Foundations of the Optical Model for Nuclei and Direct Interaction

G. E. BROWN

Department of Mathematical Physics, University of Birmingham, Brimingham, England

I. INTRODUCTION

THIS paper gives a description of the scattering of particles of energies in the kev and Mev range by complex nuclei. Starting from exact many-level formulas for the scattering amplitude, we show that several phenomena which were inexplicable in the old compound-nucleus theory arise from phase relations between the nuclear levels, whereas terms in which phase relations would not seem to be important correspond to the scattering predicted by the old compound-nucleus picture, supplemented by statistical assumptions.

This paper might be described as a treatment of the theory of nuclear reactions in the region in which many nuclear levels participate in the scattering. Thus, it is complementary to the comprehensive article of Lane and Thomas[1] where the treatment is particularly appropriate to the case in which few levels contribute. These authors employed the R-matrix theory of Wigner and Eisenbud[2] which is especially well adapted to the low-energy case because it makes the energy dependence of all expressions as explicit as possible. We employ here the formalism of Kapur and Peierls[3] which has the great advantage for the many-level case that the sum over levels enters linearly into the scattering amplitude. The formalism employed is only an intermediate step to the final results, and, therefore, we choose the one that seems simplest for the particular development here.

Surveying first the old compound-nucleus picture, we note the points at which it must be revised. It was realized long ago that this was not a complete description; in fact N. Bohr[4] already suggested in 1938 that a particle, upon entering the nucleus, might go directly to a final state without forming a compound nucleus. However, most calculations were carried out with the extreme form of the model, and most physical pictures tended to follow in this way. By extreme form we mean the form in which it is assumed that the phases of contributions from different compound levels are random, and that they therefore do not interfere. Such assumptions are generally implicit in statistical calculations.

The compound-nucleus picture formulated by N. Bohr[5] explained the very narrow resonances observed in the elastic scattering of slow neutrons by assuming that the incident neutron, once inside the nucleus, shares its energy with many other nucleons through its strong interaction with them; the resulting compound nucleus then lasts a long time, until one of the nucleons acquires sufficient energy to escape from the nucleus. Because of the long lifetime of the state, the uncertainty principle shows that its energy can be well determined, and hence, that its width is small. The observed resonances were of the order of electron volts wide, about a million times narrower than the single-particle levels formed in the scattering of particles by the potential wells earlier assumed to represent the nucleus.

Since the observed compound-nucleus resonances were so long-lived, it was usually assumed that their characteristics are independent of the way in which they are formed, aside from restrictions resulting from conservation laws, and are also independent of the neighboring states. It seemed natural then to introduce statistical assumptions which neglected interference between the scattering from different levels. Then, in the neighborhood of a compound state, one could write the cross section $\sigma_{\alpha\beta,j}$ for a particle incident in "channel" α to emerge in "channel" β (we give precise definitions of all terms later), leaving the residual nucleus in excited state j as a product of factors

$$\sigma_{\alpha\beta,j}=\frac{\pi}{k^2}(2l+1)\frac{\gamma_{p\alpha}\gamma_{p\beta j}}{(\epsilon_p-E)^2+\alpha_p^2/4},$$

where p labels the compound state, $\gamma_{p\alpha}$ is the partial width[6] for formation of it by a particle in channel α, $\gamma_{p\beta j}$ is the partial width for its decay, α_p is the total width of the state, and ϵ_p is the real part of its energy. This type of formula, valid in the region of a single isolated level, was given by Breit and Wigner[7] and bears their names. The above expression for σ as a product of factors implies the independence of the processes of formation and decay of the level both from each other and from the properties of other levels.

It is clear that the number of compound states per unit energy interval increases rapidly with the energy of the incident particle. Already at energies in the kev range, the density of levels observed in the scattering of slow neutrons by medium and heavy nuclei is $\sim 10^4$

[1] A. M. Lane and R. G. Thomas, Revs. Modern Phys. **30**, 257 (1958).
[2] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).
[3] P. L. Kapur and R. E. Peierls, Proc. Roy. Soc. (London) **A166**, 277 (1938).
[4] N. Bohr, Nature **141**, 326, 1096 (1938); Bohr, Peierls, and Placzek, Nature **144**, 200 (1939).
[5] N. Bohr, Nature **137**, 344 (1936); Science **86**, 161 (1947).
[6] Usually the widths are denoted by capital letters Γ, e.g., $\Gamma_{p\alpha}$. However, we reserve these for the widths of the single-particle levels in a potential well.
[7] G. Breit and E. P. Wigner, Phys. Rev. **49**, 579 (1936).

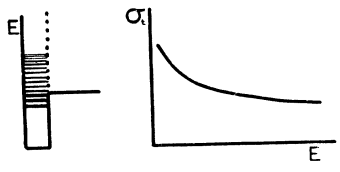


FIG. 1. The increasing density of nuclear levels with excitation energy is sketched on the left. Predictions of the continuum model for the behavior of the total cross section are shown on the right.

to 10^6 levels per Mev. Therefore, in the scattering of beams of neutrons which generally includes a spread in neutron energies of several Mev because of the experimental difficulty of producing monoenergetic neutrons of arbitrary energy, one might expect the cross section to approach that of a completely absorbing system with increasing neutron energy. A specific model incorporating these features was formulated by Feshbach, Peaslee, and Weisskopf[8] and is known as the continuum model. Here, the wave function of the neutron incident on the nucleus is subjected to the requirement that it have only an incoming part at the nuclear radius, i.e., that

$$d/dr[r\psi(r)]|_{r=R} = -iKR\psi(R),$$

where $\psi(r)$ is the wave function for the incident neutron, R is the nuclear radius, and K is an "internal" wave number that the neutron is supposed to possess inside the nucleus as a result of its interaction with the other nucleons. The predictions of such a theory for the total cross section are illustrated in Fig. 1 which also gives an indication of how the number of compound states increases with increasing excitation energy. The negative energy states of the nucleus are shown schematically as bound states in a potential well. At high energies the total cross section approaches $2\pi R^2$. The rise in the curve at lower energies is due to the fact that, in this model, the particle "feels" the nucleus already at distance $R+\lambda\!\!\!^{-}$, where $\lambda\!\!\!^{-}=\lambda/2\pi$ and λ is the wavelength, because of quantum-mechanical effects.

Contrary to expectations, the experimental cross sections, for beams in which the neutrons had a spread of energies, showed giant maxima of widths of the order of one or two Mev, as indicated schematically in Fig. 2. These were reproduced theoretically in the later theory of Feshbach, Porter, and Weisskopf[9] by the scattering from a complex potential well in which the real part represents some average interaction of the incoming nucleon with the nuclear particles, and the imaginary part, the disappearance of particles out of the incident

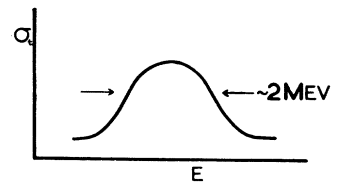


FIG. 2. Sketch of a giant resonance of the type observed in the scattering of neutrons by complex nuclei.

beam into compound states. There is no natural explanation for such giant resonances in the extreme compound-nucleus picture, and the representation of the nucleus by a complex well indicates a major modification.

Other phenomena which demanded modification of the extreme compound-nucleus picture were observed in the inelastic scattering of nucleons by complex nuclei. The predictions of the compound-nucleus model, supplemented by statistical assumptions, are shown as the solid line in Fig. 3. In this picture, the probability of leaving the nucleus in the various excited levels is assumed the same for all levels, aside from kinematical factors. The curve therefore rises as the energy taken off by the scattered particle decreases (and the energy left to the nucleus increases) and is cut off only when the wave number of the escaping nucleon (or nucleons) is so small that it will be reflected at the nuclear surface. Experimentally, an anomalously large number of fast particles was observed, as indicated by the dotted line in Fig. 3. Processes responsible for the high-energy particles were qualitatively well described by the direct interaction formalism of Austern, Butler, and

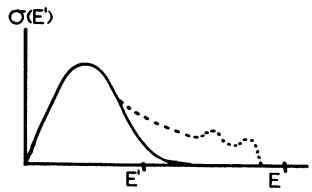


FIG. 3. The solid line shows typical predictions of the statistical model for the spectrum of inelastically-scattered neutrons. The energy of the emitted neutron is denoted by E'. The dashed line indicates the type of spectrum observed experimentally.

McManus.[10] Here, the incident particle is assumed to "chip" off particles from the nuclear surface without formation of a compound state. For example, the transition element for the nucleus to go from state i to state f is assumed to be

$$M_{if} = -\frac{2M}{\hbar^2}\int_{r_0}^{\infty} \chi_f(\xi)e^{-i\mathbf{k}_f\cdot\mathbf{r}}V(\mathbf{r}-\xi)e^{i\mathbf{k}_i\cdot\mathbf{r}}\chi_i(\xi)d^3\xi d^3r,$$

where $\exp i\mathbf{k}_i\cdot\mathbf{r}$ and $\exp i\mathbf{k}_f\cdot\mathbf{r}$ are the wave functions of the incident and scattered nucleon, and $\chi_0(\xi)$ and $\chi_f(\xi)$ those of the initial and residual nucleus. Here $\mathbf{r}$ represents the coordinate of the incident nucleon and ξ the totality of nuclear coordinates, conventions that are used throughout. The radius r_0 is supposed to define the point at which the "surface" begins and was chosen so as to fit the experimental angular distributions.

The form of this theory was suggested by the earlier description by Butler[11] of deuteron stripping which is also a form of "direct interaction." However, for sim-

[8] Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 145 (1947).
[9] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).
[10] Austern, Butler, and McManus, Phys. Rev. **92**, 350 (1953).
[11] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951).

plicity, we discuss the scattering of nucleons since complications are introduced by the composite structure of deuterons or α-particles.

Many experiments have shown that the high-energy particles in these inelastic processes have angular distributions, usually peaked towards small angles, whereas the low-energy nucleons have comparatively structureless, symmetrical angular distributions, as would be predicted by the statistical theory, and this is a further indication of the direct nature of the high-energy processes.

A further phenomenon which violates the early compound-nucleus picture occurs in the dipole photoeffect. Here, the absorption cross section for all complex nuclei shows a gaint maximum in the region of 15 Mev for heavy nuclei. This maximum was explained by Goldhaber and Teller[12] and Steinwedel and Jensen[13] in terms of a collective oscillation of proton density. How-

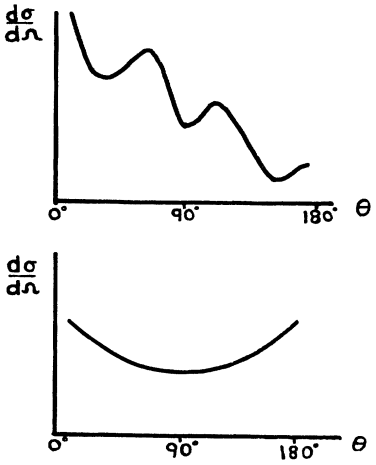


FIG. 4. In the upper figure, the angular distribution of the high-energy part of the inelastic spectrum, Fig. 3, is sketched; in the lower, that of the low-energy part of the spectrum.

ever, the decay of such a system, in which the energy is almost evenly distributed over all particles, ought to be adequately represented by the statistical theory. In the case of medium and heavy nuclei, the number of high-energy protons observed is far in excess of that predicted by this theory, often by several orders of magnitude. Such high-energy protons are predicted naturally by Wilkinson's[14] picture of the giant dipole photoeffect, in which the γ-ray is absorbed by a nucleon which then goes into a single-particle state in a complex well, and consequently has an appreciable chance of escaping with the full energy of the γ-ray before being absorbed into compound states. (This absorption is again described through the imaginary part of the well.)

Thus, although the old compound-nucleus picture

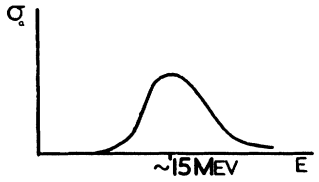


FIG. 5. The absorption cross section for γ-rays showing the giant dipole resonance.

had to be modified, the new phenomena could be reproduced by simple physical models which employed a complex well. However, conceptually, the description was not complete because it was known from cases where the cross sections could be investigated in detail that the giant resonances actually consist of thousands or millions of compound levels, and the relation between the detailed behavior—which often could not be investigated experimentally, but could be inferred—and the average behavior as predicted by the complex well had to be clarified. Further, the connection of the parameters of the optical well with more fundamental quantities such as the nucleon-nucleon force had to be made.

A unified description of the above phenomena begins from an exact description in terms of nuclear dispersion theory. The scattering amplitude is separated into terms corresponding to direct interactions and compound-nucleus processes. Before going into such a description, however, we review the dispersion theory of Kapur and Peierls,[3] which is used in the later development, and present a simplified picture illustrating the physical assumptions employed later in the more formal arguments.

II. KAPUR-PEIERLS DISPERSION FORMALISM

1. Scattering by a Potential Well

We begin by treating the scattering of an S-wave neutron by a potential well; this simple example illustrates the main features of the theory. The Schrödinger equation for this case is

$$\frac{h^2}{2M}\frac{d^2\phi}{dr^2}+[E-V(r)]\phi=0, \tag{1}$$

where $V(r)$ is the potential and $\phi(r)=r\psi(r)$ with ψ the wave function of the neutron. For radii r greater than some radius R, beyond which the potential $V(r)$ is zero,

$$(d^2\phi/dr^2)+k^2\phi=0, \quad r>R, \tag{2}$$

with

$$k^2=2ME/\hbar^2. \tag{2.1}$$

The solution of Eq. (2) is

$$\phi=(\sin kr/k)+Se^{ikr}, \tag{2.2}$$

where the normalization of ϕ has been chosen so that the first term on the right-hand side corresponds to the S-wave part of a plane wave of unit amplitude, i.e., $\exp i\mathbf{kr}\cong\sin kr/kr$ for $kr\ll 1$; S is the amplitude of the scattered (outgoing) wave.

[12] M. Goldhaber and E. Teller, Phys. Rev. **74**, 1046 (1948).
[13] H. Steinwedel and J. H. D. Jensen, Z. Naturforsch. **5a**, 413 (1950).
[14] D. H. Wilkinson, Physica **22**, 1039 (1956).

In the internal region $r<R$, ϕ can be obtained as

$$\phi(r)=\sum a_m\phi_m(r), \quad r<R, \tag{3}$$

where the ϕ_m are eigenfunctions satisfying the equation

$$\frac{\hbar^2}{2M}\frac{d^2\phi_m}{dr^2}+(E_m-V)\phi_m=0, \tag{3.1}$$

and the boundary condition

$$\left\{\frac{d\phi_m}{dr}=ik\phi_m\right\}\Bigg|_{r=R}. \tag{3.2}$$

The eigenvalues E_m are complex because of this imaginary boundary condition, the boundary condition depending explicitly on the wave number and hence, the energy of the incident neutron. We discuss the consequences of this later. These eigenfunctions form a complete set.[15] Orthogonality of the ϕ_m is easily established by considering the equation for ϕ_n,

$$\frac{\hbar^2}{2M}\frac{d^2\phi_n}{dr^2}+(E_n-V)\phi_n=0, \tag{3.3}$$

multiplying Eq. (3.1) on the left by ϕ_n and Eq. (3.3) on the left by ϕ_m and subtracting. Thus,

$$\frac{\hbar^2}{2M}\left(\phi_n\frac{d^2\phi_m}{dr^2}-\phi_m\frac{d^2\phi_n}{dr^2}\right)+(E_m-E_n)\phi_n\phi_m=0. \tag{3.4}$$

By integrating this equation from 0 to R and using Green's theorem, one finds that

$$\frac{\hbar^2}{2M}\left(\phi_n\frac{d\phi_m}{dr}-\phi_m\frac{d\phi_n}{dr}\right)_{r=R}=(E_n-E_m)\int_0^R\phi_n\phi_m dr. \tag{3.5}$$

From Eq. (3.2) it can be seen that the left-hand side vanishes and hence

$$\int_0^R\phi_n(r)\phi_m(r)dr=0, \quad E_m\neq E_n. \tag{3.6}$$

In general, $E_m\neq E_n$ implies $m\neq n$; the exceptional case $E_n=E_m$ for $m\neq n$ can be handled by special methods, but it does not occur for the type of potentials we consider. For $m=n$, we choose the normalization of the ϕ_m's so that the integral is equal to unity, i.e.,

$$\int_0^R\phi_m^2(r)dr=1. \tag{3.7}$$

Two important features are first, the orthogonality is between ϕ_n and ϕ_m, not ϕ_n^* and ϕ_m as is usually the case, and secondly, the orthogonality depends essentially on the fact that ϕ_m and ϕ_n obey the same boundary condition.

[15] R. E. Peierls, Proc. Cambridge Phil. Soc. 44, 242 (1948).

By using the orthogonality of the ϕ_m we can obtain the a_m of Eq. (3) by again using Green's theorem.

$$\int_0^R(\phi_m\mathfrak{L}\phi-\phi\mathfrak{L}\phi_m)dr=-\frac{\hbar^2}{2M}\left(\phi_m\frac{d\phi}{dr}-\phi\frac{d\phi_m}{dr}\right)_R, \tag{4}$$

with

$$\mathfrak{L}=-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+V(r). \tag{4.1}$$

From Eqs. (1) and (3.1), one obtains

$$(E-E_m)a_m=-\frac{\hbar^2}{2M}\left[\left(\frac{d\phi}{dr}-ik\phi\right)\phi_m\right]_R$$
$$=-\frac{\hbar^2}{2M}e^{-ikR}\phi_m(R), \tag{4.2}$$

giving

$$a_m=\frac{\hbar^2}{2M}\frac{e^{-ikR}}{E_m-E}\phi_m(R). \tag{4.3}$$

The joining of the inside and outside solutions at $r=R$ gives

$$\sum a_m\phi_m(R)=\frac{\sin kR}{k}+Se^{ikR}, \tag{5}$$

giving

$$S=\frac{1}{ke^{ikR}}\left\{\sum_m\frac{k\hbar^2}{2M}e^{-ikR}\frac{[\phi_m(R)]^2}{E_m-E}-\sin kR\right\}. \tag{6}$$

This can be put into familiar form by defining the width

$$\Gamma_m\equiv\frac{k\hbar^2}{M}[\phi_m(R)]^2. \tag{6.1}$$

Then

$$S=\frac{e^{-2ikR}}{2k}\sum_m\frac{\Gamma_m}{E_m-E}-\frac{\sin kR}{k}e^{-ikR}; \tag{7}$$

and

$$\sigma=4\pi|S|^2. \tag{7.1}$$

Because of the imaginary boundary condition, the ϕ_m and hence the Γ_m are complex. For low bombarding energies, the imaginary part is small as appears later.

The imaginary part of E_m can be found by the development

$$\int_0^R(\phi_m\mathfrak{L}\phi_m^*-\phi_m^*\mathfrak{L}\phi_m)dr$$
$$=(E_m^*-E_m)\int_0^R\phi_m\phi_m^*dr. \tag{8}$$

From Green's theorem, the left-hand side is

$$(ik\hbar^2/M)\phi_m^*\phi_m(R),$$

and if the imaginary part of E_m is denoted by $-\beta_m/2$, i.e.,

$$E_m=\epsilon_m-i\beta_m/2, \tag{8.1}$$

then from Eq. (8) we find that

$$\beta_m=\frac{k\hbar^2}{M}\frac{|\phi_m(R)|^2}{\int_0^R|\phi_m(r)|^2dr}. \tag{8.2}$$

This has a simple interpretation. The numerator is proportional to the escape velocity of the particle $k\hbar/M$ multiplied by the probability of the particle being at the surface, whereas the denominator represents the probability of the particle being in the nucleus. At low energies where the ϕ_m are mainly real, as already indicated, the denominator is nearly unity [see Eq. (3.7)] and comparing Eqs. (8.2) and (6.1), we see that

$$\beta_m\cong\Gamma_m \quad \text{(low energies)}. \tag{8.3}$$

To obtain a better idea of the sizes of the various quantities, it is useful to illustrate the formalism by evaluating them for a square well. We choose a well $V=-U$, $r<R_0$; $V=0$, $r>R_0$; with $U=42$ Mev and $R_0=1.45$ A$^{\frac{1}{3}}$ $\times10^{-13}$ cm, where A is the atomic number; these are the parameters of the well employed by Feshbach, Porter, and Weisskopf,[9] aside from the imaginary part employed by them which is introduced later. Consequently, these parameters are typical of those considered later. The solution of Eqs. (3.1) and (3.2) are compared later with $\phi_m^{(0)}$ defined by

$$\frac{\hbar^2}{2M}\frac{d^2}{dr^2}\phi_m^{(0)}(r)+(E_m^{(0)}-V)\phi_m^{(0)}(r)=0, \tag{9}$$

$$\left.\frac{d\phi_m^{(0)}}{dr}\right|_{r=R}=0. \tag{9.1}$$

This real boundary condition is that of the Wigner-Eisenbud theory, and the solutions $\phi_m^{(0)}$ are real. The functions $\phi_m^{(0)}$ are considered because it is easier to calculate these real functions and, at low energies, they are a good zero-order approximation to ϕ_m as we show.

In the square well ϕ_m is equal to

$$\phi_m=A_m\sin(K_m-i\kappa_m)r, \tag{9.2}$$

with K_m and κ_m defined by

$$(K_m-i\kappa_m)^2=2M(E_m+U)/\hbar^2. \tag{9.3}$$

For simplicity, the joining radius R in Eq. (3.2) has been chosen to be the edge of the well, R_0. It can be chosen differently and is generally chosen somewhat larger, but this is immaterial for the qualitative features of the results derived here. The boundary condition, Eq. (3.2), can now be written

$$(K_m-i\kappa_m)\cos(K_m-i\kappa_m)R=ik\sin(K_m-i\kappa_m)R. \tag{9.4}$$

Now κ_mR is shown to be small at low energies; Eq. (9.4) is expanded to first order in this quantity. Then

$$(K_m-i\kappa_m)(\cos K_mR+i\kappa_mR\sin K_mR)\cong ik(\sin K_mR-i\kappa_mR\cos K_mR). \tag{9.5}$$

This gives

$$\kappa_m=k\sin K_mR/(K_mR\sin K_mR-\cos K_mR). \tag{9.6}$$

Since $\cos K_mR\ll K_mR\sin K_mR$ [see Eq. (9.8)], it follows that

$$\kappa_m\cong k/K_mR. \tag{9.7}$$

By using this value for κ_m, one obtains from Eq. (9.5)

$$\cos K_mR\cong-(kR)^2\sin K_mR/(K_mR)^3 \tag{9.8}$$

to lowest order in κ_mR. Now K_mR is large, of the order of 10 even for medium A and zero incident energy, and so κ_m and $\cos K_mR$ are small. Equation (9.7) shows that κ_mR is small—which is necessary for the rapid convergence of the expansion above—as long as $E\ll U$, that is, as long as the bombarding energy is much less than the depth of the well.

We now let $K_mR=K_m^{(0)}R+\delta K_mR$, where $\phi_m^{(0)}=(2/R)^{\frac{1}{2}}\sin K_m^{(0)}R$, and $K_m^{(0)}R=(n+\frac{1}{2})\pi$ gives a solution to Eqs. (9) and (9.1). To lowest order in κ_mR, Eq. (9.8) shows that

$$\delta K_m=k[kR/(K_mR)^3]. \tag{9.9}$$

Although this is a second-order correction (in the sense that δK_mR is of second order in kR), it is given correctly to this order by our first-order expression Eq. (9.5), since further terms enter only in third order. We find, further, that

$$\delta\epsilon_m\equiv\epsilon_m-E_m^{(0)}=(\hbar^2/2M)(2K_m\delta K_m-\kappa_m^2)\cong[1/(K_mR)^2]\epsilon_m, \tag{9.10}$$

where the energies are measured from the bottom of the well. For a typical value of K_mR of ~10, this means that calculation of the real part of the resonance energy from the boundary condition Eq. (9.1) gives results accurate to $\sim1\%$.

The imaginary part β_m of E_m is given to first order in κ_mR by $(\hbar^2k/M)[\phi_m^{(0)}(R)]^2$ which is equal to

$$\beta_m\cong2\hbar^2k/MR\cong\Gamma_m, \tag{9.11}$$

since

$$\phi_m^{(0)}(r)=(2/R)^{\frac{1}{2}}\sin K_mr. \tag{9.12}$$

Thus, to lowest order in κ_mR, the Γ_m are real and the dependence of the real parts of the resonance energies ϵ_m on the bombarding energy enters only in the second order of this quantity.

Generalization of these results to the case of a

complex square well is trivial. The introduction of $\tilde{V}(r)=-U-iW$ (use $\tilde{V}$ to denote a complex well; eigenfunctions in this complex well are denoted by $\tilde{\phi}_m(r)$, etc.) so that W simply shifts the resonance energy by the constant amount $-iW$. Hence,

$$\tilde{E}_m=E_m-iW=\epsilon_m-i\beta_m/2-iW. \tag{9.13}$$

Consequently, introduction of an imaginary part broadens the resonance levels. The eigenfunctions are unchanged, i.e., $\tilde{\phi}_m(r)=\phi_m$ in the special case of the square well. The development for the case in which the real and imaginary parts vary with r is easily carried out. This development can also be carried out for complex velocity-dependent potentials[16]; this is important since the parameters of the optical model wells occurring in practice depend on the bombarding energy. It is convenient to take the depth of the well to be different for the different eigenenergies at a given bombarding energy. In discussing a velocity-dependent well $\tilde{V}(r)$, one can consider $\tilde{V}$ to be a nonlocal operator $\tilde{V}(\mathbf{r}',\mathbf{r})$, which is equivalent to a velocity-dependent potential. The wave equation is

$$\frac{\hbar^2}{2M}\frac{d^2}{dr'^2}\tilde{\phi}_m(r')+\tilde{E}_m\tilde{\phi}_m(r')-\int\tilde{V}(\mathbf{r}',\mathbf{r})\tilde{\phi}_m(r)dr=0. \tag{10}$$

Orthogonality of the $\tilde{\phi}_m$ requires $\tilde{V}(\mathbf{r}',\mathbf{r})=\tilde{V}(\mathbf{r},\mathbf{r}')$; potentials not obeying this relation can be shown to be physically unreasonable.[16] Proof of orthogonality assumes $\tilde{V}(\mathbf{r},\mathbf{r}')$ to be zero unless both r and r' are less than R.

The resonance treatment for a neutron in the complex potential has been developed in some detail, not only to illustrate the formalism, but also because knowledge of the positions, widths, and spacings of the levels is often useful in theoretical estimates.

Generalization of these results to the case of nonzero angular momentum is easy. In this case we express the function, regular at the origin, $\psi_\alpha(\mathbf{r})$ as

$$r\psi_\alpha(\mathbf{r})=\Theta_\alpha(\theta,\varphi)\phi_\alpha(r), \tag{11}$$

where $\Theta_\alpha(\theta,\varphi)$ is the normalized function of angles, e.g., in the case of spinless particles, $\Theta_\alpha(\theta,\varphi)$ is equal to $Y_l^m(\theta,\varphi)$, where Y_l^m is the normalized spherical harmonic, and ϕ_α is the radial function in the potential well, regular at the origin and asymptotic to

$$e^{i\delta_l}\sin[kr-(l\pi/2)+\delta_l],$$

where δ_l is the phase shift for the lth wave. The lower index α labels all angular momentum quantum numbers, i.e., in the terminology of Wigner and co-workers, it is the "channel" index. Further, solutions asymptotic to outgoing waves are denoted by $\psi^+(\mathbf{r})$, i.e.,

$$r\psi_\alpha^+(\mathbf{r})=\Theta_\alpha(\theta,\varphi)\phi_\alpha^+(r), \tag{11.1}$$

where $\phi_\alpha^+(r)$ is asymptotic to $\exp(ikr-il\pi/2)$ in the case of neutrons. Generalization to the case of charged particles is easily made. Here $\phi_\alpha^+(r)$ is asymptotic to $\exp(ikr-\eta\ln 2kr-l\pi/2+\sigma_l)$ where $\eta=Ze^2/\hbar v$, with v the velocity of the proton, and σ_l is the Coulomb phase shift. Solution of the radial Schrödinger equation can be expressed for $r>R$, as

$$\phi_\alpha(r)=[\phi_\alpha^+(r)-\phi_\alpha^-(r)]/2ik+S_\alpha\phi_\alpha^+(r), \tag{12}$$

with $\phi_\alpha^-=(\phi_\alpha^+)^*$, the solution asymptotic to incoming waves. The eigenfunctions ϕ_m (these have the same angular dependence as the $\phi_\alpha(r)$, so we do not carry the subscript α on them) are then determined by the boundary condition

$$\left[\frac{d\phi_m}{dr}=f_\alpha^+(r)\phi_m\right]_{r=R} \tag{12.1}$$

where

$$f_\alpha^+(r)=\frac{1}{\phi_\alpha^+(r)}\frac{d\phi_\alpha^+(r)}{dr} \tag{12.2}$$

is the logarithmic derivative for an outgoing wave at the joining radius. Again, $\phi_\alpha(r)$ can be expanded in terms of the ϕ_m,

$$\phi_\alpha(r)=\sum_m a_m\phi_m(r), \quad r<R, \tag{12.3}$$

and the development analogous to that of Eqs. (4) to (4.3) now gives

$$(E-E_m)a_m=-\frac{\hbar^2}{2M}\left[\left(\frac{d\phi_\alpha}{dr}-f^+\phi_\alpha\right)\phi_m\right]_R. \tag{12.4}$$

The use of Eq. (12) for ϕ_α leaves only the term in ϕ_α^- on the right-hand side of Eq. (12.4), and we obtain

$$(E-E_m)a_m=\frac{\hbar^2}{4Mik}[(f_\alpha^--f_\alpha^+)\phi_\alpha^-\phi_m]_R. \tag{12.5}$$

The Wronskian is

$$[f_\alpha^+(r)-f_\alpha^-(r)]\phi_\alpha^+(r)\phi_\alpha^-(r)=2ik, \tag{12.6}$$

so that we obtain

$$a_m=\frac{\hbar^2}{2M}\frac{1}{\phi_\alpha^+(R)}\frac{\phi_m(R)}{(E_m-E)}. \tag{12.7}$$

The joining of the inside and outside solutions at $r=R$ now gives

$$S_\alpha=\frac{1}{k[\phi_\alpha^+(R)]^2}\sum_m\frac{k\hbar^2}{2M}\frac{\phi_m^2(R)}{E_m-E}-\frac{\phi_\alpha^+(R)-\phi_\alpha^-(R)}{2ik\phi_\alpha^+(R)}. \tag{12.8}$$

We define

$$\Gamma_m\equiv\frac{k\hbar^2}{M}P(R)\phi_m^2(R), \tag{12.9}$$

[16] See Appendix C of G. E. Brown and C. T. De Dominicis, Proc. Phys. Soc. (London) **A72**, 70 (1958).

where P is the penetrability,

$$P=\frac{1}{\phi_\alpha^+(R)\phi_\alpha^-(R)}=\frac{1}{|\phi_\alpha^+(R)|^2}. \tag{12.10}$$

We then have

$$S_\alpha=\frac{1}{2k}\frac{\phi_\alpha^-(R)}{\phi_\alpha^+(R)}\sum_m\frac{\Gamma_m}{E_m-E}-\frac{1}{2ik}\left(1-\frac{\phi_\alpha^-(R)}{\phi_\alpha^+(R)}\right). \tag{12.11}$$

The cross section is given by

$$\sigma_\alpha=4\pi|S_\alpha|^2$$
$$=\frac{\pi}{k^2}\left|\frac{\phi_\alpha^-(R)}{\phi_\alpha^+(R)}\sum_m\frac{\Gamma_m}{E_m-E}+i\left(1-\frac{\phi_\alpha^-(R)}{\phi_\alpha^+(R)}\right)\right|^2. \tag{12.12}$$

2. The Many-Body Case

We consider the case in which an incident nucleon is scattered either elastically or inelastically by a nucleus of A particles.[17] Once this particle is inside the radius R —which is chosen so that the interaction between the incident particle and the nucleus vanishes outside this radius—the wave function can again be expanded in compound states $\Phi^{(p)}(\mathbf{r},\xi)$. Here we use $\mathbf{r}$ for the coordinate of the incident nucleon and ξ to label the totality of coordinates of the A particles inside the nucleus. For the moment the incident particle is treated as distinguishable from those in the nucleus, but later the generalization to the case where it is identical with particles in the nucleus is indicated.

The compound states $\Phi^{(p)}(\mathbf{r},\xi)$ obey the equations

$$H\Phi^{(p)}(\mathbf{r},\xi)=W_p\Phi^{(p)}(\mathbf{r},\xi),$$
$$H=H_\xi+T(r)+V(\mathbf{r},\xi). \tag{13}$$

H_ξ is the Hamiltonian of the A nuclear particles, $T(r)$ is the kinetic energy of the incident particle, and $V(\mathbf{r},\xi)$ is the potential interaction between the A nuclear particles and the incident nucleon. It is a sum of nucleon-nucleon potentials,

$$V(\mathbf{r},\xi)=\sum_{i=1}^{A}V(\mathbf{r},\mathbf{r}_i),$$

the nucleon to which the coordinate $\mathbf{r}$ refers being the incident particle. It is assumed that $V(\mathbf{r},\xi)$ is a well-behaved potential, i.e., a potential without a strong repulsive core and other singularities; generalization to the case that it is not can be made using techniques developed by Watson and Brueckner, but this only introduces nonessential complications for the points considered here.

[17] The emission of composite particles is dropped from the dispersion-theory description later on when one makes the assumption that the interaction between the incident particle and all of the A particles vanishes beyond the joining radius R. These processes can presumably be included by modifying the formalism.

States $\Phi^{(p)}$ and $\Phi^{(q)}$ are not now orthogonal, but rather $\Phi^{(p)}$ and $\bar{\Phi}^{(q)}$, where $\bar{\Phi}^{(q)}$ is obtained from $\Phi^{(q)}$ by taking the complex conjugate of all functions of angles, in particular, of the $\Theta_\alpha(\theta,\varphi)$ occurring in the single-particle function. Equivalently, to within an arbitrary phase factor, $\bar{\Phi}^{(q)}$ is obtained from $\Phi^{(q)}$ by rotating the wave function so that the total angular momentum component M is changed into $-M$ without taking the complex conjugate of the function of intrinsic structure (e.g., the radial wave function in the single-particle case). We choose

$$\int_0^R\bar{\Phi}^{(q)}\Phi^{(p)}d^3\xi d^3r=1,\quad(p=q), \tag{13.1}$$

and the integral is zero for $p\neq q$. In using the orthogonality later on, we do not indicate the bar over the Φ on the left since the change in the angular momentum functions is a trivial one to introduce once we introduce the expansion, Eq. (14), and all integrals over angles are easy to carry out.

We introduce a complete set of states $\chi_j(\xi)$ for the A particles. Boundary conditions for these states could be chosen in various ways. However, the tightly bound states close to the ground state in which we are mainly interested are insensitive to the precise boundary condition, e.g., if we choose the joining radius R fairly far out, then both the wave function and its derivative are small for a bound state defined in any sensible way, and it would not matter much whether we chose the former or the latter to be zero, or to have some small finite value at that radius as our boundary condition. Difficulties enter when the excitation energy ϵ_j, which we measure from ϵ_0 as origin, of the state χ_j is sufficiently large so that one of the A particles can escape from the nucleus. However, since the only property of these highly excited states that we use is that they, together with those of low excitation, form a complete set, we believe our results to be independent of the way in which these boundary conditions are chosen.

In order to make the connection with the optical model later, it is convenient to employ states $\tilde{\psi}_m(\mathbf{r})$ in a complex well of the type discussed in the last section for the incident particle. We can then expand

$$\Phi^{(p)}(\mathbf{r},\xi)=\sum_{j,m}a_{jm}{}^p\chi_j(\xi)\tilde{\psi}_m{}^j(\mathbf{r}). \tag{14}$$

The boundary condition on $\Phi^{(p)}$ at $r=R$ can now be simply stated in terms of the boundary conditions on the $\tilde{\psi}_m{}^j$. It is convenient, in analogy with Eq. (11), to introduce radial functions $\tilde{\phi}_m$ through

$$r\tilde{\psi}_m{}^j(\mathbf{r})=\Theta_\alpha(\theta,\varphi)\tilde{\phi}_m{}^j(r). \tag{14.1}$$

The boundary condition is then

$$\left[\frac{d\tilde{\phi}_m{}^j}{dr}=f_\alpha^+(r,E-\epsilon_j)\tilde{\phi}_m{}^j(r)\right]_{r=R}. \tag{14.2}$$

The $E-\epsilon_j$ in the argument of f_α^+ indicates that it is to be taken for that energy. The boundary condition is chosen to be that of an outgoing wave at the reduced energy available to the outgoing particle after leaving the residual nucleus in excited state j. The channel index α on the $\tilde{\psi}_m{}^j$ is again suppressed. Because of the dependence of f_α^+ on ϵ_j, the $\tilde{\psi}$'s also depend on j, as indicated; however, this dependence is weak. The fact that the boundary condition on the $\tilde{\psi}$'s depends on j does not upset the orthogonality of the $\Phi^{(p)}$'s as long as one always integrates over the ξ coordinates first.

We can express the wave function $\Psi(\mathbf{r},\xi)$, which is the solution to the Schrödinger equation,

$$H\Psi=E\Psi, \tag{15}$$

as

$$\Psi(\mathbf{r},\xi)=\sum a_p\Phi^{(p)}(\mathbf{r},\xi) \quad \text{for } r<R, \tag{15.1}$$

and as

$$\Psi(\mathbf{r},\xi)=I_\alpha\left(\frac{\psi_\alpha^+(\mathbf{r})-\psi_\alpha^-(\mathbf{r})}{2ik}\right)\chi_0(\xi)+S_\alpha\psi_\alpha^+(\mathbf{r})\chi_0(\xi)$$
$$+\sum_{\alpha\neq\alpha'} S_{\alpha\alpha',j}\psi_{\alpha'}{}^{j+}(\mathbf{r})\chi_j(\xi) \quad \text{for } r>R, \tag{15.2}$$

where S_α is the amplitude for elastic scattering in channel α and $S_{\alpha\alpha',j}$ is the amplitude for inelastic scattering from channel α into channel α', leaving the nucleus excited in state χ_j. The $\psi_{\alpha'}{}^{j+}$ represents an outgoing wave at energy $E-\epsilon_j$ in channel α'. The coefficient I_α represents the amplitude for channel α in the plane wave. In the case of spinless particles,

$$e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{l=0}^{\infty} i^l[4\pi(2l+1)]^{\frac{1}{2}}Y_l{}^0(\theta,\varphi)j_l(kr). \tag{15.3}$$

If we set $\alpha=l$, i.e., identify $Y_l{}^0$ with Θ_α for this case, and use

$$j_l\sim\frac{\sin(kr-l\pi/2)}{kr}, \tag{15.4}$$

we choose

$$I_l=i^l(2l+1)^{\frac{1}{2}} \tag{15.5}$$

for this special case. The $(4\pi)^{\frac{1}{2}}$ is taken into account at a later stage by multiplying by 4π to obtain the cross section, as in Eqs. (7.1) and (12.12); this convention makes our general treatment correspond to our S-wave case when $l=0$.

We can again find the a_p by Green's theorem,

$$\int_0^R(\Phi^{(p)}H\Psi-\Psi H\Phi^{(p)})d^3\xi d^3r=(E-W_p)a_p$$
$$=-\frac{\hbar^2}{2M}\sum_{j,m}a_{jm}{}^p\int_0^R d^3\xi\chi_j(\xi)$$
$$\times\int d\Omega\left\{R\tilde{\psi}_m{}^j(\mathbf{R})\frac{d}{dr}[r\Psi(\mathbf{r},\xi)]\right.$$
$$\left.-R\Psi(\mathbf{R},\xi)\frac{d}{dr}[r\tilde{\psi}_m{}^j(\mathbf{r})]\right\}_{r=R}. \tag{16}$$

With the use of Eq. (14.2), we find

$$a_p=\frac{\hbar^2}{2M(W_p-E)}\sum_{j,m}a_{jm}{}^p\int_0^R d^3\xi\chi_j(\xi)\int d\Omega$$
$$\times\left\{R\tilde{\psi}_m{}^j(\mathbf{R})\left[\frac{d}{dr}-f_\alpha^+(R,E-\epsilon_j)\right]r\Psi(\mathbf{r},\xi)\right\}_{r=R}, \tag{16.1}$$

where we have indicated the energy dependence of the f_α^+ explicitly. The operator in [ ]'s in Eq. (16.1) gives zero operating on any "outgoing" part of Ψ, and the only term that contributes in Eq. (15.2) is the term in $\psi_\alpha^-(\mathbf{r})\chi_0(\xi)$. Hence, using the orthogonality of the χ's,

$$a_p=-I_\alpha\frac{\hbar^2}{2M(W_p-E)}\sum_m a_{0m}{}^p\left\{\frac{f_\alpha^-(R)-f_\alpha^+(R)}{2ik}\right\}$$
$$\times\int d\Omega R\tilde{\psi}_m(\mathbf{R})R\psi_\alpha^-(\mathbf{R}). \tag{16.2}$$

The integral over Ω guarantees that the angular dependence of the $\tilde{\psi}_m$ is the same as that of the incident channel α. With this restriction understood, we can write a_p in terms of the radial part of the wave functions,

$$a_p=I_\alpha\frac{\hbar^2}{2M(W_p-E)}\left\{\frac{f_\alpha^--f_\alpha^+}{2ik}\right\}\phi_\alpha^-(R)\sum_m a_{0m}{}^p\tilde{\phi}_m(R). \tag{16.3}$$

We can simplify Eq. (16.3) by using the Wronskian, Eq. (12.6), giving

$$a_p=I_\alpha\frac{\hbar^2}{2M(W_p-E)}\frac{1}{\phi_\alpha^+(R)}\sum_m a_{0m}{}^p\tilde{\phi}_m(R). \tag{16.4}$$

By having a_p, we can now obtain S_α and $S_{\alpha\alpha',j}$ by matching the internal and external wave functions, Eqs. (15.1) and (15.2), at the joining radius. If both equations are multiplied by $\chi_0(\xi)$ and $R\Theta_\alpha$ and integrated over $d^3\xi$ and $d\Omega$, one obtains

$$S_\alpha\phi_\alpha^+(R)=\sum_p a_p\sum_{m'}a_{0m'}{}^p\tilde{\phi}_{m'}(R)$$
$$-I_\alpha(\phi_\alpha^+(R)-\phi_\alpha^-(R))/2ik. \tag{17}$$

If we introduce the width,

$$\gamma_p=\frac{k\hbar^2}{M}\frac{1}{\phi_\alpha^+(R)\phi_\alpha^-(R)}\sum_m\sum_{m'}a_{0m}{}^p a_{0m'}{}^p\tilde{\phi}_m(R)\tilde{\phi}_{m'}(R), \tag{17.1}$$

we can write

$$S_\alpha=\frac{I_\alpha\phi_\alpha^-(R)}{2k\,\phi_\alpha^+(R)}\sum_p\frac{\gamma_p}{W_p-E}-\frac{I_\alpha}{2ik}\left(1-\frac{\phi_\alpha^-(R)}{\phi_\alpha^+(R)}\right), \tag{18}$$

which is of the same form as Eq. (12.11). The cross

section σ_α for elastic scattering is given by

$$\sigma_\alpha=4\pi|S_\alpha|^2. \tag{18.1}$$

For spinless particles, α can be replaced by the label l, and with the value of I_l given by Eq. (15.5), one obtains

$$\sigma_l=(2l+1)\frac{\pi}{k^2}\left|\frac{\phi_l^-(R)}{\phi_l^+(R)}\sum_p\frac{\gamma_p}{W_p-E}+i\left(1-\frac{\phi_l^-(R)}{\phi_l^+(R)}\right)\right|^2, \tag{18.2}$$

for the total scattering through channel l. Since the incident particle is described by a plane wave, Eq. (15.3), the differential cross section is given as the square of a sum of terms involving different l,[18]

$$\frac{d\sigma(\theta)}{d\Omega}=4\pi|\sum_l i^{-l}S_lY_l^0|^2. \tag{18.3}$$

Similarly, the amplitude for inelastic scattering is

$$S_{\alpha\alpha',j}=\frac{1}{\phi_{\alpha'}{}^{j+}(R)}\sum_p a_p\sum_m a_{jm}{}^p\tilde{\phi}_m{}^j(R). \tag{19}$$

In the inelastic case, the widths do not occur naturally in the amplitude, but if quantities u_p and u_{pj} are defined by

$$u_p=\left(\frac{k\hbar^2}{M}\right)^{\frac{1}{2}}\frac{1}{|\phi_\alpha{}^+(R)|}\sum_m a_{om}{}^p\tilde{\phi}_m(R), \tag{19.1}$$

and

$$u_{pj}=\left(\frac{k_j\hbar^2}{M}\right)^{\frac{1}{2}}\frac{1}{|\phi_{\alpha'}{}^{j+}(R)|}\sum_m a_{jm}{}^p\tilde{\phi}_m{}^j(R), \tag{19.1}$$

then

$$\sigma_{\alpha\alpha',j}=\frac{4\pi k_j}{k}|S_{\alpha\alpha',j}|^2=\frac{\pi}{k^2}\left|I_\alpha\sum_p\frac{u_pu_{pj}}{W_p-E}\right|^2. \tag{19.2}$$

The ratio k_j/k, with k_j defined by $k_j{}^2=2M(E-\epsilon_j)/\hbar^2$, is the ratio of the velocity of the outgoing particle to that of the incident one, which enters into the cross section.

In case one term, p, in the sum in Eq. (19.2) contributes the main part of the sum, the inelastic cross section can be put into the form of the Breit-Wigner formula

$$\sigma_{\alpha\alpha',j}=\frac{\pi}{k^2}|I_\alpha|^2\frac{\gamma_p\gamma_{pj}}{|W_p-E|^2}. \tag{19.3}$$

(See the Breit-Wigner formula given in the introduction; there we indicated the channel index on the γ's explicitly.) In Eq. (19.3)

$$\gamma_p=u_p{}^2,$$

as previously, and

$$\gamma_{pj}=u_{pj}{}^2 \tag{19.4}$$

is the width for the inelastic process.

Now $W_p=\epsilon_p-i\alpha_p/2$, where ϵ_p and α_p are real, and we are interested in the width α_p. This can be obtained from the original and conjugate equations,

$$(H-W_p)\Phi^{(p)}=0, \tag{20}$$
$$(H-W_p{}^*)\Phi^{(p)*}=0.$$

If the first equation is multiplied on the left by $\Phi^{(p)*}$ and the second by $\Phi^{(p)}$, one obtains, after subtracting and integrating,

$$i\alpha_p\int_0^R\Phi^{(p)*}\Phi^{(p)}d^3\xi d^3r=\frac{\hbar^2}{2M}\int d^3\xi\int d\Omega \times\left[r\Phi^{(p)*}\frac{d}{dr}(r\Phi^{(p)})-r\Phi^{(p)}\frac{d}{dr}(r\Phi^{(p)*})\right]_{r=R}. \tag{21}$$

By using the expansion Eq. (14) and the boundary conditions Eq. (14.2), this becomes

$$i\alpha_p\int_0^R\Phi^{(p)*}\Phi^{(p)}d^3\xi d^3r =\frac{\hbar^2}{2M}\sum_{j,m,m'}a_{jm}{}^{p*}a_{jm'}{}^p\tilde{\phi}_m{}^{j*}(R)\tilde{\phi}_m{}^j(R) \times[f_{\alpha'}{}^+(E-\epsilon_j,R)-f_{\alpha'}{}^-(E-\epsilon_j,R)]. \tag{22}$$

Use of the Wronskian, Eq. (12.6), results in

$$\alpha_p=\frac{\hbar^2}{M}\sum_{j,m,m'}\frac{k_ja_{jm}{}^{p*}a_{jm'}{}^p\tilde{\phi}_m{}^{j*}(R)\tilde{\phi}_{m'}{}^j(R)/\phi_{\alpha'}{}^{j+}(R)\phi_{\alpha'}{}^{j-}(R)}{\int_0^R\Phi^{(p)*}\Phi^{(p)}d^3\xi d^3r}, \tag{22.1}$$

where the sum over j includes the term $j=0$. The right-hand side of Eq. (22.1) has a simple interpretation similar to that of Eq. (8.2). In the numerator the jth term in the sum is proportional to the probability of the particle being at the radius R, multiplied by its escape velocity and the relevant penetrability. The integral in the denominator represents the probability of the particle in compound state p being in the nucleus. (The normalization is such that this probability is not

[18] The factor i^{-l} enters because S_l is the coefficient of $\exp(ikr-il\pi/2)$, so that the coefficient of $\exp ikr$ is

$$S_l\exp(-il\pi/2)=S_li^{-l}.$$

necessarily unity.) In the low-energy region where all of the quantities on the right-hand side of Eq. (22.1) are essentially real, as is shown later, this reduces to

$$\alpha_p=\sum_j \gamma_{pj}. \tag{22.2}$$

This is interpreted as the total width being equal to the sum of partial widths for the various processes.

Equation (18) is similar to a many-level Breit-Wigner formula. This relationship between the scattering amplitude and the sum of resonance terms is linear, and this is why it is often simpler to employ in the many-level case than the Wigner-Eisenbud expressions. However, before it can be applied to the case in which only one or two levels contribute, as in the application to low-energy reactions with light nuclei, it is sometimes necessary to make the energy dependence of the γ_p's (which results from the dependence of the boundary conditions on E) explicit, and to determine the imaginary part [the γ_p's defined by Eq. (17.1) are complex]. The determination of these is equivalent to relating the formalism back to the Wigner-Eisenbud one in which the widths are real and independent of energy. The approximate relationship was developed in perturbation theory by Kapur and Peierls,[3] but we prefer to use a procedure developed by A. M. Lane.[19] States $\tilde{\psi}_m{}^{j(0)}(\mathbf{r})$ are defined for which the radial part, multiplied by r, obeys the energy-independent boundary condition

$$\left[\frac{d}{dr}r\tilde{\phi}_m{}^{j(0)}(r)=0\right]_{r=R}, \tag{23}$$

in analogy with Eq. (9.1). By building up $\Phi_0{}^{(q)}$ as

$$\Phi_0{}^{(q)}(\mathbf{r},\xi)=\sum_{j,m} a_{jm}{}^{q(0)}\chi_j(\xi)\tilde{\psi}_m{}^{j(0)}(\mathbf{r}), \tag{23.1}$$

we define energy-independent compound states. The overlap between such states and our energy-dependent compound states can be found by employing Green's theorem again as in Eq. (16). This gives

$$(W_p-W_q{}^{(0)})\int_0^R \Phi_0{}^{(q)}\Phi^{(p)}d^3\xi d^3r = -\frac{\hbar^2}{2M}\int_0^R d^3\xi\int d\Omega\left\{R\Phi_0{}^{(q)}(\mathbf{R},\xi)\times\frac{d}{dr}[r\Phi^{(p)}(\mathbf{r},\xi)]\right\}_{r=R}. \tag{23.2}$$

For the case in which the resonances p are well separated and only S-wave scattering can occur, we have

$$(W_p-W_q{}^{(0)})\int_0^R \Phi_0{}^{(q)}\Phi^{(p)}d^3\xi d^3r = -\frac{i\hbar^2}{2M}\sum_{j,m,n} k_j a_{jm}{}^{q(0)}a_{jn}{}^{p}\tilde{\phi}_m{}^{j(0)}(R)\tilde{\phi}_n{}^{j}(R). \tag{24}$$

[19] A. M. Lane (private communication).

This equation is exact, but we now use perturbation theory starting with the zero-order approximation $\Phi^{(p)}=\Phi_0{}^{(p)}$. To obtain the first-order correction to the energy, we let $p=q$ and set $\Phi_0{}^{(p)}$ equal to $\Phi^{(p)}$ under the integral. This gives

$$W_p-W_p{}^{(0)}\cong -i\alpha_p/2,$$

where we have used Eq. (22.1) and remembered that the $a_{jn}{}^{(p)}$ and $\tilde{\phi}_m{}^j$ are real in this approximation. Hence, the first-order correction to the energy simply adds an imaginary part which is just the width of the state $\Phi^{(p)}$ and which is independent of energy in this order. We can obtain the first-order correction to the wave function by expanding

$$\Phi^{(p)}=\Phi_0{}^{(p)}+\sum c_q{}^p\Phi_0{}^{(q)}, \tag{24.1}$$

and we obtain

$$(W_p-W_q{}^{(0)})c_q{}^p=-\frac{i\hbar^2}{2M}\sum_{j,m,n} k_j a_{jm}{}^{q(0)}a_{jn}{}^{p}\times\tilde{\phi}_m{}^{j(0)}(R)\tilde{\phi}_n{}^{j}(R), \quad (p\neq q). \tag{24.2}$$

The right-hand side is of order $-i\alpha_p/2$, its approximate size for $p=q$, or less. (The signs of the $a_{jm}{}^{q(0)}$ and the $a_{jn}{}^{p}$ can be both positive and negative, and this tends to make the right-hand side smaller than $-i\alpha_p/2$. If a random-phase approximation were applicable, the right-hand side would be zero.) Therefore, for order of magnitude we have

$$c_q{}^p\sim -i\alpha_p/(W_p-W_q{}^{(0)}). \tag{24.3}$$

Thus in the case of well-separated levels, when

$$|W_p-W_q{}^{(0)}|\gg\alpha_p, \tag{24.4}$$

the $c_q{}^p$ and, consequently, the imaginary part of $\Phi^{(p)}$ are small, and the latter is of the order of the ratio of the width to the spacing of the levels. Hence, to a good approximation, the energy dependence of the various parameters and the imaginary parts of the widths are absent in the region of well-separated resonances, so that, except for effects especially sensitive to these features such as the "Thomas Shift," one can usually ignore them.

The arguments in the next section do not neglect the energy dependence or the imaginary parts of the widths; these are included correctly, although we may often use the case of well-separated levels, which is especially simple, as an illustration. These features do not essentially complicate the arguments.

III. THE OPTICAL MODEL AND DIRECT INTERACTION

1. The Physical Picture

The physical picture underlying the optical model was developed chiefly by Feshbach, Porter, and Weisskopf[9]

and Friedman and Weisskopf.[20] These authors split the scattering amplitude S_α into

$$S_\alpha=\langle S_\alpha\rangle_{\text{Av}}+(S_\alpha-\langle S_\alpha\rangle_{\text{Av}}), \tag{25}$$

where $\langle S_\alpha\rangle_{\text{Av}}$ is the value of S_α averaged over an energy interval I. The averaging process is defined more precisely later. The average elastic cross section can now be expressed as

$$\langle\sigma_\alpha\rangle_{\text{Av}}=|\langle S_\alpha\rangle_{\text{Av}}|^2+\langle|S_\alpha-\langle S_\alpha\rangle_{\text{Av}}|^2\rangle_{\text{Av}}, \tag{26}$$

since the cross term between $\langle S_\alpha\rangle_{\text{Av}}$ and $S_\alpha-\langle S_\alpha\rangle_{\text{Av}}$ averages out. Although both terms on the right-hand side represent elastic scattering, they are physically quite different in nature.

Friedman and Weisskopf,[20] who used a time-dependent treatment, showed that if one constructs a wave packet of width ΔE in energy, then the scattered particles corresponding to the average amplitude $\langle S_\alpha\rangle_{\text{Av}}$, i.e., those given by the first term on the right-hand side of Eq. (26), pass over the nucleus in time $\tau=\hbar/\Delta E=\hbar/I$. This is, in fact, clearly allowed by the uncertainty principle, because one has, by averaging, constructed a scattering amplitude $\langle S_\alpha\rangle_{\text{Av}}$ which varies appreciably only over an energy interval I; since the energy of particles associated with this scattering is undefined within this interval, the time τ which the particle spends in the nucleus can be defined to within $\hbar/I$. Hence, if the energy interval I is very large, the particles corresponding to the average amplitude go over the nucleus very quickly. It is therefore reasonable that for sufficiently large intervals I, the particles make only a few interactions with the nuclear particles—even though the nucleon-nucleon potentials are relatively strong—and that we can describe them in the weak-coupling type picture indicated by the optical model. This scattering, which corresponds to the average phase, is termed "shape-elastic." Later we find precisely how large the interval I must be for various weak-coupling type descriptions of the average scattering to be valid.

On the other hand, the fluctuation scattering (often called compound-elastic scattering), described by the second term on the right-hand side of Eq. (26), varies over energy intervals of the order of the width of the compound nuclear states ($\sim$1 ev at low energies). Hence, the corresponding particles stay in the nucleus the order of a million times longer than those of the shape-elastic scattering that would correspond to a wave packet of the order of 1 Mev wide. If the particles corresponding to the shape-elastic scattering have time for one or two collisions, then those corresponding to the fluctuation scattering stay in the nucleus long enough to make the order of a million collisions. The latter particles can be regarded as those forming compound states of the type envisaged in the old compound-nucleus picture, whereas the shape-elastic scattering corresponds to that from the complex well, the imaginary part of which describes the disappearance of particles out of the incident beam into the long-lived component.

We define the complex well $\tilde{V}(r)$ so that it reproduces the average scattering phase, i.e., so that $\tilde{S}_\alpha=\langle S_\alpha\rangle_{\text{Av}}$. The physical picture indicates that this is the most reasonable procedure. We then go on to discuss the characteristics of this well $\tilde{V}(r)$ and relate the parameters back to nucleon-nucleon forces.

2. The Picture of Lane, Thomas, and Wigner

Possibly the most striking feature of experimental data on elastic neutron scattering was the appearance of the giant resonances in the total cross section in the energy region $\sim$0 to 3 Mev, as discussed in Sec. I. In order to understand these, we develop the picture of Lane, Thomas, and Wigner.[21] In this development we assume that the well $\tilde{V}(r)$, which reproduces the average scattering amplitude ($\tilde{S}_\alpha=\langle S_\alpha\rangle_{\text{Av}}$), has the characteristics of the optical model potential of Feshbach, Porter, and Weisskopf[9]; that is, if $\tilde{V}=-U-iW$ then $U\cong 42$ Mev, W lies in the range 1 to 2 Mev, and the radius of the potential is $R\cong 1.45\,A^{\frac{1}{3}}\times 10^{-13}$ cm. These parameters are used only for order-of-magnitude estimates; knowledge of their precise values is not necessary for understanding the conceptual points considered here. Later in this section we show how to relate these parameters back to nucleon-nucleon forces.

In the picture by Lane, Thomas, and Wigner,[21] for $W_p\cong\epsilon_n$, i.e., for energies of the compound states in the neighborhood of the single-particle energy ϵ_n, only terms $m=m'=n$ in the width γ_p of Eq. (17.1) are important, i.e., only the $(a_{on}{}^p)^2$ are large if W_p lies in the neighborhood of ϵ_n. We now make these criteria more quantitative. For simplicity, we consider the scattering of S-wave neutrons in the region of well-separated levels where the γ_p can be taken to be real.

We write the amplitude for compound-elastic scattering, defined by

$$S_\alpha{}^{ce}\equiv S_\alpha-\langle S_\alpha\rangle_{\text{Av}}, \tag{27}$$

in dispersion theory, using the fact that we choose $\tilde{V}$ so that $\tilde{S}_\alpha=\langle S_\alpha\rangle_{\text{Av}}$. Then,

$$S_\alpha{}^{ce}=S_\alpha-\tilde{S}_\alpha=\frac{e^{-2ikR}}{2k}\left\{\sum_p\frac{\gamma_p}{W_p-E}-\sum_m\frac{\Gamma_m}{\tilde{E}_m-E}\right\}, \tag{27.1}$$

where the second term in brackets gives the scattering amplitude in the complex well, and $\tilde{E}_m$ is the eigenvalue in the complex well, which may be taken to have the form, Eq. (9.13), $\tilde{E}_m=\epsilon_m-i\beta_m/2-iW$. (We are considering the complex well to be a square well here, although generalization to the case where it varies with r is easy.) From our definition, $\langle S_\alpha{}^{ce}\rangle_{\text{Av}}$, the average of

[20] F. L. Friedman and V. F. Weisskopf, *Niels Bohr and the Development of Physics* (Pergamon Press, London, 1955), p. 134.

[21] Lane, Thomas, and Wigner, Phys. Rev. **98**, 693 (1955).

S^{ce}, is zero. Hence, the average of the quantity in brackets must vanish.[22] The intervals over which we average are always small compared with the widths of the single-particle resonances, so the second term in brackets need not be averaged. We can write

$$\left\{\sum \frac{\gamma_p}{W_p-E}\right\}_{\text{Av}}=P(E)+iQ(E)=\sum_m \frac{\Gamma_m}{\tilde{E}_m-E}, \quad (28)$$

where P and Q are real. We find

$$P(E)=\sum_m \frac{(\epsilon_m-E)\Gamma_m}{(\epsilon_m-E)^2+(W+\beta_m/2)^2}, \quad (28.1)$$

and

$$Q(E)=\sum_m \frac{\Gamma_m(W+\beta_m/2)}{(\epsilon_m-E)^2+(W+\beta_m/2)^2},$$

where Γ_m is assumed to be real, which implies specialization to the low-energy region. We cannot express $P(E)$ more simply in terms of γ_p, but the imaginary part of the sum $Q(E)$ can be put into a more useful form.

The average $\langle F(E)\rangle_{\text{Av}}$ of a quantity $F(E)$ is defined by

$$\langle F(E)\rangle_{\text{Av}}=\int_{-\infty}^{\infty}\rho(E-E')F(E')dE'. \quad (29)$$

The weighting function ρ employed by Feshbach, Porter, and Weisskopf[9] was a square one,

$$\rho(x)=\begin{cases}0, & x<-I/2\\ 1/I, & -I/2<x<I/2,\\ 0, & x>I/2\end{cases} \quad (29.1)$$

so that

$$\langle F(E)\rangle_{\text{Av}}=\frac{1}{I}\int_{E-I/2}^{E+I/2}F(E')dE'.$$

With such integrals, one has end effects coming from resonances where ϵ_p lies near either $E-I/2$ or $E+I/2$, and these must be disposed of, somewhat inelegantly. Physical results must be independent of the precise form of weighting function employed, as long as it is not an unreasonable one, and we find it convenient to use

$$\rho(E-E')=\frac{I}{\pi}\frac{1}{(E-E')^2+I^2},$$

in which case

$$\left\{\sum_p \frac{\gamma_p}{W_p-E}\right\}_{\text{Av}}=\frac{I}{\pi}\int_{-\infty}^{\infty}\sum_p \frac{\gamma_p}{W_p-E'}\frac{1}{(E-E')^2+I^2}dE'$$

$$=\sum_p \frac{\gamma_p}{W_p-E-iI}, \quad (30)$$

where the integral can be evaluated by contour integration. Quite generally, averaging with this weighting function a quantity $F(E)$ which has poles only in the lower half of the complex E plane (such functions are often called R functions following Wigner), one finds

$$\langle F(E)\rangle_{\text{Av}}=F(E+iI), \quad (31)$$

where I corresponds to the interval over which the average is carried out. We always assume this interva to contain many resonances p.

We find, then,

$$Q(E)=\text{Im}\sum_p \frac{\gamma_p}{W_p-E-iI}. \quad (32)$$

By defining the average density of the levels in the region of interest by $1/D$ (where $1/D=N$ gives the number of levels per unit energy, which is assumed to be large), one can convert the sum in Eq. (32) into an integral, obtaining

$$Q(E)=\frac{1}{D}\int_{-\infty}^{\infty}\frac{\bar{\gamma}I}{(\epsilon_p-E)^2+I^2}d\epsilon_p=\frac{\pi\bar{\gamma}}{D}, \quad (32.1)$$

where we have neglected α_p compared with I, and where $\bar{\gamma}$ is the average width. The function $\pi\bar{\gamma}/D$ is called the "strength function." Our assumption of well-separated levels is equivalent to $\pi\bar{\gamma}/D\ll 1$.

Finally, we find

$$\pi\frac{\bar{\gamma}}{D}=\text{Im}\sum_m \frac{\Gamma_m}{\tilde{E}_m-E}, \quad (32.2)$$

when we make the average value of the imaginary part of the quantity in brackets, Eq. (27.1), vanish, using $\langle S^{ce}\rangle_{\text{Av}}=0$. In the special case where the energy E is in the neighborhood of ϵ_n, the main contribution to the sum on the right-hand side comes from the nth term provided that the single-particle resonances are sufficiently far apart. Then,

$$\pi\frac{\bar{\gamma}}{D}\cong\frac{\Gamma_n(W+\beta_n/2)}{(\epsilon_n-E)^2+(W+\beta_n/2)^2}, \quad E\cong\epsilon_n. \quad (33)$$

The strength function, for constant W, Γ_n, and β_n, is thus of the Lorentz form. The contribution of the other levels is at least of order $W/\Delta\epsilon$ smaller,[23] where $\Delta\epsilon$ is

[22] In the low-energy region, where the k-dependent factor $k^{-1}\exp(-2ikR)$ and the penetrabilities in the widths vary rapidly with bombarding energy, these factors should be divided out before the average is taken. This is done explicitly in the next section.

[23] Since the sum in Eq. (32.2) relates only to the single-particle well, one can carry out the calculation of the sum explicity for special cases, such as a square well, and verify that if $\epsilon\cong\epsilon_n$, the sum of terms $m\neq n$, contributes in order $W/\Delta\epsilon$ (see Appendix A, reference 16).

the distance between single-particle levels of the *same angular momentum*, i.e., of the same channel α. This distance is of the order of U, the real part of $\tilde{V}$.

The validity of the picture depends on $W \ll U$, which is also one of the criteria for the existence of marked resonances in $\tilde{V}$. In the range of separated compound-nucleus resonances, the condition $kR \ll 1$ is fulfilled, as appears later, and then the maxima in the strength function of resonance shape given by Eq. (33) are simply related to the total cross section σ_T, as can be seen by using the optical relation

$$\sigma_T = (4\pi/k)\ \mathrm{Im}\ S_\alpha. \tag{34}$$

Hence

$$\langle \sigma_T \rangle_{\mathrm{Av}} = \frac{4\pi}{k}\ \mathrm{Im}\langle S_\alpha \rangle_{\mathrm{Av}} = \frac{4\pi}{k}\ \mathrm{Im}\ \tilde{S}_\alpha \cong \frac{2\pi}{k^2}\pi\frac{\bar{\gamma}}{D}, \tag{34.1}$$

which we have obtained from Eq. (32.2).

This situation just discussed is illustrated in Fig. 6, where the distance between levels $\epsilon_{n'} - \epsilon_n$ is assumed to be much greater than W, so that the picture of Lane, Thomas, and Wigner is valid. The $4s$ and $5s$ single-particle levels are shown on the left-hand side, and it is illustrated on the right-hand side how—although each single-particle level is split up into many compound states—there is an appreciable probability of finding them only within an energy region of width $\sim W$ bracketing the single-particle energy.[24] Hence, although the interaction $V(\mathbf{r},\xi)$ is strong enough to change the wave function completely so as to split the single particle level $\tilde{\psi}_n$ into thousands or millions of compound states, it is not strong enough to mix the single-particle level $\tilde{\psi}_{n'}$, which is a distance of order $\epsilon_{n'} - \epsilon_n$ away, into the compound state in the neighborhood of ϵ_n with appreciable probability.

We might ask how well the condition $(\epsilon_{n'} - \epsilon_n) \gg W$ is fulfilled in the actual physical case, taking the parameters of Feshbach, Porter, and Weisskopf[9] for $\tilde{V}$.

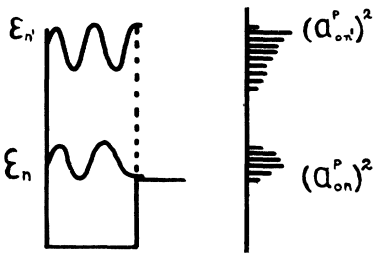


FIG. 6. On the left, S-wave single-particle resonances in the complex well are shown; on the right, the behavior of the square of the expansion coefficients. Behavior of the strength function is essentially the same as this.

By way of example, consider the nucleus A = 160 where the $4s$ single-particle resonance occurs near zero energy experimentally, as in Fig. 6. The occurrence of the $4s$ resonance requires $K_n R = (7/2)\pi$, where K_n is the wave number for a particle of zero energy, measured from the bottom of the well. The $5s$ resonance occurs for $K_{n'} R = (9/2)\pi$. Hence, $\epsilon_{n'}/\epsilon_n = 81/49$, where ϵ_n and $\epsilon_{n'}$ are now measured from the bottom of the well. For a well depth of 42 Mev, $\epsilon_{n'} - \epsilon_n \equiv \Delta\epsilon = 27$ Mev. Thus there is no doubt that $W \ll \Delta\epsilon$ is the low-energy region, where W is of order 1–2 Mev.[25]

Although the single-particle width Γ_n is split up among thousands or millions of compound states, some features of the single-particle resonance remain. In particular, since to a very good approximation,

$$\gamma_p \cong (a_{on}{}^p)^2 \Gamma_n, \quad E \cong \epsilon_n,$$

and since it follows from the completeness of our two representations connected by the coefficients $a_{jm}{}^p$ that $\sum_p (a_{on}{}^p)^2 = 1$, then

$$\sum_p{}' \gamma_p \cong \Gamma_n,$$

where the prime on the sum indicates that it is extended only over compound states in the neighborhood of ϵ_n. In other words, the single-particle width is split up among many levels, but the sum of widths of these many levels is just equal to the single-particle width.

At this stage we have an understanding of why the compound resonances p are so narrow without resorting to the classical picture which explains this in terms of the energy being shared among all of the nucleons. The interaction $V(\mathbf{r},\xi)$ is sufficiently strong so as to mix many states $\chi_j(\xi)\tilde{\psi}_m{}^j(\mathbf{r})$ into the state $\Phi^{(p)}$. Since

$$\sum_{j,m} (a_{jm}{}^p)^2 = 1,$$

following from the normalization of the $\Phi^{(p)}$, and since there are many terms which contribute in this sum, then $(a_{on}{}^p)^2 \ll 1$ for any given p and the width of the compound resonance is much less than that of the single-particle one.

Nonetheless, the interaction $V(\mathbf{r},\xi)$ is not strong enough to break down the underlying single-particle structure completely and, in particular, it does not mix different single-particle levels n and n' into the same $\Phi^{(p)}$, but only spreads the single-particle resonance locally. Consequently, it is not surprising that some of the single-particle features remain in experiments carried out with wide beams.

For completeness, we remark that the width of the strength function is given essentially by W as shown

[24] We have made arguments only for the case of separated levels where the $a_{on}{}^p$ are essentially real, and therefore our arguments do not apply in detail to the higher single-particle resonance n'. However, it is clear from them that the $(a_{on'}{}^p)^2$ has a spread of order W, where W is the imaginary part of $\tilde{V}$ (which is velocity-dependent) at the excitation energy $\epsilon_{n'}$.

[25] The discussion of the contribution of terms $m \neq n$ in Eq. (32.2) was carried out taking the same W for all terms m, contrary to the spirit of the other development where we think of $\tilde{V}$ as a velocity-dependent potential, or, equivalently, a nonlocal potential $\tilde{V}(\mathbf{r}',\mathbf{r})$, in which case the W's entering into the widths of different single-particle levels are quite different. However, the sum of terms $m \neq n$ must be the same in either method.

in Eq. (33) (although this equation is a reasonable approximation only when E is in the neighborhood of ϵ_n), and not by the square root of the second moment of the perturbing potential, as Lane, Thomas, and Wigner[21] indicated might be the case. Bloch[26] has pointed out, using some rough quantitative estimates, that one should expect the root-mean-square moment to be much larger than the width owing to large contributions from the wings of the strength function which are weighted heavily. Use of the form Eq. (33) for all energies would lead to an infinite second moment which would contradict the finite expression for it in terms of the potential,[27] so that this form is clearly not valid far from the resonance energy.

3. Low-Energy Behavior of the Scattering[28]

At low energies, where $(\pi\bar{\gamma}/D)\ll 1$ and $kR\ll 1$, the interpretation of average quantities is especially simple. Equation (34.1) shows that the average cross section is proportional to the strength function in this region. The maxima in the strength function indicated by Eq. (33) would be difficult to observe experimentally by measuring the strength function at various energies for a given nucleus because the single-particle resonance is broad, of width $2W+\beta_n$, and even if the same experimental technique could be used over this broad energy interval, the interpretation would become difficult, both because waves of higher angular momenta would be mixed into the experiments and because our simple consideration—which applied only to the region of well-separated resonances where $kR\ll 1$—would no longer be valid. Consequently, the strength function is usually measured at low energies for different nuclei and plotted as a function of A. In this way a variation in the quantity $E-\epsilon_n$ is obtained which depends on KR, where K is the wave number measured from the bottom of the well. Some experimental data, plotted in this way, are shown in Fig. 7.[29] The theoretical curve is not computed here from dispersion formalism but is obtained from

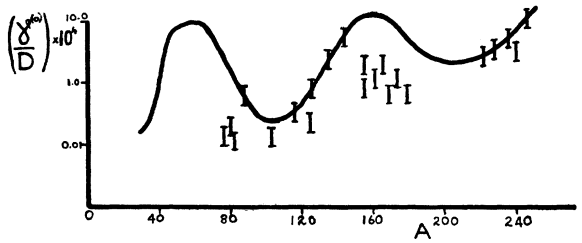


FIG. 7. Experimental data for the strength function. The solid line indicates typical theoretical predictions for a complex-well model. A measured width γ_p is reduced to a $\gamma^{(0)}$ by using $\gamma^{(0)}=\gamma_p(k_0/k)$, where k_0 is taken to be the wave number for a neutron of energy 1 ev. A more refined theory[29] takes into account nuclear deformation, and the resulting agreement between theory and experiment is then better.

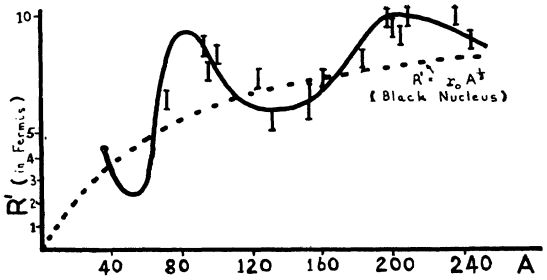


FIG. 8. Experimental data compared with theoretical predictions for R' (solid line) from the complex-well model. A more refined theory[31] takes into account nuclear deformations, and the resulting agreement between theory and experiment is then better.

$$\pi\frac{\bar{\gamma}}{D}=2k\ \mathrm{Im}\ \tilde{S}_\alpha, \tag{34.2}$$

[see Eq. (34.1)], where $\tilde{S}_\alpha$ is calculated directly by integrating the Schrödinger equation with potential $\tilde{V}$. Nevertheless, our formulas are useful in understanding the qualitative behavior in the resonance region.

Historically the first striking observations of the nonmonotonic behavior of average cross sections with changing A referred to measurement[30] of a quantity sometimes called the "potential scattering." To obtain this quantity, we expand in k, considering k to be small, but must carry the expansion one step further, i.e., we express the average cross section $\langle\sigma_T\rangle_{Av}$ as

$$\langle\sigma_T\rangle_{Av}=(a/k)+b, \tag{35}$$

which can be done at low energies where terms of higher order in k are negligible. The first term on the right-hand side is the familiar $1/v$ term in the cross section which is related to the strength function and which was already discussed in Eqs. (34.1) and (34.2), and the second term is often expressed as

$$b=4\pi(R')^2, \tag{35.1}$$

where R' is interpreted as a radius. The observed behavior of R' with atomic number A is shown in Fig. 8.[31]

A simple description of b is difficult to give in the Kapur-Peierls theory in so far as the imaginary parts of the width γ_p [see Eq. (23) ff.] are of order k compared with the real parts, and these must therefore be taken into account. It is consequently more straightforward to give this description in the Wigner-Eisenbud for-

[26] C. Bloch, Nuclear Phys. **3**, 137 (1957).
[27] The second moment is clearly finite for well-behaved potentials which are used here.
[28] Many of the results in this section were first derived in the R-matrix formalism by R. G. Thomas, Phys. Rev. **97**, 224 (1955).
[29] Hughes, Zimmerman, and Chrien, Phys. Rev. Letters **1**, 461 (1958).
[30] Fields, Russell, Sachs, and Wattenberg, Phys. Rev. **71**, 308 (1947).
[31] Seth, Hughes, Zimmerman, and Garth, Phys. Rev. **110**, 692 (1958).

malism. We sketch this briefly, following the treatment of Lane and Lynn,[32] and refer to the review article of Lane and Thomas[1] for a detailed discussion.

The expansion is actually in powers of $\rho=kR$, as in Eq. (9.5) ff., because ρ is a good expansion parameter at the energies of interest. To make this explicit, we write

$$2ik\langle S\rangle_{Av}=e^{-2i\rho}-1+2i\rho e^{-2i\rho}\langle R_{KP}\rangle_{Av}, \tag{36}$$

where

$$R_{KP}=\sum_p \chi_p/(W_p-E), \tag{36.1}$$

R_{KP} being the Kapur-Peierls R function. (It has poles only in the lower half of the complex plane.) χ_p is the reduced width,

$$\chi_p=(2\rho)^{-1}\gamma_p. \tag{36.2}$$

The relation between R_{KP} and the Wigner R function R_W is easily obtained by looking at the scattering matrix in the two theories. It is

$$R_{KP}=R_W/(1-i\rho R_W). \tag{37}$$

In averaging R_{KP} by adding iI to E, we find that

$$\langle R_{KP}\rangle_{Av}=\bar{R}_W/(1-i\rho\bar{R}_W), \tag{37.1}$$

(see pp. 306–309 of Lane and Thomas), where

$$\bar{R}_W=R_W{}^{\infty}+i\pi s_W \tag{37.2}$$

with $R_W{}^{\infty}$ and s_W now real, and given by

$$s_W=\frac{\bar{\chi}^{(0)}}{D}, \quad R_W{}^{\infty}=\int\frac{s_W(E')}{E'-E}dE'. \tag{37.3}$$

Here $\bar{\chi}^{(0)}$ is the average of $\chi_p{}^{(0)}$, the width with the Wigner-Eisenbud boundary conditions [see Eq. (23)]. The expansion to any required order is now easily made.

$$\frac{4\pi}{k}\operatorname{Im}\langle S_\alpha\rangle_{Av}=\frac{4\pi\rho}{k^2}(\pi s_W)+\frac{4\pi\rho^2}{k^2}\left[(1-R_W{}^{\infty})^2-(\pi s_W)^2\right]+O(\rho^3). \tag{38}$$

The left-hand side of Eq. (38) is equal to $\langle\sigma_T\rangle_{Av}$ by the relation Eq. (34.1). We now can identify easily the a and b of Eq. (35); thus,

$$a=\frac{4\pi\rho}{k^2}(\pi s_W),$$

and

$$b=4\pi\rho^2[(1-R_W{}^{\infty})^2-(\pi s_W)^2]. \tag{38.1}$$

To lowest order in ρ, $2\rho\pi s_W=\pi\bar{\gamma}/D$ and, retaining only a, we see that Eq. (38) reduces to our earlier expression, Eq. (34.1).

This derivation is independent of any particular model, although the model introduced in the last section is helpful in understanding the qualitative behavior of s_W and $R_W{}^{\infty}$ in the actual physical case. We have already indicated the resonance-type behavior in the strength function in Eq. (33). We see immediately from Eq. (37.3) that to the extent that the strength function is symmetrical about its maximum, $R_W{}^{\infty}$ is zero at this maximum which occurs at the real part of the single-particle resonance energy. Further, $R_W{}^{\infty}$ becomes negative on one side of the maximum and positive on the other so that one can understand the qualitative behavior of R' shown in Fig. 8.

Theoretical curves, such as those shown by the solid lines in Figs. 7 and 8, are obtained by calculating $\tilde{S}_\alpha$ and then using $\langle S_\alpha\rangle_{Av}=\tilde{S}_\alpha$.

In past work the term $(\pi s_W)^2$ has often been neglected. However, near the single-particle resonance at A=50, s_W is appreciable, as R. K. Seth has pointed out,[33] and it is vital to include the term $(\pi s_W)^2$ in analyzing data near A=50. Otherwise $(1-R_W{}^{\infty})$ comes out to be too large.

This treatment illustrates the type of problem in which the Wigner-Eisenbud formalism is useful. It is advantageous here to deal with the real quantities of this theory in which the energy dependence is explicit.

Reverting to the Kapur-Peierls formalism, we can show the relation between the average cross section for forming a compound nucleus and the W in this energy region, to lowest order in k. The former is defined as

$$\langle\sigma_c\rangle_{Av}\equiv\langle\sigma_T\rangle_{Av}-\sigma_{se},$$

where σ_{se} is the cross section for shape-elastic scattering,

$$\sigma_{se}=4\pi|\langle S_\alpha\rangle_{Av}|^2=4\pi|\tilde{S}_\alpha|^2.$$

Use of the relation between $\langle\sigma_T\rangle_{Av}$ and $\operatorname{Im}\tilde{S}_\alpha$ given in Eq. (34.1) results in

$$\langle\sigma_c\rangle_{Av}=4\pi\left\{\frac{1}{k}\operatorname{Im}\tilde{S}_\alpha-|\tilde{S}_\alpha|^2\right\}. \tag{38.2}$$

By using the resonance expansion for $\tilde{S}_\alpha$ [obtained from Eq. (7) by replacing E_m by $\tilde{E}_m$] and remembering that Γ_n is of order kR, we find, to lowest order in kR,

$$\langle\sigma_c\rangle_{Av}\cong\frac{4\pi}{k^2}\sum_m\frac{W\Gamma_m/2}{(E-\epsilon_m)^2+W^2}, \tag{38.3}$$

where we see that the shape-elastic scattering does not contribute in this order. To interpret this formula we consider $\psi_\alpha(\mathbf{r})$, the solution of the Schrödinger equation in the complex well. Since we are dealing with S waves we can expand ϕ_α, the radial part of ψ_α multiplied by r, in terms of eigenstates $\tilde{\phi}_m$. The coefficient has already been derived in Eq. (4.3). Thus

$$\phi_\alpha=\frac{\hbar^2}{2M}e^{-ikR}\sum_m\frac{\tilde{\phi}_m(R)\tilde{\phi}_m(r)}{\tilde{E}_m-E}. \tag{38.4}$$

[32] A. M. Lane and J. E. Lynn, "The widths and spacings of resonance levels," Harwell Report T/R 2210 (1957, unpublished).

[33] R. K. Seth, Optical Model Conference, Tallahassee, Florida, March, 1959.

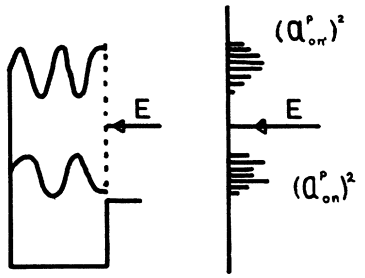


FIG. 9. Illustration of the position of the neutron energy with respect to the single-particle resonances in our "simple case of scattering."

This formula clearly shows the resonance behavior of the probability

$$\mathcal{P}_\alpha \equiv \int_0^R |\phi_\alpha(r)|^2 dr = \left(\frac{\hbar^2}{2M}\right)^2 \sum_m \frac{[\tilde{\phi}_m(R)]^2}{(E-\epsilon_m)^2+W^2}, \quad (39)$$

of finding a particle inside the nucleus whose wave function has unit amplitude asymptotically. Here we have dropped β_m which is of order kR smaller than W. It is clear that

$$\langle \sigma_c \rangle_{Av} = \frac{8\pi M}{\hbar^2 k} \mathcal{P}_\alpha W. \quad (40)$$

In other words, the average cross section for compound-nucleus formation just defined is proportional to the absorption W times the probability $\mathcal{P}_\alpha$ of finding the nucleon inside the nucleus. In a somewhat classical way, one might interpret W as the absorption per unit nucleon in the nucleus.

4. A Simple Case of Scattering

Much of the relationship of our detailed picture, where we make a complete description in terms of the $\Phi^{(p)}$ to the simplified complex-well model, can be understood by considering an especially simple case[34,35] in which the energy of the incident particle falls between single-particle resonance energies. Let us suppose, referring to Fig. 6, that the energy E of the incident particle lies midway between ϵ_n and $\epsilon_{n'}$, as in Fig. 9. The amplitude for elastic scattering coming from the lower group of levels is

$$(S_\alpha)_n = \frac{e^{-2ikR}}{2k} \sum_p \frac{(a_{on}{}^p)^2 \Gamma_n}{W_p - E} - \frac{\sin kR}{k} e^{-ikR}, \quad (41)$$

where the lower suffix n on S_α indicates that this is the S_α that would result if only the lower group of levels existed. We have here used the model developed in the last section and dropped the $a_{on'}{}^p$ in the widths γ_p referring to the set of compound-nucleus levels lying around ϵ_n.

The W_p are spread a distance W about ϵ_n. Consequently, if $E-\epsilon_n \gg W$, then a good first approximation is to set $E-W_p$ equal to $E-\epsilon_n$ in the denominator of Eq. (41). Evaluation of the sum over p gives in this approximation,

$$(S_\alpha)_n = \frac{e^{-2ikR}}{2k} \frac{\Gamma_n}{\epsilon_n - E} - \frac{\sin kR}{k} e^{-ikR}, \quad (42)$$

which is just the contribution to S_α from the nth resonance in the complex-well picture if we neglect W compared with $E-\epsilon_n$ in the denominator.

Therefore, if the spread of the γ_p about the single-particle resonance energy is small, then the whole set of them act together just as the single-particle resonance. This is related to the principle of spectroscopic stability, from which it follows that—although the wave functions may be drastically changed by the perturbation—certain simple features come out just as in the unperturbed problem.

The main oversimplification is in the neglect of the widths γ_p which lie close to E, i.e., the γ_p are not so well concentrated around ϵ_n and $\epsilon_{n'}$ as we have shown. However, our approximations are good in so far as the "shape-elastic" scattering is concerned. Our neglect of the near levels has simply resulted in our dropping the compound-elastic scattering. We now make the development more precise in such a way that the near levels can also be treated. Such a treatment is clearly necessary in order to consider general E which may be in the region of a single-particle resonance.

5. Expression of the Scattering in the Green's Function Formalism

In defining $\tilde{V}(r)$, it is convenient to express S^{ce} in terms of the Green's function as was done by Bloch.[36] Actually, Bloch's elegant formalism is much more general than that we use, and by specialization to specific representations he can obtain either the Kapur-Peierls or Wigner-Eisenbud expression. However, our simpler formalism is adequate for illustrating the points considered here. We again specialize our consideration to the scattering of S-wave neutrons; generalization of our results to other cases is obvious.

We use the notation

$$\Phi^{(p)}(\mathbf{r},\xi) = |p\rangle = \langle p|, \quad \chi_j(\xi)\tilde{\psi}_m{}^j(\mathbf{r}) = |jm\rangle = \langle jm|, \quad (43)$$

and

$$\langle p|jm\rangle = \int_0^R \Phi^{(p)}(\mathbf{r},\xi)\chi_j(\xi)\tilde{\psi}_m{}^j(\mathbf{r}) d^3r d^3\xi = a_{jm}{}^p.$$

No distinction is made between $|p\rangle$ and $\langle p|$ since expansion and orthogonality conditions in this theory involve only the wave function and not its complex conjugate, aside from the trivial changes in functions of angles mentioned in part 2 of this section.

[34] J. Bowcock, Proc. Phys. Soc. (London) **A70**, 515 (1957).
[35] G. E. Brown and C. T. De Dominicis, Proc. Phys. Soc. (London) **A70**, 668 (1957).
[36] C. Bloch, Nuclear Phys. **4**, 503 (1957).

We then have

$$S_\alpha{}^{ce}=\frac{e^{-2ikR}}{2k}\left\{\sum_p\frac{\gamma_p}{W_p-E}-\sum_m\frac{\Gamma_m}{\tilde{E}_m-E}\right\},$$

$$=\frac{\hbar^2}{2M}e^{-2ikR}\sum_{m,m'}\left\{\sum_p\langle p\,|\,om\rangle\langle p\,|\,om'\rangle\left\langle p\left|\frac{1}{H-E}\right|p\right\rangle\right.$$

$$\left.-\left\langle om\left|\frac{1}{\tilde{H}-E}\right|om'\right\rangle\right\}\tilde{\phi}_m(R)\tilde{\phi}_{m'}(R), \quad (44)$$

$$=\frac{\hbar^2}{2M}e^{-2ikR}\sum_{m,m'}\left\{\left\langle om\left|\frac{1}{H-E}\right|om'\right\rangle\right.$$

$$\left.-\left\langle om\left|\frac{1}{\tilde{H}-E}\right|om'\right\rangle\right\}\tilde{\phi}_m(R)\tilde{\phi}_{m'}(R),$$

where $\tilde{H}$ is obtained from H by replacing $V(\mathbf{r},\xi)$ by $\tilde{V}$, i.e.,

$$H=\tilde{H}+(V(\mathbf{r},\xi)-\tilde{V}(r))=\tilde{H}+\delta V. \quad (44.1)$$

By using the identity

$$\frac{1}{H-E}=\frac{1}{\tilde{H}-E}-\frac{1}{\tilde{H}-E}(V-\tilde{V})\frac{1}{\tilde{H}-E}$$

$$+\frac{1}{\tilde{H}-E}(V-\tilde{V})\frac{1}{H-E}(V-\tilde{V})\frac{1}{\tilde{H}-E}, \quad (45)$$

which is easily checked by multiplying both sides of the equation by $H-E$, we obtain

$$S_\alpha{}^{ce}=-\frac{\hbar^2}{2M}e^{-2ikR}\sum_{m,m'}\frac{1}{\tilde{E}_m-E}\frac{1}{\tilde{E}_{m'}-E}$$

$$\times\left\{\langle om\,|\,V-\tilde{V}\,|\,om'\rangle-\left\langle om\left|(V-\tilde{V})\frac{1}{H-E}\right.\right.\right.$$

$$\left.\left.\left.\times(V-\tilde{V})\right|om'\right\rangle\right\}\tilde{\phi}_m(R)\tilde{\phi}_{m'}(R). \quad (46)$$

At this stage, by using Eq. (38.4) we obtain $S_\alpha{}^{ce}$ in a convenient form,

$$S_\alpha{}^{ce}=-\frac{2M}{\hbar^2}\left\{\langle o\alpha\,|\,V-\tilde{V}\,|\,o\alpha\rangle\right.$$

$$\left.-\left\langle o\alpha\left|(V-\tilde{V})\frac{1}{H-E}(V-\tilde{V})\right|o\alpha\right\rangle\right\}, \quad (47)$$

where ψ_α is the scattering state defined by Eq. (11) and the discussion following it. Further expansion of $1/(H-E)$ by iteration with Eq. (45) would give the Born expansion for $S_\alpha{}^{ce}$, i.e., the expansion in successive powers of the perturbing potential δV. Equation (47) can be derived directly from scattering theory without recourse to dispersion theory.

Insertion of the unit operator $\sum_p|p\rangle\langle p|$ in the second term on the right-hand sides of Eqs. (46) and (47) allows replacement of $(H-E)^{-1}$ by $(W_p-E)^{-1}$, indicating that all of the poles of this term lie in the lower half of the complex energy plane. Thus we can average this term with respect to energy by replacing E by $E+iI$, following the procedure of Eqs. (30) and (31).

$$\langle S_\alpha{}^{ce}(E)\rangle_{Av}=S_\alpha{}^{ce}(E+iI). \quad (48)$$

The complex potential $\tilde{V}(r)$ is defined by the condition $\tilde{S}_\alpha(E)=\langle S_\alpha(E)\rangle_{Av}$, or by $\langle S_\alpha{}^{ce}(E)\rangle_{Av}=0$. This can be accomplished by requiring

$$\langle o\,|\,V-\tilde{V}\,|\,o\rangle$$

$$-\left\langle o\left|(V-\tilde{V})\frac{1}{H-E-iI}(V-\tilde{V})\right|o\right\rangle=0. \quad (49)$$

One can therefore define an optical potential $\tilde{V}(r)$ which reproduces the average scattering phase. At this stage, not very much has been proved because the formalism does not make it clear that $\tilde{V}$ does not vary rapidly and possibly nonmonotonically with energy; its usefulness empirically comes from the fact that it varies slowly and regularly with energy and with atomic number A. We now go as far as we can towards demonstrating that $\tilde{V}$ has these characteristics. In order to do this, we make the connection between the parameters of $\tilde{V}$ and the nucleon-nucleon potentials.

6. Perturbation Theory of the First Kind

In the perturbation theory to be developed, we will have to carry out an expansion in $\delta V=V(\mathbf{r},\xi)-\tilde{V}(r)$ since we wish to obtain expressions for quantities which do not contain the complicated $\Phi^{(p)}$. The smallness of the widths of the $\Phi^{(p)}$ compared with those of the single-particle states $\tilde{\psi}_m$ which, together with χ_j we employ as unperturbed wave functions, indicates that a large number of terms are necessary in the expansion Eq. (14) of $\Phi^{(p)}$ in terms of $\chi_j(\xi)\tilde{\psi}_m{}^j(\mathbf{r})$. Thus, an expansion of $\Phi^{(p)}$ in perturbation theory, starting from $\chi_0(\xi)\tilde{\psi}_m{}^i(\mathbf{r})$ as zero-order function, would converge very slowly, if at all. However, the perturbation expansions of the average phase involve less stringent criteria for convergence than those for the wave functions as might be suspected from the physical arguments of the first part of this section. We now formulate these criteria, following the development of Brown, De Dominicis, and Langer.[37]

$\mathfrak{W}$ is defined as

$$\tilde{V}=\bar{V}-\mathfrak{W}, \quad (49.1)$$

where

$$\bar{V}\equiv\langle o\,|\,V\,|\,o\rangle. \quad (49.2)$$

[37] Brown, De Dominicis, and Langer, Ann. Phys. (N. Y.) **6**, 209 (1959).

Our defining equation, Eq. (49), for $\tilde{V}$ becomes

$$\left\langle o\left|(V-\bar{V}+\mathfrak{W})\left[1-\frac{1}{H-E-iI}(V-\bar{V}+\mathfrak{W})\right]\right|o\right\rangle=0. \tag{49.3}$$

Equation (49.3) can be explicitly solved for $\mathfrak{W}$.[38] For brevity,

$$e=H-E-iI, \tag{49.4}$$

and

$$\bar{e}=\bar{H}-E-iI=e-(V-\bar{V}).$$

Then Eq. (49.3) can be written

$$\begin{aligned}&\left\langle o\left|(e-\bar{e}+\mathfrak{W})\left[1-\frac{1}{e}(e-\bar{e}+\mathfrak{W})\right]\right|o\right\rangle=0,\\&=\left\langle o\left|(e-\bar{e}+\mathfrak{W})\frac{1}{e}(\bar{e}-\mathfrak{W})\right|o\right\rangle,\\&=\left\langle o\left|(e-\bar{e})\frac{1}{e}\bar{e}\right|o\right\rangle-\left\langle o\left|(e-\bar{e})\frac{1}{e}\right|o\right\rangle\mathfrak{W}\\&\qquad+\mathfrak{W}\left\langle o\left|\frac{1}{e}(\bar{e}-\mathfrak{W}\right|o\right\rangle.\end{aligned} \tag{49.5}$$

Since $H_\xi|O\rangle=0$, the factor $(\bar{e}-\mathfrak{W})$ in the final term commutes with $|o\rangle$ and can be taken out to the right. Equations (45)–(49) imply

$$\left\langle o\left|\frac{1}{e}\right|o\right\rangle=\left\langle o\left|\frac{1}{\bar{e}-\mathfrak{W}}\right|o\right\rangle, \tag{49.6}$$

and from this we see that the final term in Eq. (49.5) is equal to $\mathfrak{W}$. We can then solve the equation for $\mathfrak{W}$, obtaining

$$\mathfrak{W}=\frac{1}{1-\left\langle o\left|(V-\bar{V})\frac{1}{e}\right|o\right\rangle}\left\langle o\left|(V-\bar{V})\frac{1}{e}(V-\bar{V})\right|o\right\rangle, \tag{49.7}$$

where we have added to the final factor the term

$$\left\langle o\left|(e-\bar{e})\frac{1}{e}e\right|o\right\rangle,$$

for symmetry. This term is zero, since $\langle o|e-\bar{e}|o\rangle=0$. Equation (49.7) is equivalent to

$$\mathfrak{W}=\left\langle o\left|(V-\bar{V})\frac{1}{e-\Lambda_o(V-\bar{V})}(V-\bar{V})\right|o\right\rangle, \tag{50}$$

where

$$\Lambda_o=|o\rangle\langle o|, \tag{50.1}$$

as can be checked by expanding in powers of $(V-\bar{V})1/e$. One can derive this expression without recourse to perturbation theory as an intermediate step.[39,40] For purposes of perturbation theory it is more useful to think of the denominator of Eq. (50) as $\bar{e}+(1-\Lambda_o)(V-\bar{V})$, since the expansion is in powers of $(1-\Lambda_o)(V-\bar{V})$.

In working with Eq. (50) it is advantageous to use a representation for the extra particle which is diagonal in $T+\bar{V}$ rather than in $T+\tilde{V}$. The importance of considering $\mathfrak{W}$ comes from the fact that it contains the entire imaginary part of $\tilde{V}$; its contribution to the real part of $\tilde{V}$ would be important only in case one was trying to calculate this quantity fairly accurately. For the simple case in which $\tilde{V}$ is a square well, changing the imaginary part of it by iW does not change th eeigenfunctions but only shifts the resonance energies by iW. Our new energy eigenvalues are, therefore,

$$E_m=\epsilon_m-i\beta_m/2. \tag{50.2}$$

We now investigate the eigenfunctions and eigenvalues of the operator

$$\mathfrak{IC}=H-\Lambda_o(V-\bar{V}). \tag{51}$$

We label them by Ω_p and W_p', i.e.,

$$\mathfrak{IC}\Omega_p=W_p'\Omega_p. \tag{51.1}$$

Now, $-\Lambda_o(V-\bar{V})$ can be considered as a small perturbation since the Ω_p are not very different from the $\Phi^{(p)}$. Thus, we choose

$$\Omega_p=\Phi^{(p)}+\delta\Omega_p, \tag{52}$$

and

$$\delta\Omega_p=\textstyle\sum_n a_p{}^n\Phi^{(n)}.$$

To lowest order

$$W_p'=W_p. \tag{52.1}$$

The use of Green's theorem as in Eq. (16) and the assumption[41] that Ω_p satisfies the same boundary condition as $\Phi^{(p)}$, gives the result

$$\begin{aligned}&\int_0^R(\Phi^{(n)}\mathfrak{IC}\Omega_p-\Omega_p\mathfrak{IC}\Phi^{(n)})d^3\xi d^3r=0\\&\quad=(W_p-W_n)a_p{}^n+\sum_m\langle p|(V-\bar{V})|om\rangle\langle om|n\rangle,\end{aligned} \tag{53}$$

where $W_p'=W_p$ and $\Omega_p=\Phi^{(p)}$ on the right-hand side since we wish to compute $a_p{}^n$ only to lowest order. By employing

$$\begin{aligned}\langle p|V-\bar{V}|om\rangle&=\langle p|H-\bar{H}|om\rangle\\&=(W_p-\epsilon_m+i\beta_m/2)\langle p|om\rangle,\end{aligned} \tag{54}$$

[38] C. T. De Dominicis, J. phys. radium **19**, 1 (1958).

[39] J. S. Langer, thesis, University of Birmingham, 1958.

[40] H. Feshbach, Ann. Rev. Nuclear Sci. **8**, 44 (1958).

[41] Actually, the functions Ω_p obey slightly different boundary conditions from the $\Phi^{(n)}$, because, as is clear from Eq. (51), the Ω_p contain no components χ_0, these being projected out by the operator Λ_0. However, such components of $\Phi^{(n)}$ do not contribute to the left-hand side of Eq. (53) because they are orthogonal to Ω_p, and the equality given there holds.

we obtain

$$a_p{}^n=\sum_m \frac{(W_p-\epsilon_m+i\beta_m/2)}{W_n-W_p}\langle p|\,om\rangle\langle om|\,n\rangle. \quad (55)$$

For order-of-magnitude estimates,

$$\langle p|\,om\rangle\langle om|\,n\rangle\beta_m\sim\langle p|\,om\rangle^2\beta_m\sim\gamma_p,$$

and on the giant resonance

$$W_p-\epsilon_m\lesssim W.$$

As a result the $a_p{}^n$ are large only if $(W_p-W_n)\sim W\gamma_p/\Gamma_m$ and, therefore, because of the smallness of γ_p, only a few of the neighboring Φ's are mixed into each Ω by the perturbation.

We can expand the Ω's in terms of the χ's and the ψ's just as with the $\Phi^{(p)}$'s. Then

$$\Omega_p=\sum_{j,m} b_{jm}{}^p\chi_j(\xi)\bar{\psi}_m{}^j(\mathbf{r}). \quad (55.1)$$

One important difference between the right-hand side of Eq. (55.1) and that of the expansion of the $\Phi^{(p)}$, Eq. (14), is that the right-hand side of Eq. (55.1) contains no terms with $j=0$. These have been projected out by the operator Λ_o, so that the eigenfunctions of $\mathcal{H}$ form a complete set in the space orthogonal to χ_0.

Since the eigenvalues and eigenfunctions of $\mathcal{H}$ differ only slightly from those of H, aside from the fact that the former do not contain components in $|o\rangle$, our arguments about the distribution of the $a_{jm}{}^p$ apply equally well to the distribution of the $b_{jm}{}^p$. However, it is important to retain the Λ_o in the denominator of Eq. (50), i.e., not to approximate the Ω_p by the $\Phi^{(p)}$,[42] because this would allow states $|o\rangle$ to occur as intermediate states in the expansion which would give large spurious contributions. Retaining the Λ_o is related to the need to eliminate unlinked clusters in expansions of the Brueckner type.

We are now ready to discuss the criteria for perturbation theory. We can express the matrix element as

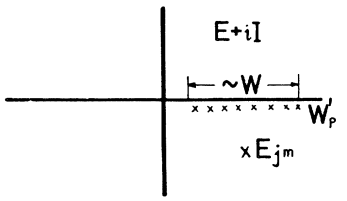
$$\langle on|\,\mathcal{W}(E)\,|on\rangle=\langle on|\,V-\bar{V}\,|p'\rangle \times\frac{1}{W_{p'}-E-iI}\langle p'|\,V-\bar{V}\,|on\rangle, \quad (56)$$

where we label the eigenfunctions of $\mathcal{H}$ by $|p'\rangle$. In a perturbation expansion of the type

$$\mathcal{W}(E)=\left\langle o\left|(V-\bar{V})\frac{1}{\bar{H}-E-iI}\sum_{n=0}^{\infty}(-1)^n \times\left[(1-\Lambda_0)(V-\bar{V})\frac{1}{\bar{H}-E-iI}\right]^n(V-\bar{V})\right|o\right\rangle, \quad (57)$$

[42] This was pointed out to the author by Dr. A. M. Lane. The partial summation of Bloch[26] takes this into account properly.

FIG. 10. Positions of poles in the complex-energy plane. The scattering amplitude, averaged over energy interval I, is obtained by taking its value for the complex energy $E+iI$.



the lowest-order term of the expansion can be expressed as

$$\langle on|\,\mathcal{W}(E)\,|on\rangle\cong\sum_{j,m}\frac{\langle on|\,V-\bar{V}\,|jm\rangle\langle jm|\,V-\bar{V}\,|on\rangle}{E_{jm}-E-iI}. \quad (58)$$

The same result can be obtained by replacing $|p'\rangle$ by $\sum_{j,m}|\,jm\rangle\langle jm|\,p'\rangle$ in Eq. (56) and then approximating W_p' by E_{jm}. This approximation is justified if I is large, because the $\langle jm|\,p'\rangle$ are large only if $W_p'-\epsilon_{jm}\lesssim W$ by the arguments of the preceding section. The situation is illustrated in Fig. 10. In the complex energy plane, the state $|\,jm\rangle$ has an appreciable probability of being found in states $|\,p'\rangle$ distributed over a region of width W. In going from Eq. (56) to the approximation Eq. (58), we are approximating the distances $E+iI-W_p'$ in the energy plane by the common distance $E+iI-E_{jm}$. This is justified if $|E+iI-E_{jm}|\gg W$, W giving the spread of the W_p' as indicated in Fig. 10. In the worst case for the fulfillment of this criterion, in which $E=\epsilon_{jm}$, which is illustrated in Fig. 10, we can satisfy this criterion by making I sufficiently large, so we must have $I\gg W$.

This could be phrased in the following way: Calculating only the average phase means that we can evaluate it for the imaginary energy $\mathcal{E}=E+iI$. The approximation of taking only the first term in the expansion is clearly better, the larger $\mathcal{E}-E_{jm}$. In the simple case of scattering considered earlier (where we actually approximated at a different stage), a large value for the distance between $\mathcal{E}$ and the single-particle resonance was obtained by taking E, the real part of $\mathcal{E}$, to be far away. The same result can be achieved by pushing $\mathcal{E}$ up in the complex plane, which means averaging over larger intervals.

We have thus formulated mathematically the physical considerations of part 1 of this section. There it was seen that the time spent in the nucleus by the particles corresponding to the average phase decreased as the interval I, over which the average was carried out, increased. This development shows that the larger I is, the faster the series Eq. (57) converges. This series in powers of δV can be interpreted as an expansion in the number of collisions the incident particle makes with the target nucleons. The larger I is, the more quickly the particle passes over the nucleus and the fewer the collisions it makes.

The expansion of $\mathcal{W}$ in this way has been carried out by Bloch.[36] After expansion, he makes assumptions

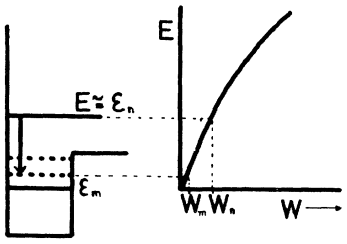


FIG. 11. Variation of absorption W with energy of the single-particle excitation.

about the randomness of signs of the $\langle kl|\delta V|jm\rangle$. We do not, however, make such assumptions, and show later that they are not justified. The expansion is, however, convergent without these assumptions provided I is sufficiently large.

It might appear that the condition $I\gg W$ is difficult to achieve. The decisive point here is that the W to be employed is the W appropriate to the state $|jm\rangle$ which corresponds to the nucleus excited by energy ϵ_j and a single particle of energy roughly $E-\epsilon_j$ in the well. This is illustrated in Fig. 11. From our earlier developments, we can assume energy conservation to within the width of the excitation $|jm\rangle$. Thus $E\cong\epsilon_j+\epsilon_m$. The width of the state $|j\rangle$, which is a true compound state, can be neglected relative to that of the single-particle excitation $|m\rangle$. But the W corresponding to state $|m\rangle$ is much smaller than that corresponding to state $|n\rangle$ because the former is far down in the well, and W is a rapidly increasing function of energy, as shown in Fig. 11. Taking the quadratic dependence of W given by simple theories, as discussed later,

$$W_m=\left(\frac{E-\epsilon_j}{E}\right)^2 W_n, \tag{59}$$

where we have put a lower suffix m on the W on the left-hand side to indicate that it refers to the state $|m\rangle$, and we now denote the W referring to state $|n\rangle$ by W_n. In other words, the width of the state into which the single-particle excitation decays is less than the width of the original single-particle excitation.

Usually, the state j is a highly excited one since the number of states per Mev available increases exponentially with excitation,[43] and in this case $W_m\ll W_n$. Of course, the more highly excited states also tend to have a more complicated structure so that the matrix elements for excitation of the A particles become smaller. In the next part of this section it is shown that the ratio $(E-\epsilon_j)/E$ tends to be $\sim 1/3$ or, $W_m/W_n\sim 1/9$. However, in the case of easily deformable nuclei, there are large matrix elements to low-lying states j, the collective ones.

From the foregoing, we see that the inequalities

$$W_m<I<W_n, \tag{60}$$

can usually be satisfied, although W_m is smaller than W_n only by a numerical factor and not by orders of magnitude. When these inequalities are satisfied, the average can be carried out over an interval which is large enough for the convergence of perturbation theory but small enough so that one can still obtain information about the shape of the single-particle resonance of width $\sim W_n$. In fact, neglect of the variation in the $|n\rangle$ and other quantities in the various averages constrains us to average over a distance $<W_n$.

The same arguments apply, with slight modification, to the case of inelastic processes going into a low-excited final state. Here the transition amplitude is

$$\begin{aligned}S_{\alpha\alpha',j}=&-\frac{2M}{\hbar^2}\Big\{\langle j\alpha'|V-\tilde{V}|o\alpha\rangle\\&-\Big\langle j\alpha'\Big|(V-\tilde{V})\frac{1}{H-E}(V-\tilde{V})\Big|o\alpha\Big\rangle\Big\},\\=&-\frac{\hbar^2}{2M}e^{-i(k+k_j)R}\Big]\sum_{m,m'}\frac{1}{\tilde{E}_m-E}\frac{1}{\tilde{E}_{m'}-E}\\&\times\Big\{\langle jm'|V-\tilde{V}|om\rangle-\Big\langle jm'\Big|(V-\tilde{V})\frac{1}{H-E}\\&\times(V-\tilde{V})\Big|om\Big\rangle\Big\}\tilde{\phi}_m(R)\tilde{\phi}_{m'}{}^{i}(R),\end{aligned}$$

where j is assumed to be a low-excited state. Perturbation theory can be used to calculate the $\langle S_{\alpha\alpha',j}\rangle_{\text{Av}}$, since the states that m and m' decay into are, in general, of substantially lower energy and consequently have a smaller width. This gives the result, in the first approximation,

$$\langle S_{\alpha\alpha',j}\rangle_{\text{Av}}\cong-\frac{2M}{\hbar^2}\langle j\alpha'|V-\bar{V}|o\alpha\rangle, \tag{62}$$

which has been used extensively in calculating direct interaction processes. This is just the same result as follows from lowest-order Born approximation, i.e., keeping only the lowest term in δV with initial and final states distorted by the complex well. Such a weak interaction picture is only applicable to the average phase and then only if it is averaged over a large energy interval.

7. Perturbation Theory of the Second Kind

Relation (62) is an important one in that it justifies the use of perturbation theory—in the usual terminology, Born approximation with distorted waves—in direct interaction calculations. We found also, in part 6, that the average elastic scattering amplitude was reproduced by $\tilde{V}=\bar{V}-\mathfrak{W}$ with $\mathfrak{W}$ given to lowest order by Eq. (58). This latter relation is interesting but not well adapted for the calculation of $\mathfrak{W}$ since it still

[43] The energy of excitation is limited since the initially excited particle does not drop into a state already occupied. This is inconsistent with our neglect of antisymmetrization, but this does not seem to be an essential difficulty.

involves the highly complicated nuclear states $\chi_j(\xi)$. We therefore develop a perturbation expansion of the type used by Bruckner, Eden, and Francis[44] which essentially relates quantities back to shell-model states. This expansion gives an expression for $\mathcal{W}$ which is practical for calculation; however, to satisfy the criterion for its validity, one must average over a large energy interval as we shall see.

In this treatment the unperturbed Hamiltonian is

$$\bar{H}=\sum_{i=0}^{A}(T_i+V_i)-\tfrac{1}{2}\sum_{i,j=0}^{A}\bar{V}_{ij}, \tag{63}$$

with eigenstates $|\bar{q}\rangle$ such that

$$\bar{H}|\bar{q}\rangle=E_{\bar{q}}|\bar{q}\rangle. \tag{63.1}$$

Here V_i is the self-consistent potential defined in the state $|\bar{q}\rangle$ felt by the ith particle, as in the Hartree-Fock theory, and

$$\bar{V}_{ij}=\langle\bar{q}|V_{ij}|\bar{q}\rangle. \tag{63.2}$$

The summations cover the incident particle ($i=0$) as well as the A particles in the nucleus. Thus, the A+1 particles are here treated symmetrically. The c number, $\frac{1}{2}\sum_{i,j}\bar{V}_{ij}$, has been subtracted to ensure that $\langle\bar{q}|H|\bar{q}\rangle$ is equal to the energy in the Hartree-Fock approximation. The crucial question in using the $|\bar{q}\rangle$ as zero-order functions relates to the extent of the spread of the strength function $\langle\bar{q}|p\rangle^2$ in energy. This can be obtained from

$$\frac{\pi}{D}\{\langle\bar{q}|p\rangle^2\}_{Av}=\mathrm{Im}\left\{\left\langle\bar{q}\left|\frac{1}{H-E}\right|\bar{q}\right\rangle\right\}_{Av}$$
$$=\mathrm{Im}\left\langle\bar{q}\left|\frac{1}{\tilde{H}-E}\right|\bar{q}\right\rangle, \tag{64}$$

where $\tilde{H}$ is a comparison Hamiltonian which has been introduced as an artifice to calculate this strength function.[45] Its significance is seen later. It is given by

$$\tilde{H}=\bar{H}-\mathcal{W}, \tag{64.1}$$

where $\mathcal{W}$ is then defined by

$$0=\left\langle\bar{q}\left|\mathcal{W}\right|\bar{q}\right\rangle$$
$$-\left\langle\bar{q}\left|(V-\bar{V}+\mathcal{W})\frac{1}{H-E-iI}(V-\bar{V}+\mathcal{W})\right|\bar{q}\right\rangle, \tag{64.2}$$

which guarantees satisfaction of Eq. (64). [See the similar development, Eqs. (44)–(49).] This equation is formally similar to Eq. (56), and by the same procedure as employed to solve the latter, we can find that Eq. (64.2) can be satisfied by choosing $\mathcal{W}$ to be diagonal in the $|\bar{q}\rangle$ representation, with matrix elements

$$\langle\bar{q}|\mathcal{W}|\bar{q}\rangle$$
$$=\left\langle\bar{q}\left|(H-\bar{H})\frac{1}{H-\Lambda_{\bar{q}}(V-\bar{V})-E-iI}(H-\bar{H})\right|\bar{q}\right\rangle, \tag{65}$$

where

$$\Lambda_{\bar{q}}=|\bar{q}\rangle\langle\bar{q}|. \tag{65.1}$$

In evaluating $\langle\bar{q}|\mathcal{W}|\bar{q}\rangle$ by lowest-order perturbation theory which now means replacing

$$\{H-\Lambda_{\bar{q}}(H-\bar{H})-E-iI\}^{-1}$$

by

$$\sum_{\bar{s}\neq\bar{q}}|\bar{s}\rangle\{E_{\bar{s}}-E-iI\}^{-1}\langle\bar{s}|,$$

we find that

$$\mathcal{W}=\sum_{i<j=0}^{A}\mathcal{W}_{ij}, \tag{65.3}$$

with

$$\langle\bar{q}|\mathcal{W}_{ij}|\bar{q}\rangle=\sum_{\bar{s}\neq\bar{q}}\langle\bar{q}|V_{ij}|\bar{s}\rangle\frac{1}{E_{\bar{s}}-E-iI}\langle\bar{s}|V_{ij}|\bar{q}\rangle. \tag{65.4}$$

The half-width of the strength function $\pi\{\langle\bar{q}|p\rangle^2\}_{Av}/D$ is given by

$$\sum_{i<j=0}^{A}\mathrm{Im}\langle\bar{q}|\mathcal{W}_{ij}|\bar{q}\rangle, \tag{65.5}$$

if we neglect the natural width of the state $|\bar{q}\rangle$.

If we identify $|\bar{q}\rangle$ with the state in which A+1 particles form a state, wherein the A levels from the bottom of the well are filled except for one hole in the state we label by k' and in which two particles are excited in states labeled by k and l, then it is easy to identify the various parts of the width:

1. $\mathrm{Im}\sum_{j=0}^{A}\langle\bar{q}|\mathcal{W}_{kj}|\bar{q}\rangle$ and $\mathrm{Im}\sum_{i=0}^{A}\langle\bar{q}|\mathcal{W}_{li}|\bar{q}\rangle$,

correspond to the widths for the particles k and l to interact with the other particles so that the two-particle excitation, in perturbation theory language, decays into a three-particle excitation. (One should omit $j=k'$ and $i=k'$ in the sums.)

2. $\mathrm{Im}\sum_{j}{}'\langle\bar{q}|\mathcal{W}_{k'j}|\bar{q}\rangle$ corresponds to the width for the decay of the shell model state with a hole in it, i.e., the width for absorption of the hole. (The prime indicates that the term $j=k'$ should be omitted. But $j=k$ and l should, for completeness, be included in the sum.)

By using the approximate symmetry of holes and particles[46] near the Fermi surface E_F, we find that the

[44] Brueckner, Eden, and Francis, Phys. Rev. **100**, 891 (1955). See also M. Cini and S. Fubini, Nuovo cimento **2**, 75 (1955) and A. M. Lane and C. F. Wandel, Phys. Rev. **98**, 1524 (1955).

[45] We use the symbols $\tilde{H}$ and $\mathcal{W}$ because, although they stand now for different quantities than previously, there is a close analogy between these and the previous ones.

[46] Calculations by E. P. Pendlebury, analogous to those of Lane and Wandel,[44] indicate that this symmetry holds only near the Fermi surface; the absorption for holes then rises more steeply than that for particles (private communication).

total width of $|\bar{q}\rangle$ is

$$W_{\bar{q}}=W_k+W_l+W_{k'}$$
$$\cong\left\{\frac{(\epsilon_k-E_F)^2+(\epsilon_l-E_F)^2+(E_F-\epsilon_{k'})^2}{(\epsilon_n-E_F)^2}\right\}W_n, \quad (66)$$

from the simple theories referred to previously. We label the width of the original single-particle excitation $|n\rangle$ by W_n as previously, and we employ the W's to denote the imaginary parts of the $\mathcal{W}_{ij}$.

The width of our original single-particle excitation is given in the lowest approximation by

$$W_n=\mathrm{Im}\sum_{\bar{q}}$$
$$\frac{\langle on|\sum_{i=1}^{A}V_{oi}-\sum\bar{V}_{oi}|\bar{q}\rangle\langle\bar{q}|\sum V_{oi}-\sum\bar{V}_{oi}|on\rangle}{E_{\bar{q}}-E-iI}. \quad (67)$$

The largest phase space is available when the energy is equally distributed among the particles and the hole so that W_k, W_l, and $W_{k'}$ are each $\sim(1/9)W_n$. Since $W_k+W_l+W_{k'}<W$, we can again satisfy

$$W_{\bar{q}}<I<W_n, \quad (68)$$

and employ perturbation theory, although all three of these quantities are of the same order of magnitude, so that the average must be carried out over an interval I that is a good fraction of the width of the single-particle resonance.

In case $|o\rangle$ can be described as a shell-model state, i.e., as an antisymmetrized wave function of independent particles moving in a well, or can be obtained from such a state by successive application of the two-body interaction, the A+1 particles can be treated symmetrically, and the identity of the particles can be taken into account easily. By using such an iteration in which the evaluation is carried out starting from the Fermi sea, which is assumed to be a reasonable approximation for the interior of a large nucleus, Brueckner[47] finds that the value of W is increased considerably over that given by the simple theory and introduces both a term linear in energy and a constant term in the dependence of W. This would mean that the criterion, Eq. (67), might not be satisfied by any I since the sum $W_k+W_l+W_{k'}$ might be as large as W_n. Whereas Brueckner's calculation brings out the sensitivity of W to correlations in the ground-state wave function near the Fermi surface, there are good indications that these cannot be calculated in perturbation theory, i.e., that the ground-state wave function $\chi_o(\xi)$ cannot be obtained in the region of the Fermi surface by iterating the shell-model state.[48] In fact, the parts of these correlations responsible for the pairing forces tend to make the Fermi surface stiffer against perturbations, working in the opposite direction from the effects calculated by Brueckner.[47] Consequently, a consistent evaluation of the corrections to the lowest-order theory undoubtedly gives results smaller than Brueckner's calculation.

In the discussion following Eq. (59) we promised to return to the question of easily deformable nuclei where there are large matrix elements from χ_o to low-lying rotational states so that the initial single-particle excitation has a high probability of decaying into an excitation where the initial particle has only slightly less energy. This state, therefore, has a width almost as large as the initial width. In this case, one should separate out the strong transitions to the low-lying collective states which, together with the initial channel, we term the "chosen channels" and treat them separately in a system of coupled equations as has been done by Sano, Yoshida, and Terasawa,[49] (references to extensive earlier work by Yoshida and others are given in this article) and by Chase, Wilets, and Edmonds.[50]

In fact, the preceding development can be generalized so that $V-\bar{V}$ is a matrix between chosen channels, $|o\rangle$ becoming the space of the chosen channels and Λ_o excluding these from occurring in intermediate states, essentially reproducing Yoshida's formalism. This generalization is within the spirit of the optical model where the distortion provided by the central potential is supposed to represent the average effects from the great number of channels and correspondingly large number of degrees of freedom which cannot be conveniently treated in detail.

8. Special Models

It would not be possible to evaluate $\mathcal{W}$ from either Eq. (56) or Eq. (58) since the matrix elements $\langle on|V-\bar{V}|p\rangle$ or $\langle on|V-\bar{V}|jm\rangle$ cannot be calculated owing to the complexity of the $\Phi^{(p)}(\mathbf{r},\xi)$ and the $\chi_j(\xi)$. It would be difficult to evaluate $\mathcal{W}$ even from the equations of the last section since these would involve using wave functions in a complex well of finite extent. Consequently, somewhat idealized models have been introduced to carry out this evaluation and we now discuss their relationship to the development here.

The interior of a large nucleus has been represented as a Fermi gas. In this case, the absorption, Eq. (67), is approximated by

$$W_n=\frac{\pi}{D}\{\langle on|\sum V_{oi}-\sum\bar{V}_{oi}|\bar{q}\rangle^2\}_{\mathrm{Av}}, \quad (69)$$

where the Fermi sea $|o\rangle$ is taken as an antisymmetrized product function of plane waves, $|n\rangle$ is a plane wave of energy above the top of the Fermi sea, and $|\bar{q}\rangle$ is a

[47] K. A. Brueckner, Phys. Rev. **103**, 172 (1956).
[48] Bohr, Mottelson, and Pines, Phys. Rev. **110**, 936 (1958).
[49] Sano, Yoshida, and Terasawa, Nuclear Phys. **6**, 20 (1958).
[50] Chase, Wilets, and Edmonds, Phys. Rev. **110**, 1080 (1958).

state of the same energy as $|on\rangle$ in which a second particle has been lifted above the sea.

In such a model it is immaterial how large the interval I over which the average is taken in calculating $\mathfrak{W}$ or W. No features are left in the model capable of giving rapid variations in the quantities calculated, so that the scattering amplitude is a smoothly varying function of energy and the average phase, for reasonable intervals, is very close to the actual phase. Consequently, one does not need to carry the iI in the energy denominator in the actual evaluations. However, in establishing the validity of Eq. (67), it was necessary to carry the iI in the compound-state picture.

A second model, which is not so useful in obtaining numerical values, but is instructive, has been used by Wigner.[51] He assumes that levels $|jm\rangle$ are evenly spaced in energy and that the matrix elements $\langle jm|\delta V|kl\rangle$, where δV is the perturbation, are all equal in absolute value but of random sign as long as $E_{jm}-E_{kl}$ is less than a certain quantity (in the Wigner formalism the energy eigenvalues are real) and zero for greater energy differences. He then finds, under certain conditions which are probably satisfied in the nuclear case, a Lorentzian form, i.e., the shape Eq. (33), for the strength function in the region of the single-particle resonance. This implies, according to Eq. (33), that W is constant with respect to energy.

A third model, used by Bloch,[36] is similar to Wigner's except that the point at which he makes his assumption of random signs and equality of matrix elements is in the perturbation expansion, Eq. (57). This means that only the even powers of the expansion contribute. His analysis shows that the largest number of nonvanishing terms comes from the lowest-order term, corresponding to Eq. (58). However, such assumptions of random signs are unjustified in the actual physical case, and for the case of a large nucleus, when approximated by a Fermi sea, it is found that the third term in the series,[52] which would be zero under the above assumptions, actually contributes about as much to the imaginary part of the potential as the second term. Furthermore, the second term appears to be much less predominant when the exclusion principle is taken into account since the second-order processes are severely inhibited.

Once one has adopted a simplified model, any mechanism that might be present in the actual nucleus to give rapid variations in the parameters has been dropped, and consequently, the result is insensitive to the interval over which one averages since the actual phase is very nearly the same as the average phase. However, the inclusion of iI when the compound-state energies W_p were present was necessary to establish the relevance of these models to the calculation of the average phase.

It seems reasonable that at high bombarding energies, where the widths of the compound states are much greater than their spacings, the actual scattering amplitude varies smoothly with energy and that fluctuations are small. Consequently, one might expect the actual amplitude to be the same as the average one here, and again, the calculated amplitude would be insensitive to the distances over which one averaged. This would explain why our formulas for the average amplitude have the same form as those of Watson and collaborators for higher energy scattering even though no average over energy is carried out in their work.

However, in the intermediate region where fluctuations may still be important and in the low-energy region, it is necessary to carry out the averages over energy as we have done.

We have not discussed the fluctuation scattering in much detail. This is because it is not possible to evaluate this directly; since the corresponding particles stay in the nucleus a long time, any expansion in terms of successive collisions is bound to fail. In fact, because of their long duration in the nucleus, these particles would seem to correspond to those described by the original extreme compound-nucleus model. Consequently, it is reasonable to neglect phase relations here on the average, and this results in angular distributions symmetrical about 90°. There is always some ambiguity about adding in the fluctuation scattering, but if one can calculate the average scattering amplitude so as to obtain the first term on the right-hand side of Eq. (26), one obtains a lower limit on the cross section for the relevant process. This is especially useful in inelastic scattering where similar considerations apply. An application of this is indicated in the next section. Furthermore, if the above assumptions are justified, the fluctuation scattering is rather structureless and simply tends to make the minima in the shape-elastic scattering less pronounced without altering the general picture.

IV. DIRECT INTERACTION IN THE DIPOLE PHOTOEFFECT AND IN RADIATIVE CAPTURE

We treat one case in which the incident and emerging particles are different, namely, that in which one of the particles is a photon. As discussed in Sec. I, a giant resonance is observed in the absorption of photons by complex nuclei in the region of $\sim$15 Mev for medium weight nuclei. Collective models give a natural explanation of the absorption mechanism but not of the number of fast protons emerging which is orders of magnitude in excess of the predictions of the statistical theory. Direct interactions are necessary to account for them. This does not mean that emission of fast protons is the dominant decay process; this constitutes only $\sim$1% of the total decay processes in heavy nuclei where such decays are strongly inhibited by the Coulomb barrier. However, this is several orders of magnitude above the predictions of 10^{-4} or 10^{-5} for the probability given by the statistical theory. The

[51] E. P. Wigner, Ann. Math. **62**, 548 (1955).
[52] L. Verlet and J. Gavoret, Nuovo cimento **10**, 505 (1958).

inverse (p,γ) processes violate the predictions of the statistical theory in just as striking a fashion.[53,54]

Whereas this theory would predict a strong decrease in the cross section with increasing energy or increasing atomic number A, the experimental results of Cohen[55] show no such decreases for proton energies in excess of 5 Mev. This is an added indication that direct processes play an important role.

Wilkinson[14] has proposed a model for the (γ,p) reaction which incorporates the physical features that follow from the detailed theory. He views the absorption of the γ-ray by one of the nucleons as leading first to an excited state of the single particle of width W in the complex well. This is just the single-particle excitation discussed earlier. One then associates with this level the natural width Γ for escape from the nucleus and the width $2W$ for absorption into compound states. Consequently, the proportion $\Gamma/(\Gamma+2W)$ escape, carrying off essentially the full energy of the γ-ray. This reproduces the observed order-of-magnitude of fast protons when averaged over the relevant single-particle levels.

We now extend the formalism to treat the absorption of radiation following Brown and Levinger.[56] This extension is similar to that carried out in the framework of the continuum theory of nuclear reactions by Peaslee.[57] The wave function of the system in which γ-rays are absorbed consists only of outgoing waves and is described by

$$\Psi(\mathbf{r},\xi)=\sum_{\alpha',j} S_{\gamma\alpha',j}\chi_j(\xi)\psi_{\alpha'}{}^{j+}(\mathbf{r}), \quad r>R, \tag{70}$$

where $S_{\gamma\alpha',j}$ is the amplitude for γ-ray absorption with emission of the nucleon into channel α', leaving the residual nucleus in state j. For $r<R$ we have

$$\Psi=\Psi_o(\mathbf{r},\xi)+\textstyle\sum_p a_p\Phi^{(p)}(\mathbf{r},\xi), \tag{71}$$

where the $\Phi^{(p)}$ are the compound-state functions, Eq. (13). Here Ψ_o is the wave function of the initial nucleus (it being understood that a γ-ray is present). It can often be adequately represented as a shell-model wave function. The coordinate of the particle to be emitted is labeled by $\mathbf{r}$ whereas those of all the other A$-$1 particles are labeled by ξ. Thus, although there is a symmetry between the A particles, we choose to describe the absorption by a particular particle for simplicity, and our description consequently singles out this particle. The interaction with the radiation field (electric field $\mathcal{E}$ along the z axis normalized to a flux of one photon per cm^2 sec) is described by H_I where

$$H_I=e\mathcal{E}_z=e(2\pi\hbar\omega)^{\frac{1}{2}}z \tag{72}$$

on restricting the consideration to the dipole interaction. Since the coupling to the radiation field is weak, the a_p's are needed only to first order which can easily be done by requiring Ψ to satisfy the Schrödinger equation to that order. We have

$$H_I\Psi_o=(E-H)\textstyle\sum a_p\Phi^{(p)}=\sum a_p(E-W_p)\Phi^{(p)}. \tag{73}$$

By multiplying on the left by $\Phi^{(p)}$ and integrating, we find

$$a_p=\langle p|H_I|o\rangle/(E-W_p), \tag{73.1}$$

as expected from perturbation theory. Here we use $|o\rangle$ for Ψ_o; E is the full energy of the system. We now measure it from the ground state of the initial nucleus, $E=\hbar\omega$, where $\hbar\omega$ is the γ-ray energy.[58]

We easily obtain $S_{\gamma\alpha',j}$ by equating Ψ from Eqs. (71) and (73) at $r=R$, multiplying by $\chi_j(\xi)\Theta_\alpha(\theta,\varphi)$ and integrating over $d^3\xi$ and $d\Omega$.

$$S_{\gamma\alpha',j}=\frac{1}{\phi_{\alpha'}{}^{j+}(R)}\sum_p\int_0^R d^3\xi\int d\Omega\Phi^{(p)}(\xi,\mathbf{R})\chi_j(\xi) \times\Theta_\alpha(\theta,\varphi)\frac{\langle p|H_I|o\rangle}{W_p-\hbar\omega}. \tag{74}$$

From the expansion, Eq. (14), we find

$$S_{\gamma\alpha',j}=\frac{1}{\phi_{\alpha'}{}^{j+}(R)}\sum_{p,m}\tilde{\phi}_m{}^j(R)\langle jm|p\rangle\frac{\langle p|H_I|o\rangle}{W_p-\hbar\omega}, \tag{74.1}$$

This gives a cross section

$$\sigma_{\gamma\alpha',j}=\frac{hk_j}{Mc}|S_{\gamma\alpha',j}|^2. \tag{75}$$

Once again we are concerned with the average scattering amplitude $\langle S_{\gamma\alpha',j}\rangle_{\text{Av}}$ which can be obtained from Eq. (74.1) by replacing $\hbar\omega$ by $\hbar\omega+iI$ in the denominator, i.e.,

$$\langle S_{\gamma\alpha',j}\rangle_{\text{Av}}=\frac{1}{\phi_{\alpha'}{}^{j+}(R)}\sum_{p,m}\tilde{\phi}_m{}^j(R)\langle jm|p\rangle\frac{\langle p|H_I|o\rangle}{W_p-E-iI}, \tag{76}$$

which in the equivalent Green's function description is

$$\langle S_{\gamma\alpha',j}\rangle_{\text{Av}}=\frac{1}{\phi_{\alpha'}{}^{j+}(R)}\sum_m\tilde{\phi}_m{}^j(R)\left(jm\left|\frac{1}{H-E-iI}H_I\right|o\right). \tag{76.1}$$

The equivalence of Eqs. (76) and (76.1) is easily shown by inserting the unit operator $|p\rangle\langle p|$ to the left of H_I in the latter. By employing an expansion similar to that, Eq. (45) for $(H-\hbar\omega-iI)^{-1}$, one has

[53] F. Beck, Nuclear Phys. **9**, 140 (1958/9).
[54] A. M. Lane and J. E. Lynn, Nuclear Phys. **11**, 646 (1959).
[55] B. L. Cohen, Phys. Rev. **100**, 206 (1955).
[56] G. E. Brown and J. S. Levinger, Proc. Phys. Soc. (London) **A71**, 733 (1958).
[57] D. C. Peaslee, Phys. Rev. **88**, 812 (1952).

[58] This means that the energy of the excited particle, in terms of single-particle excitations, is measured from the energy of the ground state of the original nucleus, whereas earlier, the E associated with the single particle in our discussion of elastic scattering was its energy at $r=\infty$.

$$\langle S_{\gamma\alpha',j}\rangle_{\mathrm{Av}}=\frac{1}{\phi_{\alpha'}{}^{i+}(R)}\sum_m \tilde{\phi}_m{}^{i}(R)\Big\{\frac{\langle jm|H_I|o\rangle}{\bar{E}_{jm}-\hbar\omega-iI}$$
$$-\frac{1}{\bar{E}_{jm}-\hbar\omega-iI}\frac{1}{\bar{E}_{kl}-\hbar\omega-iI}$$
$$\times\Big[\langle jm|V-\tilde{V}|kl\rangle$$
$$-\frac{\langle jm|V-\tilde{V}|p\rangle\langle p|V-\tilde{V}|kl\rangle}{W_p-\hbar\omega-iI}\Big]$$
$$\times\langle kl|H_I|o\rangle\Big\}, \quad (77)$$

where the label k refers to the excited states of the A$-$1 particles as does j. We now limit consideration to the case in which $\epsilon_{jm}\cong E$, i.e., the single particle takes off approximately the full energy of the γ-ray. The other case, in which $\epsilon_{jm}-E\gg W_m$ where W_m is the absorption relating to the single-particle state $|m\rangle$, can be treated just as our simple case of scattering, Sec. III, 4.

Perturbation theory here amounts to expanding the $(W_p-\hbar\omega-iI)^{-1}$ and the first term is obtained by replacing $|p\rangle(W_p-\hbar\omega-iI)^{-1}\langle p|$ by

$$\sum_{i,n}|in\rangle(E_{in}-\hbar\omega-iI)^{-1}\langle in|.$$

As discussed in Sec. III, 6, this is a good approximation because the states $|n\rangle$ are mainly at a lower excitation than the original $|l\rangle$ which are the states reached by the single particle absorbing approximately the full γ-ray energy, and consequently the spread of the $\langle p|in\rangle^2$ is small compared with W_l, the absorption pertaining to the state $|l\rangle$.

We can expect the first term on the right-hand side of Eq. (77) to be a good approximation to $\langle S_{\gamma\alpha',j}\rangle_{\mathrm{Av}}$. We consider first the case $|jm\rangle=|kl\rangle$. In this case, it is useful to consider both terms in square brackets together. If it is remembered that the overlap between $|j\rangle$ and the part of Ψ_o relating to the ξ-particles is large only if the states of the ξ-particles are similar in both (the final factor $\langle kl|H_I|o\rangle$ ensures this will be as shown), and if $\tilde{V}$ is chosen so as to satisfy Eq. (49), then the term in square brackets here very nearly vanishes, any remaining effects coming from the small differences between $|j\rangle$ and the ξ-variable part of Ψ_o. Since the two terms in the square brackets are separately of order W_m, and since the denominators in front never become smaller than W_m+I, then if the two terms cancel to a good approximation, their contribution can be neglected. Estimates of the first term in square brackets for the case $|jm\rangle\neq|kl\rangle$ indicate that it is of order $(\bar{V}/\mathrm{A})/(W_l+I)$ relative to the first term. Corrections for this term may have to be made in some cases. We do not pursue this further but assume that the first term gives a sufficiently good approximation. In any case, we have indicated that the series expansion converges for large enough I so that we could calculate further terms if necessary for good numerical results.

The first term on the right-hand side of Eq. (77) looks very much like the matrix element that would describe the photonuclear process for a particle in a complex well, since the states $|p\rangle$ have disappeared. Remnants of the fact that the process is really not a single-particle one remains however, in that states $|j\rangle$ are employed rather than the shell-model excited states which are just the states in which particles fill all but one of the A levels from the bottom of the well. To show how we can dispose of the details of the states $|j\rangle$, let us assume that the initial state Ψ_o is well represented by the shell-model state so that we can write

$$\Psi_o(\mathbf{r},\xi)=\Omega(\xi)\psi_{m'}(\mathbf{r}), \quad (78)$$

where $\psi_{m'}(\mathbf{r})$ is the bound state occupied initially by the $\mathbf{r}$-particle. Then we can split the matrix element coming into the first term of Eq. (77) into

$$\langle jm|H_I|o\rangle=\langle m|H_I|m'\rangle\int_0^R\Omega(\xi)\chi_j(\xi)d^3\xi$$
$$=\langle m|H_I|m'\rangle Q_{\Omega j}, \quad (78.1)$$

defining $Q_{\Omega j}$. Let us assume that only one single-particle level contributes essentially to a given channel which is often a good approximation. The cross section arising from the first term on the right-hand side of Eq. (77) is

$$\sum_j\frac{\hbar k_j}{Mc}|\langle S_{\gamma\alpha',j}\rangle_{\mathrm{Av}}|^2\cong\sum_j\Gamma_{\alpha'j}Q_{\Omega j}{}^2\frac{1}{\hbar c}\left|\frac{\langle m|H_I|m'\rangle}{\bar{E}_{jm}-\hbar\omega-iI}\right|^2, \quad (79)$$

where

$$\Gamma_{\alpha'j}\equiv\frac{\hbar k_j}{M}\left|\frac{1}{\phi_{\alpha'}{}^{i+}(R)}\right|^2|\tilde{\phi}_m{}^{i}(R)|^2, \quad (79.1)$$

is the full single-particle escape width taken at energy $\hbar\omega-\epsilon_j$ where ϵ_j is the excitation energy of the state j. From completeness, the probability of finding the state Ω somewhere in the excited states $|j\rangle$ is unity, i.e.,

$$\sum_j Q_{\Omega j}{}^2=1. \quad (79.2)$$

From the development in Sec. III, 7, the width of the distribution of ϵ_j about $|\epsilon_{m'}|$ is $W_{m'}$, where $W_{m'}$ can be interpreted as the width of the hole in state m'. Consequently, we see that the width of the total distribution of emitted fast particles is the sum of the width for the excited particle and of the width of the hole in the shell-model states.

These arguments do not add any new physical features to Wilkinson's model but show how this simple description relates to the many-body description.

Similar equations can be used to estimate the cross section for direct radiative capture of neutrons or

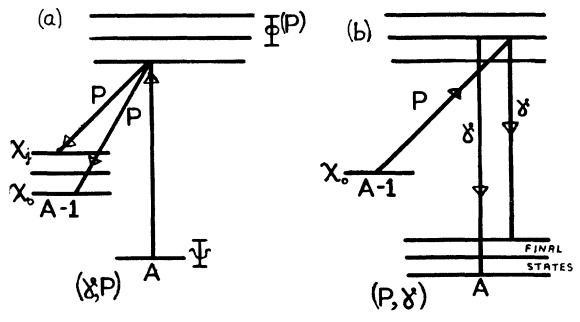


FIG. 12. Relation of the (γ,p) to (p,γ) reactions.

protons. An extensive description of this is contained in work by Lane and Lynn.[54]

The cross sections for (γ,p) and (p,γ) reactions can of course easily be related. We have the (γ,p) cross section by summing over all members of the configuration of the final A−1 nucleons (see Fig. 12). The (γ,p) reaction goes from the ground state to all final states j. The (p,γ) reaction may go from the ground state $\chi_o(\xi)$ to the various final states [see Fig. 12(b)] which contain an appreciable component of $\chi_o(\xi)$. (The wave function for the ξ-particles must remain the same since H_I is a function only of $\mathbf{r}$.) Further, the final state must contain an appreciable component of $\psi_{m'}(\mathbf{r})$, the relevant single-particle state in the well. From arguments similar to those already used the probability of finding $\chi_o\psi_{m'}$ somewhere in the ground or low-excited states is essentially unity. Thus, the cross section for the (p,γ) reaction, summed over the ground and low-excited states, is given by Eq. (79) with two minor changes: (i) we multiply by the usual factor k_γ^2/k_p^2 from detailed balance or dispersion theory where k_γ and k_p are γ-ray and proton wave numbers; (ii) we use $\Gamma_{\alpha' j}$ without the averaging implied in the sum over j in Eq. (79) since the incident particle now has a unique energy.

V. DISCUSSION OF OTHER FORMALISMS

Recently, a unified theory of nuclear reactions has been formulated by Feshbach,[59] and many of the relations between compound-nucleus parameters and those of the optical model have been derived by him, although he does not develop the perturbation theory described in Sec. III. Conceptually, his formalism has the advantage that it avoids the introduction of a joining radius. However, his formalism does not seem to be as convenient for making many of the arguments of Sec. III and Sec. IV as the one used here. In any case, there is a close connection between his formalism and the one employed here even though they appear quite different at first sight. We briefly demonstrate some of the relationship, trying to preserve most of Feshbach's notation so that one can easily compare our formulas with his papers.

[59] H. Feshbach, Ann. Phys. 5, 357 (1958).

Feshbach employs functions $\Phi_n(\mathbf{r},\xi)$ which are eigenfunctions of the operator $\mathfrak{H}$, Eq. (51), and which are, consequently, just the Ω_p of Eq. (51). The Λ_o in Eq. (51) effectively projects off the incident channel so that the Φ_n form a complete set of states in the space orthogonal to that of the incident channel. Hence, one can write the solution of the Schrödinger equation as

$$\Psi(\mathbf{r},\xi)=\chi_o(\xi)u_o(\mathbf{r})+\sum a^{(n)}\Phi_n(\mathbf{r},\xi), \tag{80}$$

and this equation holds for all values of r since the boundary conditions on the Φ_n are taken to be outgoing waves at infinity in case the particle can escape by a channel other than the incident one. In this case, the Φ's form a continuum, and the sum in Eq. (80) is to be understood in a generalized sense to include both a sum over discrete states and an integral over the continuum states. Since the boundary conditions are imposed at infinity, introduction of a channel radius is not necessary.

The $a^{(n)}$ are determined by requiring Ψ to satisfy the Schrödinger equation $H\Psi=E\Psi$, and one finds

$$a^{(n)}=\frac{\int\Phi_n(\mathbf{r},\xi)V(\mathbf{r},\xi)\chi_0(\xi)u_0(\mathbf{r})d^3\xi d^3r}{E-\mathcal{E}_n}, \tag{81}$$

where we have used Feshbach's notation in denoting the eigenvalues of Φ_n by $\mathcal{E}_n$. Our $a^{(n)}$ are the same as his Λ_n. One also finds, by multiplying the Schrödinger equation on the left by $\chi_o(\xi)$ and integrating over $d^3\xi$, that

$$(T-\bar{V}-E)u_0(\mathbf{r})+\sum a^{(n)}\int\chi_o(\xi)V(\mathbf{r},\xi)\Phi_n d^3\xi=0. \tag{82}$$

By substituting the $a^{(n)}$ from Eq. (81), one finds that $u_o(\mathbf{r})$ obeys the equation

$$(T+\mathcal{U}-E)u_o(\mathbf{r})=0, \tag{83}$$

where $\mathcal{U}$ is Feshbach's generalized optical potential

$$\mathcal{U}=\bar{V}(r)+\sum_n\frac{\int\chi_0(\xi)V(\mathbf{r},\xi)\Phi_n(\mathbf{r},\xi)d^3\xi\int\Phi_n(\mathbf{r}',\xi)V(\mathbf{r}',\xi)\chi_0(\xi)d^3\xi}{E-\mathcal{E}_n}, \tag{83.1}$$

and $\mathcal{U}$ is an integral (nonlocal) operator. One can easily make further connections between our formalism and his by noting that his matrix H_{ij} is just

$$H_{ij}=\int\chi_i(\xi)H\chi_j(\xi)d^3\xi.$$

In the very low energy region in which only elastic scattering is possible, the main difference between the two formalisms results from the fact that the incident channel occurs in our $\Phi^{(p)}$ but not in Feshbach's Φ_n. The width corresponding to this channel is contained in our complex W_p, whereas one must essentially invert a matrix—which can be done in a straightforward fashion—to introduce the widths into the denominators of the scattering amplitude in the formalism of Feshbach. The discussion following Eq. (51) shows that the eigenfunctions $\Phi^{(p)}$ and Φ_n are, aside from the above difference, very nearly the same.

The difference between the formalisms is that one can start from a formalism without joining radius, as Feshbach does, in which case the expression for the wave function is simple, but the transition to the scattering amplitude is somewhat complicated in detail. On the other hand, one can begin with a joining radius, in which case the scattering amplitude is simply obtained, and then carry out various sums as we have done to obtain results in which this radius does not appear. Whereas the former procedure has conceptual advantages, the intermediate formulas seem simpler in the latter procedure and the arguments about the spectrum of the compound states, which we have made to establish perturbation theory, appear to be simpler.

A more general formalism for nuclear reactions has been given by Bloch,[36] and we have referred to his work at several points. By appropriate choice of representation, one can obtain either the Wigner-Eisenbud or Kapur-Peierls theory from his formalism. Making the arguments of this article in his formalism would be completely equivalent to the treatment here, and we have chosen the simpler notation of the less general treatment.

Finally, many of the arguments of this article were first formulated in the Wigner-Eisenbud formalism. This is especially true of the development of Sec. III, 3 as shown by a given reference. However, for uniformity we have chosen to restate most of these arguments in our formalism which is just as suitable for them and is probably simpler for other considerations as discussed in Sec. I.

ACKNOWLEDGMENTS

The author would like to acknowledge his debt to Professor R. E. Peierls for countless helpful suggestions and criticisms relevant to the development presented here. He is also greatly indebted to Dr. C. T. De Dominicis. Many other colleagues at Birmingham have been helpful, and Dr. A. M. Lane has contributed numerous helpful suggestions.