Energy Balance and Confinement of a Magnetized Plasma

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I. INTRODUCTION

COSMICAL phenomena usually take place in a highly ionized gas. In spite of this fact, there exist a number of important cases where an influence of neutral gas particles on the motion of the charged constituents has to be taken into account. Examples are given by the heating of the solar chromosphere by magnetohydrodynamic waves,[1,2] the decay of magnetic fields in cool interstellar clouds,[3,4] motions in the ionosphere,[5–9] and in the theory on the origin of the solar system.[10,11]

Even at very high temperatures, the gas in a high-current discharge experiment may have to be treated as a partially ionized gas. The reason for this is that the particles which are lost from the plasma region to the walls of a surrounding vessel are not always absorbed at the wall but produce a flux of neutral particles which reenters the plasma region. The backscatter of neutrals has been discussed by Luce[12] and the adsorption of gases by Miller.[13]

Here the conservation theorems for mass, charge, momentum, and energy are considered for a partially ionized gas where ionization and charge exchange phenomena are included. The theory is applied to a rotating plasma, i.e., to a situation which is of interest both in connection with rotating stars, in cosmogony, and in the research on controlled thermonuclear fusion.[14–19]

II. CONSERVATION LAWS

The treatment here is simplified such as not to include the spontaneous emission of photons by excitation, bremsstrahlung, and synchrotron radiation as well as two-stage ionization, volume recombination, and multiple charge exchange collisions. An electrically quasi-neutral plasma with n ions and electrons per unit volume is assumed. Define σ_i and σ_e as the cross sections for ionization by means of ions and electrons moving with total velocities u_i and u_e in a neutral gas of local density n_n. The number of ionizing events per unit volume and time is

$$nn_n\langle\sigma_i u_i\rangle + nn_n\langle\sigma_e u_e\rangle = n\zeta, \quad (1)$$

where ζ expresses the number of charged particles created per unit volume and time per charged particle being present, and $\langle\ \rangle$ indicates effective mean values. In an analogous way, σ_{ex} and ν_{ex} define the cross section and frequency of charge exchange collisions and

$$nn_n\langle\sigma_{ex} u_i\rangle = n\nu_{ex} = nf\zeta, \quad (2)$$

where Eq. (2) defines f. Finally, elastic collisions with cross sections σ_{in} and σ_{en} and frequencies ν_{in} and ν_{en} give the rates

$$nn_n\langle\sigma_{in} u_i\rangle = 2n\nu_{in}, \quad (3)$$

$$nn_n\langle\sigma_{en} u_e\rangle = n\nu_{en}. \quad (4)$$

In a collision ions lose, on the average, half of their momentum relative to the neutrals. Also introduce the number $nh\zeta$ per unit volume and time of neutral particles which are scattered elastically from the plasma, thereby obtaining a momentum comparable to that of the ions and being lost to a vessel wall.

Thermal diffusion effects are not included here.

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With these starting points, the conservation laws are established in the following parts of this section.

1. Conservation of Mass and Charge

The mass density of the ion gas is defined by $\rho_i = nm_i$, the temperature by T_i, the pressure, which is assumed to be isotropic, by $p_i = nkT_i$, and the mass velocity by $\mathbf{v}_i$. Corresponding quantities for the electron gas are indicated by the subscript e. Also introduce the total plasma density

$$\rho = n(m_i + m_e) = nm = \rho_i + \rho_e, \tag{5}$$

the plasma pressure

$$p = p_i + p_e, \tag{6}$$

the mass velocity $\mathbf{v}$ of the plasma, and the electric current density $\mathbf{i}$. The two latter quantities obey the relations

$$\mathbf{v}_i = \mathbf{v} + m_e\mathbf{i}/e\rho; \quad \mathbf{v}_e = \mathbf{v} - m_i\mathbf{i}/e\rho. \tag{7}$$

Since charged particles are created only by ionization, conservation of mass and charge is expressed by

$$\mathrm{div}(n\mathbf{v}_i) = \zeta n - (\partial n/\partial t) = \mathrm{div}(n\mathbf{v}_e). \tag{8}$$

Introduce the derivative $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ and combination of Eqs. (7) and (8) gives

$$\mathrm{div}\mathbf{v} = \zeta - \rho^{-1}d\rho/dt \tag{9}$$

and

$$\mathrm{div}\mathbf{i} = 0. \tag{10}$$

Consequently, the starting points also require that the displacement current has to be neglected and

$$\mathrm{curl}\mathbf{B} = \mu_0\mathbf{i}, \tag{11}$$

where $\mathbf{B}$ is the magnetic field and μ_0 the permeability in vacuum. This is a good approximation when the characteristic velocities of the phenomena to be studied are much less than the velocity of light.

2. Conservation of Momentum

If the mass velocity of the neutral gas is assumed to be $\mathbf{v}_n$, the gravitation potential ϕ_g, and the electric field $\mathbf{E}$, conservation of momentum for ions and electrons requires that

$$\begin{aligned}\rho_i(d_i/dt)\mathbf{v}_i = en(\mathbf{E} + \mathbf{v}_i\times\mathbf{B}) - \nabla p_i - nm_i\nabla\phi_g \\ - \rho_e\nu_{ei}(\mathbf{v}_i - \mathbf{v}_e) - \rho_i(\zeta + \nu_{ex} + \nu_{in})(\mathbf{v}_i - \mathbf{v}_n),\end{aligned} \tag{12}$$

and

$$\begin{aligned}\rho_e(d_e/dt)\mathbf{v}_e = -en(\mathbf{E} + \mathbf{v}_e\times\mathbf{B}) - \nabla p_e - nm_e\nabla\phi_g \\ + \rho_e\nu_{ei}(\mathbf{v}_i - \mathbf{v}_e) - \rho_e(\zeta + \nu_{en})(\mathbf{v}_e - \mathbf{v}_n),\end{aligned} \tag{13}$$

where ν_{ei} is the frequency of Coulomb collisions and $d_i/dt = \partial/\partial t + \mathbf{v}_i\cdot\nabla$; $d_e/dt = \partial/\partial t + \mathbf{v}_e\cdot\nabla$.

By means of the substitutions (5)–(7), expressions (12) and (13) can be written in terms of $\mathbf{v}$ and $\mathbf{i}$.[9] The sum of the resulting equations is

$$\rho\frac{d\mathbf{v}}{dt} + \frac{m_im_e}{e^2}(\mathbf{i}\cdot\nabla)(\mathbf{i}/\rho) = \mathbf{i}\times\mathbf{B} - \nabla p - \rho\nabla\phi_g - \alpha\rho(\mathbf{v} - \mathbf{v}_n) + \beta\mathbf{i}, \tag{14}$$

where

$$\alpha = (m\zeta + m_i\nu_{ex} + m_i\nu_{in} + m_e\nu_{en})/m \tag{15}$$

and

$$\beta = m_im_e(\nu_{en} - \nu_{in} - \nu_{ex})/em \tag{16}$$

are "frictional coefficients" related to the loss of momentum by collision processes.

If, instead, Eq. (12) is multiplied by $m_e/e\rho$ and Eq. (16) by $-m_i/e\rho$, the sum of the resulting equations becomes

$$\begin{aligned}\frac{m_im_e}{e^2\rho}\left[\rho\frac{d}{dt}(\mathbf{i}/\rho) + (\mathbf{i}\cdot\nabla)\mathbf{v} - \frac{m_i - m_e}{e}(\mathbf{i}\cdot\nabla)(\mathbf{i}/\rho)\right] \\ = \mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{m_i - m_e}{e\rho}\mathbf{i}\times\mathbf{B} + \frac{1}{e\rho}(m_i\nabla p_e - m_e\nabla p_i) \\ + \beta(\mathbf{v} - \mathbf{v}_n) - \eta\mathbf{i},\end{aligned} \tag{17}$$

where

$$\eta = \frac{mm_e}{e^2\rho}\left[\nu_{ei} + \frac{m_i}{m}\zeta + \frac{m_im_e}{m^2}(\nu_{in} + \nu_{ex}) + \left(\frac{m_i}{m}\right)^2\nu_{en}\right]. \tag{18}$$

Equation (14) expresses conservation of momentum of the mean motion of the plasma. Equation (17) is a generalization of Ohm's law and expresses the conservation of momentum of the "slip" motion between the ions and electrons; η is a generalized electric resistivity. Observe that ionization and charge exchange correspond to a frictional brake on the plasma. For further discussions on the significance of the different terms in Eqs. (14) and (17), reference is made to an earlier paper.[9]

3. Conservation of Energy

If scalar multiplication of Eq. (14) by $\mathbf{v}$ and of Eq. (17) by $\mathbf{i}$ is performed, the sum of the resulting equations can be written

$$\mathbf{E}\cdot\mathbf{i} - \mathrm{div}\left(p\mathbf{v} - \frac{m_ip_e - m_ep_i}{e\rho}\mathbf{i}\right) - \mathbf{v}\cdot\rho\nabla\phi_g + w_{nm} = w_a + w_c + w_f. \tag{19}$$

The first term in the left-hand member represents the electric power input per unit volume. The second term in the same member is equal to $-\mathrm{div}(p_i\mathbf{v}_i + p_e\mathbf{v}_e)$ and represents the total work done by the pressure on a volume element of the plasma. Input of energy by the gravitation field is given by the third term and

$$\begin{aligned}w_{nm} &= -\rho_i(\zeta + \nu_{ex} + \nu_{in})(\mathbf{v}_i - \mathbf{v}_n)\cdot\mathbf{v}_n \\ &\quad - \rho_e(\zeta + \nu_{en})(\mathbf{v}_e - \mathbf{v}_n)\cdot\mathbf{v}_n \\ &= -\alpha\rho(\mathbf{v} - \mathbf{v}_n)\cdot\mathbf{v}_n + \beta\mathbf{i}\cdot\mathbf{v}_n\end{aligned} \tag{20}$$

is the mechanical work done on the plasma by the moving neutral gas. Thus, the left-hand member represents the total energy input per unit volume and time. The right-hand member tells how this energy is consumed. Here

$$w_a=\rho_i\mathbf{v}_i\frac{d_i}{dt}\mathbf{v}_i+\rho_e\mathbf{v}_e\frac{d_e}{dt}\mathbf{v}_e$$
$$=\mathbf{v}\cdot\left[\rho\frac{d\mathbf{v}}{dt}+\frac{m_im_e}{e^2}(\mathbf{i}\cdot\nabla)(\mathbf{i}/\rho)\right]+\frac{m_im_e}{e^2\rho}\mathbf{i}\cdot\left[\rho\frac{d}{dt}(\mathbf{i}/\rho)+(\mathbf{i}\cdot\nabla)\mathbf{v}-\frac{m_i-m_e}{e}(\mathbf{i}\cdot\nabla)(\mathbf{i}/\rho)\right] \quad (21)$$

is the total acceleration work,

$$w_c=-p_i\,\mathrm{div}\mathbf{v}_i-p_e\,\mathrm{div}\mathbf{v}_e=-p\,\mathrm{div}\mathbf{v}+(m_ip_e-m_ep_i)\,\mathrm{div}(\mathbf{i}/e\rho) \quad (22)$$

the compression work, and

$$w_f=w_{fi}+w_{fe}=\alpha\rho(\mathbf{v}-\mathbf{v}_n)^2-2\beta\mathbf{i}\cdot(\mathbf{v}-\mathbf{v}_n)+\eta\mathbf{i}^2, \quad (23)$$

where

$$w_{fi}=\rho_i(\zeta+\nu_{ex}+\nu_{in})(\mathbf{v}_i-\mathbf{v}_n)^2+\rho_e\nu_{ei}(\mathbf{v}_i-\mathbf{v}_e)\cdot\mathbf{v}_i, \quad (24)$$

$$w_{fe}=\rho_e(\zeta+\nu_{en})(\mathbf{v}_e-\mathbf{v}_n)^2-\rho_e\nu_{ei}(\mathbf{v}_i-\mathbf{v}_e)\cdot\mathbf{v}_e \quad (25)$$

expresses the frictional work done on ions and electrons, respectively.

It should be observed that Eq. (19) can be rewritten in such a way that the pressure p is everywhere replaced by $p^*=p+B^2/2\mu_0$, where the second term is the "magnetic pressure." The right-hand side of Eq. (19) then contains a "magnetic compression work" $-(B^2/2\mu_0)$ div**v**, an acceleration work $-\mathbf{v}\cdot\nabla(B^2/2\mu_0)$ done by the "magnetic pressure," and an additional acceleration work $\mathbf{v}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}/\mu_0]$, which is due to the fact that the magnetic field lines also may act like elastic strings. The force $(\mathbf{B}\cdot\nabla)\mathbf{B}/\mu_0$ usually does not have the character of a pressure gradient. However, this way of describing the energy balance is unnecessarily complicated and does not give any result beyond that of Eq. (19), which is written in terms of the real pressures p_i and p_e and the total electrodynamic force $\mathbf{i}\times\mathbf{B}$.

It should be observed that the relation (19) does not include more information than the conservation law of momentum from which it is deduced. Information is still required about the coupling between thermal motion and mass motion, and this can be obtained from the principle of conservation of energy.[8,20–23]

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First consider the energy balance *separately* for the ion gas inside a volume element which is fixed in space. The increase in total energy is

$$(\partial/\partial t)(\tfrac{3}{2}p_i+\tfrac{1}{2}\rho_i\mathbf{v}_i^2).$$

The energy input consists of the contributions

$$en\mathbf{E}\cdot\mathbf{v}_i-\mathrm{div}(p_i\mathbf{v}_i)-nm_i\mathbf{v}_i\cdot\nabla\phi_g$$

from the electric field, the work done by the pressure and the gravitation fields, and by

$$-\rho_i(\zeta+\nu_{ex}+\nu_{in})(\mathbf{v}_i-\mathbf{v}_n)\cdot\mathbf{v}_n+w_{ni},$$

where the first term is the mechanical work done by the moving neutral gas on ions and the second term $w_{ni}=\tfrac{3}{2}\zeta nkT_n+\tfrac{1}{2}\rho_i\zeta\mathbf{v}_n^2$ is the energy which is brought from the neutral gas of temperature T_n into the ion gas when particles are ionized. Convection produces a power loss

$$\mathrm{div}(\tfrac{3}{2}p_i\mathbf{v}_i+\tfrac{1}{2}\rho_i\mathbf{v}_i^2\cdot\mathbf{v}_i)+[\tfrac{3}{2}(p_i-nkT_n)+\tfrac{1}{2}\rho_i(\mathbf{v}_i^2-\mathbf{v}_n^2)]\times(\nu_{ex}+h\zeta),$$

where the first term is due to the motion of the ion gas and the second to energetic neutrals which escape from the plasma region after having been heated from T_n to T_i and accelerated from $\mathbf{v}_n$ to $\mathbf{v}_i$. The definition of h has been given at the beginning of Sec. II. The total power loss which arises from the ionization work is $Wn\zeta$, where W/e is the ionization potential. W_i and W_e are defined as the corresponding contributions by the ion and electron gases. Finally, a transfer of mass motion into heat takes place due to the mechanical work w_{fi}, which the ions perform against frictional forces. Thereby they gain the fraction $(1-\delta_i)$ of w_{fi} in the form of frictional heat. Conservation of energy requires

$$(\partial/\partial t)(\tfrac{3}{2}p_i+\tfrac{1}{2}\rho_i\mathbf{v}_i^2)=en\mathbf{E}\cdot\mathbf{v}_i-\mathrm{div}(p_i\mathbf{v}_i)-nm_i\mathbf{v}_i\nabla\phi_g-\rho_i(\zeta+\nu_{ex}+\nu_{in})\times(\mathbf{v}_i-\mathbf{v}_n)\cdot\mathbf{v}_n-\mathrm{div}(\tfrac{3}{2}p_i\mathbf{v}_i+\tfrac{1}{2}\rho_i\mathbf{v}_i^2\cdot\mathbf{v}_i)-W_in\zeta-[\tfrac{3}{2}(p_i-nkT_n)+\tfrac{1}{2}\rho_i(\mathbf{v}_i^2-\mathbf{v}_n^2)](\nu_{ex}+h\zeta)+w_{ni}-\mathrm{div}\mathbf{Q}_i-\delta_iw_{fi}, \quad (26)$$

where heat conduction is described by the flow vector $\mathbf{Q}_i$. It should be stressed that $e(\mathbf{E}+\mathbf{u}_i\times\mathbf{B})\cdot\mathbf{u}_i=e\mathbf{E}\cdot\mathbf{u}_i$ is the total energy input from the electromagnetic field into the motion of an ion which has the total velocity $\mathbf{u}_i$. If the magnetic field $\mathbf{B}=\mathrm{curl}\mathbf{A}$ changes in time, this produces an energy input by means of the induced electric field part $-\partial\mathbf{A}/\partial t$. Thus, $en\mathbf{E}\cdot\mathbf{v}_i$ is the only link by which energy can be fed into the ion gas by the electromagnetic field, irrespective of the kind of electromagnetic processes being in operation.[8]

For the electron gas an analogous relation is obtained:

$$(\partial/\partial t)(\tfrac{3}{2}p_e+\tfrac{1}{2}\rho_e\mathbf{v}_e^2) = -en\mathbf{E}\cdot\mathbf{v}_e-\operatorname{div}(p_e\mathbf{v}_e)-nm_e\mathbf{v}_e\cdot\nabla\phi_g-\rho_e(\zeta+\nu_{en}) \times(\mathbf{v}_e-\mathbf{v}_n)\cdot\mathbf{v}_n-\operatorname{div}(\tfrac{3}{2}p_e\mathbf{v}_e+\tfrac{1}{2}\rho_e\mathbf{v}_e^2\cdot\mathbf{v}_e)-W_en\zeta +w_{ne}-\operatorname{div}\mathbf{Q}_e-\delta_e w_{fe}, \quad (27)$$

where $w_{ne}=\frac{3}{2}\zeta nkT_n+\frac{1}{2}\rho_e\zeta\mathbf{v}_n^2$.

Equations (26) and (27) express the conservation of total energy as seen from the point of view of the charged particles. The net change in total energy of the plasma is caused by an outflow of particles, by heat conduction, and by energy which is transformed into other forms than thermal and kinetic energy. In this case the latter form of energy is electromagnetic and the first term of the right-hand sides of Eqs. (26) and (27) gives the rate at which this energy is supplied from the electromagnetic field. "Radiation" of magnetohydrodynamic waves is also included in Eqs. (26) and (27) since these waves are characterized by the work done on the charged *particles* by the induced electric field. A deduction of the energy equation in terms of the joint action of electromagnetic and mechanical changes of state has been given by Chu[23] for a one-fluid model. As seen from the present results, an approach of this kind is not necessary, at least in the case of an ionized gas.

After scalar multiplication by $\mathbf{v}_i$ and $\mathbf{v}_e$, Eqs. (12) and (13) are subtracted from Eqs. (26) and (27) and $\operatorname{div}\mathbf{v}_i$ and $\operatorname{div}\mathbf{v}_e$ are substituted from Eq. (8) into the obtained relations. After some deductions the result becomes

$$\rho_i(d_i/dt)[\tfrac{3}{2}(p_i/\rho_i)]-(p_i/\rho_i)(d_i\rho_i/dt) = -\tfrac{5}{2}\zeta p_i-\tfrac{3}{2}(f+h)\zeta(p_i-nkT_n)+\rho_e\nu_{ei}(\mathbf{v}_i-\mathbf{v}_e)\cdot\mathbf{v}_i +\rho_i(\zeta+\nu_{ex}+\nu_{in})(\mathbf{v}_i-\mathbf{v}_n)^2-\tfrac{1}{2}\rho_i(\nu_{ex}+h\zeta) \times(\mathbf{v}_i^2-\mathbf{v}_n^2)-\tfrac{1}{2}\rho_i\zeta\mathbf{v}_i^2+w_{ni}-\operatorname{div}\mathbf{Q}_i-W_in\zeta -\delta_i w_{fi}; \quad (28)$$

$$\rho_e(d_e/dt)[\tfrac{3}{2}(p_e/\rho_e)]-(p_e/\rho_e)\cdot(d_e\rho_e/dt) = -\tfrac{5}{2}\zeta p_e-\rho_e\nu_{ei}(\mathbf{v}_i-\mathbf{v}_e)\cdot\mathbf{v}_e+\rho_e(\zeta+\nu_{en})(\mathbf{v}_e-\mathbf{v}_n)^2 -\tfrac{1}{2}\rho_e\zeta\mathbf{v}_e^2+w_{ne}-\operatorname{div}\mathbf{Q}_e-W_en\zeta-\delta_e w_{fe}. \quad (29)$$

Without dissipation and particle losses these equations reduce to the simple adiabatic relations between the pressure and the density. By means of the substitutions (5)–(7), the sum of Eqs. (28) and (29) gives the energy theorem for the plasma,

$$(1-\delta)w_f-\operatorname{div}\mathbf{Q}-\frac{3}{2}(1+f+h)\zeta(p_i-nkT_n) -\frac{3}{2}\zeta(p_e-nkT_n) -Wn\zeta=\left[\rho\frac{d}{dt}\left(\frac{3}{2}\frac{p}{\rho}\right)-\frac{3}{2}\mathbf{i}\cdot\nabla\left(\frac{m_ip_e-m_ep_i}{e\rho}\right)\right] -\left[\frac{pd\rho}{\rho dt}-\frac{m_ip_e-m_ep_i}{e\rho^2}(\mathbf{i}\cdot\nabla\rho)-\zeta p-w_{na}\right], \quad (30)$$

where $\mathbf{Q}=\mathbf{Q}_i+\mathbf{Q}_e$ is the total heat flow by conduction, w_f is the total heat created by friction as given by Eq. (23), and

$$w_{na}=\tfrac{1}{2}(1+f+h)\zeta\rho_i(\mathbf{v}_i^2-\mathbf{v}_n^2)+\tfrac{1}{2}\zeta\rho_e(\mathbf{v}_e^2-\mathbf{v}_n^2) \quad (31)$$

is the mechanical work performed to accelerate the charged particles to the full plasma velocities after having been in interaction with the neutral gas. The heat lost from the plasma to the neutral gas which does not escape directly to the walls with energetic neutrals is given by δw_f. The left-hand member of Eq. (30) gives the net inflow of heat subtracted by the net heat loss due to escaping charged and neutral particles. This is balanced by the first bracket of the right-hand member, which gives the rate of growth of the internal energy and the second bracket which represents the total mechanical work. The first two terms inside the second bracket give the compression work, the third the work to "force" $n\zeta$ new particles into the plasma volume with the pressure p.

Finally, introduce the total flow of energy $\mathbf{q}$ caused by convection and heat conduction by means of charged particles:

$$\mathbf{q}=\tfrac{3}{2}(p_i\mathbf{v}_i+p_e\mathbf{v}_e)+\tfrac{1}{2}(\rho_i\mathbf{v}_i^2\cdot\mathbf{v}_i+\rho_e\mathbf{v}_e^2\cdot\mathbf{v}_e)+\mathbf{Q}. \quad (32)$$

The energy theorem (30) may then be rewritten by taking the sum of Eqs. (26) and (27) and combining it with Eqs. (19) and (8). The result is

$$(\partial/\partial t)[\tfrac{3}{2}p+\tfrac{1}{2}\rho\mathbf{v}^2+(m_im_e/2e^2\rho)\mathbf{i}^2] =w_a+w_c+(1-\delta)w_f-\operatorname{div}\mathbf{q}-[\tfrac{3}{2}(p_i-nkT_n)-\tfrac{1}{2}\rho_i \times(\mathbf{v}_i^2-\mathbf{v}_n^2)](f+h)\zeta-Wn\zeta+3\zeta nkT_n+\tfrac{1}{2}\rho\zeta\mathbf{v}_n^2. \quad (33)$$

The present section is concluded by a comment on the total frictional work which is given by Eqs. (23), (15), (16), and (18). The ratio between the last and the first terms in Eq. (23) has a modulus $\eta e^2n^2|\mathbf{v}_i-\mathbf{v}_e|^2/\alpha\rho|\mathbf{v}-\mathbf{v}_n|^2$, and the ratio between the second and first terms has a modulus less than or equal to $2\beta en|\mathbf{v}_i-\mathbf{v}_e|/\alpha\rho|\mathbf{v}-\mathbf{v}_n|$. The coupling between the plasma and the neutral gas has been investigated earlier.[9] For cosmical phenomena such as the propagation of magnetohydrodynamic waves in the solar chromosphere and motions in the ionosphere, as well as for discharges on laboratory scale, the "slip" $\mathbf{v}-\mathbf{v}_n$ was found to be of considerable importance. When $|\mathbf{v}_i-\mathbf{v}_e|$ does not exceed $|\mathbf{v}-\mathbf{v}_n|$ very much, Eqs. (15) and (16) show that the second term in the expression (23) for w_f can be neglected. Further, when the ionization degree is not extremely high, there exist situations where the last term of Eq. (23) also can be neglected, in spite of the fact that it contains the frequency ν_{ei} for Coulomb collisions. This implies that the heating of the plasma is provided mainly by collisions between ions and neutral particles, and not by Coulomb collisions. A mechanism of this kind has earlier been suggested by Piddington[1,2] for the heating of the solar chromosphere.

III. ROTATING PLASMA

As an application to the theory of the previous section, the stationary rotation of an axially symmetric plasma is now investigated. The rotation may be produced by magnetohydrodynamic coupling between a central, electrically conducting body and a surrounding electrically conducting fluid, or by a transverse electric field imposed upon a magnetized plasma. A cylindrical coordinate system (r,φ,z) with z along the axis of symmetry is introduced. The part of the magnetic field **B** which is generated by sources outside of the plasma region is supposed to be purely poloidal. In addition, an induced field may exist, and it is assumed that the toroidal part of this field is relatively small, but not necessarily the poloidal part of the same field. The electric field is

$$\mathbf{E}=|-\partial\phi/\partial r,\, 0,\, -\partial\phi/\partial z|. \qquad (34)$$

The main part of **E** is associated with the rotation, as is seen later. Some considerations are first made in terms of single-particle motion, after which a macroscopic theory is established.

1. Motion of an Individual Particle

In a study of the forbidden regions for a single charged particle moving in the magnetic field of a current loop, it has been found[19] that a high degree of particle confinement can be achieved inside a toroidal shell such as that given by the shaded area in Fig. 1. This occurs when a transverse electric field is applied which produces a drift motion and a rotation around the axis of symmetry.

The magnetic field of Fig. 1 is now assumed to be a vacuum field and an electric field is applied across the narrow space between two toroidal surfaces generated by the field lines. **E** and **B**=curl**A** then become orthogonal all over the space between the surfaces. The shaded strip in the figure contains the magnetic flux

$$d\Phi=2\pi rBdl=2\pi d(rA_\varphi), \qquad (35)$$

and the potential difference between its inner and outer surface is

$$d\phi=-Edl. \qquad (36)$$

The angular velocity of rotation inside the strip becomes

$$\Omega=-E/Br=2\pi d\phi/d\Phi=d\phi/d(rA_\varphi). \qquad (37)$$

Since $d\phi$ and $d\Phi$ are constant along the strip, this shows that Ω is constant along a magnetic field line. With $|u_r,u_\varphi+\Omega r,u_z|$ as the total velocity of a charged particle of mass m_ν and charge q_ν and index 0 indicating the starting point,

$$\begin{aligned}u_r^2+u_z^2 &= u_0^2-(r_0^2-r^2)\Omega^2-2(q_\nu/m_\nu)(\phi-\phi_0)+2\Omega(q_\nu/m_\nu)\\ &\quad\times(rA_\varphi-r_0A_{\varphi0})-[r_0u_{\varphi0}-(q_\nu/m_\nu)(rA_\varphi-r_0A_{\varphi0})\\ &\quad+(r_0^2-r^2)\Omega]^2/r^2, \end{aligned}\qquad (38)$$

as shown by Bonnevier and Lehnert.[19] For a very strong magnetic field, the particle is guided along a very narrow strip centered along a field line in the rz plane. Relation (37) then shows that the third and fourth terms in Eq. (38) cancel and the particle, in any case, is turned back at a point r when the centrifugal work $(m_\nu/2)(r_0^2-r^2)\Omega^2$ becomes equal to the particle energy $(m_\nu/2)u_0^2$ at the starting point $(r_0,0)$.

From the macroscopic point of view, the situation resembles that of a planetary atmosphere, where the particles are trapped by a gravitation field, provided that their thermal velocity falls below the escape velocity. In this case, the gravitation force is substituted by the centrifugal force.[15] For a plasma, there are interactions between particles and a coupling between the thermal motion and the mass motion which have not been taken into account in the single-particle theory. This is done in the following macroscopic treatment.

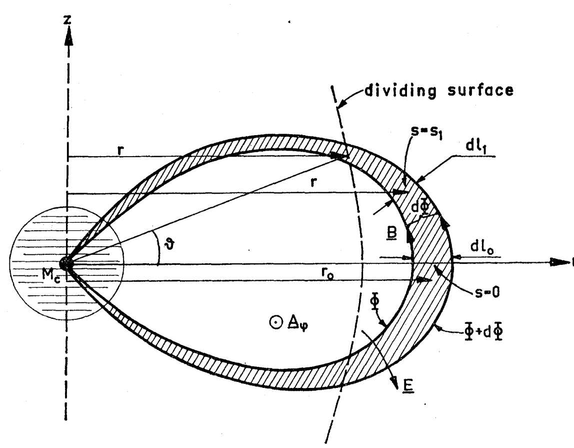

FIG. 1. The shaded area represents the cross section of a thin toroidal shell generated by the field lines of a strong poloidal magnetic field **B** which is symmetric around the z axis of the figure. Plasma is situated inside the shell and rotates around the z axis, the rotation being associated with an electric field **E**, essentially at right angles to **B**. *Note.* Underlined letters in the figure correspond to boldface letters in the text.

2. Macroscopic Theory

(a) Basic Considerations

In applications to cosmical physics and to laboratory experiments, the macroscopic theory can very often be simplified by a number of approximations. In the equation of motion (14), the second term of the left-hand side represents the difference in acceleration of the ion and electron gases. When this difference is neglected the plasma is said to be in a state of "creeping diffusion."[24,25] In Ohm's law (17) the left-hand side represents the inertia of the moving charges. Through-

[24] A. Schlüter, Z. Naturforsch. 6a, 73 (1951).

[25] A. Schlüter, *Electromagnetic Phenomena in Cosmical Physics*, B. Lehnert, Editor (Cambridge University Press, New York, 1956), p. 71.

out this paper, surface currents and pressure jumps across discontinuity surfaces are excluded. Then, a rough estimation of orders of magnitude can be made by replacing space and time derivatives by inverted characteristic lengths and times. In doing so, it is easily seen that the terms just mentioned can be neglected, provided that the time scale of the macroscopic phenomenon is much longer than the gyro period of an electron.[9]

A further assumption to be made henceforth is that of small losses and ionization rates, i.e., the dissipation terms in Eqs. (14), (17), (30), and (33) should be small.

With these starting points, scalar multiplication of Ohm's law (17) by **B** gives

$$\mathbf{E}\cdot\mathbf{B}\approx-\mathbf{B}\cdot(m_i\nabla p_e-m_e\nabla p_i)/e\rho, \tag{39}$$

which shows that the electric potential difference along a field line and along the macroscopic configuration in question becomes of the order of the thermal particle energy. It is now supposed that a strong electric field part $\mathbf{E}_\perp$, perpendicular to **B**, exists which makes the transverse potential difference across the configuration much greater than that corresponding to the thermal energy. This is possible if a sufficiently strong magnetic field is applied to make the radius of gyration of a charged particle much smaller than the macroscopic dimensions. Thus, the situation to be studied is characterized by $E_\perp \gg E_{\parallel}$, where $\mathbf{E}_{\parallel}$ is the longitudinal electric field.

Combination of Eqs. (14) and (17) gives

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}\approx[(m_i-m_e)/e](d\mathbf{v}/dt+\nabla\phi_g)+(m_i\nabla p_i-m_e\nabla p_e)/e\rho+[\eta-(m_i-m_e)\beta/e\rho]\mathbf{i}-[\beta-(m_i-m_e)\alpha/e](\mathbf{v}-\mathbf{v}_n). \tag{40}$$

A rough estimation of orders of magnitude shows that for a very strong magnetic field the right-hand member of Eq. (40) becomes much smaller than the second term of the left-hand member, i.e., when the gyro periods of the charged particles are much shorter than the characteristic times of the macroscopic motion and the collision times and when the change in ϕ_g across the configuration does not exceed v^2 too much. The consequence of this is that the velocity of rotation becomes

$$v_\varphi=\Omega r\approx(\mathbf{E}\times\mathbf{B})_\varphi/B^2. \tag{41}$$

Since **E** is almost perpendicular to **B**, Eqs. (35)–(37) show that Ω becomes nearly constant along a field line. This is consistent with the conclusion earlier drawn by Ferraro[26] about the isorotation of a magnetized star.

Finally, a few comments should also be made on the conduction of heat by charged particles. According to elementary kinetic theory,[27] the heat conductivity in a gas of density n, mean thermal velocity $\bar{v}$, and a mean free path L is about $\frac{1}{2}nk\bar{v}L$; nL is nearly a constant. In the case of a magnetized plasma, the heat conduction across a strong magnetic field is probably reduced considerably, whereas it is unaltered in the longitudinal direction.[28] At the long mean free paths which are actual both in a number of cosmical applications and in gas discharge experiments at high temperatures, the temperature gradient along the magnetic field lines is likely to be very small. Otherwise a very large heat flow would be set up which equalizes the temperature distribution. Similar conclusions have been drawn by Spitzer for the exosphere.[29–31] Thus, the situation of a plasma "atmosphere" which is isothermal along the magnetic field lines seems to be of some interest.

(b) *Pressure Balance along the Magnetic Field Lines*

With the present approximations, the pressure balance is easily deduced from Eq. (14) in the stationary state. Introduce the coordinate s along a magnetic field line and choose the directions given by Fig. 1; for displacements along a field line,

$$d/ds=(B_r/B)d/dr;\quad ds/dr=B/B_r. \tag{42}$$

When Eq. (14) is multiplied scalarly by **B**, the result becomes

$$dp/ds=\rho[(B_r/B)\Omega^2 r-d\phi_g/ds]. \tag{43}$$

Especially for an atmosphere which is isothermal along the magnetic field lines with a temperature T_0, Eq. (43) is easily integrated to

$$\rho=\rho_0\exp\{-(m/2kT_0)[\tfrac{1}{2}\Omega^2(r_0^2-r^2)+\phi_g-\phi_{g0}]\}, \tag{44}$$

where index 0 indicates values at the starting point, and the density distribution is given along a field line starting at r_0. A similar relation has been derived earlier by Block.[32]

In a theory on the origin of the solar system, Alfvén[11] considers a cosmic cloud of ionized gas, rotating around a central mass M_c in a magnetic dipole field which has a dipole moment coinciding with the axis of rotation (see Fig. 1). In the theory is introduced a dividing surface. At the inner side of this surface, the parts of the cloud are attracted and fall down to the central body. Outside of the same surface, the centrifugal force brings matter out along the magnetic field lines down to the equatorial plane where planets or satellites may be formed. Alfvén determines the dividing surface from the balance between the gravitation and centrifugal forces. This corresponds to $dp/ds=0$ which from

[26] V. C. A. Ferraro, Monthly Notices Roy. Astron. Soc. **97**, 458 (1937).
[27] E. H. Kennard, *Kinetic Theory of Gases* (McGraw-Hill Book Company, Inc., New York, 1938).
[28] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), p. 88.
[29] L. Spitzer, Jr., *The Atmospheres of the Earth and Planets*, G. P. Kuiper, Editor (University of Chicago Press, Chicago, 1949), p. 213.
[30] D. R. Bates in *The Earth as a Planet*, G. P. Kuiper, Editor (University of Chicago Press, Chicago, 1954), p. 635.
[31] M. Nicolet in *The Earth as a Planet*, G. P. Kuiper, Editor (University of Chicago Press, Chicago, 1954), p. 655.
[32] L. Block, Arkiv Fysik **14**, 179 (1958).

Eq. (43) gives Alfvén's result,

$$r(\cos\vartheta)^{-\frac{1}{3}}=(2\kappa M_c/3\Omega^2)^{\frac{1}{3}}, \quad (45)$$

for a dipole field and with κ as the gravitation constant and $tg\vartheta=z/r$. If the temperature would be zero, the points on the surface (45) would be the only equilibrium points and matter outside the surface would move all the way down to the equatorial plane. That matter can be trapped in a flat disk in the equatorial plane, provided that the temperature is low, is also seen from the forbidden regions given by Eq. (38). If the temperature differs from zero, a "tail" of the cloud is extended across the dividing surface in the stationary state.

To make a rough estimation of the magnitude of this tail, the temperature is assumed to be constant along a field line, at least in the region outside of the dividing surface, and Eq. (44) can be used. The solar system is taken as an example with $\Omega\approx 2\times 10^{-6}$ sec^{-1}, $M_c\approx 2\times 10^{30}$ kg, $r_0\approx 1.5\times 10^{11}$ m for the earth, and $\phi_{g0}\approx 7\times 10^8$ m^2/sec^2, $\Omega^2 r_0^2=9\times 10^{10}$ m^2/sec^2. With a temperature $T_0=10^{5}$°K for a cosmic cloud of protons, $kT_0/m\approx 10^9$ m^2/sec^2. In static equilibrium the cloud is then expanded only very little above and below the equatorial plane at the earth's orbit and forms a flat disk.

In connection with this problem should also be pointed out that a weak magnetic field of the type discussed in this section is expanded radially by the centrifugal force. The present deductions merely serve as an illustration to the situation where the magnetic field is strong enough to balance this force.

(c) *Energy Balance in a Discharge Experiment*

The present results can also be applied to gas discharge experiments. A number of approximations are made here. The most unfavorable situation is discussed where charged particles which escape to the vessel walls are assumed to recombine there and to reenter the plasma volume in the form of neutral particles with a thermal velocity distribution corresponding to the wall temperature. The plasma is assumed to have a much higher temperature than the vessel walls, and both T_n and $\mathbf{v}_n$ can be neglected in the following deductions. The gravitation field is not taken into account, the ion and electron pressures are assumed to be equal, $p_i=p_e=p/2$, and the temperature is assumed to be high enough for the ionization work to be negligible. Further, all the reentering neutral gas flux is assumed to be "absorbed" in the plasma in the sense that it experiences only ionization and charge exchange, and that no neutral particles are "kicked out" to the walls by elastic collisions. This implies that the energy losses defined by h and δ and the momentum loss given by ν_{in} and ν_{en} are not retained in the equations.

From Eqs. (40) and (41) is easily concluded that the losses of charged particles by convection take place along the field lines in a strong magnetic field. Thus, the equation of continuity (9) gives

$$\rho\zeta=\operatorname{div}(\rho\mathbf{v})\approx\operatorname{div}(\rho\mathbf{v}_{\parallel})\approx\pm\operatorname{div}(\mathbf{B}\rho v_{\parallel}/B)=\pm\mathbf{B}\cdot\nabla(\rho v_{\parallel}/B), \quad (46)$$

where $\mathbf{v}_{\parallel}$ is the mass velocity along the magnetic field $\mathbf{B}$, and it is observed that $\operatorname{div}\mathbf{B}=0$. The plus sign is chosen when $\mathbf{v}_{\parallel}$ is in the positive direction of $\mathbf{B}$. With the notation (42), Eq. (46) is integrated along a field line,

$$v_{\parallel}\approx\frac{B}{\rho}\int_0^s\frac{\zeta\rho}{B}ds\equiv\frac{B}{\rho}I(s);\quad (z\geqslant 0), \quad (47)$$

where $s=0$ corresponds to $r=r_0$, which gives $v_{\parallel}=0$ by reasons of symmetry. From Eqs. (41) and (47) is seen that $v_{\parallel}\ll v_\varphi$ when a charged particle is lost to the walls first after having traveled many times around the axis of symmetry. This is assumed henceforth.

From an examination of Eq. (14) can be shown[18] that the second and third terms of the right-hand member of expression (23) can be neglected, provided that the thermal and rotation velocities are of the same order of magnitude and that the radius of gyration of ions is much smaller than the macroscopic dimensions of the plasma. Then,

$$w_f\approx\alpha\rho v^2\approx\alpha\rho\Omega^2 r^2 \quad (48)$$

in agreement with the general discussion in Sec. II.3 on the heating by collisions with the neutral gas.

First study the limiting case of negligible heat flow $\mathbf{Q}$. Since the convection in a strong poloidal magnetic field is forced to take place along the field lines

$$(d/dt)(p/\rho)=[v_r(\partial/\partial r)+v_z(\partial/\partial z)](p/\rho)\approx v_{\parallel}(d/ds)(p/\rho), \quad (49)$$

where the derivative is taken along a field line. A similar expression is obtained for $d\rho/dt$. With the approximations applied (see Lehnert[18]) and from Eq. (48), the energy theorem (30) reduces to

$$(1+f)\zeta\rho\Omega^2 r^2-\tfrac{5}{2}\zeta p-\tfrac{3}{4}f\zeta p\approx\tfrac{3}{2}\rho v_{\parallel}(d/ds)(p/\rho)-(p/\rho)v_{\parallel}d\rho/ds+\tfrac{1}{2}(1+f)\zeta\rho\Omega^2 r^2, \quad (50)$$

where m_i/m has been put equal to unity. Introduce the variable

$$\tau=2kT/m\Omega^2. \quad (51)$$

Equations (43), (42), and (50) give

$$\rho r dr/ds=(d/ds)(\rho\tau) \quad (52)$$

and

$$(1+f)r^2-(5+\tfrac{3}{2}f)\tau=(v_{\parallel}/\zeta)(d/ds)(5\tau-r^2). \quad (53)$$

Two special solutions are obtained immediately. If charge exchange is negligible ($f\ll 1$),

$$\tau=\tfrac{1}{5}r^2; \quad (54)$$

$$\rho/\rho_0=(r/r_0)^3;\quad p/p_0=(r/r_0)^5. \quad (55)$$

If, instead, charge exchange dominates over ionization ($f \gg 1$), the corresponding solutions become

$$\tau = \tfrac{2}{3}r^2; \tag{56}$$

$$\rho/\rho_0 = (r/r_0)^{-\frac{1}{2}}; \quad p/p_0 = (r/r_0)^{\frac{3}{2}}. \tag{57}$$

Another limiting case to be studied is that of constant temperature along a magnetic field line. The density distribution is given by Eq. (44) with $\phi_\theta = \phi_{\theta 0} = 0$. The temperature T_0 has to be determined from the energy balance. Since heat conduction and convection cannot be separated in this case, the energy balance has to be considered for a complete strip like that given in Fig. 1, which is extended from the equatorial plane at $r = r_0$ up to a point $r = r_1$, where charged particles are assumed to hit a cylindrically symmetric wall. With the notations of the figure, conservation of magnetic flux requires that

$$rBdl = r_0 B_0 dl_0, \tag{58}$$

and a volume element of the strip at r becomes $dw = 2\pi(B_0/B)r_0 dl_0 ds$. Then, the equation of continuity (9) can be integrated over the volume of the strip to give the total number of charged particles of either sign lost per unit time at r_1,

$$dN_1 = (2\pi/m)r_0 B_0 dl_0 \cdot I_1, \tag{59}$$

where the index 1 refers to values at $r = r_1$ and use has been made of relations (47) and (58). The bounding material wall at r_1 is assumed to be nonconductive and is charged negatively in such a way that ions and electrons escape at the same rate through the thin Debye sheet close to r_1. This rate is given by Eq. (59). The corresponding energy loss is

$$dW_1 = \pi r_0 B_0 dl_0 I_1 \Omega^2 (3\tau_0 + r_1^2), \tag{60}$$

where the substitution (51) has been used. The corresponding energy loss due to charge exchange is

$$dW_{ex1} = \pi r_0 B_0 dl_0 \Omega^2 \int_0^{s_1} f\left(\frac{3}{2}\tau_0 + r^2\right)\frac{\rho\zeta}{B} ds. \tag{61}$$

With the energy theorem given by expressions (32) and (33), the balance requires

$$\int \operatorname{div}\mathbf{q}\, dw = dW_1 = \int (w_a + w_c + w_f) dw - dW_{ex1}. \tag{62}$$

From Eqs. (21), (22), (41), (46), (47), and (9), the relations

$$w_a \approx \rho(\mathbf{v}\cdot\nabla)(\tfrac{1}{2}v^2) \approx \rho r v_{\parallel} \Omega^2 dr/ds \tag{63}$$

and

$$w_c \approx -p \operatorname{div}\mathbf{v} \approx -p\zeta + (p/\rho) v_{\parallel} d\rho/ds \tag{64}$$

are obtained. Combination of these relations and Eqs. (48), (47), (43), and (61) gives, after some deductions, the following result:

$$\left[\left(5 + \frac{3}{2}f\right)\tau_0 - (1+f)r_1^2\right] I_1 = \int_0^{s_1} I \frac{d}{ds}\left[\left(\frac{3}{2}\tau_0 - r^2\right) f\right] ds. \tag{65}$$

The modulus of the right-hand member of this equation is less than $fI_1(\frac{3}{2}\tau_0 + |r_0^2 - r_1^2|)$. If the discussion is restricted to cases where ionization dominates over charge exchange ($f \ll 1$), the result becomes

$$\tau_0 = \tfrac{1}{5} r_1^2, \tag{66}$$

and

$$\rho/\rho_0 = \exp[-5(r_0^2 - r^2)/2r_1^2] = p/p_0. \tag{67}$$

This shows that the density distribution is concentrated mainly around the equatorial plane at r_0. The difference between the results (54) and (66) is caused by the high thermal conductivity along the field lines in the latter case, which reduces the temperature to a value associated with the cold material wall at $r = r_1$. Thereby it should be observed that, according to elementary kinetic theory, the heat conductivity is independent of density and is not affected by a steep density gradient. However, care is necessary in the application of these results when the mean free path becomes large.

IV. CONCLUDING REMARKS

The problems treated in Sec. III serve as simplified illustrations to the theory of the present paper. There still remain a number of important phenomena to be taken into account in order to give a true picture of the complicated situations which usually arise both in cosmical physics and in laboratory experiments. Important effects which have not been treated are such as the interaction with radiation fields by excitation, bremsstrahlung and synchrotron radiation as well as two-stage ionization, volume recombination, multiple charge exchange collisions, plasma oscillations due to electric charge separation, and the production of energy by nuclear reactions. Further, the flux of particles hitting the vessel walls in a laboratory experiment may be partly absorbed by the walls. This latter effect may be taken into account in the theory by assuming only a certain fraction of the outgoing particle flux to be scattered back into the plasma. The backscatter of neutral particles into the plasma may possibly be reduced by chosing a wall material with high absorption ability.

Great difficulties arise in the treatment of problems where the mean free path is too large for a macroscopic theory to be valid, and still too small for collisions to be neglected in the calculation of individual particle orbits. Finally, instabilities of macroscopic as well as of microscopic character may complicate the situation even more.