v_0\pm[(\omega_e^2+\omega_i^2)/k^2]^{\frac{1}{2}}. \tag{72}$$

In the moving frame the phase velocity is the sum of a part due to the constant translation ($-v_0$, in the direction of the negative x axis) and the value of the phase velocity in the frame at rest, since there

$$\omega^2=\omega_e^2+\omega_i^2. \tag{73}$$

The group velocity has the value of the translatory velocity.

As a result of the preceding, we conclude that: (1) the moving observer measures the whole frequency range between 0 and ∞ as a consequence of the different wave numbers; (2) instead of the whole frequency range, an observer at rest would measure *only* the plasma frequency; and (3) we find nothing like the phenomena discussed in Sec. I.4 for the case of transverse oscillations, since, as can be seen at once from Eq. (71), there is no complex or imaginary k which corresponds to a real value of ω.

7. Effect of the Pressure Gradients

The next step towards the generalization of our equations is to take into account the variation of pressure accompanying the longitudinal waves. We consider the ions and electrons separately, but make use of a reference frame at rest.

Instead of Eqs. (54) and (55), one uses [see Eqs.

(18) and (19)]

$$\frac{\partial}{\partial t}\mathbf{v}_e+\frac{e}{m_e}\mathbf{E}+\frac{u_e^2}{N_0}\nabla n_e=0, \tag{74}$$

$$\frac{\partial}{\partial t}\mathbf{v}_i-\frac{e}{m_i}\mathbf{E}+\frac{u_i^2}{N_0}\nabla n_i=0, \tag{75}$$

where we have used the sound velocities u_e and u_i as noted in the discussions following Eq. (30). As expected, the transverse oscillations are not affected because the pressure gradient has a component only in the direction of the x axis.

The dispersion equation for the longitudinal oscillations becomes

$$(\omega^2-\omega_e^2-k^2u_e^2)(\omega^2-\omega_i^2-u_i^2k^2)-\omega_e^2\omega_i^2=0. \tag{76}$$

Equation (76) is quadratic in k^2 and describes two coupled wave pairs, one of them excited by the ions, the other by the electrons. Hence, by taking into account the pressure variations together with the ion motions, we find two wave types instead of one, as previously [see Eqs. (43) and (57)].

For a discussion of the physical conditions underlying Eq. (76), we introduce new dimensionless variables

$$\xi=(\omega_i^2/\omega^2)>0,\quad \eta=(u_i^2/v_{\text{ph}}^2)>0, \tag{77}$$

and use for the mass ratio the symbol

$$m_i/m_e=M\gg 1. \tag{78}$$

We then find

$$\omega_e^2=M\omega_i^2,\quad u_e^2=Mu_i^2. \tag{79}$$

In this notation the dispersion relation becomes

$$\eta^2+\eta\left[2\xi-\left(1+\frac{1}{M}\right)\right]-\left(1+\frac{1}{M}\right)\xi+\frac{1}{M}=0 \tag{80}$$

with solutions (neglecting terms of the order M^{-2})

$$\eta=-\xi+\tfrac{1}{2}+(1/2M)\pm[\xi^2+\tfrac{1}{4}-(1/2M)]^{\frac{1}{2}}. \tag{81}$$

The solutions (81) are illustrated, in Fig. 1, schematically and on an arbitrary scale.

Equation (81) represents a pair of hyperbolas with

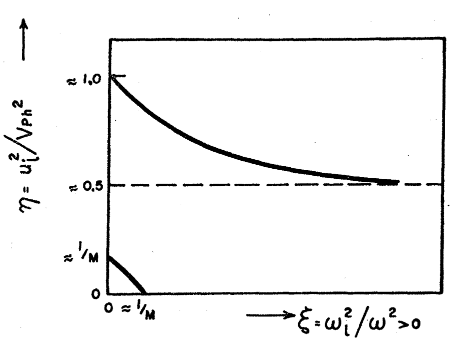

FIG. 1. The dispersion of longitudinal oscillations from Eq. (76), Sec. I. The ordinate and abscissa scales are arbitrary.

shifted axis. Figure 1 shows only those parts of the curves which lie in the first quadrant: $\xi>0$, $\eta>0$.

The upper branch corresponds to the positive sign and represents essentially ionic waves; we come back to this point later. All frequencies between 0 and ∞ are possible. The phase velocities have values between

$$v_{\text{ph}}{}^i\approx u_i(\omega=\infty)\quad\text{and}\quad v_{\text{ph}}{}^i\approx\sqrt{2}u_i(\omega=0). \tag{82}$$

The lower branch corresponds to the negative sign and represents the electronic waves. Here, only frequencies

$$\omega\geq\omega_e \tag{83}$$

are possible, because for $\omega<\omega_e$ the phase velocity becomes imaginary. The phase velocity itself varies between

$$v_{\text{ph}}{}^e\approx u_e(\omega=\infty)\quad\text{and}\quad v_{\text{ph}}{}^e=\infty\,(\omega\approx\omega_e), \tag{84}$$

and therefore is always higher than the corresponding sound velocity.

The properties of the electron waves are essentially unchanged by the presence of the ions. To see this it is sufficient to set the quantities referring to ions in Eq. (76), i.e., ω_i and u_i, equal to zero. Then we find the simple dispersion equation for electron waves,

$$\omega^2-\omega_e^2-k^2u_e^2=0. \tag{85}$$

The discussion of the longitudinal-wave dispersion equation, Eq. (85), is the same as the one in Sec. I.4 for the transverse waves, substituting the sound velocity u_e for the velocity of light c. We again obtain the frequency condition Eq. (83) [see Eq. (48)], but now strictly, while in deriving Eq. (83) we neglected quantities of the order of magnitude M^{-2} against quantities M^{-1}. For the phase velocity we find, corresponding to Eq. (84),

$$v_{\text{ph}}^2=u_e^2[\omega^2/(\omega^2-\omega_e^2)]>u_e^2, \tag{86}$$

and for the group velocity,

$$g^2=u_e^2(u_e^2/v_{\text{ph}}^2)<u_e^2. \tag{87}$$

For the frequency range $\omega<\omega_e$, we again use the remarks from Sec. I.4. From Eqs. (86) and (87), we see that for almost the whole frequency range, the value of $k^2u_e^2$ is not negligible with respect of ω_e^2; therefore, the neglect of pressure gradients, which leads to neglecting $k^2u_e^2$, is *not allowable*.

In the preceding discussion we called, without further proof, the two wave types, corresponding to the negative and positive sign in front of the root in Eq. (81), the "electronic" and "ionic" ones. This distinction can be made evident by a consideration of the amplitudes in both cases.

From the fundamental equations the connections between velocities and densities are obtained,

$$v_e=-v_{\text{ph}}\frac{n_e}{N_0},\quad v_i=-v_{\text{ph}}\frac{n_i}{N_0}, \tag{88}$$

and the electric field and the densities are connected by

$$E=(4\pi e/ik)(n_i-n_e). \tag{89}$$

Furthermore, one finds, with the help of the dispersion relation, the two densities,

$$n_i=-\frac{\omega^2-\omega_e^2-u_e^2k^2}{\omega_e^2}n_e=\frac{-\omega_i^2}{\omega^2-\omega_i^2-u_i^2k^2}n_e. \tag{90}$$

To draw a qualitative picture of the different amplitudes, we take, for example, $v_{\mathrm{ph}}\approx u_e$ as the phase velocities, which is strictly correct only for frequencies $\omega\gg\omega_e$, and corresponds to the "electron waves." One finds for the vector components, in terms of the electric field,

$$v_e\approx i\frac{e}{m_e}E\frac{1}{\omega}, \tag{91}$$

$$v_i\approx -i\frac{e}{m_e}E\frac{1}{\omega}\cdot\frac{m_e}{m_i}, \tag{92}$$

$$n_e\approx -iN_0\frac{eE}{\omega}(m_eKT)^{-\frac{1}{2}}, \tag{93}$$

$$n_i\approx iN_0\frac{eE}{\omega}(m_eKT)^{-\frac{1}{2}}\frac{m_e}{m_i}. \tag{94}$$

In the case of the "ionic waves" with $v_{\mathrm{ph}}\approx u_i$, which is correct for almost the entire frequency range, one finds

$$v_e\approx -i\frac{e}{m_i}E\frac{1}{\omega}, \tag{95}$$

$$v_i\approx +i\frac{e}{m_i}E\frac{1}{\omega}, \tag{96}$$

$$n_e\approx iN_0\frac{eE}{\omega}(m_iKT)^{-\frac{1}{2}}, \tag{97}$$

$$n_i\approx -iN_0\frac{eE}{\omega}(m_iKT)^{-\frac{1}{2}}. \tag{98}$$

From Eqs. (91)–(94) we first conclude that the ions practically do not take part in the oscillations of the electronic type, because both densities and velocities are smaller than the corresponding quantities of the electrons by a factor $m_e/m_i\approx 1/1800$.

On the other hand, densities and velocities in the second case [Eqs. (95)–(98)] are about equal for electrons and ions but considerably smaller than before.

Finally, all quantities have opposite signs in the two cases, and furthermore, the signs once again are opposite for quantities referring to electrons and ions, thus giving a net current N_0v_e in the case of electronic waves and zero (in first order) in the case of ionic waves.

By taking into account a constant translatory motion with velocity v_0 of ions and electrons together, we get, instead of Eq. (76), an equation of the fourth degree in ω instead of the quadratic equation in ω^2:

$$[(\omega+v_0k)^2-\omega_e^2-k^2u_e^2][(\omega+v_0k)^2-\omega_i^2-k^2u_i^2]-\omega_e^2\omega_i^2=0. \tag{99}$$

The change is obvious: The constant velocities split off the wave pairs, distinguished from each other by their direction of propagation, into four separate waves, which for a stationary observer show different frequencies.

8. Isothermal and Adiabatic Treatment

In Sec. I.2 we supposed that in a linearized theory the isothermal expression differs from the adiabatic one only by an unimportant factor of the order of magnitude unity. This is clear for the case of "separate adiabaticity" of electrons and ions [see Eqs. (21) and (25)]. We now show that this is also true for the two-term expressions Eqs. (29) and (30) by starting from the discussion of the longitudinal waves in the last section; the transverse oscillations are unaffected. It is evident that we have to take into account the motions of the ions for this purpose.

Instead of Eqs. (74) and (75), in which we used the isothermal condition Eq. (21), we now use as equations of motion [see Eq. (30)],

$$\frac{\partial}{\partial t}\mathbf{v}_e+\frac{e}{m_e}\mathbf{E}+\left(\frac{\gamma+1}{2}\right)\frac{u_e^2}{N_0}\nabla n_e+\left(\frac{\gamma-1}{2}\right)\frac{u_e^2}{N_0}\nabla n_i=0, \tag{100}$$

$$\frac{\partial}{\partial t}\mathbf{v}_i-\frac{e}{m_i}\mathbf{E}+\left(\frac{\gamma+1}{2}\right)\frac{u_i^2}{N_0}\nabla n_i+\left(\frac{\gamma-1}{2}\right)\frac{u_i^2}{N_0}\nabla n_e=0, \tag{101}$$

and obtain, instead of Eq. (76), the dispersion relation

$$(\omega^2-\omega_e^2-\gamma k^2u_e^2)(\omega^2-\omega_i^2-\gamma k^2u_i^2)-\omega_e^2\omega_i^2=\gamma(\gamma-1)k^4u_e^2u_i^2-\left(\frac{\gamma-1}{2}\right)(u_e^2+u_i^2)k^2\omega^2. \tag{102}$$

The distinctive differences between Eqs. (76) and (102) are the factors γ in front of the sound velocities u_e and u_i and the terms on the right-hand side. These too have only the character of a small correction as may be shown by multiplying the brackets on the left side and adding the right-side terms: for example, for $\gamma=5/3$, the factor 25/9 in front of $k^4u_e^2u_i^2$ (on the left) becomes $15/9=5/3$, while the factor 5/3 in front of $k^2u_e^2\omega^2$ and $k^2u_i^2\omega^2$ (on the left) becomes 4/3.

9. Influence of Damping Terms in the Equation of Motion

We now try to understand what happens in general when damping terms are introduced in the equations of motion. Instead of becoming involved in the intricacies of a completely general case, we consider a damping force proportional to the product of the electron and ion densities and the relative velocity, corresponding to our statement in Sec. I.1. Other treatments are discussed briefly at the end of this section.

By starting with the longitudinal oscillations and neglecting ion oscillations and pressure terms in the equations of motion, we get

$$\frac{\partial}{\partial t}\mathbf{v}+\frac{\alpha}{m}N_0\mathbf{v}+\frac{e}{m}\mathbf{E}=0, \tag{103}$$

$$N_0\nabla\cdot\mathbf{v}-\frac{\partial}{\partial t}\left[\frac{\nabla\cdot\mathbf{E}}{4\pi e}\right]=0. \tag{104}$$

The dispersion equation is

$$\omega^2-i\frac{\alpha}{m}N_0\omega-\omega_e^2=0. \tag{105}$$

Solving for ω, we find

$$\omega=\frac{1}{2}i\frac{\alpha}{m}N_0\pm[\omega_e^2-\tfrac{1}{4}(N_0^2\alpha^2/m^2)]^{\frac{1}{2}}. \tag{106}$$

From this equation we can understand the two main consequences of our damping terms: all waves are damped in time with a damping proportional to the electron density, and all frequencies are shifted, but this shift has no significance. We can understand this statement even without a knowledge of the numerical values of the damping constant α. For the case in which the mean collision frequency is equal to the plasma frequency,

$$\nu'^2\equiv\left(\frac{N_0\alpha}{m}\right)^2\approx\omega_e^2, \tag{107}$$

the wave would be almost completely damped over a time corresponding to one whole oscillation; this result is well known from mechanics.

By taking into account the pressure gradient we do not find important differences. Instead of Eq. (105), we obtain a somewhat more general expression, but with the same structure,

$$\omega^2-i\nu'\omega-\omega_e^2-k^2u^2=0, \tag{108}$$

which could have been derived directly from Eqs. (106) and (85).

On the other hand, taking into account the ion oscillations results in two equations of motion,

$$\frac{\partial}{\partial t}\mathbf{v}_e+\frac{e}{m_e}\mathbf{E}+\nu'(\mathbf{v}_e-\mathbf{v}_i)=0, \tag{109}$$

$$\frac{\partial}{\partial t}\mathbf{v}_i-\frac{e}{m_i}\mathbf{E}+\frac{m_e}{m_i}\nu'(\mathbf{v}_i-\mathbf{v}_e)=0, \tag{110}$$

and we obtain, instead of (105),

$$\omega^2-i\nu'\left(1+\frac{m_e}{m_i}\right)\omega-(\omega_e^2+\omega_i^2)=0. \tag{111}$$

For the transverse waves the damping effects are of greater interest. We combine the y and z components of Eqs. (109) and (110) with Maxwell's equations, and obtain, as the dispersion relation,

$$\omega^3-i\nu'\left(1+\frac{m_e}{m_i}\right)\omega^2-(\omega_e^2+\omega_i^2+k^2c^2)\omega +i\nu'\left(1+\frac{m_e}{m_i}\right)k^2c^2=0 \tag{112}$$

or, neglecting the ion oscillations, the somewhat less complicated relation

$$\omega^3-i\nu'\omega^2-(\omega_e^2+k^2c^2)\omega+i\nu'k^2c^2=0. \tag{113}$$

This dispersion equation is of the third degree in ω. This does *not* mean the appearance of a third type of wave. By solving for the phase velocity or the complex index of refraction, we obtain

$$n^2=\frac{c^2}{v_{\text{ph}}^2}=\frac{\omega^2-\omega_e^2-\nu'^2}{\omega^2+\nu'^2}-i\frac{\nu'}{\omega}\frac{\omega_e^2}{\omega^2+\nu'^2}. \tag{114}$$

Here the real part again contains a frequency shift and is not of special interest. The value of the damping constant is the same as for the case of longitudinal oscillations according to Eqs. (106) and (111).

The derivation of Eq. (114) is well known from optics. One generally handles the problem of line broadening in spectroscopy in this manner. [For further details see, e.g., A. Unsoeld, (1955), Sec. 68, p. 269.] One has only to note that there is no eigenfrequency (ω_0 in Unsoeld's book) for the free electrons in a plasma. The same expression is used in the treatments of free-free-radiation in the radio-frequency range, as, e.g., by L. Oster (1959).

One should not go into more detail here concerning the questions of damping because of the "first approximation" character of our equations. A somewhat different kind of treatment was given by H. C. van de Hulst (1951). For the damping term, he assumes a form similar to that used in the Navier-Stokes equations,

$$\nabla(\nabla\cdot\mathbf{v})+\nabla^2\mathbf{v}, \tag{115}$$

while we started from a direct proportionality to the velocity. Thus, van de Hulst gets an expression $k^2\alpha^*$ with k as wave number instead of our α. This is the reason for the additional type of wave ("viscosity wave") which he finds.

PLASMAS WITH CONSTANT MAGNETIC FIELDS

10. Basic Equations for the General Case

Interest was first aroused in this field by problems occurring in ionospheric physics. The basic phenomena were explained around 1930 by E. V. Appleton and others; there is an extensive review of the older literature by R. Rompe and M. Steenbeck (1939). A more modern description of this field has been published by K. Rawer (1953). Piddington (1955) has dealt with some astrophysically interesting problems considering the oscillations in a plasma with nonzero magnetic fields and neglecting the ionic motions.

The essential difference between a plasma with zero magnetic field and a plasma with nonzero magnetic field is the anisotropy in the propagation of waves. This is the reason for the many analogous features in plasma physics and crystal optics: in the plasma the axis defined by the magnetic field corresponds to one of the principal axes of the dielectric constant. We do not go into more detail concerning this analogy because these things are well known and often discussed in connection with ionospheric physics; instead, we emphasize the connection with the Zeeman effect. Here one must have in mind the fact that the normal Zeeman effect deals with the splitting of spectral *lines* in the magnetic fields, rather than with the continuous radiation we are here considering.

We start with a plasma with zero mean velocities (we come back to this question in Sec. I.13) but allow for ion motions and pressure gradients. We choose the axes in such a way that the magnetic field lies in the xy plane. Thus,

$$\mathbf{H}^0=(H_L,H_T,0). \tag{116}$$

We have the basic equations

$$\frac{\partial}{\partial t}\mathbf{v}_e+\frac{e}{m_e}\mathbf{E}+\frac{e}{m_e c}\mathbf{v}_e\times\mathbf{H}^0+\frac{u_e^2}{N_0}\nabla n_e=0, \tag{117}$$

$$\frac{\partial}{\partial t}\mathbf{v}_i-\frac{e}{m_i}\mathbf{E}-\frac{e}{m_i c}\mathbf{v}_i\times\mathbf{H}^0+\frac{u_i^2}{N_0}\nabla n_i=0, \tag{118}$$

$$N_0\nabla\cdot\mathbf{v}_e+\frac{\partial}{\partial t}n_e=0, \tag{119}$$

$$N_0\nabla\cdot\mathbf{v}_i+\frac{\partial}{\partial t}n_i=0, \tag{120}$$

$$c\cdot\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{h}, \tag{121}$$

$$c\cdot\nabla\times\mathbf{h}=4\pi eN_0(\mathbf{v}_i-\mathbf{v}_e)+\frac{\partial}{\partial t}\mathbf{E}. \tag{122}$$

The basic equations treated in ionospheric physics are included in our system, Eqs. (117) to (122), except for the formulation of damping effects: we obtain them at once by neglecting the motion of ions and the pressure gradients for the electrons in the equation of motion, i.e., considering the case of an incompressible plasma. This does not affect the situation for transverse oscillations.

The derivation of a dispersion relation in the most general case is quite lengthy and the resulting equation too complicated to give any idea of the physical features. The best way to represent it is as a determinant:

$$\begin{vmatrix}
\omega^2-\omega_e^2-k^2u_e^2 & 0 & i\frac{\omega_T^e}{\omega}(\omega^2-k^2c^2) & \omega_e^2 & 0 & 0\\
0 & \omega^2-\omega_e^2-k^2c^2 & -i\frac{\omega_L^e}{\omega}(\omega^2-k^2c^2) & 0 & \omega_e^2 & 0\\
-i\omega\omega_T^e & i\frac{\omega_L^e}{\omega}(\omega^2-k^2c^2) & \omega^2-\omega_e^2-k^2c^2 & 0 & 0 & \omega_e^2\\
\omega_i^2 & 0 & 0 & \omega^2-\omega_i^2-k^2u_i^2 & 0 & -i\frac{\omega_T^i}{\omega}(\omega^2-k^2c^2)\\
0 & \omega_i^2 & 0 & 0 & \omega^2-\omega_i^2-k^2c^2 & i\frac{\omega_L^i}{\omega}(\omega^2-k^2c^2)\\
0 & 0 & \omega_i^2 & i\omega\omega_T^i & -i\frac{\omega_L^i}{\omega}(\omega^2-k^2c^2) & \omega^2-\omega_i^2-k^2c^2
\end{vmatrix} \tag{123}$$

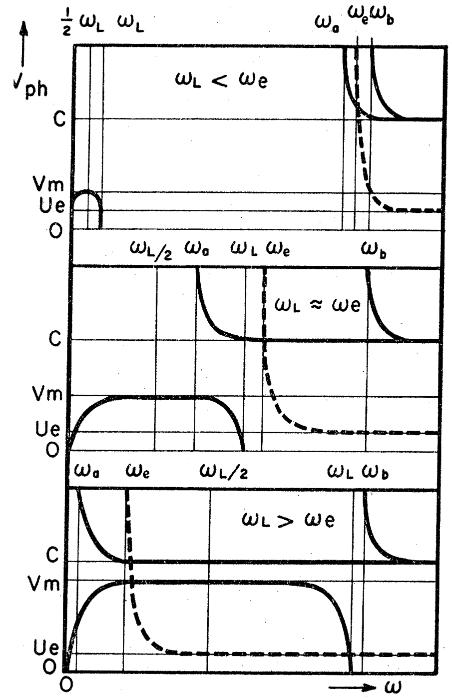

FIG. 2. The dispersion of a plasma in the case of a purely longitudinal magnetic field. The phase velocity (in an arbitrary scale) is plotted against the frequency. The three sets of curves correspond to different strengths of the magnetic field ($\sim\omega_L$) at a given density ($\sim\omega_e$). The solid lines correspond to the transverse oscillations, and the broken line to the longitudinal oscillation, independent of the magnetic field.

Besides the notations previously used, we have for the gyrofrequencies

$$\omega_{T,L}{}^{e}=eH_{T,L}/m_e c, \quad \omega_{T,L}{}^{i}=eH_{T,L}/m_i c. \tag{124}$$

We understand that the determinant (123) is composed of four clearly distinct groups of elements: the upper left group obviously deals with only the electrons; the lower right with only the ions; the remaining two groups provide the coupling between the electronic and ionic motions.

The first mentioned groups are the physically interesting ones. They contain two wave pairs corresponding to the longitudinal type from Sec. I.7 [Eq. (85)],

$$\omega^2-\omega_{e,i}{}^2-u_{e,i}{}^2k^2, \tag{125}$$

and two pairs

$$\omega^2-\omega_{e,i}{}^2-c^2k^2, \tag{126}$$

corresponding to the usual transverse type of oscillation [Sec. I.4, Eq. (45)]. This division does not determine immediately the *physical* type of oscillation, i.e., the oscillations of different components of field vectors, but the mathematical form of the dispersion equation. We want to emphasize that this should not be confused, at least *a priori*, as has been done frequently in previous work.

The coupling of the two types of oscillations (longitudinal and transverse) is effected by the magnetic field represented by the off-diagonal terms, a consequence of our special choice of the coordinate system in terms of the direction of the magnetic field which causes many of the off-diagonal terms to go to zero. Thus, the characteristic phase velocities and frequencies of both wave types are coupled together; this is the main consequence of a constant magnetic field. The second consequence is the splitting of the two transverse waves, which are discussed later in more detail. The coupling terms have the form of "light waves," mathematically represented by the well-known expression

$$\omega^2-k^2c^2=0. \tag{127}$$

Summing up our discussion of Eq. (123), we find that in the presence of a nonzero magnetic field there are four wave pairs. At a fixed frequency there may be different phase velocities corresponding to the number of (real) solutions of the dispersion equation in question. The two transverse oscillations appear separately, thus corresponding to the ordinary and the extraordinary ray in crystal optics.

We do not use Eq. (123) in all detail to describe the behavior of a plasma in the presence of a constant magnetic field. For this purpose we specialize, following Piddington (1955), to the case of immobile ions. We then try to survey the essential physical properties, emphasizing especially the points not taken into account in previous work.

11. Electronic Oscillations in a Constant Magnetic Field

Neglecting the motions of the ions, we have to consider in Eq. (123) only the upper left square. By again dropping the index e wherever it is not necessary, we get the following dispersion relation:

$$(\omega^2-\omega_e{}^2-k^2u_e{}^2)[(\omega^2-\omega_e{}^2-k^2c^2)^2-(\omega_L{}^2/\omega^2)(\omega^2-k^2c^2)^2] \\ -(\omega^2-k^2c^2)\omega_T{}^2(\omega^2-\omega_e{}^2-k^2c^2)=0. \tag{128}$$

Equation (128) is of the third degree in k^2. In the case of zero magnetic field ($\omega_L=\omega_T=0$), the longitudinal and transverse waves are no longer coupled. This means that there is one dispersion relation for the longitudinal waves (which involve E_x and v_x), and one common dispersion equation for both the transverse oscillations.

Instead of discussing Eq. (128) we specialize once more and consider the limiting cases of a purely longitudinal magnetic field ($\omega_T=0$) and a purely transverse magnetic field ($\omega_L=0$). The case of an arbitrary direction for the magnetic field is mathematically very complicated, and probably will not give rise to new results.

(a) *Longitudinal Magnetic Field:* $\omega_T=0$

In a longitudinal magnetic field there is no coupling between the longitudinal oscillations (E_x, v_x), which are not affected by the magnetic field, and the two transverse oscillations (v_y, E_y, h_z and v_z, E_z, h_y), which are coupled together by the magnetic field. Thus the dispersion equation for the longitudinal oscillations,

$$\omega^2-\omega_e^2-k^2u_e^2=0, \tag{129}$$

is identical with our former relation (85).

The two transverse oscillations are described by

$$\omega^2(\omega^2-\omega_e^2-k^2c^2)^2-(\omega^2-k^2c^2)^2\omega_L^2=0, \tag{130}$$

showing no pressure dependence and thus being identical with the well-known expressions in the ionospheric theory. We briefly note the main features in order to make possible a comparison with the case of the transverse magnetic field, where the dispersion relations differ essentially from the one derived in ionspheric theory.

Equation (130) yields two roots; by solving for the phase velocity or the refractive index, we obtain

$$n_a^2=\frac{c^2}{v_{\rm ph}^2}=1-\frac{\omega_e}{\omega}\frac{\omega_e}{\omega+\omega_L}, \tag{131}$$

called the "ordinary ray" in ionospheric theory in analogy to crystal optics, and

$$n_b^2=\frac{c^2}{v_{\rm ph}^2}=1-\frac{\omega_e}{\omega}\frac{\omega_e}{\omega-\omega_L}, \tag{132}$$

the "extraordinary ray" of ionospheric theory.

On the other hand, both solutions are formed like the corresponding ones in the case of the Zeeman effect in spectral lines; there is no wave which is not influenced by the magnetic field, and the solutions are somewhat symmetrical to the (nonexistent) unshifted component

$$n^2=c^2/v_{\rm ph}^2=1-(\omega_e^2/\omega^2). \tag{133}$$

For zero magnetic field $\omega_L\rightarrow 0$ the expressions (131) and (132) converge to Eq. (133).

Propagation of waves is possible only when the square of the refractive index is >0, i.e., when the frequency of the ordinary ray,

$$\omega>(\omega_L/2)+[(\omega_L^2/4)+\omega_e^2]^{\frac{1}{2}}\equiv\omega_a, \tag{134}$$

and of the extraordinary ray,

$$\omega>+(\omega_L/2)+[(\omega_L^2/4)+\omega_e^2]^{\frac{1}{2}}\equiv\omega_b. \tag{135}$$

The negative square root is excluded in both cases. The magnetic field causes a shift of the lowest frequency able to propagate through the plasma as compared with the unshifted case (123).

In the case of the extraordinary ray (132) there is a second range of frequencies for which wave propagation is allowed. Since the refractive index becomes positive again for $\omega=\omega_L$, propagation is possible for all frequencies

$$\omega<\omega_L. \tag{136}$$

This feature is well known from ionospheric physics.

With increasing magnetic field strength, the range

$$\omega_L<\omega<(\omega_L/2)+[(\omega_L^2/4)+\omega_e^2]^{\frac{1}{2}} \tag{137}$$

becomes smaller and ends, for $\omega_L\gg\omega_e$, in a "prohibited line" at ω_L, but there is no way to get a continuous frequency range of propagation. In the case of zero magnetic field the forbidden region of (127) cancels out, in accordance with the results of Sec. I.4.

Discussing now the values of the phase velocities, Eqs. (131) and (132) show that $v_{\rm ph}>c$ for the ranges (134) and (135), again in accordance with the results of Sec. I.4.

On the other hand, the phase velocity in the range (136) of frequencies is always $<c$ and goes to zero at the borders of the transmission range $\omega=0$ and $\omega=\omega_L$; for $\omega=\omega_L/2$ there is a maximum for the phase velocity. Its value v_m is obtained from the relation

$$v_m^2=c^2\omega_L^2/(\omega_L^2+4\omega_e^2). \tag{138}$$

A schematic picture of the situation is drawn in Fig. 2. For each case we have to choose the ratio of ω_e to ω_L. We take $\omega_L=0.1\omega_e$, $\omega_L=0.9\omega_e$, and $\omega_L=5.0\omega_e$ as samples. The limiting case of zero magnetic field is seen easily from this sequence. Finally, ω_L^2 must always be less than ω_b^2.

We now consider the different vector components of the velocity $\mathbf{v}$, the electrical field $\mathbf{E}$, and the alternating magnetic field $\mathbf{h}$. Between each pair of these quantities, there exist two homogeneous equations which must be identical for all solutions compatible with the dispersion equation. We write down only the mathematically simpler form.

There are two relations between the alternating magnetic field components and the electric ones:

$$h_y=(kc/\omega)E_z, \tag{139}$$

$$h_z=-(kc/\omega)E_y. \tag{140}$$

On the other hand, the electric field components are related to the velocity components:

$$\frac{e}{m}E_x=-i\frac{\omega_e^2}{\omega^2}v_x, \tag{141}$$

$$\frac{e}{m}E_y=-i\frac{\omega_e^2\omega}{\omega^2-k^2c^2}v_y, \tag{142}$$

$$\frac{e}{m}E_z=-i\frac{\omega_e^2\omega}{\omega^2-k^2c^2}v_z. \tag{143}$$

There is just one relation (141) for the *longitudinal* components v_x and E_x whether or not there is a magnetic field. This does not hold for the transverse components E_y and v_y, and E_z and v_z, according to Eqs. (142) and (143); these relations contain the wave number and thus the refractive index or the phase velocity.

Thus in the preceding case of purely longitudinal magnetic field, there exist two different connections between the amplitudes of E_y (or h_z) and v_y, and E_z (or h_y) and v_z. We obtain them by introducing our dispersion relations (131) and (132) into the equations for the components (142) and (143). The two oscillations then correspond to the ordinary and extraordinary ray, but are, as previously stated, not allowable for all frequencies.

(b) *Transverse Magnetic Field:* $\omega_L=0$

In the other limiting case, when the magnetic field is purely transverse, one of the transverse waves separates out and is called the ordinary ray in ionospheric theory:

$$\omega^2-\omega_e^2-k^2c^2=0. \tag{144}$$

If we solve Eq. (144) for the phase velocity, we obtain

$$v_{\text{ph}}^2=c^2\omega^2/(\omega^2-\omega_e^2). \tag{145}$$

There is no dependence on the pressure as was the case for *both* transverse waves for purely longitudinal magnetic field. Our solution (145) corresponds to the unshifted component in the transverse Zeeman effect. This case is discussed in Sec. I.4.

The relations between the vector components are

$$h_z=-\frac{1}{\omega}(\omega^2-\omega_e^2)^{\frac{1}{2}}E_y=+i\frac{m}{e}(\omega^2-\omega_e^2)^{\frac{1}{2}}v_y, \tag{146}$$

and

$$\frac{e}{m}E_y=-i\omega v_y. \tag{147}$$

The second part of the dispersion equation yields

$$(\omega^2-\omega_e^2-k^2u_e^2)(\omega^2-\omega_e^2-k^2c^2)-\omega_T^2(\omega^2-k^2c^2)=0. \tag{148}$$

In the case of a transverse magnetic field, the frequencies and phase velocities of the longitudinal (v_x,E_x) and one of the transverse waves (v_z,E_z,h_y) are coupled together. This is the reason for the speculation concerning the possibility of *energy* transfer between the longitudinal and the transverse oscillations as may be the case in some astrophysically-interesting situations; for example, between shock waves and the observed radio-frequency radiation of the sun.

If we introduce the new variable into Eq. (148), we obtain

$$y^2=c^2u_e^2/v_{\text{ph}}^2, \tag{149}$$

and then

$$y=-\tfrac{1}{2}B\pm\left[\tfrac{1}{4}B^2-\left(\frac{\omega_e^2}{\omega^2}-1\right)^2+\frac{\omega_T^2}{\omega^2}\right]^{\frac{1}{2}},$$
$$B\equiv\frac{c}{u_e}\left(\frac{\omega_e^2+\omega_T^2}{\omega^2}-1\right)+\frac{u_e}{c}\left(\frac{\omega_e^2}{\omega^2}-1\right). \tag{150}$$

The dispersion relation, Eq. (148), is valid for the longitudinal oscillations (v_x,E_x) as well as the transverse one (v_z,E_z,h_y); hence, these two solutions of the dispersion relation are identical for both types of oscillations.

It is evident that $u_e^2\ll c^2$; thus the second term in the expression for B is always small compared with the first:

$$B\approx\frac{c}{u_e}\cdot\frac{1}{\omega^2}(\omega_e^2+\omega_T^2-\omega^2). \tag{151}$$

Also,

$$\tfrac{1}{4}B^2\gg\left(\frac{\omega_e^2}{\omega^2}-1\right)^2+\frac{\omega_T^2}{\omega^2} \tag{152}$$

always holds. Hence, on expanding the square root, this leads to

$$y\approx-\tfrac{1}{2}B\pm\tfrac{1}{2}B\pm B\left[-\left(\frac{\omega_e^2}{\omega}-1\right)^2+\frac{\omega_T^2}{\omega^2}\right]. \tag{153}$$

By considering first the oscillation corresponding to the *negative* sign in front of the root, we get

$$y\approx-\frac{c}{u_e}\cdot\frac{1}{\omega^2}(\omega_e^2+\omega_T^2-\omega^2) \tag{154}$$

or, by solving for the phase velocity,

$$v_{\text{ph}}^2=u_e^2\frac{\omega^2}{\omega^2-\omega_e^2-\omega_T^2}. \tag{155}$$

The value of the phase velocity, Eq. (155), corresponds closely to the phase velocity for longitudinal oscillations for zero magnetic field; here, however, this velocity applies to both waves. The waves propagate for all frequencies

$$\omega^2>\omega_e^2+\omega_T^2\equiv\omega_0^2. \tag{156}$$

Comparison with analogous results in Sec. I.7 reveals that the magnetic field increases the critical frequency which corresponds to the zero refractive index. For $\omega_T\to 0$, the field free case of Sec. I.7 results, while at the same time, the coupling between longitudinal and transverse vector components vanishes.

For the discussion of Eq. (155) we refer to Sec. I.4, substituting u_e^2 for c^2 and ω_0^2 for ω_e^2. The group velocity comes out in a manner analogous to Eq. (87):

$$g=u_e[1-(1/\omega^2)(\omega_e^2+\omega_T^2)]^{\frac{1}{2}}<u_e, \tag{157}$$

i.e., the velocity associated with the energy flow is smaller than the velocity of sound.

In the case of the solution of Eq. (150) with the *positive* sign in front of the root, the first two terms in Eq. (153) compensate each other and we obtain for the phase velocity

$$v_{\text{ph}}^2=-c^2\frac{\omega^2(\omega_e^2+\omega_T^2-\omega^2)}{(\omega_e^2-\omega^2)^2-\omega^2\omega_T^2}. \tag{158}$$

It is convenient to consider the numerator and denominator separately, putting

$$v_{\mathrm{ph}}^2 = -c^2 \frac{Z}{N}. \qquad (159)$$

Transmission is possible only when

$$\mathrm{sng}Z = -\mathrm{sgn}N. \qquad (160)$$

The numerator for the frequency

$$\omega_0^2 \equiv \omega_e^2 + \omega_T^2 \qquad (161)$$

is zero, with

$$\begin{array}{lcl} \mathrm{sgn}Z = +1 & \text{when} & \omega^2 < \omega_0^2 \\ \mathrm{sgn}Z = -1 & & \omega^2 > \omega_0^2. \end{array} \qquad (162)$$

The denominator equals zero for two values of the frequency, again analogous to the transverse Zeeman case with *two* shifted components. The denominator vanishes for the frequencies

$$\omega_{1,2}^2 \equiv \omega_e^2 + \tfrac{1}{2}\omega_T^2 \pm (\omega_e^2\omega_T^2 + \tfrac{1}{4}\omega_T^4)^{\frac{1}{2}}. \qquad (163)$$

It is easy to see that

$$\omega_2^2 < \omega_0^2 < \omega_1^2. \qquad (164)$$

Because the denominator is negative for ω_0^2,

$$N(\omega_0^2) = -\omega_T^2\omega_e^2 < 0, \qquad (165)$$

we have

$$\begin{array}{lcl} \mathrm{sgn}N = +1 & \text{when} & \omega^2 \begin{cases} < \omega_2^2 \\ > \omega_1^2 \end{cases} \\ \mathrm{sgn}N = -1 & & \omega^2 < \omega^2 < \omega_1^2. \end{array} \qquad (166)$$

We find from Eqs. (162) and (166) the transmission behavior of the plasma: propagation of waves is possible in the frequency ranges α,

$$\omega_2^2 < \omega^2 < \omega_0^2, \qquad (167)$$

and β,

$$\omega^2 > \omega_1^2. \qquad (168)$$

By letting $\omega_T \to 0$ the critical frequency becomes ω_e^2 as expected.

Finally, we have to determine the values of the phase velocity, illustrated in Fig. 3. v_{ph} is plotted on an arbitrary scale *vs* the frequency for the three cases mentioned in connection with Fig. 2: $\omega_T = 0.1\omega_e$, $\omega_T = 0.9\omega_e$, and $\omega_T = 5.0\omega_e$.

In the transmission range α there are all phase velocities between ∞ ($\omega = \omega_2$) and 0 ($\omega = \omega_0$), with

$$v_{\mathrm{ph}} = c \quad \text{for} \quad \omega = \omega_e. \qquad (169)$$

In the range β the phase velocity varies from ∞ ($\omega = \omega_1$) to c ($\omega \to \infty$).

By briefly reviewing the relations between the vector components, we again find the relation (141) for the longitudinal quantities v_x and E_x; thus, the amplitude relations are identical for both solutions (155) and (158).

On the other hand, these solutions yield different relations between v_z, E_z, and h_y; as stated at the beginning of this section, v_y, E_y, and h_z are not affected by the magnetic field. A rough calculation shows that we have to consider terms up to the order u_e^2/c^2 and thus the unabridged solution (150) (the root, however, may be expanded) instead of Eqs. (155) and (158).

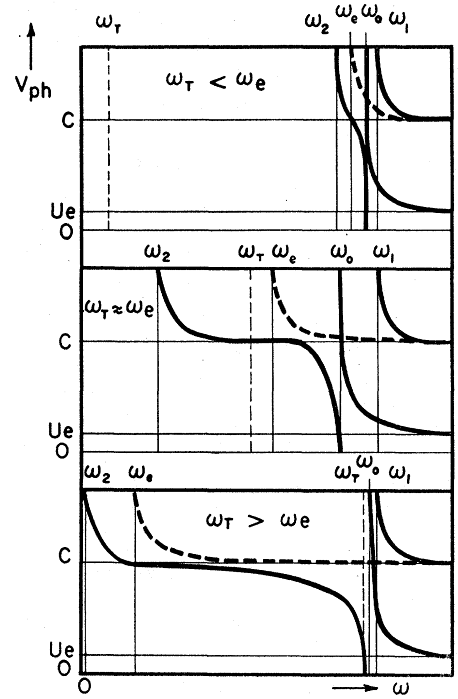

FIG. 3. The dispersion of a plasma in the case of a purely *transverse* magnetic field. The values of ω_T/ω_e are chosen as in Fig. 2. The solid lines correspond to the oscillations dependent on the magnetic field (v_x, E_x; v_z, E_z, h_y); the broken line corresponds to the transverse oscillation independent of the magnetic field.

After some calculation we find for the two solutions,

$$v_z = -\frac{i}{\omega}\frac{\omega_e^2 - \omega^2}{\omega_e^2 + \omega_T^2 - \omega^2}\frac{e}{m}E_z, \qquad (170)$$

$$v_z = -\frac{i}{\omega}\left[\frac{\omega_T^2(\omega^2 - \omega_e^2 - \omega_T^2)}{(\omega^2 - \omega_e^2)^2 - \omega^2\omega_T^2}\frac{c^2}{u^2} + 1\right]\frac{e}{m}E_z. \qquad (171)$$

The expressions for the amplitude relations are as involved as the ones for the phase velocities and depend as well on three parameters: mean density (ω_e), temperature (u_e), and magnetic field strength (ω_T). A detailed discussion is out of the scope of this article. However, one easily sees that the solutions (155) and (171) are physically as important as the usually quoted solutions, (158) and (170). Thus, there are transverse waves in the plasma traveling with nearly the sound velocity, their amplitudes being not at all negligible. It seems to be of major interest to check whether or not these waves may continue into a field-free region; such an investigation, which must include macroscopic density and field gradients, cannot be undertaken in the frame of our equations.

Finally, we find for vanishing magnetic field ($\omega_T \to 0$) that both solutions (170) and (171) converge to the relation for ordinary transverse waves.

12. Alfvén's Magnetohydrodynamics

Another special case of the treatment of plasmas with nonzero magnetic fields is known as "magnetohydrodynamics," which was initiated by the work of H. Alfvén and his collaborators.

Alfvén's treatment is still closer to the concept of a continuum than is the case for the system of hydrodynamic equations used in the last sections. He reduces electrons and ions to a *uniform* continuum which he describes by the phenomenological (constant) quantities ρ (density), σ (electrical conductivity), and μ (magnetic permeability). This model of the plasma again is electrically neutral, but now, for each volume element no charge separation at all is possible. Finally, he assumes incompressibility[||] for the whole plasma, thus obtaining only transverse and no longitudinal waves. We considered the incompressible case for a pure electron plasma in Sec. I.4. At present it is not possible to decide if Alfvén's model is well suited for handling the gaseous plasmas occurring in nature in stellar matter.

There are many papers dealing with magnetohydrodynamic problems. We do not review them here but try to clarify the common points between our treatment and the usual magnetohydrodynamic one. We cite, besides the original papers of Alfvén and his collaborators [cf. Alfvén (1950)], the books by Spitzer (1956) and T. G. Cowling (1957). A somewhat specialized choice, but with more details and a large list of papers, is given in the review by Cole (1956).

The whole Alfvén theory is included in our dispersion determinant (123) as long as one is not interested in damping processes, and we could develop this theory by first making $u_e^2 = u_i^2 = 0$ according to the concept of complete incompressibility. The usual magnetohydrodynamic formulas then result from further simplifications, e.g., neglect of the displacement current as compared with the conduction current.

Instead of proceeding along these lines, we use the fundamental equations given by Spitzer (1956), which seem to be more convenient in many cases, and obtain the connection between this treatment and the one used hitherto.

For this purpose we introduce with A. Schlueter (1950) the so-called "mass velocity"

$$\mathbf{V} = \frac{m_i \mathbf{v}_i + m_e \mathbf{v}_e}{m_i + m_e}, \tag{172}$$

and the "diffusion velocity"

$$\mathbf{d} = \mathbf{v}_i - \mathbf{v}_e. \tag{173}$$

[||] In recent work, this restriction sometimes has been removed; cf. the reviews by L. Spitzer (1956) and G. H. A. Cole (1956).

This mass velocity $\mathbf{V}$ corresponds exactly to the velocity of the uniform plasma used in magnetohydrodynamics.

With the assumption of incompressibility, we use

$$N_e = N_i = N_0 = \text{const.} \tag{174}$$

The electric current is then defined by

$$\mathbf{j} = eN_0 \mathbf{d}. \tag{175}$$

By neglecting the electronic mass compared with the mass of the ions, which always may be done, we obtain for the density

$$\rho = N_0 m_i. \tag{176}$$

By letting the permeability $\mu = 1$, which again is no restriction, we obtain, by summing and subtracting the equations of motion for electrons and ions,

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \frac{1}{c}(\mathbf{j} \times \mathbf{H}^0), \tag{177}$$

$$\frac{m_e}{N_0 e^2} \frac{\partial \mathbf{j}}{\partial t} = \mathbf{E} + \frac{1}{c}(\mathbf{V} \times \mathbf{H}^0) - \frac{1}{ecN_0}(\mathbf{j} \times \mathbf{H}^0) - \frac{1}{\sigma}\mathbf{j}. \tag{178}$$

The term proportional to $\mathbf{j}$ results from the consideration of damping effects in the equations of motion and yields a connection between the damping constant (some kind of mean collision frequency) and the conductivity σ. Equation (178) reduces to the normal Ohm's law by putting the change in time of the electric current and the magnetic field equal to zero.

As usually is done in the Alfvén theory, we took into account only a scalar conductivity. Piddington [(a) (1955)] has investigated the influence of the magnetic field upon the electric conductivity.

By using Eqs. (177) and (178) together with Maxwell's equations, the same dispersion relation is obtained as would be by using the determinant (123) together with the approximations just mentioned and assuming infinite conductivity, i.e., neglecting damping.

The usually quoted magnetohydrodynamic equations follow from Eqs. (177) and (178); by using $\sigma = \infty$,

$$\frac{m_e}{N_0 e^2} \frac{\partial \mathbf{j}}{\partial t} = 0, \quad \frac{1}{ecN_0} \mathbf{j} \times \mathbf{H}^0 \tag{179}$$

By comparing the resulting dispersion relation so derived with the general one, we find that the assumptions (179) correspond to the neglect of terms of the order

$$\sim (\omega^2/\omega_H^e \omega_H^i) \quad \text{and} \quad \sim (\omega/\omega_H^i) \tag{180}$$

as previously stated by Spitzer (1956).

This means that for Alfvén's basic equations to be valid, the frequency ω must be *small* compared with the gyrofrequency of the ions ω_H^i. In the last section we saw that this kind of wave is transmitted through a plasma only under certain conditions. It seems necessary to investigate this point further in the usual magnetohydrodynamic theory.

Finally, the displacement current in Maxwell's equations usually is neglected, and so we obtain the well-known relation

$$v_{\mathrm{ph}}^2=\mu H_0^2/4\pi\rho=c^2(\omega_H{}^{i2}/\omega_i{}^2). \tag{181}$$

13. Constant Motions across the Propagation Direction of Waves

In Sec. I.6 we discussed the influence of the choice of the reference frame upon the results for longitudinal waves. We restricted our considerations to the case of translations parallel to the direction of wave propagation. We delayed the discussion of motions *across* the direction of propagation because, in this case, we have to use the formulas for the electrodynamics of moving matter, which are to be derived from the special theory of relativity. While until now the formulas of the Galilean transformation were enough for our purposes, we have to consider in the following—directly or indirectly—the full Lorentz transformation, in spite of the fact that we neglect all strictly relativistic effects, i.e., terms of the order v^2/c^2. Following the discussion of motions perpendicular to the propagation direction of longitudinal waves, we consider the case of transverse waves and motions parallel and perpendicular to their direction of propagation.

We again start from the usual model of an infinite plasma and a longitudinal density wave, taking into account the variation of pressure in the equations of motion. Throughout this section we neglect the motions of ions and assume that there is no constant magnetic field; otherwise, we would have to use terms of the form

$$\mathbf{v}_0\times\mathbf{H}^0 \tag{182}$$

in the equations of motion, which make the equations inhomogeneous. In Sec. I.6 we showed that such a case can not be handled by the methods used in this investigation. Furthermore, it is well known that a plasma moving as a whole across a magnetic field changes the field itself. As a matter of fact, for infinite conductivity ($\sigma=\infty$) the field lines are frozen in the plasma.

Consider a reference frame K' in which one observes the phenomena described in Sec. I.7, i.e., the observer measures an electric field

$$E_x'\neq 0,\quad E_y'=E_z'=0, \tag{183}$$

velocities

$$v_x'\neq 0,\quad v_y'=v_k'=0, \tag{184}$$

and, besides the mean density N_0, density variations

$$n_e\neq 0. \tag{185}$$

There are no magnetic fields

$$\mathbf{h}'=0. \tag{186}$$

Equations (183)–(186) have nothing to do with the assumption of a harmonic shape of the traveling waves (except the general restrictions upon the use of hydrodynamic methods for the description of nonequilibrium features such as harmonic waves); therefore, they are valid whenever there is a stationary situation with a uniform and constant phase velocity. In general, we still use harmonic waves, but want to make this point clear in connection with the paper by Larenz (1955) discussed later in this section.

Furthermore, we still assume that there are no gradients besides the one in a direction defining the x axis:

$$\frac{\partial}{\partial y'}=\frac{\partial}{\partial z'}\equiv 0 \tag{187}$$

We now go over to a reference frame K. The plasma is supposed to move in K with the constant velocity v_0 in the direction of the positive z axis. The velocities measured in K are to be calculated from the values measured in K' by means of the Lorentz transformation, for which we still can use the Galilean transformation in the case $v\ll c$ to be treated here. For uniformity of treatment—as stated previously, we have to use the formulas of the special theory of relativity for the calculation of the electric and magnetic field components—we use the unabridged formulas of the Lorentz transformation.

From Eq. (184) we obtain

$$v_z=\frac{v_z'+v_0}{1+(v_0v_z'/c^2)}=v_0=\text{const.} \tag{188}$$

With the usual meaning of

$$\beta^2=v_0^2/c^2, \tag{189}$$

we obtain for the x and y components

$$v_x=v_x'\frac{(1-\beta^2)^{\frac{1}{2}}}{1+(v_0v_z'/c^2)}=v_x'(1-\beta^2)^{\frac{1}{2}}\approx v_x', \tag{190}$$

and

$$v_y=v_y'\frac{(1-\beta^2)^{\frac{1}{2}}}{1+(v_0v_z'/c^2)}=0 \tag{191}$$

From Eq. (188) it follows that no accelerations are possible in the direction of constant motion (z axis), a result which is obvious from a physical point of view.

The transformation formulas for the electric and magnetic quantities—cf. A. Sommerfeld (1949), whose equations we transformed from the mks to the Gaussian system of units—are ($\|$ corresponds to the z direction, $\perp$ to the x and y direction)

$$E_{\|}'=[\mathbf{E}+(1/c)\mathbf{v}_0\times\mathbf{h}]_{\|},\qquad E_{\perp}'(1-\beta^2)^{\frac{1}{2}}=[\mathbf{E}+(1/c)\mathbf{v}_0\times\mathbf{h}]_{\perp}, \tag{192}$$

$$h_{\|}'=[\mathbf{h}-(1/c)\mathbf{v}_0\times\mathbf{E}]_{\|},\qquad h_{\perp}'(1-\beta^2)^{\frac{1}{2}}=[\mathbf{h}-(1/c)\mathbf{v}_0\times\mathbf{E}]_{\perp}. \tag{193}$$

By using the relation

$$\rho^* = -ne \tag{194}$$

for the variable electric charge density (the mean electric charge density is zero), we obtain for the electric current density

$$(1-\beta^2)^{\frac{1}{2}} j_{||}' = j_{||} + nev_0, \quad j_\perp' = j_\perp. \tag{195}$$

By using Eqs. (183) and (186) we find

$$E_z = 0, \tag{196}$$

$$h_z = 0, \tag{197}$$

$$E_x - (v_0/c)h_y = (1-\beta^2)^{\frac{1}{2}} E_x', \tag{198}$$

$$h_y - (v_0/c)E_x = 0, \tag{199}$$

$$E_y + (v_0/c)h_x = 0, \tag{200}$$

$$h_x + (v_0/c)E_y = 0. \tag{201}$$

From Eqs. (200) and (201) it follows that

$$E_y = h_x = 0, \tag{202}$$

since

$$1 - (v_0^2/c^2) = 1 - \beta^2 \neq 0. \tag{203}$$

Furthermore, from Eqs. (198) and (199), we obtain

$$E_x = E_x'/(1-\beta^2)^{\frac{1}{2}}. \tag{204}$$

This means that an observer moving across the gas measures the same electric field in the longitudinal oscillations as the observer at rest, except for a relativistic correction. On the other hand, the moving observer finds, according to Eq. (199), a magnetic field component

$$h_y = [\beta/(1-\beta^2)^{\frac{1}{2}}] E_x', \tag{205}$$

while the observer at rest finds *no* magnetic field.

The equations for the electric current, according to (195), are

$$j_x = j_x' = -nev_x', \tag{206}$$

$$j_y = j_y' = 0, \tag{207}$$

and taking into account $j_z' = 0$,

$$j_z = -nev_0. \tag{208}$$

Finally, we still have in the moving system

$$\frac{\partial}{\partial y} = 0, \tag{209}$$

but

$$\frac{\partial}{\partial z} \neq 0. \tag{210}$$

This result is not unexpected. It is nothing else than the well-known aberration of light. By using the concept of harmonic oscillations, we find, from the Lorentz transformation,

$$x = x', \quad z' = \frac{z - v_0 t}{(1-\beta^2)^{\frac{1}{2}}}, \quad t' = \frac{t - (1/c^2)v_0 z}{(1-\beta^2)^{\frac{1}{2}}}, \tag{211}$$

that a plane wave propagating in the x' direction

$$e^{i\omega t' + ikx'} \tag{212}$$

becomes a wave for which the planes of equal phase are no longer perpendicular to the $x = x'$ axis:

$$\exp\left[i\frac{\omega}{(1-\beta^2)^{\frac{1}{2}}} t + ikx - i\omega \frac{v_0}{c^2(1-\beta^2)^{\frac{1}{2}}} z\right]. \tag{213}$$

Thus, the moving observer finds a plane wave having a propagation vector with a component in the z direction:

$$\mathbf{k} = \left[k, 0, -\omega \frac{v_0}{c^2(1-\beta^2)^{\frac{1}{2}}}\right] = k\left[1, 0, -\frac{v_0}{c} \cdot \frac{v_{\text{ph}}}{c(1-\beta^2)^{\frac{1}{2}}}\right]. \tag{214}$$

This "shift of the plane of equal phases" is an effect of the first order, i.e., $\sim v_0/c$, as is well known from the theory of aberration.

From these considerations we have obtained a fairly complete answer concerning the behavior of longitudinal waves as seen by a moving observer.

These conclusions do not agree with those obtained by Larenz [(c) (1955)]. By starting from phenomena for an observer in the system K' at rest in the gas (Sec. 1), he goes over to the system K without explicitly transforming the electrical and mechanical quantities.

Larenz at once writes down the fundamental equations in the system K (Sec. 2), introducing an acceleration in the direction of the constant translation $\bar{v}$ [Larenz: $dv_z/dt \neq 0$, Eq. (8), p. 903]; furthermore, the z component of the electric field vector is finite. The shift of the phase planes, cf. our Eq. (210), is not considered.

Interpreting his fundamental equations correctly leads, according to Eqs. (188) and (196), to transverse waves as well as longitudinal waves even in the reference frame at rest in the plasma.¶

Consequently, it seems rather unnecessary to carry on the already quite complicated calculations (as mentioned in the introduction, Larenz does *not* linearize and uses an adiabatic connection between density and pressure which again complicates the equations a great deal) in a coordinate system moving through the gas.

As we have seen, the equations used by Larenz do not hold for the reference frame he has in mind since he neglects our condition (210). We now wish to obtain from his results as much information as possible concerning the effect of his nonlinear terms. It is possible to transform the equations [Larenz: Eqs. (7)–(11), p. 903] back into a reference frame at rest by

¶ I am indebted to Dr. Larenz and Dr. Burckhardt, Hannover, Germany, for helpful criticisms of this point of view.

setting his $\bar{v}=0$. The equations then are complete and without contradiction. However, this method fails later, because he introduces new variables

$$u\equiv\frac{v_z}{\bar{v}}\to\infty,\quad \zeta\equiv(1+u)^2\to\infty,\quad \varphi\equiv\frac{u^2-1}{\text{const}}\to\infty \tag{215}$$

for

$$\epsilon\equiv\frac{\bar{v}}{c}\to 0. \tag{216}$$

By using his numerical results without further discussion for the case $\epsilon\to 0$, we are inclined to assume that even in the nonlinear case treated by Larenz and interpreted along our lines, there is *no* coupling between longitudinal and transverse waves. This would indicate that there is no possibility of an energy transfer from longitudinal waves (e.g., shock waves) to transverse waves (e.g., electromagnetic waves) without a nonzero constant magnetic field. For a final decision concerning this most important question, more exact calculations are urgently needed. D. A. Tidman and E. N. Parker (1959) have come to the same conclusion; they showed that macroscopic temperature or density gradients are needed for the generation of electromagnetic radiation by longitudinal plasma waves.

In discussing Larenz's paper, there is still a second possibility in interpreting his fundamental equations: By starting from our concept and taking into account the aberration (210), we can define the axes in such a way that the direction of *propagation* coincides with the x axis. Then the electric field vector as well as the velocity vector has a "transverse" component in z direction. Again, as a consequence of Eq. (210), the condition

$$\left|\frac{E_z}{E_x}\right|=\left|\frac{v_z}{v_x}\right|=\frac{v_0v_{\text{ph}}}{c^2(1-\beta^2)^{\frac{1}{2}}} \tag{217}$$

holds. Thus, in the limiting case $v_0\to 0$ (in Larenz's notation $\epsilon\to 0$), i.e., in the frame at rest, we should expect a purely longitudinal wave. This does not hold because Larenz neglects condition (217) (he obtains a transverse wave even for $\epsilon\to 0$), and thus this second interpretation is not possible.

We now briefly sketch the effects of motions on the behavior of *transverse* waves. We do not go into much detail, because for a complete investigation one would have to write down the equation of motion in a relativistically invariant form. We do not think that the results are worth this labor because new physical phenomena are not to be expected. We write, for the sake of completeness, the field components measured by a moving observer.

By starting with a constant motion parallel to the direction of propagation (again chosen as the x axis) and determining the transverse wave by its field components E_z' and h_y', we conclude from Eqs. (172) and (173) that in the moving system too, there are only two field components. Between the old and the new quantities, the following conditions hold:

$$\begin{aligned}(1-\beta^2)^{\frac{1}{2}}E_z&=E_z'-(v_0/c)h_y',\\(1-\beta^2)^{\frac{1}{2}}h_y&=h_y'-(v_0/c)E_z'.\end{aligned} \tag{218}$$

In the case of a motion perpendicular to the direction of propagation we find a slightly different result. Let the motion take place along the z axis. We first split off the transverse oscillation into the linearly polarized components, the first one determined in the frame at rest by the quantities E_z' and h_y'. From Eqs. (192) and (193) we now obtain, in the moving frame,

$$E_z=E_z',\quad (1-\beta^2)^{\frac{1}{2}}h_y=h_y',\quad (1-\beta^2)^{\frac{1}{2}}E_x=(v_0/c)h_y', \tag{219}$$

and for the second wave (with E_y' and h_z' in the frame at rest) we obtain

$$h_z=h_z',\ (1-\beta^2)^{\frac{1}{2}}E_y=E_y',\ (1-\beta^2)^{\frac{1}{2}}h_x=-(v_0/c)E_y'. \tag{220}$$

This means that the moving observer measures components E_x and h_x, while the observer at rest does not; this again is the effect of aberration. The additional field components make the moving observer find a somewhat different direction for the source of the radiation than does the observer at rest. The situation is exactly the one well known in astronomy, optics, etc., since in deriving Eqs. (218)–(220) we have not made use of the special plasma character of the medium. Finally, the plane of equal phases is inclined, as we already have shown in the beginning of Sec. I.13, the amount being of the order v_0/c. These considerations complete our investigation concerning the situation found by a moving observer.

14. Conclusion

Some points which seem to be of importance to later, especially nonlinear, investigations require emphasis. We tried to show that the normally accepted theory of electron wave tubes is not correct for various reasons and should be reconsidered. We think that for this purpose it is necessary to take as a model a *finite* plasma and the proper boundary conditions for the problem at hand.

For an investigation of a plasma with nonzero magnetic field, progress towards a decision of whether or not there is radiation excited by primarily longitudinal waves seems to be possible only by nonlinear methods. On the other hand, there still may be some problems to be solved by linearized treatments such as the transmissivity of a plasma for Alfvén waves.

Finally, there seems to be no reason to take into account constant velocities of the plasma as a whole, which complicate the mathematical treatment a great deal but apparently without new physical results. The same may be inferred from our discussions of the statement that an adiabatic formulation is necessary and has to replace the usual isothermal one.

II. Kinetic Theory Treatment

INTRODUCTION

In Sec. I the longitudinal and transverse oscillations of a plasma were investigated by using a linearized hydrodynamic treatment, and the existence of a certain number of different wave types and their mutual interaction was discussed.

It is well known that the fundamental equations of hydrodynamics can be obtained from the Boltzmann equation. To a certain extent, the basic hydrodynamic equations are an approach to the correct kinetic treatment. This exact treatment using the kinetic equations is at present too complicated for mathematical reasons to be used in the description of the general behavior of a plasma undergoing all types of wave motions. Thus, until now, the kinetic treatment has been used only in more or less narrow fields of plasma physics, especially in those instances in which the hydrodynamic treatment fails for intrinsic reasons. This is the case for all questions involving a more than phenomenological approach to damping effects caused by collisions of the plasma particles and for problems in which the velocity distribution function is specifically involved. The first of these problems is closely connected with the calculation of the scattering cross sections of charged particles and thus with the electric and thermal conductivity, etc. A survey of these problems has been given by Oster (1957). Some aspects of the effects of collisions on plasma oscillations have been treated recently by Bhatnagar, Gross, and Krook (1954) and Gross and Krook (1956). We discuss them briefly in Sec. II.6.

We are interested rather in showing what restrictions on the shape of the velocity distribution of electrons are necessary for the hydrodynamic treatment to be valid. In a recent paper N. G. van Kampen (1957) has discussed this question but from a somewhat different point of view. Finally, we explain why Gross (1951) came to the conclusion that the hydrodynamic treatment conceals the essential features of a plasma.

1. Boltzmann Equation

The kinetic approach must begin with the Boltzmann transport equation. We write the Boltzmann equation for the *electrons* only, assuming as we usually did in Sec. I that the ions as a whole are at rest compared with the electrons as an assembly. We thus neglect the oscillations of the ions. In the kinetic picture this means that an undisturbed Maxwell distribution always holds for the ions.

We take the Boltzmann equation in the form

$$\frac{\partial f}{\partial t}+(\mathbf{v}\cdot\nabla)f+\frac{1}{m}(\mathbf{F}\cdot\nabla_v)f=\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \qquad (1)$$

where f is the velocity distribution of the electrons and is a function of time, space, and the velocities themselves. $\nabla_v f$ is a vector with the components $\partial f/\partial v_n$ (v_n are the velocity components in the directions $n=1$, 2, 3), and $(\partial f/\partial t)_{\text{coll}}$ is the change in the velocity distribution due to collisions with other particles. $\mathbf{F}/m$ is the external force per unit mass, which is identified later with the electric field set up by the oscillations.

The essential feature of Eq. (1) is the collision term on the right side. We are interested in knowing under what conditions this term vanishes, i.e., when the collision do not destroy an initial velocity distribution. This obviously happens when there are no collision at all, e.g., when the density is extremely small. As van Kampen (1957) pointed out, most of the previously published papers deal with this case.

For the purpose of a comparison with hydrodynamic treatments we have to deal with conditions in which the right side vanishes for another reason, i.e., when a stationary state is reached. This takes place when collisions *reproduce* a given velocity distribution.

We cannot expect from the beginning that this is true for all types of plasma oscillations, because they correspond to an ordered motion of many particles, and collisions usually try to establish a disordered state of motion. We therefore first determine the exact conditions under which there are no effects of collisions on the oscillations, and then use this knowledge to work out an approximation method for the general case.

2. Locally Maxwellian Distribution

It is a known result of the kinetic theory of gases that a steady state [right side of Eq. (1) equals zero] is exactly possible only in the case of a "locally Maxwellian" distribution of velocities, i.e., in the case of a distribution function

$$f(v)=a\cdot\exp[-\Gamma(\mathbf{v}-\boldsymbol{\xi})^2], \qquad (2)$$

where a, Γ, and ξ are parameters depending on the space coordinates and the time but not on the velocity components v_n. The physical interpretation of these parameters becomes clear by comparison with a normal Maxwellian distribution

$$f_0=N_0\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}\exp\left[-\frac{m}{2KT}(v_x^2+v_y^2+v_z^2)\right], \qquad (3)$$

where N_0 is the density, m the mass, K the Boltzmann constant, and T the temperature.

Equation (2) is derived under the assumption [A. Sommerfeld (1952)] that the "collisions" are understood as two-body elastic collisions of identical particles. This means, in our case, collisions of electrons with themselves. The statement that collisions restore the distribution function (2) therefore refers only to this type of collision. Besides that, there is still the interaction of the electrons with heavy ions which is neglected in the derivation of Eq. (2) and which gives rise to an additional damping of all types of oscillations, even of those compatible with condition (2). We take that into account in Sec. II.6 and derive an appropriate expression for the resulting attenuation of the oscillations.

The dependence of the distribution function (2) on the space and time coordinates is not arbitrary, but is restricted by the left-hand side of Eq. (1). We first consider the case of a velocity independent force **F**, i.e., we exclude the presence of magnetic fields which is considered in Sec. II.5.

By introducing expression (2) for the distribution function f into Eq. (1), we obtain

$$\frac{\partial a}{\partial t}\cdot\frac{1}{a}-(\mathbf{v}-\boldsymbol{\xi})^2\frac{\partial\Gamma}{\partial t}+2\Gamma\left[(\mathbf{v}-\boldsymbol{\xi})\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}\right]+(\mathbf{v}\cdot\nabla a)/a$$
$$-(\mathbf{v}-\boldsymbol{\xi})^2(\mathbf{v}\cdot\nabla\Gamma)+2\Gamma[\mathbf{v}\cdot\nabla(\boldsymbol{\xi}\cdot\mathbf{v})]-\Gamma[\mathbf{v}\cdot\nabla(\boldsymbol{\xi}^2)]$$
$$-2\Gamma\mathbf{F}\cdot(\mathbf{v}-\boldsymbol{\xi})/m=0. \quad (4)$$

Since a, Γ, and $\boldsymbol{\xi}$ do not depend on the velocity, we may separate Eq. (4) into terms of the zeroth, first, second, and third degree in $\mathbf{v}$ and equate the coefficients to zero. One then obtains

$$\nabla\Gamma=0, \quad (5)$$

$$\frac{\partial a}{\partial t}\cdot\frac{1}{a}=-(\boldsymbol{\xi}\cdot\nabla a)/a+\Gamma[\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}^2)]-\frac{\partial\Gamma}{\partial t}\boldsymbol{\xi}^2, \quad (6)$$

$$\mathbf{F}/m=\frac{\partial\boldsymbol{\xi}}{\partial t}+\nabla a/2\Gamma a-\tfrac{1}{2}\nabla(\boldsymbol{\xi}^2)+\frac{\partial\Gamma}{\partial t}\boldsymbol{\xi}/\Gamma, \quad (7)$$

$$\frac{\partial u_i}{\partial k}+\frac{\partial u_k}{\partial i}=\frac{\partial\Gamma}{\partial t}\cdot\frac{1}{\Gamma}\delta_{ik},\quad i,k=1,2,3, \quad (8)$$

with $\delta_{ik}=1$ for $i=k$, $\delta_{ik}=0$ for $i\neq k$, and $\boldsymbol{\xi}=(u_x,u_y,u_z)$.

By comparing the distribution (3) with the Maxwellian (2), we notice that the only variable quantity in Γ is the temperature. Restricting our considerations as in Sec. I to the case of an isothermal plasma, we fulfill condition (5) automatically. With

$$\Gamma=\text{const}, \quad (9)$$

Eqs. (6)–(8) become somewhat less involved.

From Sec. I we know that the result of the adiabatic treatment is the same as for the isothermal one, provided one uses the appropriate expression for the sound velocity. We come back to this statement from the kinetic point of view in Sec. II.3(a).

(a) *Case of Plane Harmonical Waves*

We limit the discussion to the case of plane harmonic waves. The first type of waves of interest to us is the longitudinal type. We define the reference frame as in Sec. I, assuming for all space variable quantities a dependence on only the x coordinate. From Eqs. (8) and (9) we then have, for the longitudinal oscillations,

$$\partial u_x/\partial x=0,\quad u_y=u_z=0. \quad (10)$$

Instead of Eqs. (6) and (7) we obtain

$$\frac{\partial a}{\partial t}=-u_x\frac{\partial a}{\partial x} \quad (11)$$

and

$$\frac{F_x}{m}=-\frac{e}{m}E_x=\frac{\partial u_x}{\partial t}+\frac{1}{2\Gamma a}\frac{\partial a}{\partial x}. \quad (12)$$

Because "a" essentially describes the space and time behavior of the density (the temperature which affects "a" also has been assumed to be constant), harmonic oscillations are compatable with Eq. (11) only if

$$u_x=v_0=\text{const}. \quad (13)$$

Equation (10) is always fulfilled.

Equation (11) describes a density wave traveling with constant velocity in front of an observer. On the other hand, for an observer moving with constant velocity v_0 along the x axis, this means a standing wave in the plasma. The electric field associated with the density distribution is, from Eq. (12), given by

$$E_x=-\frac{m}{2e\Gamma a}\frac{\partial a}{\partial x}. \quad (14)$$

This solution does not interest us because there is really no traveling wave involved.

However, there is a second solution,

$$a=\text{const}, \quad (15)$$

compatible with the assumption of harmonic waves and with Eq. (11): In this case the density would be always and everywhere constant. We would be dealing with an incompressible plasma which, from Eq. (10), would be oscillating synchronously over the whole space.

By introducing the notation

$$u_x\equiv\xi=\xi_0e^{i\omega t} \quad (16)$$

for the velocity component, according to Eq. (12) we obtain, for the amplitude of the electrical field,

$$E_0=-i\omega\frac{m}{e}\xi_0 \quad (17)$$

by using

$$E_x=E_0e^{i\omega t} \quad (18)$$

for the electric field.

We conclude, strictly speaking, that oscillations in a plasma with a stationary velocity distribution for the electrons are possible only in the case of synchronous oscillations of the whole incompressible medium. Only in this case is there a purely Maxwellian distribution which is indeed variable in time.

(b) *Shape of the Velocity Distribution*

The derived velocity distribution has the shape

$$f=N_0\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}\exp\left[-\frac{m}{2KT}\left\{\left(v_x-\frac{e}{m}E_0\frac{e^{i\omega t}}{i\omega}\right)^2+v_y^2+v_z^2\right\}\right]. \quad (19)$$

For the case of small amplitudes, in addition to a factor of the dimension of a length, the ratio

$$eE_0/KT \quad (20)$$

between the electric field energy and the thermal energy is small. Thus we may expand the exponential function and find an expression corresponding to the one derived by Lorentz (1916):

$$f=f_0+f_1=N_0\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}\exp\left\{-\frac{m}{2KT}(v_x^2+v_y^2+v_z^2)\right\}\times\left[1+\frac{eE_0}{KT}\frac{e^{i\omega t}}{i\omega}v_x\right]. \quad (21)$$

The factor in front of the term in brackets describes the undisturbed Maxwellian distribution f_0. We note that

$$\int f_1 dv_x \sim \int_{-\infty}^{+\infty}\exp(-\text{const}\cdot y^2)y\,dy=0 \quad (22)$$

(constant density), but

$$\int f_1 v_x dv_x \sim \int_{-\infty}^{+\infty}\exp(-\text{const}\cdot y^2)y^2dy\neq 0 \quad (23)$$

(nonzero momentum transport).

3. Distribution Function in the Case of Longitudinal Oscillations

In the last section we saw that the assumption of a steady-state distribution holds strictly only for synchronous oscillations of an incompressible plasma. We now take this model as an approximation for the general case of longitudinal oscillations. The conditions justifying such a procedure are clear: The wavelength of the plasma oscillation has to be large compared with the mean free path of the particles and, accordingly, the period of the oscillation large compared with the mean time between two collisions.**

In what follows, we link together the space and time coordinates which always appear in the mutual relation

$$x+v_{\text{ph}}t=x^* \quad (24)$$

writing

$$\omega/k=v_{\text{ph}} \quad (25)$$

for the phase velocity. We always assume the validity of the restrictions discussed above on the frequency ω and the wave number k.

Instead of Eqs. (16) and (18), we now have

$$\xi=\xi_0 e^{ikx^*},\quad E_x=E_0 e^{ikx^*}, \quad (26)$$

while the connection between ξ_0 and E_0 has to be calculated separately. This is done later in this section.

We start from a locally Maxwellian distribution of the form

$$f=N_0\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}\left[1+\frac{n_0}{N_0}e^{ikx^*}\right]\times\exp\left\{-\frac{m}{2KT}[(v_x-\xi)^2+v_y^2+v_z^2]\right\}. \quad (27)$$

Since ξ is supposed to be small compared to v_x, we can expand the last factor in (27) in a Taylor series and obtain

$$f=N_0\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}\left[1+\frac{n_0}{N_0}e^{ikx^*}\right]\times\exp\left\{-\frac{m}{2KT}(v_x^2+v_y^2+v_z^2)\right\}\times\left[1+\frac{m}{KT}v_x\xi+\cdots\right], \quad (28)$$

which, on keeping only the lower-order terms and introducing

$$n_1=N_0\frac{m}{KT}v_{\text{ph}}\xi_0, \quad (29)$$

may be written

$$f=N_0\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}\left[1+\frac{n_0}{N_0}e^{ikx^*}+\frac{n_1}{N_0}\frac{v_x}{v_{\text{ph}}}e^{ikx^*}\right]\times\exp\left\{-\frac{m}{2KT}(v_x^2+v_y^2+v_z^2)\right\}. \quad (30)$$

The quantities n_0 and n_1 are parameters of the dimension cm^{-3}.

By introducing the equilibrium distribution f_0 and the derivatives with respect to v_x, we can write Eq. (30) in the slightly different form which is well known in kinetic theory [see, e.g., A. Sommerfeld (1952)]:

$$f=f_0+c_1(x,t)\frac{\partial f_0}{\partial v_x}+c_2(x,t)\frac{\partial^2 f_0}{\partial v_x^2}. \quad (31)$$

The term with the first-order derivative corresponds to the distribution we derived in the incompressible case (21), while the essential new feature of Eq. (27), the density fluctuations independent of v_x, comes into the picture by the term with the second-order derivative.

** Remember, however, that by collisions we understand only the electron-electron interactions.

The latter includes a term proportional to v_x^2 which is of interest when adiabatic conditions are considered.††

For the determination of n_0 and n_1 the Boltzmann equation, which is now written

$$\frac{\partial f}{\partial t}+v_x\frac{\partial f}{\partial x}-\frac{eE_0}{m}e^{ikx^*}\cdot\frac{\partial f}{\partial v_x}=0, \tag{32}$$

may be integrated, once directly (leading to the continuity equation) and once multiplied by v_x (leading to the momentum equation). Multiplying by v_x^2 and integrating yields the energy sum which gives a third relation compatible with previous relations only when use is made of the adiabatic approximation; see subsection of Sec. II.3.

By the integration one replaces essentially the odd powers of v_x by their average of zero, while the even powers of v_x are replaced by powers of the average thermal velocity. This procedure may be justified by the following argument: We have chosen a solution to satisfy the right-hand side of the Boltzmann transport equation. We then know from the last section that, strictly speaking, our solution violates the left-hand side. However, the difference can be minimized if we determine the free parameters in such a way that the conservation laws are fulfilled. This is done by the integration process.

From the integrated equations, on dropping the higher-order terms in e^{ikx^*}, we obtain

$$n_0+\frac{u^2}{v_{\text{ph}}^2}n_1=0 \tag{33}$$

and

$$n_0+n_1=i\frac{eE_0}{m}\frac{N_0}{ku^2}, \tag{34}$$

where we use, for Newton's sound velocity, the symbol

$$u=(KT/m)^{\frac{1}{2}}. \tag{35}$$

For the parameters n_0 and n_1 we find

$$n_0=-i\frac{eE_0}{KT}\frac{N_0}{k}\frac{u^2}{v_{\text{ph}}^2-u^2}, \tag{36}$$

$$n_1=+i\frac{eE_0}{KT}\frac{N_0}{k}\frac{v_{\text{ph}}^2}{v_{\text{ph}}^2-u^2}. \tag{37}$$

The quantity $v_{\text{ph}}^2-u^2$ is discussed later in this section.

We now return to our problem of comparing the hydrodynamic and the kinetic treatment. If we are correct in supposing that both treatments yield the same results, we ought to obtain, on using the velocity distribution derived previously, the dispersion equation (85) from Sec. I.

†† To obtain the complete expressions in the adiabatic case, however, we should include second-order derivatives with respect to v_y and v_z; see Sec. II. 3(a).

We find the dispersion relation from the x component of Maxwell's equation

$$\frac{\partial E_x}{\partial t}-4\pi e\int_{-\infty}^{+\infty}v_xf(v_x,v_y,v_z)dv_xdv_ydv_z=0 \tag{38}$$

to be

$$i\omega E_0-4\pi e(u^2/v_{\text{ph}})n_1=0 \tag{39}$$

or, using the value of n_1 from Eq. (37),

$$\omega^2-\omega_e^2-k^2u^2=0, \tag{40}$$

with

$$\omega_e^2=4\pi e^2N_0/m \tag{41}$$

as the plasma frequency.

At first sight it is very astonishing that all terms with n_0 vanish on neglecting the terms of second order; hence, they do not appear at all in the dispersion equation (39). This suggests the possibility of neglecting them completely in the distribution function.

This neglect of n_0 underlies the derivation of a dispersion equation by van Kampen [(1957), p. 644]. It is of no importance that he is referring to the adiabatic and not to the isothermal case. However, this is not allowed, as one would conclude from the fact that we (as does also van Kampen) allow for a wave number $k\neq 0$ and thus for space variable components. In this case, consequently, one expects density variations $\sim n_0$.

There is also a direct argument that the term proportional to n_0 is the essential feature for the correct distribution function: By transforming to a moving reference frame, one obtains an incorrect dispersion equation from $n_0=0$.

Finally we have to verify that the space variable quantities may be arbitrarily small according to our basic assumptions. It follows from Eqs. (36) and (37) that for this purpose we have to keep only the actual alternating electrical field E_0 below a certain limit. The quantity $v_{\text{ph}}^2-u^2$ does not disturb this concept because, according to Eq. (40), it is never zero.

To investigate the dependence on the wave number k we first rewrite the parameters n_0 and n_1, making use of the dispersion relation (40),

$$n_0=-i\frac{eE_0}{KT}\frac{N_0}{k}\frac{k^2u^2}{\omega_e^2}, \tag{42}$$

$$n_1=+i\frac{eE_0}{KT}\frac{N_0}{k}\frac{\omega^2}{\omega_e^2}, \tag{43}$$

and the velocity distribution

$$f=f_0\left[1-ie^{ikx^*}\frac{eE_0}{KT}\frac{1}{\omega_e^2}(ku^2-\omega v_x)\right]. \tag{44}$$

The term proportional to v_x corresponds to a distribution which is independent of the wave number k. As expected, it is common to the compressible and incompressible case. On the other hand, the term proportional to n_0 depends linearly on k; since this distribution is derived on the basis of small deviations from equilibrium, we see that the k value determines the maximum charge separations (proportional to E_0) and the maximum density fluctuations (proportional to n_0). Going over to an incompressible plasma ($k\rightarrow 0$) finally, we again find the distribution function (21).

Longitudinal Oscillations Under Adiabatic Conditions

We want to prove the correctness of our previous statement that in the case of adiabatic conditions the only feature to change is a numerical factor close to unity in the expression for the sound velocity.‡‡ This has been done already from the hydrodynamic point of view in Sec. I.2. The analogous kinetic calculations are as follows:

From Eq. (5) we know that the temperature must be space independent to obtain an exact solution of the Boltzmann transport equation with collision terms canceling. Thus our treatment is, for one more reason, an approximation. We start at once from a suitably generalized expression for the velocity distribution function analogous to Eq. (27).

The adiabatic concept means that the temperature is composed of a constant average T_0 and small fluctuation T_1,

$$T=T_0+T_1,\quad T_0\gg T_1, \tag{45}$$

with

$$T_1\sim e^{ikx^*}. \tag{46}$$

Therefore, the exponential function for the velocity distribution is

$$\exp\left\{-\frac{m}{2KT}\left(1-\frac{T_1}{T_0}\right)(v_x^2+v_y^2+v_z^2)\right.$$
$$\approx\exp\left\{-\frac{m}{2KT_0}(v_x^2+v_y^2+v_z^2)\right\}$$
$$\times\left[1+\frac{m}{2KT_0}(v_x^2+v_y^2+v_z^2)\frac{T_1}{T_0}\right]. \tag{47}$$

Introducing the bracket term into Eq. (28) and canceling all terms of second and higher order in e^{ikx^*} yields

$$f=f_0\left[1+\frac{n_0}{N_0}e^{ikx^*}+\frac{n_1}{N_0}\frac{v_x}{v_{\mathrm{ph}}}e^{ikx^*}\right.$$
$$\left.+\frac{n_2}{N_0}\frac{v_x^2+v_y^2+v_z^2}{v_{\mathrm{ph}}^2}e^{ikx^*}\right] \tag{48}$$

in the parameter representation analogous to Eq. (30). The novel feature is the term proportional to the square of the velocity components. This term necessarily is isotropic in the three dimensions.

Introducing the distribution function (48) and its derivatives into the Boltzmann transport equation and integrating over the velocity space (we now also have to use the integration with the weight v_x^2) together with Maxwell's Eq. (38) yields the well-known dispersion relation under adiabatic conditions

$$\omega^2-\omega_e^2-3k^2u^2=0. \tag{49}$$

The factor 3 in front of the sound velocities is a consequence of the one-dimensional nature of the waves: In the distribution function (48) there are no terms proportional to v_y or v_z. In the case of three-dimensional waves one expects a factor 5/3 instead of 3, as has been discussed by van Kampen (1957).

With the derivation of Eq. (49), we proved by kinetic means that the adiabatic results differ from the isothermal ones only by a numerical factor close to one. Hence one always can correct the isothermal calculations by using the correct sound velocities without farther changes.

‡‡ *Note added in proof.*—The differences between the adiabatic and isothermal approach have been discussed recently with kinetic methods by K. Rawer and K. Suchy, Ann. Physik **2**, 313 (1959).

4. Velocity Distribution for Transverse Oscillations

We now can solve the corresponding problem for the transverse waves without difficulty.

As seen in Sec. I, density variations do not affect the transverse oscillations. This has the consequence that we can start at once with the density everywhere constant, and thus, from a distribution function of the form,

$$f=N_0\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}\exp\left\{-\frac{m}{2KT}(v_x^2+v_z^2)\right\}$$
$$\times\exp\left\{-\frac{m}{2KT}(v_y-\eta)^2\right\}. \tag{50}$$

The direction of propagation of the waves again may be the x axis. For η the expression is the same as above:

$$\eta=\eta_0e^{ikx^*}=(eE_0/m)(e^{ikx^*}/i\omega), \tag{51}$$

with

$$E_y=E_0e^{ikx^*}. \tag{52}$$

From Maxwell's equation,

$$[i\omega-i(c^2k^2/\omega)]E_0e^{ikx^*}$$
$$-4\pi e\int_{-\infty}^{+\infty}v_yf(v_x,v_y,v_z)dv_xdv_ydv_z=0, \tag{53}$$

we obtain the dispersion relation

$$\omega^2-\omega_e^2-k^2c^2=0, \tag{54}$$

which is the same result as Eq. (45), Sec. I.

5. Oscillations with a Constant Magnetic Field

We now have to consider the case of oscillations with a constant magnetic field present. To make sure that we can start again from distribution functions of the types (29) and (40), we have to reconsider Eqs. (6)–(8) for the case of the external force,

$$\frac{\mathbf{F}}{m}=-\frac{e}{m}\left[\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{H}^0\right], \quad (55)$$

having a velocity dependent term.§§

The general shape of the distribution function does not change. Thus, starting with Eq. (2) we find only an additional term,

$$[2\Gamma/mc][(\mathbf{v}\times\mathbf{H}^0)\cdot(\mathbf{v}-\boldsymbol{\xi})], \quad (56)$$

which enters Eq. (4). It affects Eq. (7), resulting in

$$-\frac{e}{m}[\mathbf{E}+\boldsymbol{\xi}\times\mathbf{H}^0]=\frac{d\boldsymbol{\xi}}{dt}+\frac{1}{2\Gamma a}\nabla a-\tfrac{1}{2}\nabla(\xi^2)+\left(\frac{d\Gamma}{dt}\Big/\Gamma\right)\boldsymbol{\xi}, \quad (57)$$

but leaves Eq. (9) unchanged. The reason is that

$$\mathbf{v}\times\mathbf{H}^0\cdot\mathbf{v}=0. \quad (58)$$

Equation (57) as well as Eq. (12) allows for plane waves. As discussed in Sec. I, the magnetic field couples the different wave types, represented here by the components of the velocity $\boldsymbol{\xi}$.

(a) *Longitudinal Magnetic Field*

We start with the case of a longitudinal magnetic field and expect from the discussion in Sec. I that the longitudinal oscillations separate out undisturbed, leaving us with the two transverse oscillations coupled together by the constant magnetic field.

We define the direction of propagation as before by

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial z}\equiv 0, \quad (59)$$

thus the magnetic field components are given by

$$\mathbf{H}^0=(H_L,0,0). \quad (60)$$

The symbol

$$\omega_L=eH_L/mc \quad (61)$$

is used for the gyrofrequency of the electrons as in Sec. I.

The generalized expression (27) for the distribution function becomes

$$f=N_0\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}\exp\left\{-\frac{m}{2KT}[(v_x-\xi)^2+(v_y-\eta)^2+(v_z-\zeta)^2]\right\}\left[1+\frac{n_0}{N_0}e^{ikx^*}\right], \quad (62)$$

the term in the last bracket describing the density variations in the longitudinal wave. For the velocity components we assume, as before,

$$\xi=\xi_0 e^{ikx^*},\quad \eta=\eta_0 e^{ikx^*},\quad \zeta=\zeta_0 e^{ikx^*}, \quad (63)$$

corresponding to electric field components,

$$\mathbf{E}=e^{ikx^*}(E_1,E_2,E_3). \quad (64)$$

The unknown quantities ξ, η, ζ are found in terms of the field components E_i by introducing the distribution function f into the Boltzmann equation,

$$\frac{\partial f}{\partial t}+(\mathbf{v}\cdot\nabla)f-\frac{e}{m}\left[\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{H}^0\right)\cdot\nabla_v\right]f=0, \quad (65)$$

and integrating over the velocity space, neglecting terms of second and higher order in e^{ikx^*}.

The first integration yields

$$\omega(n_0/N_0)+k\xi_0=0. \quad (66)$$

Multiplication with v_x and integration gives

$$i\omega\xi_0+ik\frac{KT}{m}\frac{n_0}{N_0}+\frac{e}{m}E_1=0, \quad (67)$$

while by multiplication with v_y or v_z and integration, ξ_0 and u_0 do not appear. Equations (66) and (67) are identical with (33) and (34) and independent of the magnetic field. This shows, as expected, that the longitudinal waves separate out.

For the quantities η_0 and ζ_0 we find the relationships

$$i\omega\eta_0+\frac{e}{m}E_2+\omega_L\zeta_0=0, \quad (68)$$

$$i\omega\zeta_0+\frac{e}{m}E_3-\omega_L\eta_0=0, \quad (69)$$

which, together with Maxwell's equations

$$i\left(\omega-\frac{k^2c^2}{\omega}\right)E_y-4\pi e\int v_y f(v_x,v_y,v_z)dv_x dv_y dv_z=0, \quad (70)$$

$$i\left(\omega-\frac{k^2c^2}{\omega}\right)E_z-4\pi e\int v_z f(v_x,v_y,v_z)dv_x dv_y dv_z=0, \quad (71)$$

finally yield the dispersion relation,

$$\omega^2(\omega^2-\omega_e^2-k^2c^2)^2-\omega_L^2(\omega^2-k^2c^2)^2=0, \quad (72)$$

§§ There is no alteration in the collision behavior due to the magnetic field which enforces a circular motion on electrons and ions as long as the gyrofrequency is small compared with the collision frequency. This may restrict our considerations to the case of moderate magnetic field strengths. For details see L. M. Tannenwald, Phys. Rev. **113**, 1396 (1959).

as derived in Sec. I, Eq. (130).

(b) *Transverse Magnetic Field*

Here we expect one of the transverse waves to separate out. We now define the direction of the constant magnetic field by letting

$$\mathbf{H}^0=(0,H_T,0), \tag{73}$$

with

$$\omega_T=eH_T/mc \tag{74}$$

for the gyrofrequency.

The distribution function with which we start is the same as in the longitudinal case. The direct integration again yields

$$\omega\eta_0+kN_0\xi_0=0. \tag{75}$$

By multiplication with v_x and v_z, respectively, we obtain, with the use of Eq. (75),

$$i\left[\omega-\frac{k^2u^2}{\omega}\right]\xi_0+\frac{e}{m}E_1+\omega_T\zeta_0=0, \tag{76}$$

$$i\omega\zeta_0+\frac{e}{m}E_3+\omega_T\xi_0=0, \tag{77}$$

while the multiplication with v_y and integration again yields the connection (57) between the y component of the velocity and the electric field, independent of the magnetic field. This means the separation of one of the transverse waves as expected.

Finally we have the two Maxwell equations,

$$i\omega E_x-4\pi e\int v_xf(v_x,v_y,v_z)dv_xdv_ydv_z=0, \tag{78}$$

$$i\left(\omega-\frac{k^2c^2}{\omega}\right)E_z-4\pi e\int v_zf(v_x,v_y,v_z)dv_xdv_ydv_z=0, \tag{79}$$

and obtain the dispersion relation,

$$(\omega^2-\omega_e^2-k^2c^2)(\omega^2-\omega_e^2-k^2u^2)-\omega_T^2(\omega^2-k^2c^2)=0, \tag{80}$$

as derived in Sec. I, Eq. (148).

This completes the investigation, showing in all cases the same results as by the hydrodynamic treatment.

(c) *Previous Treatments*

Recently, a number of papers have appeared dealing with plasma oscillations in a static magnetic field and using kinetic methods. We discuss one in detail, because here the kinetic approach was used in attempt to investigate the limitations of the hydrodynamic treatment. Gross (1951) published an investigation dealing with the distributions of electron velocities for plasma waves in a nonzero constant magnetic field. He claimed that the hydrodynamic treatment conceals a great deal of the essential features of a plasma which can be discovered only by kinetic methods. In particular, he found in the case of a nonzero magnetic field frequency ranges without transmission ("gaps") at multiples of the gyrofrequency.

Gross starts from equations which are on the same level of approximation as ours: He assumes the right-hand side of Boltzmann's equation to be zero and uses complete linearization, i.e., he adds to the main distribution of velocities a small disturbance. We have seen that there are no differences between the hydrodynamic and the kinetic results as long as use is made of the same order of approximation, and therefore, Gross's results are very confusing.

Gross starts from Boltzmann's equation without collision terms, but with additional terms representing the nonzero magnetic field:

$$\frac{\partial f}{\partial t}+(\mathbf{v}\cdot\nabla)f-\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{H}^0\right)\cdot\nabla_vf=0. \tag{81}$$

The magnetic field is directed along the z axis. We designate the gyrofrequency by the notation

$$\omega_T=eH_0/mc. \tag{82}$$

Gross now linearizes, i.e., he assumes

$$f=f_0+f_1,\quad |f_1|\ll|f_0| \tag{83}$$

for the velocity distribution, with f_0 as the average distribution, in our case the Maxwellian equilibrium distribution.[|| ||] The dependence on space and time of the disturbance terms is assumed to be of the form

$$f_1\sim e^{ikx^*}=e^{i\omega t+ikx^*}.\P\P \tag{84}$$

Gross calculates the disturbance f_1 completely independent of the average distribution f_0, and does not consider the restrictions on f_1 by the collision terms on the right-hand side of Boltzmann's transport equation at all. His procedure therefore corresponds to the complete neglect of all collision effects.*** As we tried to show in the foregoing sections, this procedure is not legitimate under conditions where hydrodynamic treatments are used.

An additional remark concerning Gross's (collision free) solution of Boltzmann's equation is this: To

|| || The other cases treated by Gross (peaked velocity distribution as average distribution) are of no interest here.

¶¶ Gross has chosen the sign in the exponential function of the harmonic oscillations differently from us. For the sake of uniformity we write his equations in our system.

*** This statement, if taken literally, is not correct. As E. A. Desloge has pointed out privately, the electric field introduced in the left-hand side of Boltzmann's equation takes care of a certain portion of the electron-electron interaction. It describes the collective force of the electrons in a wave front on an electron in a neighboring region of lower or higher density but does not describe the random short range interactions. The characteristic distance of the former type of interaction is, by definition, large compared with the Debye length, while the characteristic distances of the collisions to which we referred in Sec. I.3 are below this limit.

arrive at his solution, he introduces new variables

$$\rho^2=v_x^2+v_y^2, \quad \tan\varphi=v_y/v_x, \tag{85}$$

$$F_x=E_x-iE_y, \quad F_y=E_x+iE_y, \tag{86}$$

and finds from the differential equation

$$i(k\rho\cos\varphi+\omega)f_1 + \frac{e}{2m}\frac{df_0}{d\rho}(F_xe^{i\varphi}+F_ye^{-i\varphi})-\omega_T\frac{\partial f_1}{\partial\varphi}=0, \tag{87}$$

the disturbance distribution

$$f_1=A(\rho)\exp\left[\frac{i}{\omega_T}(k\rho\sin\varphi+\omega\varphi)\right] +\exp\left[\frac{i}{\omega_T}(k\rho\sin\varphi+\omega\varphi)\right] \times\int_0^\varphi \exp\left[-\frac{i}{\omega_T}(k\rho\sin\varphi+\omega\varphi)\right] \times\frac{e}{2m\omega_T}\frac{df_0}{d\rho}(F_xe^{i\varphi}+F_ye^{-i\varphi})d\varphi. \tag{88}$$

$A(\rho)$ is an arbitrary function of ρ, but is not a function of φ. The mathematical solution of the differential equation is, however, restricted by the physical condition that the solution f_1 must be periodic in φ with mod2π. This condition may be used to determine $A(\rho)$. For this purpose Gross expands the exponential function into Bessel functions,

$$e^{\pm(ik\rho/\omega_T)\sin\varphi}=\sum_{-\infty}^{+\infty} J_n\left(\pm\frac{k\rho}{\omega_T}\right)e^{in\varphi}, \tag{89}$$

and obtains

$$A(\rho)=\frac{e}{2m\omega_T}\frac{df_0}{d\rho_0}\sum J_n\left(-\frac{k\rho}{\omega_T}\right) \times\left\{\frac{F_x}{i[1+n-(\omega/\omega_T)]}-\frac{F_y}{i[1-n+(\omega/\omega_T)]}\right\}, \tag{90}$$

and hence, finally the disturbance,

$$f_1=\frac{e}{2m\omega_T}\frac{df_0}{d\rho}\exp\left[\frac{1}{\omega_T}k\rho\sin\varphi\right]\sum J_n\left(-\frac{k\rho}{\omega_T}\right) \times\left\{\frac{F_xe^{i(n+1)\varphi}}{i[1+n-(\omega/\omega_T)]}-\frac{F_ye^{i(n-1)\varphi}}{i[1-n+(\omega/\omega_T)]}\right\}. \tag{91}$$

There are singularities whenever the frequency ω approaches multiples of the gyrofrequency ω_T; this feature is the reason for the forbidden frequency ranges discussed by Gross.

However, when the frequency is equal to multiples of ω_T, the function $A(\rho)$ is *completely* arbitrary, since then the exponential function is always periodic in φ with mod2π as may be seen from Eq. (88). To obtain a complete solution therefore, we have to add to Eq. (90) a sum of delta functions with arbitrary coefficients $a(\rho)$,

$$\sum\nolimits_m a_m(\rho)\delta(\omega-m\cdot\omega_T). \tag{92}$$

There is no obvious physical argument as to why these additional terms can be omitted.

The same objections as against the results of Gross hold for the treatment of plasma oscillations by I. B. Bernstein (1957), because he essentially uses the same fundamental equations and procedures, although he writes all equations in the form of the Fourier and Laplace transforms: he determines the disturbance distribution independent of the main distribution by the same kind of differential equation which Gross had used [Bernstein (1957), Eq. (9), p. 11], and applies the periodicity condition to the mathematical solution. He too neglects the fact that for multiples of the gyrofrequency the solution is already periodic, and hence does not include an arbitrary function in his Eq. (11).

6. Inclusion of Dissipative Terms

In Sec. I we derived simple expressions for the attenuation of both longitudinal and transverse waves, introducing a damping term which corresponds to the assumption of a constant frequency for the inelastic collisions.

We expect these to be electron-ion collisions, where the oscillating electrons lose part of their momentum to the (fixed) ions. The corresponding collision frequency ν may be quite different from the elastic electron-electron collision frequency introduced in Sec. II.3. Our equations describe the plasma behavior correctly if there are enough elastic collisions to maintain the Maxwellian form of our distribution function but not too many inelastic ones, with the result that wave attenuation is reached only after a sufficiently large number of oscillations.

Furthermore, we expect the kinetic expression corresponding to the hydrodynamic treatment to be a linear function of the distribution functions as well as of the (constant) collision frequency ν. In other words, we expect a "relaxation term,"

$$(\partial f/\partial t)_{\text{coll}}=\nu(f_0^*-f), \tag{93}$$

to be introduced in the right-hand side of Boltzmann's equation. f is the actual distribution function, while f_0^* is determined by the condition that the integration over the velocity space must yield the continuity equation. This implies the condition

$$\nu\int(f_0^*-f)d\mathbf{v}=0, \tag{94}$$

and therefore

$$f_0{}^*=f_0\left[1+\frac{n_0}{N_0}e^{ikx^*}\right]. \tag{95}$$

A similar expression has been used by Bhatnagar, Gross, and Krook (1954). It can be easily justified, since the collision integral vanishes in first order for functions isotropic in v. The remaining term is proportional to v_x and cancels out by integrating over the velocity space. This result is in Eq. (95). [For details, see, e.g., H. Margenau (1958).]

With the help of Eqs. (94) and (95) one easily finds the dispersion relation in the case of longitudinal oscillations,

$$\omega(\omega-i\nu)-\omega_e{}^2-k^2u^2=0 \tag{96}$$

as in Sec. I.

Finally, we conclude from Eq. (95) that the attenuation behavior for longitudinal and transverse waves is the same because the collision term (94) is the same for both cases.

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