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Magnetohydrodynamic-Hypersonic Flow in the Quasi-Newtonian Approximation*

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I. INTRODUCTION

THE hypersonic flow past a body of revolution at zero angle of attack is considered, in the case where a magnetic field is present. The gas ahead of the shock front (Fig. 1) is cold, and therefore nonconducting. Inside the shock layer, thermodynamic equilibrium is assumed, and the gas is at least partially ionized. A magnetic field is applied, which is rotationally symmetric with respect to the axis of symmetry of the body. Only the inviscid flow outside the boundary layer is considered.

Previous analyses of the inviscid flow were concerned with the local behavior near the axis of symmetry, and made use of expansions valid in this neighborhood.[1,2] However, in some applications, a knowledge of the magnetohydrodynamic flow in the large is required. This is necessary, for instance, for the determination of the magnetic drag of the body.

Here the flow is considered by means of the Newtonian approximation (or rather the "Newtonian-plus-centrifugal"[3] approximation) familiar from hypersonic aerodynamic theory. The theory is developed in detail for the steady flow about a body of revolution. No applications are given.

II. BASIC EQUATIONS

In the case of steady flow of an inviscid gas, for which radiation and heat conduction can be neglected, the equations of magnetohydrodynamics are

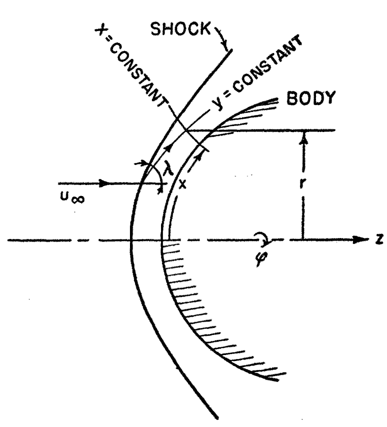

FIG. 1.

$$\nabla\times\mathbf{H}=\mathbf{j} \tag{1}$$

$$\nabla\times\mathbf{E}=0 \tag{2}$$

$$\nabla\cdot\mathbf{B}=0 \tag{3}$$

$$\nabla\cdot(\rho\mathbf{u})=0 \tag{4}$$

$$\rho D\mathbf{u}/Dt=-\nabla p+\mathbf{j}\times\mathbf{B} \tag{5}$$

$$\rho(D/Dt)(h+\tfrac{1}{2}u^2)=\mathbf{j}\cdot\mathbf{E}. \tag{6}$$

They are supplemented by constitutive equations which usually assume the form

$$\mathbf{B}=\mu\mathbf{H} \tag{7}$$

$$\mathbf{j}=\sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \tag{8}$$

together with the caloric equation of state $h=h(\rho,p)$ and an expression for the conductivity $\sigma=\sigma(h,\rho)$. In these equations, the electromagnetic quantities **B**, **H**, **E**, and μ have their usual meaning, **j** is the current density, σ the (scalar) conductivity, **u** the velocity, ρ the density, p the pressure, and h the enthalpy. The electric field strength **E** in these equations is the one observed in the stationary frame.

A recent discussion of the energy equation in magnetohydrodynamics [Eq. (6)] has been given by Chu.[4] In formulating it, the Joule heating j^2/σ must be taken into account. In many problems which are time independent and which exhibit certain symmetries, **E** can be shown to vanish identically. In particular, this is the case for rotationally symmetric problems, such as the magnetohydrodynamic flow about a symmetric body at zero angle of attack, provided that no electric-potential difference is impressed on the boundaries. In these cases, it follows from Eq. (6) that the total enthalpy is conserved along streamlines, just as in the ordinary gasdynamics of an inviscid fluid. The Joule heating is then just equal to the work done against the magnetic field.

The permeability μ can be replaced in all cases of interest here by the permeability of vacuum. The conductivity generally depends very strongly on the temperature, and—to a lesser degree—on the density. The strong temperature dependence is particularly pronounced in cases in which the gas is only weakly ionized.[5] If the degree of ionization is of the order of 1%

* Work supported by the Air Force Ballistic Missile Division Headquarters, Air Research and Development Command.

[1] W. B. Bush, J. Aero/Space Sci. 25, 685 (1958).
[2] N. H. Kemp, J. Aeronaut. Sci. 25, 405 (1958); cf. also N. C. Freeman, J. Aero/Space Sci. 26, 670 (1959).
[3] M. D. Van Dyke, "Study of hypersonic small-disturbance theory," NACA TN 3173 (1954).
[4] B. T. Chu, Wright Air Development Center, TN 57-350 (1957).
[5] L. Lamb and S. C. Lin, J. Appl. Phys. 28, 754 (1957).

or higher, the conductivity very nearly satisfies Spitzer's equation[6] derived for a fully ionized gas and increases approximately as the $\frac{3}{2}$ power of the temperature. A more complicated temperature dependence must be expected, however, in the case of an incomplete ionization of a multiply ionizable gas.

The simple form of Ohm's law represented by Eq. (8) is valid only in the limit of weak fields. In particular, if the magnetic field strength is such that the Larmor frequency of the electrons becomes comparable with or larger than their collision frequency, this equation does not apply. A more generally valid expression has been derived by Schlüter.[7] For the case of weakly ionized gases, Kemp and Petschek[8] indicate that two new effects arise, namely, a Hall current and an "ion slip."

The set of equations given in the foregoing is mathematically complete. It is not necessary to include here the expression for the net charge density as the divergence of the dielectric displacement, because the net charge can be shown to have a negligible effect in situations described by one-fluid magnetohydrodynamics.[9]

In the equations governing the motion of the gas inside the shock layer, the derivatives with respect to the azimuth φ vanish by reason of symmetry. Furthermore,

$$j_r = j_z = 0, \tag{9a}$$

$$B_\varphi = 0, \tag{9b}$$

$$u_\varphi = 0, \tag{9c}$$

$$E_r = E_z = E_\varphi = 0, \tag{9d}$$

where the subscripts indicate the components in cylindrical coordinates. The postulate of symmetry expressed by these equations is justified by showing that they satisfy Eqs. (1)-(8) and the boundary conditions of the problem. Equation (9a) means that the current flows in circular loops about the z axis. Equation (9b) follows from it by an application of Stokes' theorem to Eqs. (1) and (7). Furthermore, it follows from Eq. (9a) that the ponderomotive force $\mathbf{j}\times\mathbf{B}$ has no φ component and that therefore the gas, which ahead of the shock has no azimuthal velocity component, never acquires one. Finally, since $\mathbf{j}$ and $\mathbf{u}\times\mathbf{B}$ are purely azimuthal, it follows that, consistent with Eq. (8), $\mathbf{E}$ can have at most an azimuthal component. But even this component must vanish, as follows from applying Eq. (2).

It is convenient here to introduce a slight change in nomenclature and to put $\mathbf{B}B_0$ for the magnetic flux density, $\mathbf{j}\rho_\infty u_\infty{}^2/B_0R_0$ for the current density, $\mathbf{u}u_\infty$ for the velocity, $\rho\rho_0$ for the density, $\frac{1}{2}p\rho_\infty u_\infty{}^2$ for the pressure, $\frac{1}{2}hu_\infty{}^2$ for the enthalpy, and $\sigma\sigma_0$ for the conductivity, where $\mathbf{B}$, $\mathbf{j}$, $\mathbf{u}$, ρ, p, h, and σ are dimensionless ratios. ρ_0 and σ_0 are the density and conductivity at some fixed point inside the shock layer. Similarly, B_0 and R_0 are a characteristic flux density and a characteristic length of the problem. The free-stream velocity is designated by u_∞, and the free-stream density by ρ_∞. Similarly, the coordinates are made dimensionless by dividing through by R_0.

As preparation for the limiting process implied in the Newtonian approximation, curvilinear coordinates x and y are introduced. The quantity y is taken proportional to the Stokes stream function and is defined such that the mass-flow rate through the inside of a circle concentric with the z axis and passing through the point (r,z) equals $2\pi y\rho_\infty u_\infty R_0{}^2$. Ahead of the shock, y is constant on the surface of circular cylinders of radius $(2y)^{\frac{1}{2}}$. The quantity x is taken constant on orthogonals to the surfaces of constant y. The orthogonals are parameterized by means of the (dimensionless) arc length measured along the body (Fig. 1).

Lamé's scale factors k_x, k_y, k_φ are introduced for the curvilinear coordinate system, defined in such a manner that $k_x dx$ is the element of (dimensionless) arc length perpendicular to the surfaces of constant x, and similarly for k_y and k_φ. Clearly,

$$k_\varphi = r$$

and, from continuity [Eq. (4)],

$$k_y = (\rho_\infty/\rho_0)(r\rho u)^{-1}.$$

An expression for k_x is not required.

From Eqs. (1) and (7) follows

$$\frac{1}{k_x k_y}\left\{\frac{\partial}{\partial x}(k_y B_y) - \frac{\partial}{\partial y}(k_x B_x)\right\} = j_\varphi \frac{\rho_\infty u_\infty{}^2 \mu}{B_0{}^2},$$

where B_x and B_y are the components of $\mathbf{B}$ in the direction of the streamlines and perpendicular thereto. It is convenient to introduce

$$i = j_\varphi (r\rho u)^{-1}(\rho_\infty/\rho_0)$$

which is, except for a constant factor, the current flowing through a rectangular area, with one side being of unit length along the streamline and the other side corresponding to a unit increment of mass-flow rate. We also define the parameter

$$Q = \sigma_0 B_0{}^2 R_0/\rho_0 u_\infty$$

which is a measure for the ratio of ponderomotive to inertial forces inside the shock layer,[10] and the magnetic Reynolds number

$$R_m = u_\infty R_0 \sigma_0 \mu.$$

Consequently,

$$(\partial/\partial x)(k_y B_y) - (\partial/\partial y)(k_x B_x) = ik_x[(\rho_\infty/\rho_0)(R_m/Q)]. \tag{10}$$

[6] L. Spitzer and R. Härm, Phys. Rev. 89, 977 (1953).
[7] A. Schlüter, Z. Naturforschg. 5a, 72 (1950); 6a 73 (1951).
[8] N. H. Kemp and H. E. Petschek, J. Fluid Mech. 4, 553 (1958).
[9] W. M. Elsasser, Revs. Modern Phys. 28, 135 (1956).

[10] Since the velocity in the shock layer behind an oblique shock in general is of the order of u_∞.

Equation (3), if expressed in the curvilinear coordinates, yields

$$(\partial/\partial x)(rk_y B_x)+(\partial/\partial y)(rk_x B_y)=0. \quad (11)$$

From Eq. (5) follows

$$\rho u(\partial u/\partial x)+\tfrac{1}{2}[(\partial p/\partial x)(\rho_\infty/\rho_0)]+k_x i B_y \rho u r=0 \quad (12)$$

for the direction of increasing x, and

$$(u/R)-\tfrac{1}{2}[(\partial p/\partial y)r]+iB_x r=0 \quad (13)$$

for the direction of increasing y, where $R=R(x,y)$ is the dimensionless radius of curvature of the streamlines. The sign of R is chosen such that R taken on the surface of the body is positive if the body is convex.

For finite conductivity, the Rankine-Hugoniot equations of conservation of mass, momentum, and energy across the shock are not affected by the presence of the magnetic field. In particular, the total enthalpy is conserved, and, therefore, from Eq. (6), with $E=0$, there is for any point in the region between the shock and the body,

$$h+u^2=+1, \quad (14)$$

where use has been made of the hypersonic approximation, namely, that the enthalpy of the free stream is negligible compared with its kinetic energy.

Finally, Eq. (8) needs to be considered, resulting in the relation

$$i\rho r=Q\sigma B_y. \quad (15)$$

III. QUASI-NEWTONIAN APPROXIMATION

This approximation consists in assuming that the density at any point inside the shock layer is very large compared with the free-stream density.[3,11,12] In the case of air, and assuming that the component of the free-stream velocity perpendicular to the shock is hypersonic, this ratio actually is roughly between ten and eighteen for points directly behind the shock. The Newtonian approximation can be regarded, for instance, as the limit in which the dissociation energy per molecule is infinite compared with the product of the temperature and Boltzmann's constant,[13] but of the same order as the kinetic energy of the free stream.

As a consequence, the shock-layer thickness tends to zero and $R(x,y)\to R(x)$, which is the radius of curvature of the body.[14] With $\rho_\infty/\rho_0\to 0$, $k_x\to 1$, $r(x,y)\to r(x)$, it follows from Eq. (10) that B_x is independent of y, $B_x=B_x(x)$. Similarly, from Eq. (11), $B_y=B_y(x)$. The magnetic field is seen to be continuous across the shock layer in this approximation. Upon elimination of the current density, Eqs. (12), (13), and (15) result in

$$\partial u/\partial x+QB_y^2(x)(\sigma/\rho)=0, \quad (16)$$

and

$$[u/R(x)]-\{[r(x)/2](\partial p/\partial y)\}+QB_x(x)B_y(x)(\sigma/\rho)=0. \quad (17)$$

Together with Eq. (14), which is unchanged, and the caloric equation of state, the last two equations form a system of equations sufficient for the determination of u, p, and ρ. B_x and B_y are determined by the currents external to the flow field.

The equations apply in the region between the body ($y=0$) and the shock $[y=r^2(x)/2]$. There are two boundary conditions, namely,

$$u=\cos\lambda \quad \text{at} \quad y=r^2(x)/2,$$

from conservation of the tangential component of the momentum across the shock, and

$$p=2\sin^2\lambda \quad \text{at} \quad y=r^2(x)/2,$$

from the hypersonic approximation to the conservation of the normal component of momentum. The angle included between the z axis and the shock (which in the Newtonian approximation is replaced by the angle between z axis and body) is designated by $\lambda=\lambda(x)$.

The application of these results is particularly simple in the case of a cone at zero angle of attack. If the magnetic field on the surface is given by

$$B_x=x^{-\frac{1}{2}}\cos\beta;\quad B_y=x^{-\frac{1}{2}}\sin\beta \quad (18)$$

(where β is a given constant characterizing the direction of the magnetic field on the surface of the cone), a similarity solution exists, since the expressions

$$u=u\left(\frac{2y}{x^2}\csc^2\lambda\right);\quad p=p\left(\frac{2y}{x^2}\csc^2\lambda\right);\quad \rho=\rho\left(\frac{2y}{x^2}\csc^2\lambda\right) \quad (19)$$

satisfy all equations identically. This is the case even if the conductivity and enthalpy are arbitrary functions of the pressure and density.

[11] M. J. Lighthill, J. Fluid Mech. 2, 1 (1957).

[12] W. D. Hayes and R. F. Probstein, *Hypersonic Flow Theory* (Academic Press, Inc., New York, 1959).

[13] The latter ratio is large because of the much greater number of translational states of the dissociated gas, compared with the molecular species.

[14] In the experiments reported by R. W. Ziemer and W. B. Bush [Phys. Rev. Letters **1**, 58 (1958)], Q is very large, and the theory cannot be expected to apply.

DISCUSSION

Session Reporter: N. H. Kemp

A. R. Kantrowitz, *Avco-Everett Research Laboratory, Everett, Massachusetts:* I might just point out that the last problem—the conical problem in question—was done by one of Sedov's students in 1957. This has been published.

R. X. Meyer: In the present case the conductivity is finite and can even be temperature dependent. As I remember, the paper you are referring to assumes a constant conductivity.

H. E. Petschek, *Avco-Everett Research Laboratory, Everett, Massachusetts:* That paper assumes an infinite conductivity.

R. X. Meyer: Yes, infinite. It is interesting, however, that the present solution has conical symmetry even in the case of variable conductivity.

P. S. Lykoudis, *Purdue University, Lafayette, Indiana:* In your solution did you find that even with the existence of the magnetic field you obtained conical flows?

R. X. Meyer: Yes.

P. S. Lykoudis: Have you solved the problem of how the shock wave is distorted?

R. X. Meyer: The shock wave remains conical. If the magnetic field has the special distribution stated in the paper, there does not appear a characteristic length in the problem. While I had not said it explicitly, the derivation assumes that the magnetic Reynolds number referred to a typical body length is of order one. This is the actual situation we have in our experiments. But if one refers the magnetic Reynolds number to the shock layer thickness, then in the Newtonian limit, this magnetic Reynolds number becomes very small. Thus, it essentially drops out of the problem and this is the reason why there are no characteristic lengths left even in the case of finite conductivity.

P. S. Lykoudis: One of my students is considering the same problem but without preservation of the conical properties. The approach is the following: First of all, it seems from previous work that the pressure distribution does not change appreciably. We may assume a Newtonian pressure distribution. If you do so, then the differential equation of motion which you can write along a streamline shows you that the shock wave curves away. Unfortunately, we have not extensive experimental data to see to what degree this is true. Nevertheless, from the limited information that was given in an article by Dr. Kantrowitz,[a] we see that this is the case since you would expect the flow to decelerate more and more and you need more area for the fluid to go through.

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[a] A. R. Kantrowitz, Astronautics 3, 18 (1958).