mechanical studies is more powerful than either one alone in providing detailed information about the nature of transitions in polymers. However, there are still many unresolved questions. These include the proper assignment of the low-temperature loss peak in poly(vinyl chloride), polypropylene, and polybutene as well as the detailed mechanism of the high-temperature loss peak in polytetrafluorethylene. The very low temperature transitions (below 80°K) also need much more study both by the techniques discussed in this article and other physical methods. The fact that the experimental second moment at 80°K of many of the polymers still has not reached the theoretical rigid-lattice value indicates that low-temperature research should prove quite fruitful in uncovering new transitions and in obtaining definite assignments for those already reported.

(10) It would be helpful to our understanding of the low-temperature transitions in polymers to use the combined power of NMR and mechanical techniques to explore other types of polymers than the ones discussed here. Of particular interest are polymers in which side chain motion is known to influence the low-temperature loss spectra. It would also be helpful to have studies made by the two techniques of controlled copolymers, graft copolymers, and polymers, such as polypropylene, in which the compositions are kept constant but the crystalline-amorphous ratio varied over the widest possible limits.

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Nuclear Reaction Cross-Section Theory

R. G. MOORE, JR.

Convair, A Division of General Dynamics Corporation, Fort Worth, Texas

I. INTRODUCTION

THE problem of estimating nuclear cross sections has been discussed by many authors.[1] In spite of much excellent work on the theory of nuclear reactions,[2] there still exists no completely satisfactory method for accurately predicting nuclear cross sections. A rigorous approach to the description of nuclear reactions has been given by Wigner[3] and his collaborators,[3] and has recently been reviewed by Lane and Thomas.[4] The generality of this R-matrix theory by its very nature requires a formalism, the practical application of which is somewhat formidable. There are, however, more simplified (and thus more restricted) methods which can be adopted to estimate the magnitude and energy dependence of nuclear cross sections. The development presented in the present review is based upon the compound nucleus model, a concept first proposed by Bohr.[5] A survey of the compound nucleus picture and the points at which the picture should be revised have been given by Brown.[6] Except for qualitative mention, scattering cross sections are not discussed herein. These are discussed in the review article by Brown.[6] The purpose of the present review is to provide a systematic and concise summary of compound nucleus processes from which one can estimate nuclear reaction cross sections.

II. FORMATION OF THE COMPOUND NUCLEUS

In discussing the formation of the compound nucleus, one must concern himself not only with the direct-interaction[7] process, but with the possibility of preferential spin-state formation, i.e., the unequal statistical formation of the compound nucleus in the $I+\frac{1}{2}$ and $I-\frac{1}{2}$ states. Sailor[8] has analyzed the statistical weight factor g for $l=0$ neutron resonances. According to these data, it appears that the compound nucleus is formed preferentially in the $I+\frac{1}{2}$ spin state. He emphasizes that his study includes no measurements for weak resonances. His results certainly indicate that predominant resonances within a particular nucleus are preferentially g_+. It is believed that this effect would not affect the results of total (as opposed to differential)

[1] V. F. Weisskopf and D. H. Ewing, Phys. Rev. **57**, 472 (1939); Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 145 (1947); H. Feshbach and V. F. Weisskopf, Phys. Rev. **76**, 1559 (1949); Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954); W. Hauser and H. Feshbach, Phys. Rev. **87**, 366 (1952); B. Margolis, Phys. Rev. **88**, 327 (1952); B. Margolis, Phys. Rev. **93**, 204 (1954).

[2] C. Block, Nuclear Phys. 4, 503 (1957); H. Feshbach, Ann. Phys. **5**, 357 (1958); Brueckner, Eden, and Francis, Phys. Rev. **100**, 891 (1958); K. A. Brueckner, Phys. Rev. **103**, 172 (1956); P. L. Kapur and R. E. Peierls, Proc. Roy. Soc. (London) **A166**, 277 (1938).

[3] E. P. Wigner, Phys. Rev. **70**, 15, 606 (1946); **73**, 1002 (1948); E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947); Lane, Thomas, and Wigner, Phys. Rev. **98**, 693 (1955).

[4] A. M. Lane and R. G. Thomas, Revs. Modern Phys. **30**, 257 (1958).

[5] N. Bohr, Nature **137**, 344 (1936); Science **86**, 161 (1937).

[6] G. E. Brown, Revs. Modern Phys. **31**, 893 (1959); K. B. Mather and P. Swan, *Nuclear Scattering* (Cambridge University Press, New York, 1958).

[7] S. T. Butler, Phys. Rev. **106**, 272 (1957); Austern, Butler, and McManus, Phys. Rev. **92**, 350 (1953).

[8] V. L. Sailor, Phys. Rev. **104**, 736 (1956).

neutron cross sections, though it would have an effect on angular distribution calculations.

To deduce expressions for the cross section for reactions which proceed through the intermediary of a compound state, we can write an expression for the cross section for the reaction $X(a,b)Y$ as

$$\sigma(a,b)=\sigma_c(a)G_c(b), \tag{1}$$

where $\sigma(a,b)$ is the cross section for the (a,b) reaction, $\sigma_c(a)$ the cross section for the formation of a compound system by particle a incident upon the target nucleus X, and $G_c(b)$ the probability that the compound system c, once formed, decays by emission of particle b, leaving residual nucleus Y. $G_c(b)$ is a dimensionless quantity; the compound system c must decay eventually in some way. $G_c(b)$ is the probability for a particular mode of decay. According to the Bohr assumption,[5,9] $G_c(b)$ depends upon the excitation of a given compound nucleus, and is therefore independent of the method of formation of this compound system. The $\sigma_c(a)$ can be expressed as a weighted sum over the partial wave transmission coefficients,[9]

$$\sigma_c(a)=\pi\lambda\!\!\!^{-2}\sum_l(2l+1)T_l, \tag{2a}$$

where λ is the wavelength in the incident channel, and T_l is the lth partial wave transmission coefficient. In a somewhat more general notation, the cross section for the formation of the compound nucleus with total angular momentum J with z component m by a particle of orbital angular momentum l and energy E, combining with a target nucleus level of spin i can be written

$$\sigma(ljJmE)=\pi\lambda\!\!\!^{-2}(2l+1)T_l|(ljm_lm_j|Jm)|^2, \tag{2b}$$

where $(ljm_lm_j|Jm)$ is the Clebsch-Gordan coefficient.[10] Equations (2a) and (2b) do not take into account any resonance phenomena. They represent averages over an energy interval sufficiently large to contain many resonances. In order to obtain the σ_c near resonance, we seek a factor by which Eq. (2a) can be multiplied in order to have a maximum (whose width corresponds to the lifetime of the level formed) at the resonance energy and whose energy average is unity. We choose the following shape for this factor $y(E)$[9]:

$$y(E)=\frac{D\Gamma/2\pi}{(E-E_r)^2+(\Gamma/2)^2}, \tag{2c}$$

where D is the level spacing and Γ the total width evaluated at a suitable energy. Hence the resonance compound nucleus cross section for a particular l value is

$$\sigma_c{}^l=\pi\lambda\!\!\!^{-2}(2l+1)T_l\left[\frac{D\Gamma/2\pi}{(E-E_r)^2+(\Gamma/2)^2}\right]. \tag{2d}$$

By using a relationship between the transmission coefficient, the level spacing, and the partial width [derived later, Sec. III, Eq. (48)], we can also express Eq. (2d) as

$$\sigma_c{}^l=\pi\lambda\!\!\!^{-2}(2l+1)\frac{\Gamma_a\Gamma}{(E-E_r)^2+(\Gamma/2)^2}, \tag{2e}$$

in which Γ_a is the width for decay of the compound nucleus into the incident channel, and Γ is the total width.

A complete description of the average cross section for compound nucleus formation can be obtained provided one can obtain an adequate representation for the transmission coefficient T_l, which represents the fraction of particles incident from $r=+\infty$ which penetrate into the region $r<R$. The T_l depend upon several factors: (1) the real and imaginary parts of the logarithmic derivative of the radial wave function; (2) the shift factor[4,11]; and (3) the penetration factor.[4,11] The exact manner in which these quantities enter into an evaluation of T_l can be demonstrated in the following way.

The cross section for the formation of the compound nucleus can be written[12]

$$\sigma_c{}^l(a)=\pi\lambda\!\!\!^{-2}(2l+1)(1-|\eta_l|)^2, \tag{3}$$

where η_l is the relative amplitude of the outgoing wave with angular momentum l. Comparison of Eqs. (2) and (3) shows that $1-|\eta_l|^2$ is equivalent to the quantity T_l; to arrive at an expression for T_l, we first determine η_l.

We consider the logarithmic derivative f_l, evaluated at the nuclear surface, i.e.,

$$f_l\equiv R\left(\frac{du_l/dr}{u_l}\right)_{r=R}. \tag{4}$$

The radial wave function $u_l(r)$ [13] can be separated into incoming and outgoing waves,

$$u_l(r)=au_l{}^{(-)}(r)+bu_l{}^{(+)}(r), \tag{5}$$

where a and b are constants. But $\eta_l=-(b/a)$ (the fraction reflected), and hence

$$u_l(r)=a[u_l{}^{(-)}(r)-\eta_lu_l{}^{(+)}(r)], \tag{6}$$

therefore

$$f_l=R\frac{[u_l{}^{(-)\prime}(r)-\eta_lu_l{}^{(+)\prime}(r)]_{r=R}}{[u_l{}^{(-)}(r)-\eta_lu_l{}^{(+)}(r)]_{r=R}}, \tag{7}$$

where the primes denote differentiation with respect

[9] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

[10] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[11] R. G. Thomas, Phys. Rev. **97**, 224 (1955).

[12] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954); Office of Naval Research, Tech. Rept. 62 (1953) (unpublished).

[13] If the Schroedinger equation is separated by the substitution $R(r)Y(\theta,\phi)$, then the equation in $R(r)$ is called the radial equation. If the change of variable $R(r)=u(r)/r$ is made, then the equation which results has $u(r)$ as a solution. It is this wave function which we refer to as the "radial wave function $u(r)$." The subscript l, denotes the solution for a particular eigenvalue.

to r. After dividing both numerator and denominator of Eq. (7) by $u_l^{(-)}(r)$ and defining the phase constant ξ_l by

$$\exp(2i\xi_l)=u_l^{(-)}(R)/u_l^{(+)}(R),$$

Eq. (7) reduces to

$$f_l=R\left\{\frac{[u_l^{(-)\prime}(r)/u_l^{(-)}(r)]-\eta_l(S_l+iP_l)\times\exp(-2i\xi_l)(1/R)}{1-\eta_l\exp(-2i\xi_l)}\right\}_{r=R}, \quad (8)$$

where the quantities P_l and S_l are defined by

$$R\left[\frac{du_l^{(+)}(r)/dr}{u_l^{(+)}(r)}\right]_{r=R}=S_l+iP_l. \quad (9)$$

By making a similar substitution for the incoming wave $u_l^{(-)}(r)$ and solving for η_l, one obtains

$$\eta_l=\frac{f_l-S_l+iP_l}{f_l-S_l-iP_l}e^{+2i\xi_l}. \quad (10)$$

Equation (10) is the desired result for η_l and hence (indirectly) for T_l. If f_l is real, then $1-|\eta_l|^2$ is unity, and the cross section for formation of the compound nucleus is zero, i.e., scattering without any reaction. Thus f_l cannot be a purely real quantity if a reaction takes place. The quantity η_l is the essential feature of reaction cross section calculations[12]; it also completely determines the (shape) elastic scattering cross section through the well-known relationship[14]

$$\sigma_{sc}(\theta)d\Omega=\frac{\pi}{k^2}\left|\sum_l(2l+1)(1-\eta_l)\frac{P_l(\cos\theta)}{(4\pi)^{\frac{1}{2}}}\right|^2 d\Omega. \quad (11)$$

Equation (11) is employed in optical model analyses in which the primary objective is to determine the optical model parameters which best fit experimental data by (numerically) solving the radial wave equation, thereby determining the logarithmic derivative f_l, and hence η_l.

Upon taking the absolute value of Eq. (10), it can be shown that T_l is given by

$$T_l=\frac{-4P_l\,\mathrm{Im}(f_l)}{[\mathrm{Re}(f_l)-S_l]^2+[\mathrm{Im}(f_l)-P_l]^2}. \quad (12)$$

Equation (12) is a general result for T_l which can be used to determine both charged-particle and neutron transmission coefficients. For a complete determination of T_l, we must therefore determine four quantities: S_l, P_l, $\mathrm{Im}(f_l)$, and $\mathrm{Re}(f_l)$. P_l and S_l are called the penetration and shift factors,[4,11] respectively. The penetration factor gives a measure of the probability of the lth partial wave penetration of the neutron to the nuclear surface against the centrifugal angular momentum barrier and for charged particles against the coulomb barrier, i.e., against the nonnuclear type forces. The shift factor gives a measure of the level shift due to extranuclear interaction.

[14] A. M. Lane, Revs. Modern Phys. **29**, 191 (1957).

Shift and Penetration Factors

We discuss first the evaluation of the shift factor S_l and the penetration factor P_l, leaving until later the evaluation of the real and imaginary parts of the logarithmic derivative. In order to evaluate them we must consider the $u_l^{(+)}(r)$ and $u_l^{(-)}(r)$ of Eq. (5) in more detail. Recall that $u_l(r)$ is the radial wave function, i.e., a solution of the radial wave equation

$$\frac{d^2u_l(r)}{dr^2}+\left[k^2-\frac{l(l+1)}{r^2}\right]u_l(r)=0. \quad (13)$$

Equation (13) is the radial equation for neutrons only, since no Coulomb potential is included. To solve Eq. (13) we make the substitution $u_l(r)=(kr)R_l(r)$ to obtain

$$\frac{d^2R_l(r)}{dr^2}+\frac{2}{r}\frac{dR_l(r)}{dr}+\left(k^2-\frac{L^2}{r^2}\right)R_l(r)=0, \quad (14)$$

where $L^2=l(l+1)$; $\rho=kr$; then Eq. (14) can be written

$$\frac{d^2R_l}{d\rho^2}+\frac{2}{\rho}\frac{dR_l}{d\rho}+\left(1-\frac{L^2}{\rho^2}\right)R_l=0. \quad (15)$$

The solutions of Eq. (15) are the spherical Bessel functions $j_l(\rho)$ and the spherical Neumann functions $n_l(\rho)$, i.e.,

$$R_l^{(1)}=\mathrm{const}\,j_l(\rho)=\mathrm{const}\,j_l(kr), \quad (16a)$$

$$R_l^{(2)}=\mathrm{const}\,n_l(\rho)=\mathrm{const}\,n_l(kr), \quad (16b)$$

or

$$u_l^{(1)}=\mathrm{const}\,(kr)j_l(kr), \quad (17a)$$

$$u_l^{(2)}=\mathrm{const}\,(kr)n_l(kr). \quad (17b)$$

The spherical functions are related to the ordinary functions by the well-known relations[15]

$$j_l(X)=(\pi/2X)^{\frac{1}{2}}J_{l+\frac{1}{2}}(X), \quad (18a)$$

$$n_l(X)=(\pi/2X)^{\frac{1}{2}}N_{l+\frac{1}{2}}(X), \quad (18b)$$

with the result that

$$u_l^{(1)}=\mathrm{const}\,(\pi kr/2)^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr), \quad (19a)$$

$$u_l^{(2)}=\mathrm{const}\,(\pi kr/2)^{\frac{1}{2}}N_{l+\frac{1}{2}}(kr). \quad (19b)$$

If we define the regular and irregular solutions such that they satisfy certain asymptotic conditions,[9] then the constant coefficients in Eqs. (19) are determined. If we denote this "regular" solution by F_l and this

[15] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953).

"irregular" solution by G_l, then another solution is a linear combination of F_l and G_l. For the theory of nuclear reactions we choose the combination which gives an outgoing wave $u_l^{(+)}$ of the form[11]

$$u_l^{(+)}=G_l(r)+iF_l(r), \tag{20a}$$

and an ingoing wave

$$u_l^{(-)}=G_l(r)-iF_l(r). \tag{20b}$$

From the definitions of S_l and P_l [Eq. (9)] and using Eq. (20),

$$S_l+iP_l=R\left[\frac{G_l'(r)+iF_l'(r)}{G_l(r)+iF_l(r)}\right]_{r=R}. \tag{21}$$

By multiplying the numerator and denominator of the right-hand side of Eq. (21) by the complex conjugate of the denominator and separating real and imaginary parts, one finds that

$$S_l=R\left[\frac{G_l(r)G_l'(r)+F_l(r)F_l'(r)}{G_l^2(r)+F_l^2(r)}\right]_{r=R}, \tag{22a}$$

$$P_l=R\left[\frac{F_l'(r)G_l(r)-F_l(r)G_l'(r)}{G_l^2(r)+F_l^2(r)}\right]_{r=R}.^{16} \tag{22b}$$

Thomas[4] has obtained recursion formulas for S_l and P_l. They are

$$P_l=A_lP_{l-1}/[(b_l-S_{l-1})^2+P_{l-1}^2], \tag{23a}$$

$$S_l+b_l=A_l(b_l-S_{l-1})/[(b_l-S_{l-1})^2+P_{l-1}^2], \tag{23b}$$

where

$$A_l=\rho^2+(\rho\eta/l)^2, \tag{24}$$

$$b_l=l+(\rho\eta/l), \tag{25}$$

with

$$\rho=kr, \quad \eta=0.1574ZZ'E_{\text{Mev}}{}^{-\frac{1}{2}}M_i{}^{\frac{1}{2}}. \tag{26}$$

M_i is the reduced mass in atomic mass units, E_{Mev} the energy in Mev. Equations (22) are general expressions for the shift and penetration factors. They are valid for both neutrons and charged particles, provided one uses the proper F and G functions. As shown previously, for neutrons these functions are expressible in terms of Bessel functions:

$$F_l(r)=(\pi kr/2)^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr), \tag{27}$$

$$G_l(r)=-(\pi kr/2)^{\frac{1}{2}}N_{l+\frac{1}{2}}(kr). \tag{28}$$

Tabulations of $F_l(r)$ and $G_l(r)$ for neutrons are given by Lax and Feshbach,[17] and by Morse, Lowan, Feshbach, and Lax.[18] Tabulations of S_l and P_l for neutrons are given by Feshbach, Porter, and Weisskopf,[9] and by Monahan, Biedenharn, and Schiffer.[19]

For charged particles, the Coulomb potential must be included in the radial wave equation, in which case the F_l and G_l become the well-known Coulomb wave functions.[20] The evaluation of these functions is a lengthy and tedious numerical calculation. They are frequently evaluated by first determining the zero-order functions from asymptotic relations,[20] obtaining their derivatives, then recurring upward (i.e., increasing l) to evaluate the higher-order functions. Abramowitz[21] has discussed a method for generating $F_l(G_l)$ in decreasing (increasing) order. For our purposes their evaluation can perhaps best be accomplished by evaluating the zero-order functions from the asymptotic relations and from these evaluate S_0 and P_0, then use the recurrence relations for these functions to obtain the values for arbitrary l. The asymptotic forms are[20]

$$F_l(\rho)\cong\sin[\rho-(\tfrac{1}{2})l\pi-\eta\ln(2\rho)+\sigma_l], \tag{29}$$

$$G_l(\rho)\cong\cos[\rho-(\tfrac{1}{2})l\pi-\eta\ln(2\rho)+\sigma_l]. \tag{30}$$

The quantity σ_l is the Coulomb phase shift given by

$$\sigma_{l+1}=\sigma_l+\tan^{-1}(\eta/l+1), \tag{31}$$

where

$$\sigma_0=\eta C+\sum_\zeta\left[\frac{\eta}{\zeta}-\tan^{-1}\frac{\eta}{\zeta}\right] \tag{32}$$

is used when $\eta\leqq 2$. For $\eta>2$, σ_0 is given by

$$\sigma_0=\frac{\pi}{4}+\eta(\log\eta-1)-\left\{\frac{1}{12}\eta^{-1}+\frac{1}{360}\eta^{-3}+\frac{1}{1260}\eta^{-5}+\frac{1}{1680}\eta^{-7}+\frac{1}{1188}\eta^{-8}\right\}. \tag{33}$$

The quantity C is Euler's constant,

$$C=0.5772156649\cdots.$$

One further significant point should be emphasized. The functions F_l and G_l are functions of the radial coordinate r. The quantity η_l, however, is independent of this coordinate. Since these functions are "external"

[16] The shift factor as defined by Thomas[4,11] is

$$S=\rho[(FF'+GG')/(F^2+G^2)]$$

and the penetration factor is defined by

$$P=\rho[F^2+G^2]^{-1},$$

where the primes denote differentiation with respect to $\rho=kr$, whereas in our notation the primes denote differentiation with respect to r. The two definitions are equivalent. If the differentiation is with respect to ρ, the Wronskian is unity, while differentiation with respect to r gives a value of k for the Wronskian.

[17] M. Lax and H. Feshbach, J. Acoust. Soc. Am. **20**, 108 (1948).

[18] Morse, Lowan, Feshbach, and Lax, U. S. Navy Dept. of Research and Inventions, Rept. No. 62, IR (1945) (unpublished).

[19] Monahan, Biedenharn, and Schiffer, Argonne Natl. Laboratory Rept. ANL-5846 (1958) (unpublished).

[20] Block, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. **23**, 147 (1951); Natl. Bur. Standards, Applied Mathematics Series, No. 17 (1952); also NBS-3033 (1954); J. P. Schiffer, Argonne Natl. Laboratory Rept. ANL-5739 (1957) (unpublished); Sharp, Gove, and Paul, Atomic Energy of Canada, Ltd., AECL-268 (1953) (unpublished); A. Tubis, Los Alamos Scientific Laboratory Rept. LA-2150 (1958) (unpublished).

[21] I. A. Stegun and M. Abramowitz, Phys. Rev. **98**, 1851 (1955).

solutions (i.e., exterior to the nuclear potential), they must be evaluated at some point sufficiently removed from any effects of the nuclear potential. This is a sufficient condition for the case of neutrons, but not so using the method we have described for charged particles. Since the F_l and G_l for charged particles which we have just outlined are evaluated from asymptotic relations, one must insure that the point of evaluation is sufficiently large such that the asymptotic relations are "good," i.e., that the Wronskian is unity (or k as the case may be[16]).

Prosser and Biedenharn[22] have investigated the properties of the shift and penetration factors. In general the penetration factor is a monotonically decreasing function of η and l, a monotonically increasing function of ρ and hence energy. The shift function is a monotonically increasing function of η. Prosser and Biedenharn[22] show that for increasing ρ, S/ρ decreases monotonically.

Logarithmic Derivative

The only remaining quantity in Eq. (12) for T_l is the logarithmic derivative, of which we need both the real and imaginary parts. From the preceding discussion we see that the shift factor S_l and the penetration factor P_l are functions of the wave number k, the channel radius R, and the angular momentum l. The logarithmic derivative f_l is expressed in terms of the wave functions describing the nucleon-nucleus interaction, the wave functions being solutions of the wave equation into which is inserted the nuclear potential. Hence the f_l depends upon the nuclear potential chosen to represent the interaction. In other words, the quantity f_l must be specified by a given nuclear model.

The subject of nuclear models has received widespread attention in recent years. Since we do not have a unified description of the atomic nucleus, we are forced to adopt different nuclear models, each of which has a specific field of applicability and which is capable of reproducing results in agreement with experiment only in limited areas. The subject of nuclear models has been reviewed by Eisenbud and Wigner,[23] Hughes,[24] Frisch,[25] and Moszkowski.[26] We consider in detail two specific models (potentials) for the evaluation of the logarithmic derivative.

To obtain the logarithmic derivative for a given model, we must have a knowledge of the wave functions based on the selection of this particular model. Consider first the "strong-interaction" model.[27] Our objective is to obtain the logarithmic derivative f_l based on the potential of the strong-interaction model. In the strong-interaction model, the nucleon upon entering the nucleus moves with a high kinetic energy and immediately forms a compound nucleus in which its motion is completely integrated with the motion of all other nucleons. The potential used to describe this interaction is a real square well whose width is equal to the nuclear radius. Since the incident nucleon immediately forms a compound nucleus and does not return to the incident channel, we can assume that the wave function is that of an ingoing wave only, i.e.,

$$u_l = \text{const}\exp(-iKr) \quad r<R, \tag{34}$$

where K is the wave number of the nucleon in the interior region. Hence,

$$f_l \equiv R\left[\frac{du_l/dr}{u_l}\right]_{r=R} = -iKR. \tag{35}$$

The transmission coefficient for the strong-interaction model can be obtained by substitution of Eq. (35) into Eq. (12), the result being

$$T_l = \frac{4P_lKR}{S_l^2+(KR+P_l)^2} \tag{36}$$

such that

$$\sigma_c(a) = \pi\lambda\!\!\!^{-2}\sum_l (2l+1)\frac{4P_lKR}{S_l^2+(KR+P_l)^2}. \tag{37}$$

For $l=0$ neutrons, $S_0=0$ and $P_0=kR$, hence

$$\sigma_c(n) = \pi\lambda\!\!\!^{-2}[4kK/(K+k)^2]. \tag{38}$$

For calculational purposes it is sometimes useful to rewrite Eq. (36) for incident neutrons after introducing the definitions

$$\begin{gathered} x=kR, \quad X=KR, \\ v_l' = k^{-2}[(dG_l/dr)^2+(dF_l/dr)^2], \\ v_l = [G_l^2+F_l^2]^{-1}, \end{gathered} \tag{39}$$

from which it follows that $S_l=kRv_l$. It follows that the numerator of Eq. (36) can be written as $4xXv_l$, and further algebraic manipulation reduces the denominator to $X^2+(2xX+x^2v_l')v_l$. According to Blatt and Weiss-

[22] F. W. Prosser and L. C. Biedenharn, Phys. Rev. **109**, 413 (1958).

[23] L. Eisenbud and E. P. Wigner, *Nuclear Structure* (Princeton University Press, Princeton, New Jersey, 1958); *Handbook of Physics*, E. U. Condon and H. Odishaw, Editors (McGraw-Hill Book Company, Inc., New York, 1958).

[24] D. J. Hughes, *Neutron Cross Sections* (Pergamon Press, New York, 1957).

[25] O. R. Frisch, *Progress in Nuclear Physics* (Pergamon Press, New York, 1957), Vol. 6.

[26] S. A. Moszkowski, *Handbuch der Physik* (Springer-Verlag, Berlin, 1957), Vol. XXXIX.

[27] So called because the nucleon-nucleus interaction is assumed to be so strong that the compound nucleus is formed immediately, with no possibility for return in the incident channel. This model is also referred to as the "continuum" model or "black-nucleus" model.

kopf,[9]

$$v_0=1, \qquad v_0'=1,$$
$$v_1=\frac{x^2}{1+x^2}, \qquad v_1'=\frac{1}{x^2}+\left(1-\frac{1}{x^2}\right)^2,$$
$$v_2=\frac{x^4}{9+3x^2+x^4}, \qquad v_2'=1-\frac{6}{x^2}+\left(\frac{6}{x^3}-\frac{3}{x^2}\right)^2, \quad (40)$$
$$v_3=\frac{x^6}{225+45x^2+6x^4+x^6}, \qquad v_3'=1-\frac{21}{x^2}+\frac{45}{x^4}+\left(\frac{45}{x^3}-\frac{6}{x}\right)^2.$$

The asymptotic forms are

$$v_l\cong\frac{x^{2l}}{[(2l-1)!!]^2}, \qquad v_l'\cong\frac{l^2[(2l-1)!!]^2}{x^{2l+2}}, \quad (41)$$

where $(2l-1)!!=(2l-1)(2l-3)(2l-5)\cdots$. The asymptotic expressions are valid when $x\ll l$. Lax and Feshbach[17] have tabulated the functions v_l and v_l'. The wave number K of the neutrons inside the nuclear surface is related to the wave number k through the relationship

$$K=(K_0^2+k^2)^{\frac{1}{2}}, \quad (42a)$$

where

$$K_0^2=(9\pi/8)^{\frac{2}{3}}(1/r_0^2) \quad (42b)$$

at low energies $k\ll K$, and P_l and S_l are both small in comparison to KR. For these energies, Eq. (36) can be approximated as

$$T_l=(4k/K)v_l. \quad (43)$$

The strong-interaction model produces no resonance structure, nor does it even predict any "gross structure."

Feshbach, Porter, and Weisskopf[12] investigated the consequences of reducing the interaction between the nucleon and the nucleus. The consequence of this reduced interaction is that the incident nucleon can exist in the interior of the nucleus for a finite time without forming a compound nucleus. This necessitates the inclusion of an absorption coefficient ζ in the potential for the interaction. In the cloudy crystal-ball model we replace the nucleus by a one-body potential which is a complex well; thus,

$$V=V_0(1+i\zeta), \quad 0<\zeta<1, \quad (44)$$

where $V_0=-U$ for $r<R$, and $V_0=0$ for $r>R$. Though the potential of Eq. (44) does not reproduce the actual rapid variations in the cross section, it does describe the "gross structure" variations.

To obtain the logarithmic derivative for the cloudy crystal-ball potential, we use the definition of the logarithmic derivative [Eq. (35)] and the (interior) wave function for the cloudy crystal-ball potential. The interior solutions to the radial wave equation are[28]

$$R(r)=\text{const}\,j_l(Kr), \quad (45)$$

where

$$K=[2m(E-V)/\hbar^2]^{\frac{1}{2}}. \quad (46)$$

The expression for K can also be written

$$K=[k^2-(2mV/\hbar^2)]^{\frac{1}{2}}, \quad (47)$$

where $k^2=2mE/\hbar^2$. By denoting $2m/\hbar^2$ by $\alpha_0'^2$ and substituting the cloudy crystal-ball potential of Eq. (44) into Eq. (47), we obtain

$$K=[k^2+\alpha_0'^2U(1+i\zeta)]^{\frac{1}{2}}. \quad (48)$$

If we let $\alpha_0'^2U=\alpha_0^2$, this becomes

$$K=[k^2+\alpha_0^2(1+i\zeta)]^{\frac{1}{2}}. \quad (49)$$

The solution $u_l(r)$ [i.e., $krR(r)$] is then

$$u_l(r)=\text{const}\,krj_l(Kr), \quad (50)$$

whose derivative is

$$du_l(r)/dr=\text{const}[K(kr)j_l'(Kr)+kj_l(Kr)], \quad (51)$$

where the prime on the Bessel function denotes differentiation with respect to KR. Then, by definition,

$$f_l=R\left[\frac{K(kR)j_l'(KR)+kj_l'(KR)}{(kR)j_l(KR)}\right] \quad (52)$$

or

$$f_l=R\left[\frac{Kj_l'(KR)}{j_l(KR)}\right]+1. \quad (53)$$

If we denote KR by X, then Eq. (53) can be written

$$f_l=1+[Xj_l'(X)/j_l(X)]. \quad (54)$$

Equation (54) is the logarithmic derivative for the cloudy crystal-ball potential. In order to apply this result to a transmission coefficient calculation (and hence cross-section calculation), we must determine[12] the real and imaginary parts of Eq. (54). To accomplish this we use the trigonometric equivalence of $j_l(X)$ for $l=0$ and then use recursion relations for $j_l(X)$ to determine the results for arbitrary l. Equation (54) for $l=0$ is

$$f_0=1+[Xj_0'(X)/j_0(X)]. \quad (55)$$

Substitution of $j_0(X)=(1/X)\sin X$ into Eq. (55) yields

$$f_0=1+\left\{\frac{X[(1/X)\cos X-(1/X^2)\sin X]}{(1/X)\sin X}\right\} \quad (56)$$

which simplifies to

$$f_0=X\cot X. \quad (57)$$

To find the real and imaginary parts of f_0, we write $X=X_1+iX_2$ with the result that

$$f_0=(X_1+iX_2)\cot(X_1+iX_2). \quad (58)$$

If $\cot(X_1+iX_2)$ is expanded by making use of the trigonometric identity for the sum of two angles, Eq.

[28] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949).

(58) becomes

$$f_0=(X_1+iX_2)\left[\frac{1-i\tan X_1\tanh X_2}{\tan X_1+\tanh X_2}\right]. \quad (59)$$

By substituting sine-cosine ratios for the tangent functions and rearranging, Eq. (59) becomes

$$f_0=\mathrm{Re}(f_0)+i\,\mathrm{Im}(f_0), \quad (60a)$$

where

$$\mathrm{Re}(f_0)=\frac{X_1\cos X_1\sin X_1+X_2\cosh X_2\sinh X_2}{\cosh 2X_2-\cos 2X_1}, \quad (60b)$$

$$\mathrm{Im}(f_0)=\frac{X_2\cos X_1\sin X_1-X_1\cosh X_2\sinh X_2}{\cosh 2X_2-\cos 2X_1}. \quad (60c)$$

We now use the recurrence relations of the spherical Bessel functions to obtain $\mathrm{Re}(f_l)$ and $\mathrm{Im}(f_l)$ for arbitrary l. These are[15]

$$\frac{d}{dz}[z^{m+1}j_m(z)]=z^{m+1}j_{m-1}(z), \quad (61a)$$

$$\frac{d}{dz}[z^{-m}j_m(z)]=-z^{-m}j_{m+1}(z). \quad (61b)$$

By using Eq. (61a) and solving for $j_l'(X)$, subsequent substitution into Eq. (55) yields

$$f_l=[Xj_{l-1}(X)/j_l(X)]-l. \quad (62)$$

By using Eq. (61b), solving for $j_l(X)$ and substituting into Eq. (62), the result after rearranging gives

$$f_l=\frac{X^2}{l-1-[Xj_{l-1}'(X)/j_{l-1}(X)]}-l, \quad (63)$$

but

$$f_{l-1}=1+[Xj_{l-1}'(X)/j_{l-1}(X)],$$

hence

$$f_l=[X^2/(l-f_{l-1})]-l. \quad (64)$$

The real and imaginary parts of Eq. (64) can be obtained by writing X as X_1+iX_2 and f_l as $\mathrm{Re}(f_l)+i\,\mathrm{Im}(f_l)$. After collecting real and imaginary parts, multiplying the numerator and denominator by the complex conjugate of the denominator, and again collecting terms, one obtains

$$\mathrm{Re}(f_l)=\frac{(X_1^2-X_2^2)[l-\mathrm{Re}(f_{l-1})]-2X_1X_2\,\mathrm{Im}(f_{l-1})}{[l-\mathrm{Re}(f_{l-1})]^2+[\mathrm{Im}(f_{l-1})]^2}-l, \quad (65a)$$

$$\mathrm{Im}(f_l)=\frac{[X_1^2-X_2^2]\,\mathrm{Im}(f_{l-1})+2X_1X_2[l-\mathrm{Re}(f_{l-1})]}{[l-\mathrm{Re}(f_{l-1})]^2+[\mathrm{Im}(f_{l-1})]^2}. \quad (65b)$$

To obtain an asymptotic equation for f_l we make use of the asymptotic expression for j_l,[15]

$$j_l(z)\underset{z\to\infty}{\longrightarrow}(1/z)\cos[z-(\pi/2)(l+1)], \quad (66)$$

which is equivalent to

$$j_l(X)\underset{X\to\infty}{\longrightarrow}(1/X)\sin[X-l(\pi/2)], \quad (67)$$

which upon substitution into Eq. (54) yields

$$f_l\underset{X\to\infty}{\longrightarrow}X\cot[X-l(\pi/2)]. \quad (68)$$

The logarithmic derivative for the cloud crystal-ball potential is then given by Eqs. (65) in conjunction with Eqs. (60).

If we write the potential of Eq. (44) in the form $V+iW$, then the real part V of this complex potential represents the refractive part of the optical model, and the imaginary part W represents the absorbing part. V and W need not have the same functional dependence upon either energy[29] or radial coordinate.[30] The radial dependence of V and W is expected to be of the same form as the charge density distributions determined by the Stanford workers,[31] namely, an approximately flat central region with an appreciable diffuse surface. A closed-form expression for the logarithmic derivative for such a potential with a diffuse edge,[30] or perhaps absorbing surface,[32] is not available. Logarithmic derivatives for such potentials have been obtained only by numerical integration of the radial wave equation.

Summary

The formation of the compound nucleus is given by a weighted sum over the partial-wave transmission coefficients which are expressed in terms of the shift factor, the penetration factor, and the real and imaginary parts of the logarithmic derivative. For neutrons the shift and penetration factors are expressible in terms of Bessel functions and for charged particles in terms of Coulomb wave functions. The logarithmic derivative depends upon specific assumptions concerning the potential function chosen to represent the nucleon-nucleus interaction. In the preceding discussion we have derived expressions for the logarithmic derivative based on two different nuclear potentials: (1) strong-interaction model (real square-well potential); and (2) square-well complex potential. Logarithmic derivatives for more recent modifications of the optical-model potential are obtained by numerical integration of the radial wave equation into which has been inserted the assumed potential form with suitable parameters. The complexity of the calculation of transmission coefficients for all but the simplest cases necessarily restricts the computation to electronic computer techniques.

[29] Melkanoff, Moszkowski, Nodvik, and Saxon, Phys. Rev. **101**, 507 (1956).
[30] R. D. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954).
[31] R. Hofstadter, Ann. Rev. Nuclear Sci. **7**, 231 (1957).
[32] Bjorklund, Fernbach, and Sherman, Phys. Rev. **101**, 1832 (1956); **109**, 1295 (1958).

III. DECAY OF THE COMPOUND NUCLEUS

Particle Branching Ratio

A compound nucleus, once formed, must experience de-excitation by at least one mode of decay. In the present section we discuss the probability of particle emission, leaving until later the problem of de-excitation by gamma-ray emission.

Section II was concerned with the formation of the compound system, i.e., the σ_c factor. In the preceding section we have discussed in detail methods for evaluating σ_c by first determining the transmission coefficients. The present section is concerned with the factor $G_c(b)$, sometimes referred to as the "branching ratio" for emission of particle b from the compound nucleus. We consider the branching ratio for cases in which the statistical model[33] for nuclear reactions is applicable. $G_c(b)$ is the dimensionless probability that the compound system c, once formed, decays by emission of particle b. This probability can be expressed as the ratio of the relative probability of emission of b divided by the sum of the emission probabilities of all other possible processes. In this ratio the "probability" need not be a true probability in the sense that the sum of all possibilities must be unity. When expressed as a ratio, these quantities need be proportional only to the absolute probability. We can thus write $G_c(b)$ as

$$G_c(b)=F_b/\sum_i F_i, \tag{1}$$

where F_i is a function expressing the relative probability of emission of particle i. The problem is then reduced to determining the functional form for F_i.

One can approximate $G_c(b)$ by assuming that the neutron F function F_n is unity, and the F function for charged particles F_{cp} is the formula for the penetrability of the potential barrier[34]:

$$F_{cp}=\exp[-2g\gamma(E/B_c)], \tag{2a}$$

where

$$g=[2MzZe^2R]^{\frac{1}{2}}/\hbar, \tag{2b}$$

and

$$\gamma(X)=X^{-\frac{1}{2}}\arccos X^{\frac{1}{2}}-(1-X)^{\frac{1}{2}}, \tag{2c}$$

and B_c is the barrier height zZe^2/R, E the total energy in the center-of-mass system, M the reduced mass, R the nuclear radius, and e the charge of the electron. Equations (2) are valid only when the incident energy is less than the Coulomb barrier height. We now undertake the determination of a more realistic form for these F functions.

The probability that the system will decay by one specific channel, $G_c(\beta)$, where β denotes this channel, is the ratio of the energy widths

$$G_c(\beta)=\Gamma_\beta/\sum_i \Gamma_i, \tag{3}$$

where Γ_β is the partial width for emission of particle b into the specific channel β, and i denotes any possible exit channel. Weisskopf[35] has shown that the ratio $\sigma_c(\alpha)/\Gamma_\alpha\lambda\!\!\!^{-}_\alpha{}^2$ is independent of the channel α, and hence

$$\frac{\sigma_c(\alpha)}{\Gamma_\alpha\lambda\!\!\!^{-}_\alpha{}^2}=\frac{\sigma_c(\beta)}{\Gamma_\beta\lambda\!\!\!^{-}_\beta{}^2}, \tag{4a}$$

with the result that

$$\frac{\Gamma_\beta}{\Gamma_\alpha}=\frac{\sigma_c(\beta)\lambda\!\!\!^{-}_\alpha{}^2}{\sigma_c(\alpha)\lambda\!\!\!^{-}_\beta{}^2} \tag{4b}$$

or

$$\frac{\Gamma_\beta}{\Gamma_\alpha}=\frac{k_\beta{}^2\sigma_c(\beta)}{k_\alpha{}^2\sigma_c(\alpha)}, \tag{4c}$$

where $k=1/\lambda\!\!\!^{-}$. We wish, however, to find $G_c(\beta)$ which is simply a sum over the Γ in the denominator of Eq. (4c). Hence

$$G_c(\beta)=\frac{\Gamma_\beta}{\Gamma}=\frac{k_\beta{}^2\sigma_c(\beta)}{\sum_\alpha k_\alpha{}^2\sigma_c(\alpha)}=\frac{\sum_l(2l+1)T_l{}^\beta}{\sum_{l,i}(2l+1)T_l{}^i}. \tag{5}$$

Equation (5) gives the branching ratio for decay by particle b into the single channel β only. This equation does, in fact, describe reactions which can proceed only to a single level provided the reaction proceeds through the intermediary of a compound state; in other words, the mechanism of direct interactions is not included within the framework of the theory. Hence, satisfactory results using Eq. (5), cannot be expected for those cases in which the direct-interaction process is a significant contribution to the yield of the reaction. According to Butler,[7] it appears that this contribution is significant for those cases in which the reaction proceeds only to a low-lying level of the residual nucleus.

To determine the branching ratio for decay by particle b when several channels are open, we must multiply by the density ρ of levels of the residual nucleus into which the decay may proceed, the results being

$$G_c(b)=\frac{\int_0^{E_b'} k_b{}^2\sigma_c(b)\rho_Y(E_b'-E_b)dE_b}{\sum_\alpha\int_0^{E_\alpha'} k_\alpha{}^2\sigma_c(\alpha)\rho_Y(E_\alpha'-E_\alpha)dE_\alpha}, \tag{6}$$

where E_b' denotes the maximum energy with which particle b may be emitted, and ρ_Y denotes the nuclear level density of the residual nucleus Y at excitation $E_\alpha'-E_\alpha$. For a nuclear reaction the maximum energy E_b' of the emitted particle b is E_a+Q_{ab}, where E_a is the incident particle energy, and Q_{ab} is the Q value for the (a,b) reaction. Since $k^2=2mE/\hbar^2$, we obtain the functional form for the quantity F:

[33] K. J. LeCouteur, *Nuclear Reactions*, P. M. Endt and M. Demeur, editors (North-Holland Publishing Company, Amsterdam; Interscience Publishers, Inc., New York, 1959), Vol. I.

[34] H. A. Bethe, Revs. Modern Phys. **9**, 166 (1937).

[35] V. F. Weisskopf, Phys. Rev. **52**, 295 (1937).

$$F(E_a+Q_{ab})=\frac{2M_b}{\hbar^2}\int_0^{E_a+Q_{ab}} E_b\sigma_c(E_b)\rho_Y \times(E_a+Q_{ab}-E_b)dE_b. \quad (7)$$

In Eq. (7) M_b is the reduced mass, $\hbar$ is Planck's constant divided by 2π, E_b the energy of the exit particle b, and $\sigma_c(E_b)$ the cross section for formation of a compound nucleus by particle b on the excited nucleus Y*, and ρ_Y the nuclear level density of the residual nucleus Y* at excitation $(E_a+Q_{ab}-E_b)$.

A knowledge of the F functions completely specifies $G_c(b)$. To determine these integrals, expressions for ρ_Y and for σ_c are required. The σ_c factor is determined as described in Sec. II, namely, Eq. (2a). Input information may be expressed either in terms of the σ_c or in terms of T_l. For the case in which the input is in the form of T_l, one can rewrite Eq. (7) in the form

$$F(E_a+Q_{ab})=\pi\int_0^{E_a+Q_{ab}} T(E_b)\rho_Y(E_a+Q_{ab}-E_b)dE_b \quad (8)$$

by virtue of the second equality in Eq. (5). In Eq. (8), $T(E_b)$ has been substituted for $\sum_l(2l+1)T_l(E_b)$.

Other than the $\sigma_c(E_b)$ factor, the only unknown ingredient in the integral of Eq. (7) is the nuclear level density ρ_Y. Many attempts have been made to calculate the density of nuclear energy levels,[36–40] i.e., the number of levels per unit energy about a given energy within a particular nucleus. To discuss these in detail is beyond the scope of this review.

The level density increases rapidly with excitation energy and also with increasing mass number.[36] The density is quite small in the regions of the magic numbers. Bethe[36] used a statistical approach to obtain a crude method for calculating the nuclear level spacing. By assuming that the nucleus is a Fermi gas of A particles and calculating the "entropy," Bethe obtained for the nuclear level density

$$\rho(U)=\tfrac{1}{12}\sqrt{2}\zeta_0^{\frac{1}{4}}U^{-5/4}\mathrm{A}^{-\frac{1}{4}}\exp[\pi(AU/\zeta_0)^{\frac{1}{2}}], \quad (9a)$$

in which U is the excitation energy and A the mass number. The quantity ζ_0 is a parameter given in terms of the number of neutrons in the nucleus and a constant C by the equation

$$\zeta_0=[N/C]^{\frac{1}{2}}. \quad (9b)$$

The constant C is given by

$$C=[2^{7/9}/9\pi][mR^2/\hbar^2]^{\frac{1}{2}}, \quad (9c)$$

where m is the mass of the neutron, R the nuclear radius, and $\hbar$ Planck's constant divided by 2π.

It is of interest in the theory of neutron radiative-capture reactions to know the density of levels of the residual nucleus with a given angular momentum J. Bethe[36] calculated the probability that a nuclear level has an angular momentum J to obtain

$$\rho(U,J)\cong[\pi^4 2^{\frac{1}{2}}/432][5/\log 2]^{\frac{3}{2}} \times(2J+1)\zeta_0^{-1}X_0^{-4}e^{X_0}, \quad (10a)$$

where

$$X_0=\pi[AU/\zeta_0]^{\frac{1}{2}} \quad (10b)$$

with

$$\zeta_0=[\hbar^2/2m][9\pi A/8R^3]^{\frac{2}{3}}. \quad (10c)$$

Bethe[36] extended his original work in 1937 by obtaining an expression for the level density based on the assumption that the energy U is related to the temperature θ by

$$U=a\theta^n, \quad (11)$$

where n assumes different values for different nuclear models. For the Fermi-gas model $n=2$. For the liquid-drop model, $n=7/3$ at low excitation energies and 4 at high excitations. By using the general power law [Eq. (11)] we have, after Bethe,[36]

$$\rho(U)=(2\pi n)^{-\frac{1}{2}}[a/U^{n+1}]^{+1/2n} \times\exp[n/(n-1)a^{1/n}U^{(n-1)/n}]. \quad (12)$$

For the liquid-drop model U enters in the exponential to the 4/7 power for low excitation and to the $\frac{3}{4}$ power at high excitation. Hence, the Fermi-gas model gives a slower increase of level density than does the liquid-drop model.

The exponential depends on $U^{\frac{1}{2}}$ for the Fermi-gas model. If the coefficient can be considered constant, then Eq. (12) can be written

$$\rho(U)=C\exp[2(aU)^{\frac{1}{2}}], \quad (13)$$

where a and C are constants which depend upon the mass number of the residual nucleus. The constant a can be deduced from the work of Newton[37]:

$$a=0.062(j_p+j_n+1)A^{\frac{2}{3}}, \quad (14)$$

where j_n and j_p are appropriate averages of the total angular momenta of the single-particle states lying near the Fermi level of the gas. These quantities have been tabulated by Cameron.[41] Much effort has been given to the study of the constant a and its variation, in particular, with the nuclear mass number. A summary of the variety of a values which can be deduced from the analysis of various reaction data can be found in the work of Dostrovsky, Rabinowitz, and Bivins.[42] As yet it is not clear exactly what values this parameter should have. Preliminary analyses[43] of (n,p) and (n,α) cross-section data indicate that satisfactory fits to experimental curves for some 15 reactions can be obtained if one uses the a values reported by Blatt and Weisskopf,[9] ignoring the odd-even character of the nucleus. How-

[36] H. A. Bethe, Phys. Rev. **50**, 332 (1936).
[37] T. D. Newton, Can. J. Phys. **34**, 804 (1956).
[38] J. M. B. Lang and K. J. LeCouteur, Proc. Phys. Soc. (London) **A67**, 586 (1954).
[39] C. Block, Phys. Rev. **93**, 1094 (1954).
[40] N. Rosenzweig, Phys. Rev. **108**, 817 (1957).
[41] A. G. W. Cameron, Can. J. Phys. **35**, 666 (1957).
[42] Dostrovsky, Rabinowitz, and Bivins, Phys. Rev. **111**, 1659 (1958); G. Igo and H. E. Wegner, Phys. Rev. **102**, 1364 (1956).
[43] R. E. Bullock and R. G. Moore Jr. (to be published).

ever, it appears[43] that such an assumption for the parameter C does not fit experimental data. Preliminary results[43] indicate that the C values reported by Blatt and Weisskopf[9] are valid for odd-mass nuclei where as those for even-mass nuclei are given by the equation

$$\tfrac{1}{2}C_{\text{odd-odd}}=C_{\text{even-odd}}=C_{\text{odd-even}}=5C_{\text{even-even}}.$$

It may be that satisfactory fits to experimental data could be obtained by including the odd-even character of the nucleus in the exponent by introducing an effective excitation energy U' related to the actual excitation energy U in the following way: $U'=U-\delta$, where δ depends upon the odd-even character of the nucleus.[41]

Subsequent to Bethe's initial work, many attempts have been made to deduce expressions for the nuclear level density. Lang and LeCouteur[38] made a detailed study of the statistical behavior of nuclear levels and have calculated the level density of a Fermi gas. These results agree with those of Bethe[36] at high excitation but give a value somewhat lower than Bethe's at low excitation. Block[39] has developed a theory of nuclear level density which starts with the independent-particle model and then includes the effect of mutual interactions. Block[39] includes symmetry-dependent terms in the nuclear potential energy, in addition to the central potential. Newton[40] and Rosenzweig[40] have included the effects of nuclear shell structure in the level density.

From the foregoing we see that the cross section for a neutron-induced reaction, for example, can be expressed as

$$\sigma(n,x)=\sigma_c(n)(F_x/\textstyle\sum_i F_i), \tag{15}$$

where the symbol i denotes any possible exit particle. In general, i corresponds to n, p, and α. For $i=p$ or α, the F functions are given by Eq. (7). For neutron emission the F integrals assume the form

$$F_n=\frac{2m_n}{\hbar^2}\int_0^{E_{\max}} E_n'\sigma_c(E_n')\rho_Y(E_n-E_n')dE_n'. \tag{16}$$

At low incident energies ($<\sim$4 Mev) both compound elastic scattering and inelastic scattering may take place. Hence, for single neutron emission (i.e., F_n), $E_{\max}$ is equal to the incident neutron energy. If one assumes that compound elastic scattering does not contribute at reasonably high energies, then the neutron F function need not be integrated from zero at E_n at high E_n. If the neutrons are emitted with energy equal to E_n, then they have been (compound) elastically scattered. If compound elastic scattering does not take place, then the maximum energy with which the neutrons can be emitted is E_n-E_0, where E_0 is the energy of the lowest excited state of the residual nucleus. At high energies, however, this change in the upper limit produces negligible effect upon a calculated cross section with the result that it is permissible to integrate from zero to E_n at both low and high incident neutron energies.

For the emission of two neutrons (i.e., F_{2n}), $E_{\max}$ is taken equal to the incident neutron energy less the binding energy of a neutron in the target nucleus. In expressing F_{2n} this way, we are assuming that if the first[44] neutron is emitted with energy between zero and $E_n-(BE)_n$, then at least $(BE)_n$ remains as excitation energy in which case we assume that a second neutron is emitted.

If $F_n(E_n-E_0)$ is used as F_x in Eq. (15), the result is the statistical inelastic scattering cross section. If, however, $F_n[E_n-(BE)_n]$ is used as F_x, then the result is the $(n,2n)$ cross section. Since the residual nucleus for the emission of the first neutron must be highly excited in order to emit a second neutron, the $(n,2n)$ reaction is a high threshold process. If we assume that the $(n,2n)$ reaction is basically a neutron inelastic scattering problem and neglect the emission of charged particles, then the $(n,2n)$ cross section can be greatly simplified.

We consider the total $(n,2n)$ cross section to be the product of the cross section that one neutron is emitted (i.e., the inelastic scattering cross section) times the probability that a second neutron will be emitted:

$$\sigma(n,2n)=\sigma_{nn'}B_n, \tag{17}$$

where B_n denotes this probability that the second neutron will be emitted. To find this probability we use the assumption that if the residual nucleus after emission of the first neutron has excitation energy at least as high as the neutron binding energy, then the second neutron will be emitted. In other words, the probability that the first neutron has energy between zero and $E_n-(BE)_n$ is equivalent to the probability that a second neutron will be emitted. Hence B_n has the form

$$B_n=\frac{\displaystyle\int_0^{E_n-(BE)_n} E_n'\sigma_c(E_n')\rho_Y(E_{\text{ex}})dE_n'}{\displaystyle\int_0^{E_n} E_n'\sigma_c(E_n')\rho_Y(E_{\text{ex}})dE_n'}. \tag{18}$$

In Eq. (18) E_n is the incident neutron energy, $(BE)_n$ the neutron binding energy in the target nucleus, and E_n' the energy of the first emitted neutron. The excitation energy E_{ex} is given by

$$E_{\text{ex}}=E_n-E_n'. \tag{19}$$

If we ignore the variation of $\sigma_c(E_n')$ with E_n' and make certain assumptions about the level density ρ_Y, the integration of Eq. (18) can be carried out to obtain an approximate expression for $\sigma(n,2n)$.

To determine a suitable form for the level density $\rho_Y(E_{\text{ex}})$, we make use of the definition of nuclear

[44] The case in which the two neutrons are emitted simultaneously is very unlikely for incident energies less than 50 Mev.

entropy $S(E)$,

$$S(E)=\ln\rho(E). \tag{19a}$$

Since $E_{ex}=E_n-E_n'$, it follows that

$$\rho_Y(E_{ex})=\rho_Y(E_n-E_n')=\exp[S(E_n-E_n')]. \tag{20}$$

If we expand $S(E_n-E_n')$ in a Taylor series, then

$$S(E_n-E_n')=S(E_n)-E_n'[\partial S/\partial E_{ex}]_{E_{ex}=E_n}. \tag{21}$$

If we define the nuclear temperature $\theta(E)$ as

$$1/\theta(E)=\partial S/\partial E, \tag{22}$$

then

$$S(E_n-E_n')=S(E_n)-E_n'[1/\theta(E_n)]. \tag{23}$$

Upon substitution of Eq. (23) into Eq. (20), one obtains

$$\rho_Y(E_{ex})=\exp\{S(E_n)-E_n'[1/\theta(E_n)]\}. \tag{24}$$

The factor depending upon E_n only can be factored out and included as part of the constant coefficient with the result that

$$\rho_Y(E_{ex})=\text{const}\exp[-E_n'/\theta(E_n)]. \tag{25}$$

By ignoring the energy variation of $\sigma_c(E_n')$ with E_n' and using Eq. (25), we can write the integrands of Eq. (18) as

$$\text{const}\{E_n'\exp[-E_n'/\theta(E_n)]\}. \tag{26}$$

By substituting the integrand in (26) into Eq. (18) and using Eq. (17), one obtains

$$\sigma(n,2n)=\sigma_{nn'}\frac{\int_0^{E_n-(BE)_n}E_n'\exp[-E_n'/\theta(E_n)]dE_n'}{\int_0^{E_n}E_n'\exp[-E_n'/\theta(E_n)]dE_n'}. \tag{27}$$

If E_n is large so that we can assume $\exp[-E_n/\theta(E_n)]$ is negligibly small, then the denominator of Eq. (27) can be approximated by $\theta^2(E_n)$. Integration of the numerator by parts gives

$$\int_0^{E_n-(BE)_n}E_n'\exp[-E_n'/\theta(E_n)]dE_n' =\theta^2(E_n)-\{\theta^2(E_n)+\theta(E_n)[E_n-(BE)_n]\} \times\exp\{[-E_n+(BE)_n]/\theta(E_n)\}. \tag{28}$$

Division of Eq. (28) by $\theta^2(E_n)$ and substitution into Eq. (27) gives

$$\sigma(n,2n)=\sigma_{nn'}\left\{1-\left[1+\frac{\Delta E_n}{\theta(E_n)}\right]\exp\left[\frac{-\Delta E_n}{\theta(E_n)}\right]\right\}, \tag{29}$$

where the symbol ΔE_n has been used for $[E_n-(BE)_n]$. An estimation of $\sigma_{nn'}$ then provides a method for the estimation of the $(n,2n)$ cross section. For high energies the $\sigma_{nn'}$ factor can be approximated by the cross section for the formation of the compound nucleus at the incident neutron energy E_n.* Since the $(n,2n)$ is a high threshold reaction, we can approximate the $(n,2n)$ cross section by

$$\sigma(n,2n)=\sigma_c(n)\left[1-\left(1+\frac{\Delta E_n}{\theta(E_n)}\right)\exp\left(\frac{-\Delta E_n}{\theta(E_n)}\right)\right], \tag{30}$$

in which $\sigma_c(n)$ is the cross section for the formation of the compound nucleus by neutrons of energy E_n, ΔE_n is the incident energy above threshold, and $\theta(E_n)$ is the nuclear temperature of the target nucleus at E_n.

The quantity $\theta(E_n)$ merits further consideration. The fact that we draw these thermodynamic analogies implicitly assumes that the compound nucleus can be pictured as a gas which can be described by thermodynamic methods. We assume that the formation of the compound nucleus leaves this nucleus in a highly excited state which we interpret as the property of heat. This heating of the compound nucleus gives rise to the emission of particles which may be described as "evaporation." The energy distribution of these emitted particles follows the classical Maxwellian distribution determined by the temperature $\theta(E_n)$. If one makes the assumption that the square of the temperature $\theta(E_n)$ is proportional to the incident neutron energy, then it can be shown[9] that the proportionality constant is the reciprocal of the level density parameter a of Eq. (13), i.e.,

$$E=a\theta^2. \tag{31}$$

Equation (31) is admittedly an oversimplification of the true state of affairs. An attempt to determine a more realistic relationship between the energy and the nuclear temperature has been made by several workers. In their studies of nuclear level density, Lang and LeCouteur[38] deduced a relation between the energy and the temperature expressed as a function of the mass number. Their relationship can be written in the form

$$U=(A\theta^2/f)-\theta, \tag{32}$$

where U is the excitation energy, A the mass number, θ the nuclear temperature, and f a quantity with no regular variation with either U or A; f has a mean value of around 8 Mev.

Newton[37] has expressed the nuclear temperature in terms of the excitation energy and also in terms of the number of individual neutron and proton levels per unit energy at the neutron Fermi energy that have a given magnetic quantum number. Written in this form,

* *Note added in proof.*—Essentially three simplifying assumptions are conventionally made in estimating $(n,2n)$ cross sections: (1) charged-particle emission is negligible; (2) the inelastic scattering cross section is replaced by the cross section for compound nucleus formation; and (3) the energy variation of the compound nucleus cross section can be ignored in the high-energy region. The author and his associates recently carried out extensive IBM 704 calculations of $(n,2n)$ cross sections in an effort to examine these assumptions quantitatively. For the results of these analyses, see R. G. Moore, Jr., and R. E. Bullock (to be published).

the equation is somewhat complicated but can be reduced. Following Newton,[37] we can express the temperature by the approximate relationship

$$\theta \cong 3.2\, A^{-\frac{1}{3}} E^{\frac{1}{2}}. \tag{33}$$

Experimental studies for determining nuclear temperatures have been carried out by Gugelot[45] who measured the energy spectra of neutrons emitted during (p,n) reactions initiated by 16-Mev protons and by Ashby, Catron, Newkirk, and Taylor.[46] Their results indicate that for 14-Mev neutrons, nuclear temperatures are of the order of 2 Mev for mass numbers of around 60 and drop to less than 1.5 Mev for elements with mass numbers around 200.

The accuracy of the results predicted on the basis of Eq. (30) depends upon the energy range of the incident neutrons. For the heavier elements ($\sim A \geqq 60$) Eq. (30) predicts a value of the $(n,2n)$ cross section which is expected to be slightly high from threshold to about 18 to 25 Mev. Measured $(n,2n)$ cross-section curves reach a maximum at an incident energy of around 20 Mev. Equation (30), being exponential, never predicts this maximum with the result that the theoretical estimate becomes less accurate with increasing neutron energy.

Summary

Reactions of the form X(a,b)Y which proceed through the formation of a compound nucleus can be described by assuming that the cross section for the (a,b) reaction can be separated into two factors, one of which describes the formation of the compound nucleus; the other describes its decay. The formation of the compound nucleus has been previously described. The "branching ratio" describes the decay of the compound nucleus and is expressed as the ratio of the relative emission probability of particle b divided by the sum of the relative emission probabilities of all possible processes. These relative probabilities are expressed as integrals over the product of compound nucleus cross section times the nuclear level density evaluated at a suitable excitation energy, the most common expression for the level density being a simple exponential with constant coefficient whose exponent varies as the square root of the excitation energy. By use of simplifying assumptions, the $(n,2n)$ cross section has been simplified to an exponential form expressed as a function of the nuclear temperature.

De-excitation by Gamma-Ray Emission

In order to find the cross section for the reaction (a,b) which proceeds by decay into one specific channel, one need only multiply the formation cross section σ_c by the branching ratio G_c of Eq. (3). For the capture process the branching ratio becomes simply

$$G_c(\gamma) = \Gamma_\gamma / \Gamma, \tag{34}$$

where Γ_γ denotes the radiation width and Γ the total width. We divide the discussion of the radiative capture process into three sections: (a) resonance region; (b) thermal region; and (c) continuum region. In order to be specific, we restrict attention to the neutron capture reaction.

The resonance region for neutron cross sections is not a well-defined region of energies, but instead is characterized by the nucleus under consideration. The resonance region is defined as that region of neutron energies for which the excitation function is characterized by the appearance of resonance structure. For the lighter nuclei this structure extends to higher energies than for the heavy nuclei.

The region of energies for which the resonances are strong and well separated is of the order of 1 ev to approximately 10 kev. The light nuclei, however, still show a pronounced resonance structure up into the Mev region. For the moment we restrict attention to the region of energies for which the resonances are "isolated," i.e., for which the level width is much less than the level spacing.

To determine the capture cross section in the resonance region, we simply multiply the resonance σ_c of Eq. (2e), Sec. II, by the $G_c(\gamma)$ of Eq. (34),

$$\sigma(n,\gamma) = (2l+1)\pi\lambda\!\!\!^{-2} \frac{\Gamma_n \Gamma_\gamma}{(E_n - E_r)^2 + (\Gamma/2)^2}. \tag{35}$$

For cases in which the spins of the incident particle and target nucleus are different from zero, Eq. (35) must be reduced by a statistical factor which gives a measure of the relative probability that the incident particle and the target nucleus in an unpolarized beam have a given channel spin. For spin $\frac{1}{2}$ particles this statistical factor is given by

$$g(j) = \tfrac{1}{2}\left(1 \pm \frac{1}{2I+1}\right),$$

where I is the nuclear spin. Then for $l=0$ particles, Eq. (35) becomes

$$\sigma(n,\gamma) = \pi\lambda\!\!\!^{-2} g \frac{\Gamma_n \Gamma_\gamma}{(E_n - E_r)^2 + (\Gamma/2)^2}. \tag{36}$$

Equation (36) is the familiar Breit-Wigner one-level formula for orbital angular momentum zero.[47] It gives the variation of the cross section in the immediate neighborhood of an isolated resonance as a function of incident particle energy, and has been discussed by many authors.[48] Its derivation is presented many places in the literature.[9] It is obtained as a special case of the

[45] P. C. Gugelot, Phys. Rev. **81**, 51 (1951).
[46] Ashby, Catron, Newkirk, and Taylor, Phys. Rev. **111**, 616 (1958).
[47] For the case of particles of arbitrary angular momentum, see reference 9.
[48] See, for example, G. Breit, Phys. Rev. **69**, 472 (1946): references 1, 3, and 9.

formal theory due to Wigner and his collaborators[3] by assuming that only one level contributes appreciably to the expansion of the Wigner-Eisenbud many-level formula for the derivative matrix. The Wigner matrix formulation for the theory of nuclear reactions has been reviewed recently by Lane and Thomas.[4] In Eq. (36) the energies E_n and E_r can be expressed either as incident particle properties or as properties of the compound nucleus. It is customary to consider E_n to be the incident particle energy and E_r the incident particle energy at resonance. It is equivalent to consider E_r as the excitation energy of the level in the compound nucleus. The two representations are equivalent since for the excitation energy representation both E_n and E_r include the binding energy of the particle, but the difference E_n-E_r remains unchanged.

The widths in Eq. (36) are functions of the incident channel energy; the wavelength is also a function of this incident particle energy. The parameter E_r is independent of the incident channel energy only for $l=0$ neutron resonances. It is customary, however, to neglect the energy dependence of the radiation width[41] and the quantity E_r and to express the equation in a slightly different form such that the energy variation of $\lambda\!\!\!^{-}$ and Γ_n is included in the equation which then allows one to use constant values for $\lambda\!\!\!^{-}$ and Γ_n.

In contrast to radiation widths, neutron widths are not relatively constant from level to level. Neutron widths are appreciable only for small values of angular momentum. For $l\geqq 3$, neutron widths are quite small, and at low energies are negligibly small compared to radiation widths. Since[9]

$$\Gamma_{nl}=(4k/K)(v_l/2\pi)D^J, \tag{37}$$

then when the level spacing is small (i.e., heavy nuclei), the neutron widths are small; this observation is supported by experimental evidence.[49] This equation also shows that the neutron width contains a factor proportional to $(E)^{\frac{1}{2}}$ (through k). We can define a "reduced" neutron width $\Gamma_{nl}{}^0$ which is the width evaluated at 1 ev by the relation

$$\Gamma_{nl}{}^0=\Gamma_{nl}/[E_{(\mathrm{ev})}]^{\frac{1}{2}}, \tag{38}$$

which follows from the fact that the neutron width is proportional to the square root of the neutron energy. The neutron width can then be expressed as the product of the reduced neutron width and the square root of the neutron energy (in ev). For $l=0$ neutrons, $v_l=1$ [see Eq. (40), Sec. II], hence Eq. (37) becomes

$$\Gamma_{n0}=(4k/K)(D^J/2\pi). \tag{39}$$

Then Eq. (37) can be written

$$\Gamma_{nl}=\Gamma_{n0}v_l \tag{40a}$$

or

$$\Gamma_{nl}=\Gamma_{n0}{}^0(E)^{\frac{1}{2}}v_l. \tag{40b}$$

[49] D. J. Hughes and J. A. Harvey, Brookhaven Natl. Laboratory Rept. BNL-325 (1955); Suppl. No. 1 (1957).

Since resonance is a low energy phenomenon, Eq. (37) shows that $k\ll K$.

Following Teichmann and Wigner,[50] one can express the neutron width as

$$\Gamma_{nl}=2k_nv_l\gamma_l{}^2, \tag{41}$$

where γ_l is related to the reduced neutron width by

$$\gamma_lR^{-\frac{1}{2}}=(\Gamma_{nl}{}^0)^{\frac{1}{2}}. \tag{42}$$

Substitution of Eq. (42) into Eq. (41) yields

$$\Gamma_{nl}=2kRv_l\Gamma_{nl}{}^0. \tag{43}$$

Combination of Eq. (43) with Eq. (37) gives a result for the reduced neutron width in terms of the level spacing D^J:

$$\Gamma_{nl}{}^0=D^J/\pi KR. \tag{44}$$

Since the neutron widths are functions of the neutron energy and represent a measure of the probability of decay of the compound nucleus by neutron emission, it should be possible to express the neutron width in terms of the transmission coefficient T_l. This is indeed the case.

If τ_α denotes the lifetime of the state α, then τ_α can be expressed as the ratio of the period P of motion of the nucleons of the compound nucleus to the transmission coefficient T_l, which represents the fractional number of successful attempts to escape from the nucleus. Thus

$$\tau_\alpha=P/T_l. \tag{45}$$

But the lifetime of the level is related to the width by

$$\tau_\alpha=\hbar/\Gamma_\alpha, \tag{46}$$

thus

$$\Gamma_\alpha=\hbar T_l/P. \tag{47}$$

But $P=2\pi\hbar/D$ with the result that

$$\Gamma_\alpha=T_l[D/2\pi], \tag{48}$$

which expresses the relationship between the width and the transmission coefficients which are energy dependent quantities. For well-defined resonance levels, we require $\Gamma\ll D$ with the consequence that the transmission coefficient must be small for the resonance region.

We now return to the problem of reducing Eq. (36) to a form in which Γ_n and $\lambda\!\!\!^{-}$ are constants (in particular, their values at resonance). To rewrite Eq. (36) in this more convenient form, we make use of the fact that the neutron wavelength is inversely proportional to the square root of the energy to write

$$\lambda\!\!\!^{-}=[E_r/E_n]^{\frac{1}{2}}\lambda\!\!\!^{-}_r, \tag{49}$$

where $\lambda\!\!\!^{-}_r$ is the neutron wavelength evaluated at the resonance energy E_r. Then Eq. (36) becomes

$$\sigma(n,\gamma)=g\pi\lambda\!\!\!^{-}_r{}^2\left[\frac{E_r}{E_n}\right]\frac{\Gamma_n\Gamma_\gamma}{(E_n-E_r)^2+(\Gamma/2)^2}. \tag{50}$$

[50] T. Teichmann and E. P. Wigner, Phys. Rev. **87**, 123 (1952).

By virtue of the fact that the neutron width is proportional to the square root of the neutron energy, Γ_n can be written

$$\Gamma_n=[E_n/E_r]^{\frac{1}{2}}\Gamma_{nr}, \tag{51}$$

where Γ_{nr} is the neutron width at resonance. Substitution of Eq. (51) into Eq. (50) yields

$$\sigma(n,\gamma)=g\pi\lambda\!\!\!^{-}_r{}^2\left[\frac{E_r}{E_n}\right]^{\frac{1}{2}}\frac{\Gamma_{nr}\Gamma_\gamma}{(E_n-E_r)^2+(\Gamma/2)^2}. \tag{52}$$

In Eq. (52) all of the parameters are evaluated at resonance and hence are constants; therefore, the only way the incident neutron energy enters into Eq. (52) is through the quantity E_n.

It is also possible to express Eq. (36) in an alternate form which is given in terms of the reduced neutron width. Substitution of Eq. (38) into Eq. (36) yields

$$\sigma(n,\gamma)=g\pi\lambda\!\!\!^{-}{}^2(E_n)^{\frac{1}{2}}\frac{\Gamma_n{}^0\Gamma_\gamma}{(E_n-E_r)^2+(\Gamma/2)^2}. \tag{53}$$

This can be reduced further by combining the $(E_n)^{\frac{1}{2}}$ factor with the neutron wavelength. The result is

$$\sigma(n,\gamma)=\left[\frac{g\pi\hbar}{(2m)^{\frac{1}{2}}}\right]\lambda\!\!\!^{-}\frac{\Gamma_n{}^0\Gamma_\gamma}{(E_n-E_r)^2+(\Gamma/2)^2}. \tag{54}$$

One can also express the (n,γ) cross section in terms of the reduced width by use of Eq. (43) rather than Eq. (38) for the neutron width. This reduces Eq. (36) to

$$\sigma(n,\gamma)=2\pi\lambda\!\!\!^{-}Rg\frac{\Gamma_n{}^0\Gamma_\gamma}{(E_n-E_r)^2+(\Gamma/2)^2} \tag{55}$$

for an $l=0$ resonance. For arbitrary l the v_l factor is not unity, and the widths must be evaluated for the given value of l. In the very low-energy resonance range (of the order of a few ev) the total width is essentially equal to the radiation width, and then Eq. (35) reduces to

$$\sigma(n,\gamma)=4\pi\lambda\!\!\!^{-}{}^2[\Gamma_n/\Gamma_\gamma] \tag{56}$$

at the resonance energy $E_n=E_r$. From this relationship one can estimate the ratio of neutron width to radiation width from the maximum cross section.

It is possible to extend these resonance results to the thermal region to deduce the energy variation of the thermal-capture cross section. By thermal neutron energies we mean that region for which the incident neutrons have energies of the same order of magnitude as the average kinetic energy of the atoms or molecules of the surrounding medium. Since the average kinetic energy of these atoms is determined by the temperature of the medium, the range of neutron energies corresponding to equilibrium with these surrounding molecules is referred to as "thermal" energies. These energies are those below a few tenths of an electron volt.

The cross section for the radiative capture process in this energy range can be described by a rather simple relationship. Application of the Breit-Wigner equation in the thermal region can be expected to be only qualitatively reliable since one must assume that the resonance energy is the resonance energy nearest to the thermal region. In this case the factor $(E_n-E_r)^2$ is large and essentially constant. In fact, this term is so large as to make the $(\Gamma/2)^2$ term negligible, hence, Eq. (35) becomes

$$\sigma(n,\gamma)\cong\pi\lambda\!\!\!^{-}{}^2(\Gamma_n\Gamma_\gamma/\text{const}). \tag{57}$$

For extremely low energies the radiation width varies quite slowly with energy, and it also can be assumed constant:

$$\sigma(n,\gamma)=\text{const}\lambda\!\!\!^{-}{}^2\Gamma_n. \tag{58}$$

We recall that Γ_n is proportional to the neutron velocity whereas $\lambda\!\!\!^{-}$ shows a $1/v$ (v being the incident neutron velocity) dependence with the result that

$$\sigma(n,\gamma)=\text{const}[1/v], \tag{59}$$

which is the well-known "$1/v$ law" for the neutron radiative-capture cross section in the thermal region. At room temperature, the thermal neutron energy region is in the neighborhood of 0.0253 electron volts. Cross sections for many elements at an incident neutron energy of 0.0253 electron volts have been tabulated[51] and can be used to determine the proportionality constant of Eq. (59),

$$\sigma(n,\gamma)=(2200/v)\sigma(n,\gamma)|_{0.0253}, \tag{60}$$

in which v is in meters per second. A velocity of 2200 m/sec corresponds to an energy of 0.0253 ev. The derivation of Eq. (59) assumes that the radiation width is essentially constant. If one does not make this assumption, he may express the thermal radiative-capture cross section in terms of the radiation width and the level spacing D. In the denominator of Eq. (35), the term (E_n-E_r) is of the order of the level spacing but is not larger than $D/2$. We define the quantity δ_{th} by the equation

$$|E_n-E_r|\equiv D/2\delta_{\text{th}}, \tag{61}$$

where E_n is now a thermal neutron energy. By substituting Eqs. (44) and (61) into Eq. (55) and taking $g=\frac{1}{2}$, one obtains

$$\sigma(n,\gamma)=4\delta_{\text{th}}{}^2(\lambda\!\!\!^{-}_n/K)(\Gamma_\gamma/D), \tag{62}$$

since $(E_n-E_r)^2\gg(\Gamma/2)^2$ in the thermal region. Equation (62) is an expression for the thermal neutron radiative-capture cross section in which one does not assume the radiation width to be energy independent.

As we have seen, in the low-energy region neutron cross sections exhibit widely separated resonances which become more closely spaced with increasing

[51] D. J. Hughes, *Pile Neutron Physics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1953).

energy until they finally form a smoothly varying function of energy. The method described by Margolis[52] provides a method for estimating neutron radiative-capture cross sections which represent an average over resonances in the low-energy region and which provide the actual cross sections in the smoothly varying region. The branching ratio for gamma-ray emission is, according to Eq. (34), the ratio of radiation width to total width evaluated at a suitable energy. The cross section that the compound nucleus state J will decay by emission of a gamma ray is given by the product of the σ_c of Eq. (2b) of Sec. II and the appropriate $G_c(\gamma)$:

$$\sigma(n,\gamma)=\pi\lambda\!\!\!^{-2}(2l+1)T_l(E) \times|(ljm_lm_j|Jm)|^2(\Gamma_\gamma{}^J/\Gamma^J), \quad (63)$$

where $(ljm_lm_j|Jm)$ is the Clebsch-Gordan coefficient.[10] If the z axis is taken along the direction of the incident particle, then m_l is zero and $m=m_j$. Then the Clebsch-Gordan coefficient can be reduced to yield

$$\sigma(n,\gamma)=\pi\lambda\!\!\!^{-2}(2l+1)T_l(E) \times|(lj0m|Jm)|^2(\Gamma_\gamma{}^J/\Gamma^J). \quad (64)$$

The radiative-capture cross section for unpolarized neutrons of any value of l is obtained by summing over J and l and averaging over j and m. All terms in the j summation are not allowed. Certain terms are eliminated by selection rules. This can be included in the summation over j by use of the symbol $\epsilon_{jl}{}^J$ defined by

$$\begin{aligned}&\epsilon_{jl}{}^J=2; \quad |J-l|\leqq j\leqq|J+l| \quad \text{for } j=j^\pm\\ &\quad=1; \quad |J-l|\leqq j\leqq|J+l| \quad \text{for } j=\text{only one of } j^\pm\\ &\quad=0; \quad \text{otherwise.}\end{aligned}$$

Then Eq. (64) becomes

$$\sigma(n,\gamma)=\frac{1}{2j+1}\sum_m\sum_l\sum_J(2l+1)\pi\lambda\!\!\!^{-2}T_l(E) \times|(lj0m|Jm)|^2\epsilon_{jl}{}^J(\Gamma_\gamma{}^J/\Gamma^J). \quad (65)$$

Since[53]

$$\sum_m|(lj0m|Jm)|^2=\frac{2J+1}{2l+1}, \quad (66)$$

then

$$\sigma(n,\gamma)=\frac{1}{2j+1}\sum_{l=0}^{\infty}\sum_{J=0}^{\infty}(2J+1)\pi\lambda\!\!\!^{-2}T_l(E)\frac{\epsilon_{jl}{}^J\Gamma_\gamma{}^J}{\Gamma^J}. \quad (67)$$

The $T_l(E)$ are not functions of J and hence can be removed from the J summation. Replacing j by $i\pm\frac{1}{2}$, one obtains

$$\sigma(n,\gamma)=\frac{\pi\lambda\!\!\!^{-2}}{2(2i+1)}\sum_{l=0}^{\infty}T_l(E)\sum_{J=0}^{\infty}\frac{\epsilon_{jl}{}^J(2J+1)\Gamma_\gamma{}^J}{\Gamma^J}. \quad (68)$$

If we neglect the emission of charge particles (because of the Coulomb barrier), then the total width can be expressed as the sum of the neutron width and the radiation width. Substitution of this sum for the total width with subsequent division of both numerator and denominator by the neutron width yields

$$\sigma(n,\gamma)=\frac{\pi\lambda\!\!\!^{-2}}{2(2i+1)}\sum_{l=0}^{\infty}T_l(E) \times\sum_{J=0}^{\infty}\frac{\epsilon_{jl}{}^J(2J+1)}{1+[\Gamma_n{}^J(E)/\Gamma_\gamma{}^J(B+E)]}, \quad (69)$$

where the functional dependence of the widths is now indicated. Margolis[52] has shown that the ratio $\Gamma_\gamma{}^J(B+E)/\Gamma_\gamma{}^J(B)$ can be expressed in terms of the nuclear level spacing D and the level density ρ by the relationship

$$\frac{\Gamma_\gamma{}^J(B+E)}{\Gamma_\gamma{}^J(B)}\cong\frac{1}{f_{\Delta I}(E)}\cdot\frac{D^J(B+E)}{D^J(B)}, \quad (70)$$

where the function $f_{\Delta I}(E)$ is given by

$$f_{\Delta I}(E)=\frac{\int_0^B\epsilon^{2\Delta I+1}\rho(B-\epsilon)d\epsilon}{\int_0^{B+E}\epsilon^{2\Delta I+1}\rho(B+E-\epsilon)d\epsilon}, \quad (71)$$

where B is the neutron binding energy, $D^J(E)$ is the spacing of levels of the compound nucleus of spin J and given parity at excitation E, $\rho(E)$ is the nuclear level density at excitation E, and $2\Delta I$ is the multipole type of the radiation in the gamma decay of the compound nucleus. Since [Eq. (48)]

$$\Gamma_n(E)=[T_l(E)D^J(B+E)]/2\pi, \quad (72)$$

Eq. (70) becomes

$$\frac{\Gamma_n{}^J(E)}{\Gamma_\gamma{}^J(B+E)}\cong\frac{D^J(B)}{2\pi\Gamma_\gamma{}^J(B)}T_l'(E)f_{\Delta I}(E). \quad (73)$$

Substitution of Eq. (73) into Eq. (69) and summation over all exit neutrons yields

$$\sigma(n,\gamma)=\frac{\pi\lambda\!\!\!^{-2}}{2(2i+1)}\sum_{l=0}^{\infty}\left[T_l(E)\sum_{J=0}^{\infty}\frac{\epsilon_{jl}{}^J(2J+1)}{1+\xi_Jf_{\Delta I}(E)\sum_{l'}\sum_n\epsilon_{jnl'}{}^JT_{l'}(E-E_n)}\right]. \quad (74)$$

[52] B. Margolis, Phys. Rev. **88**, 327 (1952).
[53] W. Hauser and H. Feshbach, Phys. Rev. **87**, 366 (1958).

E_n is the energy of the nth excited state, and ξ_J is defined by

$$\xi_J = D^J(B)/2\pi\Gamma_\gamma{}^J(B). \tag{75}$$

The quantity l' denotes the angular momentum of the emitted neutrons. The sum over l' includes only those terms for which the parity of the system is conserved. If the parity of the ground state and that of the nth excited state are opposite, then only odd l' values appear, wheras if these parities are the same, only even l' values appear.

Equation (74) is the result for the neutron-capture cross section for intermediate and heavy nuclei for energies in the range 1 kev to several Mev. Several approximations[54] to this equation can be made resulting in more simplified expressions for the capture cross section. If one assumes that $f_{\Delta I}$ is unity (which is equivalent to saying that $B+E \cong B$), then Eq. (74) becomes

$$\sigma(n,\gamma) = \frac{\pi\lambda^2}{2(2i+1)} \sum_{l=0}^{\infty} \left[T_l(E) \sum_{J=0}^{\infty} \frac{[2\pi\Gamma_\gamma{}^J(B)/D^J(B)]\epsilon_{jl}{}^J(2J+1)}{[2\pi\Gamma_\gamma{}^J(B)/D^J(B)] + \sum_{l'} \sum_n \epsilon_{jnl'}{}^J T_{l'}(E-E_n)} \right] \tag{76}$$

after replacing ξ_J by its equivalent in terms of the width and spacing. If no inelastic scattering is present (i.e., $E_n=0$), then Eq. (76) reduces to

$$\sigma(n,\gamma) = \frac{\pi\lambda^2}{2(2i+1)} \left[\frac{2\pi\Gamma_\gamma(B)}{D(B)} \right] \sum_{l=0}^{\infty} \left[T_l(E) \sum_{J=0}^{\infty} \frac{\epsilon_{jl}{}^J(2J+1)}{[2\pi\Gamma_\gamma(B)/D(B)] + \sum_{l'=0}^{\infty} \epsilon_{jl'}{}^J T_{l'}(E)} \right], \tag{77}$$

in which we have now assumed that both D and Γ_γ are independent of J, and hence can be removed from the summation over J. For target nuclei with $i=0$, since J and parity must be conserved, $l=l'$, and all the $\epsilon_{jl}{}^J$ are either zero or one but not two, thus Eq. (77) further reduces to

$$\sigma(n,\gamma) = \frac{\pi\lambda^2}{2} \left[\frac{2\pi\Gamma_\gamma(B)}{D(B)} \right] \sum_{l=0}^{\infty} \left[T_l(E) \sum_{J=0}^{\infty} \frac{(2J+1)}{[2\pi\Gamma_\gamma(B)/D(B)] + T_l(E)} \right]. \tag{78}$$

The summation over J extends over the values

$$l+j, \quad l+j-1, \cdots, \quad |l-j|, \tag{79}$$

and for i equal zero, this becomes

$$J_1 = l+\tfrac{1}{2}, \quad J_2 = l-\tfrac{1}{2}, \tag{80}$$

thus

$$\sum_J (2J+1) = (2J_1+1) + (2J_2+1) = 2(2l+1). \tag{81}$$

With this result, Eq (78) becomes

$$\sigma(n,\gamma) = \pi\lambda^2 \left[\frac{2\pi\Gamma_\gamma(B)}{D(B)} \right] \sum_{l=0}^{\infty} \frac{(2l+1)T_l(E)}{[2\pi\Gamma_\gamma(B)/D(B)] + T_l(E)}. \tag{82}$$

[54] L. Dresner, Nuclear Sci. and Eng. **1**, 103 (1956).

It may be worthwhile to recapitulate the assumptions involved in the derivation of Eq. (82):

(1) The compound nucleus is formed, and no contribution from a direct process is present.
(2) The emission of charged particles is negligible.
(3) The incident energy is small compared to the neutron binding energy.
(4) No inelastic scattering is present.
(5) The level spacing and the radiation width are both independent of J.
(6) The target nucleus has spin zero.

Summary

The branching ratio for gamma-ray emission is given by the ratio of radiation width to total width. The radiative-capture cross section in the resonance region is obtained by multiplying this branching ratio by the resonance compound-nucleus cross section. The result is the well-known Breit-Wigner equation, which is used to demonstrate that the capture cross section in the thermal region is inversely proportional to the neutron velocity. With increasing energy the resonances gradually become wider until they form a continuum. The capture cross section in this region has been obtained in a general form and reduced by means of simplifying assumptions.