Masers*

J. WEBER
University of Maryland, College Park, Maryland

"It is strange that practical electronics remained untouched by these fundamental facts (electron spin) and could get along with the notion of the charged mass point or the minute charged sphere." [Arnold Sommerfeld, *Electrodynamics* (Academic Press, Inc., New York, 1948)].

I. INTRODUCTION

THE development of electronics during the first half of this century was based almost completely on the quantum states of translational energy of the free electron. Only recently did physicists recognize the fact that the spin states and the bound states of electrons in atoms and molecules could be employed for amplification. We give here an account of these technological advances together with a summary of the physics which made them possible.

The physical principles and experimental techniques required for development of paramagnetic amplifiers and molecular amplifiers were well established in the period 1945–1950. Such microwave amplifiers have much lower noise than those employing thermionic vacuum tubes. This low noise property has been the principal motivation for the large amount of work in this field.

The earliest work on molecular amplification started independently at three different laboratories.[1–3] It had a different motivation, namely, the hope that millimeter wave oscillators and amplifiers would result.

The word "maser" was coined by the Columbia group[2] as an acronym for "microwave amplification by the stimulated emission of radiation." This word appears in Webster's dictionary with the spellings "mazer" and "maser" and has the meaning "a large drinking cup, originally of a hard wood." The free electron vacuum tube amplifier may also be regarded as operating through the mechanism of the stimulated emission of radiation. In what follows, however, the word maser is taken to mean either a molecular amplifier or a paramagnetic solid state amplifier.

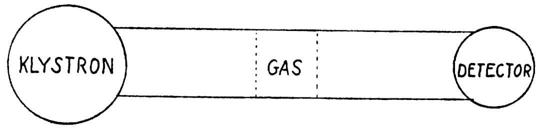


FIG. 1. Absorption of microwaves by a gas.

* This work was supported in part by the Office of Naval Research.

[1] J. Weber, Trans. Inst. Radio Engrs. Prof. Group on Electron Devices, PGED-3 (June, 1953).

[2] Gordon, Zeiger, and Townes, Phys. Rev. **95**, 282 (1954); Phys. Rev. **99**, 1264 (1955).

[3] N. G. Basov and A. M. Prokhorov, J. Exptl. Theoret. Phys. (U.S.S.R.) **27**, 431 (1954); Proc. Acad. Sci. (U.S.S.R.) **101**, 47 (1955); J. Exptl. Theoret. Phys. (U.S.S.R.) **28**, 249 (1955).

II. AMPLIFICATION BY SYSTEMS HAVING A HIGHER ENERGY STATE MORE DENSELY POPULATED THAN A LOWER STATE

A. General Principles

We follow the original[1] discussion of this principle of maser type amplification. In the experimental arrangement of Fig. 1, there is a microwave system which consists of a wave guide with gas inside, in thermal equilibrium. Such a gas may absorb microwaves, and this absorption can be described in the following terms.

Suppose the gas has a pair of energy levels (Fig. 2) with values E_1 and E_2, with $E_2 > E_1$. Let n_1 and n_2 be the numbers of particles with energies E_1 and E_2. Let the absolute temperature be denoted by T and Boltzmann's constant by k. Then we can write

$$n_2 = n_1 e^{-(E_2 - E_1)/kT}. \tag{1}$$

We suppose that electromagnetic radiation is present with frequency ν given by

$$\nu = (E_2 - E_1)/h. \tag{2}$$

The power absorbed by the gas can be written

$$P_A = W_{12} n_1 h\nu. \tag{3}$$

Here W_{12} is the transition probability for transitions from state 1 to 2 induced by radiation. Similarly the power emitted by the gas due to the stimulated emission of radiation is given by

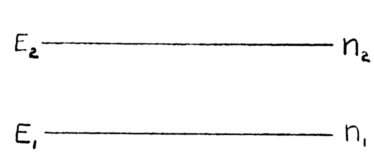
$$P_E = W_{21} n_2 h\nu. \tag{4}$$

Neglecting spontaneous emission for the moment we can write $W_{21} = W_{12}$ and for the *net* power absorbed

$$P = W_{12}(n_1 - n_2) h\nu. \tag{5}$$

P will be positive and the gas will absorb power, if n_1 exceeds n_2. This will be the case in consequence of expression (1), if the gas is in thermal equilibrium. How-

FIG. 2.



ever, the net power absorbed can be made negative, and amplification will result, if $n_2 > n_1$. For the present we consider amplification by a one-quantum process, resulting from the nonequilibrium situation $n_2 > n_1$. The same considerations which lead to coherent absorption for $n_1 > n_2$ suggest that for[4] $n_2 > n_1$ the amplified signal will be coherent with the driving signal. The energy levels which are employed may be bound states of molecules, or spin states in a magnetic field. The low-temperature solid state masers use paramagnetic ions in an externally applied magnetic field.

B. Three-Level Method

Many methods have been proposed for obtaining a higher population density in an upper energy level than in a lower one. Most of these are not generally useful, but may nonetheless have special applications. A fairly complete survey is given in Sec. IV. First we present a summary of the three-level method, first proposed by Prokhorov,[3] and developed independently by Bloembergen,[5] and also by Javan; this appears to be by far the most generally useful one. We follow Bloembergen's treatment.

In a system with three energy levels, E_1, E_2, and E_3 (Fig. 3), let the selection rules be such that transitions are allowed between each level and either of the other two. In thermal equilibrium, the numbers of particles in the different states satisfy the relations,

$$n_1 > n_2 > n_3. \qquad (6)$$

The frequencies ν_{32}, ν_{21}, ν_{31} are defined by the relations

$$\nu_{mn} = (E_m - E_n)/h. \qquad (7)$$

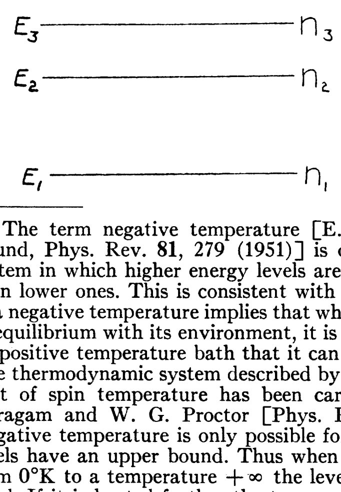


FIG. 3.

[4] The term negative temperature [E. M. Purcell and R. V. Pound, Phys. Rev. **81**, 279 (1951)] is often used to describe a system in which higher energy levels are more densely populated than lower ones. This is consistent with expression (1). The idea of a negative temperature implies that while part of a system is not in equilibrium with its environment, it is sufficiently isolated from its positive temperature bath that it can be considered as a separate thermodynamic system described by a temperature. The concept of spin temperature has been carefully considered by A. Abragam and W. G. Proctor [Phys. Rev. **109**, 1441 (1958)]. Negative temperature is only possible for a system whose energy levels have an upper bound. Thus when such a system is heated from 0°K to a temperature $+\infty$ the levels become equally populated. If it is heated further the temperature changes discontinuously to $-\infty$, and on further heating it eventually tends to minus 0°K at which point all particles are in the state of highest energy. As Bloembergen has remarked, negative temperature is hotter than hot. It has been suggested by Ramsey that the discontinuity at $+\infty$ could have been avoided if the reciprocal of temperature had been used. At negative temperatures resistances become negative and amplification is then possible.

[5] N. Bloembergen, Phys. Rev. **104**, 324 (1956).

An intense rf field of frequency ν_{31} will induce transitions between states 1 and 3. Since initially there are more particles in state 1, the particles will initially leave state 1 at a greater rate than return to it from state 3.

Saturation phenomena will result in which $n_3 \to n_1$. Under these nonequilibrium conditions we may expect either that $n_2 > n_1$ or $n_3 > n_2$. In the first case amplification is possible at frequency ν_{21} and in the second case at ν_{32}. In order to arrive at quantitative values relaxation[6]

[6] An important issue in the operation of a paramagnetic ion low-temperature maser is the mechanism of paramagnetic relaxation. We discuss briefly some pertinent aspects of this problem. It appears more complex than the corresponding nuclear magnetic resonance problem. One experimental fact is that it is possible to couple the saturation radiation to one set of spin levels and simultaneously saturate a second pair of levels at approximately the same spacing which is weakly coupled to the radiation field [Strandberg, Davis, Faughnan, Kyhl, and Wolga, Phys. Rev. **109**, 1988 (1958)]. Strandberg [Phys. Rev. **110**, 65 (1958)] has interpreted these results as evidence for the idea that there are anomalies in the phonon excitation, with the lattice modes with frequency near the spin resonance in equilibrium with the high-temperature saturated spin system. He considers this to be in part a consequence of the fact that the specific heat of the lattice vibrations at 4°K (calculated from say the Debye model) is several orders smaller than that of the paramagnetic spin system. Giordmaine, Alsop, Nash, and Townes [Phys. Rev. **109**, 302 (1958)] have described a series of experiments using paramagnetic resonance at 9000 Mc, at 1–4°K; $Gd_2Mg_3(NO_3)_{12}\cdot 24H_2O$), $K_3Cr(CN)_6$, and $Cu(NH_4)_2(SO_4)_2\cdot 6H_2O$) were used. They found that it was not possible to "burn a hole" in their resonances by saturation, indicating that their lines were homogeneously broadened. In the case of Cu, saturation of one line immediately saturated seven neighboring resonances. They were not able to observe the spin reversal by adiabatic rapid passage. No difference in absorption was noted in times as short as one msec after adiabatic rapid passage. They also observed that excitation due to a rapid sweep through a single resonance decayed very rapidly while that due to a slower sweep decayed at the normal rate, $T_1 \to 10$ sec. They interpreted these results as evidence for the idea that for some salts at low temperatures it is the lattice bath relaxation which limits the total relaxation rate. [Gorter, Van der Marel, and Bölger, Physica **21**, 103 (1955)]. They also remarked that for these salts the spin lattice relaxation time is several orders smaller than the observed values of T_1, and is of order 10^{-5} sec for Gd and 10^{-4} sec for Cu. In addition, they concluded that the breadth of the lattice modes is much larger than the width of the resonances in the diluted crystals (several hundred Mc for 1% paramagnetic concentration of the Cu salt), that the relaxation time T_1 is dependent on crystal size, and that the breadth of the lattice modes increases with increasing concentration of paramagnetic centers. They also suggested that for those salts described by the foregoing conditions, operation of an adiabatic rapid passage two-level maser would be impractical while three-level maser operation would be possible with reduced band width.

Bloembergen [Phys. Rev. **109**, 2209 (1958)] and Strandberg independently [Phys. Rev. **110**, 65 (1958)] have reached a different conclusion. Bloembergen points out that three-level maser iction is not possible for a salt whose paramagnetic relaxation rate is determined by interaction between lattice vibrations and the helium bath and for which the heat conduction between spin system and lattice is 1000 times better than between lattice and heat bath. Let ν_{31} be the saturation frequency and ν_{32} the amplification frequency and let these frequencies be widely spaced, with no overlapping of phonon bands. Then we have a negative spin temperature corresponding to levels 2 and 3. If the thermal contact with the phonons is very good then a band centered at frequency ν_{32} will start to gain energy, in order to attempt to reach equilibrium with the negative temperature spin system. Since the lattice vibration levels are those of harmonic oscillators, they have no upper bound. They cannot be saturated [R. Karplus and J. Schwinger Phys. Rev. **73**, 1020 (1948)] and the temperature associated with the ν_{32} band can never become negative. The steady state (nonequilibrium) situation would be one in which the spins of levels 2 and 3 are at a very high negative temperature and those effects must be included. Let w_{12} be the heat bath induced transition probability from state 1 to state 2, with corresponding meanings for w_{21}, w_{32}, w_{23}, w_{13}, and w_{31}. First suppose the system is in thermal equilibrium with no microwave fields applied. The number of particles leaving the state E_1 per sec must equal the number returning to it, so

$$n_1 w_{12} = n_2 w_{21}, \quad n_1 w_{13} = n_3 w_{31}. \tag{8}$$

Employing the Boltzmann factor enables us to write

$$w_{12}/w_{21} = n_2/n_1 = e^{-h\nu_{21}/kT}. \tag{9}$$

The w's are the reciprocals of the spin lattice relaxation times. Now suppose that a strong microwave field of frequency ν_{31}, and a weak microwave signal of frequency ν_{32} are present. Let W_{31} and W_{32} be the transition probabilities induced by these rf fields. The numbers of particles n_1, n_2, and n_3 occupying the three levels satisfy the relation,

$$n_1 + n_2 + n_3 = N. \tag{10}$$

For $(h\nu_{32})/(kT) \ll 1$, the populations satisfy the equations

$$\begin{aligned}\frac{dn_3}{dt} &= w_{13}\left(n_1 - n_3 - \frac{Nh\nu_{31}}{3kT}\right) + w_{23}\left(n_2 - n_3 - \frac{Nh\nu_{32}}{3kT}\right) \\ &\quad + W_{31}(n_1 - n_3) + W_{32}(n_2 - n_3), \\ \frac{dn_2}{dt} &= w_{23}\left(n_3 - n_2 + \frac{Nh\nu_{32}}{3kT}\right) \\ &\quad + w_{21}\left(n_1 - n_2 - \frac{Nh\nu_{21}}{3kT}\right) + W_{32}(n_3 - n_2), \\ \frac{dn_1}{dt} &= w_{13}\left(n_3 - n_1 + \frac{Nh\nu_{31}}{3kT}\right) \\ &\quad + w_{21}\left(n_2 - n_1 + \frac{Nh\nu_{21}}{3kT}\right) - W_{31}(n_1 - n_3).\end{aligned} \tag{11}$$

These equations have the following approximate steady-state solution for the case in mind, $W_{31} \gg W_{32}$.

$$n_1 - n_2 = n_3 - n_2 = \frac{hN}{3kT}\left[\frac{-w_{23}\nu_{32} + w_{21}\nu_{21}}{w_{23} + w_{21} + W_{32}}\right]. \tag{12}$$

In case the numerator of (12) is positive, amplification is possible at frequency ν_{32}. In case it is negative, amplification is possible at the frequency ν_{21} and the W_{32} in the denominator of (12) must be replaced by W_{21}. The signal power emitted by the material is obtained by multiplying (12) by the signal induced transition probability W_{32} and the energy of one quantum.

$$P = \frac{Nh^2\nu_{32}(w_{21}\nu_{21} - w_{32}\nu_{32})W_{32}}{3kT(w_{23} + w_{21} + W_{32})}. \tag{13}$$

The situation described by (12) and (13) is qualitatively the same, but more complex, if $(h\nu/kT)$ is not small compared to 1. The procedures for calculating the transition probability W_{32} require a knowledge of the matrix elements of the interaction with the Maxwell field, and appropriate relaxation times. For the remainder of the section we consider a paramagnetic ion type of solid state amplifier. In this case the signal induced transition probability is given by

$$W_{32} = \left(\frac{2\pi}{h}\right)^2 |\langle 2|M_x|3\rangle|^2 \langle H_s^2(\nu_{32})\rangle_{\mathrm{Av}} T_2. \tag{14}$$

Here M_x is the (x component) magnetic dipole operator, $\langle H_s^2(\nu_{32})\rangle_{\mathrm{Av}}$ is the volume average squared magnetic field at the signal microwave frequency ν_{32}, and T_2 is the spin-spin relaxation time.

Expression (13) shows that the temperature T of the heat bath must be low. The three-level solid state masers which have so far been successfully operated have employed liquid helium cooling with temperatures in the range 1.25–4°K. In addition to increasing the amplification, the low temperatures partially improve the noise performance.

The power which is absorbed from the saturation†

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lattice vibrations with which the spins are on "speaking terms" will be at a very high positive temperature. A high negative temperature means that the level populations associated with states 2 and 3 are almost equal, and therefore the maser would not operate in the observed manner. Bloembergen has solved the problem of three-level maser and phonon steady states in terms of the spin populations n_1, n_2, n_3 and the average lattice oscillator excitation quantum numbers in the three phonon bands, $n_{ph}(\nu_{32})$, $n_{ph}(\nu_{21})$, $n_{ph}(\nu_{31})$. His solution confirms that $n_3 - n_2$ is too small for maser operation if the contact between spins and phonons is better than between phonons and heat bath.

A possibility which has briefly been explored by the author is that the temperature associated with all three maser spin levels is positive, amplification occurring as a result of a two-quantum process involving absorption of a saturation frequency photon and emission of an amplification frequency photon. This is a kind of Raman process in which energy is conserved in both transitions. Direct calculation shows that this effect is too small, and that a low negative spin temperature needs to be assumed, in conjunction with single photon processes, in order to explain the operation of a maser.

It can be concluded that the paramagnetic relaxation mechanisms suggested by Giordmaine, Nash, and Townes do not apply to those salts which have been successfully used in a three-level maser.

Shapiro, Bloembergen, and Artman [Bull. Am. Phys. Soc. Ser. II, **3**, 317 (1958)] have reported additional indirect saturation experiments which they interpret as caused by higher order spin-spin interactions rather than by "hot phonons." This appears very reasonable because it is difficult to see how the lattice vibration modes could be tightly coupled to the saturated spin system without being similarly coupled to the negative temperature spin system. Further work by Bloembergen, Shapiro, Pershan, and Artman (Cruft Laboratory Technical Report No. 285, Harvard University, Cambridge, Massachusetts, October 15, 1958) has increased the evidence that the indirect saturation phenomena are indeed due to spin spin interactions. This follows the mechanism proposed by R. Kronig and C. J. Bouwkamp, Physica **5**, 521 (1938) and Physica **6**, 290 (1939).

† *Note added in proof.*—An interesting discussion of masers as heat engines has been given by H. E. D. Scovil and E. O. Schulz-

field of frequency ν_{31} is given by that needed to balance the tendency of collisions to restore equilibrium, and is

$$P_s=(h\nu_{31})^2n_1/2T_1kT. \quad (15)$$

Here T_1 is the spin-lattice relaxation time.

Expressions (13), (14), and (15) lead to the following general criteria for selection of materials for a paramagnetic ion three-level maser. If the heat bath transition probabilities w_{21} and w_{23} are nearly equal, the frequencies ν_{21} and ν_{32} should be very unequal. However, if the frequencies ν_{21} and ν_{32} are approximately equal, then w_{21} and w_{23} should be very unequal. If ν_{21} is exactly equal to ν_{32} the device cannot operate since the same signal which induces transitions between states 2 and 1 will be effective in inducing transitions between 3 and 2. Amplification between one pair of states would be annulled by absorption associated with the other pair since (12) requires that $n_1-n_2\approx n_3-n_2$. In order to obtain large gain the temperature T has to be small. The spin should be at least 1, but preferably not more than $\frac{3}{2}$, otherwise the particles will be distributed among too many states. (Gd with a spin of $\frac{7}{2}$ is a notable exception.) It is also desirable that the nuclear spin be zero. Expression (15) requires that for small T, the spin-lattice relaxation time T_1 be as large as possible, otherwise excessive saturation power will be needed with consequent difficulty in maintaining low temperatures. Expressions (13) and (14) show that the product of total number of spins, N, and the spin-spin relaxation time T_2 should be as large as possible. Inasmuch as an increase in concentration of spins tends, in general, to decrease T_2 it is necessary to attain an optimum value of the product NT_2 by suitably diluting the paramagnetic ions. The zero field splitting ought to be of the same order of magnitude as the energy level difference associated with the signal frequency. This follows because mixing of the spin states is essential in order to have transitions allowed between all three levels. This is most favorable when the Zeeman and crystalline field terms are comparable.

It is possible, and in some cases desirable, to use a four-level system. A particularly elegant solution of the problem of four-level maser design employing ruby has been given by Kikuchi,[7] Makhov, Lambe, and Terhune with $E_4-E_2=E_3-E_1$. In the latter case a pumping frequency $\nu_{42}=\nu_{31}$ populates the third level and simultaneously depopulates the second level.

A generalization of (13), given by Kikuchi, is

$$P=\frac{Nh^2}{4kT}\left[\frac{w_{21}\nu_{21}-w_{32}\nu_{32}+w_{41}\nu_{41}+w_{43}\nu_{43}}{w_{21}+w_{23}+w_{14}+w_{34}+W_{32}}\right]W_{32}\nu_{32}. \quad (16)$$

The active material of a maser can be placed in a wave-guide transmission system or in a resonant cavity. In the former case we have a traveling wave amplifier with power output given by

$$P=P_oe^{\beta l}, \quad (17)$$

where l is the distance along the amplifier.

Differentiating this with respect to l leads to the gain coefficient β given by

$$\beta=dP/Pdl. \quad (18)$$

Here (dP/dl) is the power emitted per unit length and P is the power at the point where (dP/dl) is calculated, in accordance with expressions (13) or (16). Employing (14), this can be written

$$\beta=\frac{32\pi^3(n_3-n_2)|M_{32}|^2T_2\nu f}{hv_g}, \quad (19)$$

where f is a filling factor which may approach 1 and v_g is the group velocity.

Here the power P has been set equal to the energy density times the group velocity v_g. This equation shows that a slow wave structure (small group velocity) is a desirable method of obtaining a higher gain-band-width product. Slow wave structures proposed thus far are a ruby rod with a helix wrapped around it, considered by W. W. Anderson of Stanford University, and a rectangular wave guide with conducting fingers giving circular polarization, considered by DeGrasse, Schulz-DuBois,

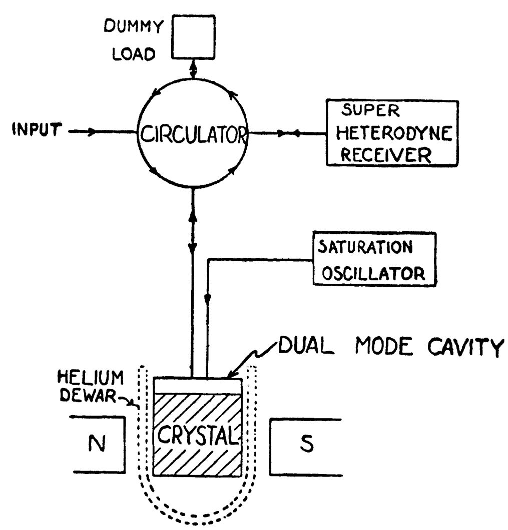

FIG. 4. Reflection cavity maser.

DuBois [Phys. Rev. Letters 2, 262 (1959)]. They consider a hot reservoir with temperature T_1 and filter allowing a band of frequencies in the vicinity of ν_{31} to pass, in thermal contact with a maser. Levels 2 and 3 are in thermal contact with a cold reservoir at temperature T_0, and coupled through a filter with pass band centered about ν_{32}. The signal frequency is ν_{21}. Ignoring relaxation processes within the maser, using the Boltzmann distribution law and requiring $n_2>n_1$ leads to the result that the maser efficiency ν_{21}/ν_{31} is that of the Carnot engine, $(T_1-T_0)/T_1$. They note the important result that instead of using a coherent microwave pump at ν_{31}, thermal excitation by two reservoirs as described above should make it possible to generate microwaves.

[7] Makhov, Kikuchi, Lambe, and Terhune, Phys. Rev. 109, 1349 (L) (1958); also Kikuchi, Lambe, Makhov, and Terhune (to be published).

and Seidel of the Bell Telephone Laboratories. These structures could be employed either as transmission or reflection devices. Means must be provided to prevent output power from being fed back to the input, causing the amplifier to oscillate, and for preventing noise from the load from getting to the maser and being amplified. Transmission devices utilizing nonreciprocity would accomplish this.

The reflection cavity type of maser has undergone most development and appears capable of giving enough gain-band-width product for most purposes. A block diagram is sketched in Fig. 4. Here a nonreciprocal device, the microwave circulator,[8] prevents power from being fed back and causing oscillation, and keeps noise from the load out of the maser. Resonant cavity expressions for gain and noise performance are usually expressed in terms of the Q's (quality factors) of a cavity. The Q is defined by

$$Q=2\pi\nu E/P_a, \qquad (20)$$

where E is the energy in the principal cavity mode and P_a is the rate of absorption of energy. The unloaded Q, usually denoted by Q_o, is defined by

$$Q_o=2\pi\nu E/P_{aw}, \qquad (21)$$

where P_{aw} is the part of the absorbed power that is absorbed in the cavity walls. The magnetic Q, denoted by Q_m, is defined by

$$Q_m=-\frac{2\pi\nu E}{P_e}=\frac{-\nu\langle H^2(\nu)\rangle V_c}{4P_e}, \qquad (22)$$

where P_e is the net power emitted by the active material.

Here V_c is the volume of the cavity and ν is the frequency being amplified. The denominator is given by expressions (13) or (16). The external Q, denoted by Q_e, is defined by

$$Q_e=2\pi\nu E/P_{en}, \qquad (23)$$

where P_{en} is the net power output.

The cavity mode may be represented by the equivalent circuit in Fig. 5. G_m, G_o, and G_e are conductances associated with the spin system, the walls of the cavity, and the output load, respectively. From Fig. 5, a loaded Q may be defined as

$$Q_l^{-1}=Q_m^{-1}+Q_o^{-1}+Q_e^{-1}. \qquad (24)$$

The voltage standing wave ratio in the transmission system which drives the cavity, B, is given in terms of

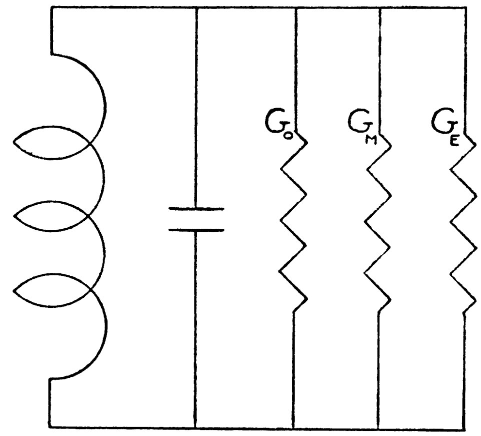

FIG. 5. Cavity equivalent circuit.

the incident voltage V_i and the reflected voltage V_r, as

$$B=\frac{|V_i|+|V_r|}{|V_i|-|V_r|}=\frac{1+g}{1-g}. \qquad (25)$$

Here g is the "voltage" gain defined by $g=|V_r/V_i|$. The power gain is g^2.

After solving (25) for g^2, we obtain

$$g^2=[(1-B)/(1+B)]^2. \qquad (26)$$

The maser will oscillate if $-Q_m^{-1}>Q_o^{-1}+Q_e^{-1}$, and will amplify if $Q_o^{-1}+Q_e^{-1}>-Q_m^{-1}>Q_o^{-1}$. Let us consider some characteristics of an amplifier.

From transmission line theory, the voltage standing wave ratio B (at resonance), in terms of the conductances shown in Fig. 5, is

$$B=(G_m+G_o)/G_e=Q_e[Q_o^{-1}+Q_m^{-1}]. \qquad (27)$$

This assumes that the input transmission line is matched to a conductance (G_m+G_o).

Use of (26) gives for the power gain,

$$g^2=\left[\frac{Q_e^{-1}-(Q_o^{-1}+Q_m^{-1})}{Q_e^{-1}+(Q_o^{-1}+Q_m^{-1})}\right]^2. \qquad (28)$$

In terms of the loaded Q this takes the form

$$g^2=[2Q_lQ_e^{-1}-1]^2. \qquad (29)$$

The band width is obtained by dividing the operating frequency by the loaded Q. For a low-temperature maser Q_o is so large that it may be neglected in (28). The product of the square root of power gain and band width is

$$g\Delta\nu=\frac{\nu[Q_e+|Q_m|]}{Q_e|Q_m|}. \qquad (30)$$

For a high-gain device it is customary to adjust Q_e so that it equals the magnitude Q_m. Under these conditions (30) is approximately a constant, given by

$$g\Delta\nu\approx 2\nu/|Q_m|. \qquad (31)$$

[8] Microwave circulators are not available below about 1400 mc/sec. Autler has proposed an ingenious arrangement [Lincoln Laboratory, MIT, Rept. M 37-27; Proc. Inst. Radio. Engrs. **46**, 1880 (1958)] using two masers in a "balanced" arrangement, which does not require use of a circulator. The two masers are at opposite ends of a coaxial or wave-guide "magic" tee. One of the maser arms is a quarter wavelength longer than the other. The antenna and receiver are connected to the other two arms. Thus the two maser outputs combine at the load. Noise from the load is amplified by the masers but is then radiated back out through the antenna.

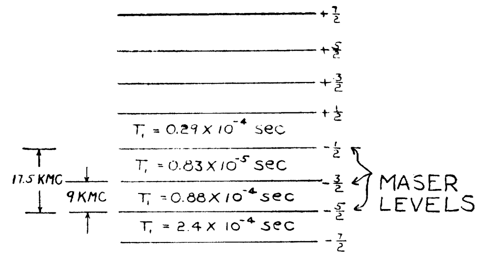

FIG. 6. Energy levels of the ground state of Gd^{+++} in ethyl sulfate, including spin-lattice relaxation times, as employed by Scovil, Feher, and Seidel.

III. SOLID STATE MASER MATERIALS AND DEVICES

The materials which have so far been successfully employed in three-level masers are[9,10] gadolinium ethyl sulfate, potassium chromicyanide,[11,12] and ruby.[7] We use the roman type g for the magnitude of the Zeeman tensor. This should not be confused with the italic g used for voltage gain.

Gadolinium ethyl sulfate was used in the first successful solid state maser.[10] The gadolinium ion is in an ^8S ground state having 7 electrons in a half-filled $4F$ shell. There is fine structure splitting into 7 lines with spacing which varies approximately as $3\cos^2\theta - 1$, where θ is the angle between the constant magnetic field, H_o, and the crystalline field axis. If the steady magnetic field is normal to the crystalline symmetry axis, the spin Hamiltonian may be written[11] approximately as

$$\mathcal{H} = g\beta_m H_o \cdot S_z - \tfrac{1}{2}D[S_z^2 - \tfrac{1}{3}S(S+1)] + \tfrac{1}{2}D[S_x^2 - S_y^2], \quad (32)$$

where β_m is the Bohr magneton, S is the spin operator, $g=1.99$, $D=.02\ \text{cm}^{-1}$, and the axis of quantization is parallel to that of the constant magnetic field, H_o. The first term of (32) represents the interaction with the field H_o which brings the transition to the required frequency. The second term makes the level spacings unequal, and the third term mixes the states, giving rise to transitions with $\Delta S_z = \pm 2$. The angle between H_o and the microwave magnetic field should be zero for the $\Delta S_z = \pm 2$ transitions and 90° for the $\Delta S_z = \pm 1$ transitions. Scovil, Feher, and Seidel used an angle of 45°. The energy levels and relaxation times of the ground state of Gd^{+++} in ethyl sulfate are shown in Fig. 6. Because the energy level separations are almost equal,

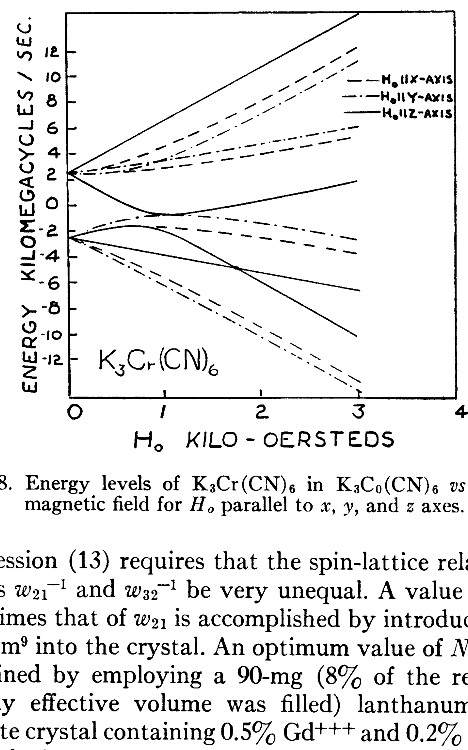

FIG. 8. Energy levels of $K_3Cr(CN)_6$ in $K_3C_0(CN)_6$ *vs* applied magnetic field for H_o parallel to x, y, and z axes.

expression (13) requires that the spin-lattice relaxation times w_{21}^{-1} and w_{32}^{-1} be very unequal. A value for w_{32} ten times that of w_{21} is accomplished by introduction of cerium[9] into the crystal. An optimum value of NT_2 was obtained by employing a 90-mg (8% of the resonant cavity effective volume was filled) lanthanum ethyl sulfate crystal containing 0.5% Gd^{+++} and 0.2% Ce^{+++}. The device was operated at 1.2°K with a saturation frequency of 17.5 kmc/sec. Oscillations were obtained at 9 kmc/sec.

The second material to be successfully used[12] in a solid state maser was potassium chromicyanide diluted in a cobalticyanide crystal [$K_3Co(CN)_6$]. In this case 0.5% Cr was used as the paramagnetic salt. This material appears to have the long spin-lattice relaxation time of 0.2 sec, at 1.25°K. The paramagnetic resonance spectrum of $K_3Cr(CN)_6$ arises from two differently oriented complexes per unit cell. The spin Hamiltonian is

$$\mathcal{H} = \beta_m \mathbf{H}\cdot\mathbf{g}\cdot\mathbf{S} + D[S_z^2 - \tfrac{1}{3}S(S+1)] + E[S_x^2 - S_y^2]. \quad (33)$$

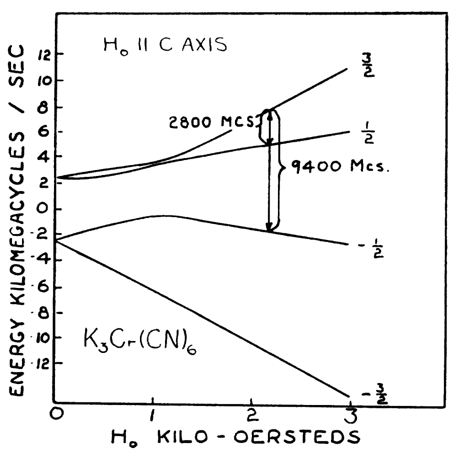

FIG. 7. Energy levels of $K_3Cr(CN)_6$ in $K_3C_0(CN)_6$ *vs* applied magnetic field, for $H_o \|$ to the c axis.

[9] G. Feher and H. E. D. Scovil, Phys. Rev. **105**, 2, 760(L) (1957).

[10] Scovil, Feher, and Seidel, Phys. Rev. **105**, 2, 762(L) (1957); H. E. D. Scovil, Trans. Inst. Radio Engrs. Prof. Group on Microwave Theory and Techniques **6**, 29 (1958).

[11] B. Bleaney and K. W. H. Stevens, Repts. Progr. in Phys. **16**, 108 (1951).

[12] A. L. McWhorter and J. W. Meyer, Phys. Rev. **109**, 312 (1958); R. H. Kingston, Lincoln Laboratory, MIT, Rept. M 35–79; Artman, Bloembergen, and Shapiro, Phys. Rev. **169**, 1392 (1958); S. H. Autler and N. McAvoy, Phys. Rev. **110**, 280(L) (1958).

For cobalt as the diluent $D=0.083$ cm^{-1}, $E=0.011$ cm^{-1}, and the Zeeman tensor $\mathbf{g}$ is approximately isotropic and equal to 1.99. The direction cosines between the magnetic axes (x, y, and z) and the pseudo-orthorhombic crystalline axes (a, b, and c) are

	x	y	z
a	0.104	0	0.994
b	±0.994	0	∓0.104
c	0	1	0

Energy level diagrams computed by McWhorter and Meyer are shown in Figs. 7 and 8. They used a dual mode coaxial cavity one half wavelength long at 2800 mc/sec operating in the *TEM* mode. For pumping, the cavity operated in the TE_{113} mode. The constant magnetic field H_o was applied, approximately parallel to the crystalline c axis, with operating conditions shown in Fig. 7. The upper two levels, labeled $+\frac{3}{2}$ and $+\frac{1}{2}$ were used for amplification and the second and fourth levels

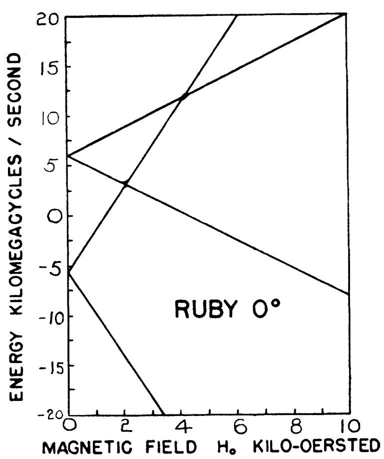

FIG. 9. Energy levels of Cr^{+++} in ruby *vs* applied magnetic field for H_o parallel to c axis of the crystal.

(labeled $+\frac{3}{2}$ and $-\frac{1}{2}$) were saturated by the rf field. Figure 7 shows that the energy level separation of the first and second levels (labeled $-\frac{3}{2}$ and $-\frac{1}{2}$) is approximately the same as that of the second and fourth levels. It has been shown[13] that for the Meyer-McWhorter arrangement (constant magnetic field parallel to the c axis and rf magnetic field parallel to the a axis) the transition probability for the first- to second-level transition is in fact 100 times greater than for the desired second and fourth levels. This is because the matrix elements are 4 times greater and the saturation radio-frequency magnetic field normal to the constant magnetic field favors the $\Delta m=1$ transitions. This implies that indirect saturation was accomplished by McWhorter and Meyer. They coupled strongly to levels 1 and 2 and succeeded in saturating levels 2 and 4 also.

This possibility of indirect saturation of one pair of levels by coupling to another pair which have the same energy separation is a useful aid in maser design. The total number of spins in the upper quartet of the chromium ion is approximately 10^{19}. A 10% filling factor was used. The calculated gain-band-width product $g\Delta\nu$, was 2.6×10^6 sec^{-1}. The measured value was 1.8×10^6 sec^{-1}. The small filling factor was chosen in order to be able to study the device without undue distortion of the cavity electromagnetic mode configurations. Practical amplifiers would therefore use a much larger filling factor, approaching unity. This gives a large increase in $g\Delta\nu$, of the order of 50. This is partly because there are more spins and partly because the increased dielectric constant enables a smaller cavity to be used.

The third material to be successfully used was ruby. Paramagnetic resonance in this material was investigated in the Soviet Union in 1955 and 1956,[14] and in the United States by Geusic.[15]

Again the Cr^{+++} ion is used, but the host crystal is Al_2O_3. The four nonequivalent sites are indistinguishable since the spin is less than two, and the crystal has trigonal symmetry. Maser action in ruby was first demonstrated by Makhov, Kikuchi, Lambe, and Terhune at the University of Michigan.[7] The spin Hamiltonian for Cr^{+++} in the Al_2O_3 lattice is given by

$$\mathcal{H}=\beta_m\mathbf{S}\cdot\mathbf{g}\cdot\mathbf{H}+D[S_z^2-\tfrac{1}{3}S(S+1)]. \tag{34}$$

$\mathbf{g}$ is the Zeeman tensor with components $g_{zz}=g_{\shortparallel}$, $g_{xx}=g_{yy}=g_{\perp}$. This Hamiltonian leads to the eigenvalue equation[7]

$$(\epsilon^2-1)^2-\tfrac{5}{2}x^2\epsilon^2+\tfrac{9}{16}x^4-\frac{x^2(5g_{\shortparallel}^2\cos^2\theta-g_{\perp}^2\sin^2\theta)}{2g^2}+\frac{2\epsilon\lambda x^2(g_{\perp}^2\sin^2\theta-2g_{\shortparallel}^2\cos^2\theta)}{g^2}=0. \tag{35}$$

Here

$$\epsilon=\frac{\text{Energy}}{|D|},\quad x=g\frac{\beta_m H}{|D|},$$

$$D=-0.1913\ \text{cm}^{-1}=3.798\times10^{-17}\ \text{ergs},$$

$$\lambda=\frac{D}{|D|},\quad g=1.986,\quad g^2=g_{\shortparallel}^2\cos^2\theta+g_{\perp}^2\sin^2\theta.$$

This equation is readily solved for the situations $\theta=0$, $\theta=\pi/2$, $g_{\perp}^2\sin^2\theta=2g_{\shortparallel}^2\cos^2\theta$. In the latter case, $\theta=\cos^{-1}(1/\sqrt{3})$ gives (assuming isotropic $\mathbf{g}$)

$$\epsilon(\pm\tfrac{3}{2})=\pm[1+(5/4)x^2+x(3+x^2)^{\frac{1}{2}}]^{\frac{1}{2}},$$
$$\epsilon(\pm\tfrac{1}{2})=\pm[1+(5/4)x^2-x(3+x^2)^{\frac{1}{2}}]^{\frac{1}{2}}.$$

[13] Standberg, Davis, Faughnan, Kyhl, and Wolga, Phys. Rev. **109**, 1988 (1958). See also Strandberg, Davis, and Kyhl, Fifth International Symposium on Low-Temperature Physics and Chemistry, Madison, Wisconsin (August 30, 1957).

[14] A. A. Manenkov and A. M. Prokhorov, Soviet Phys. JETP **1**, 611 (1955); M. M. Zaripov and Iv. Ia. Shamonin, Soviet Phys. JETP **3**, 171 (1956); J. E. Geusic, Phys. Rev. **102**, 1252 (1956).

[15] J. E. Geusic, Phys. Rev. **102**, 1252 (1956).

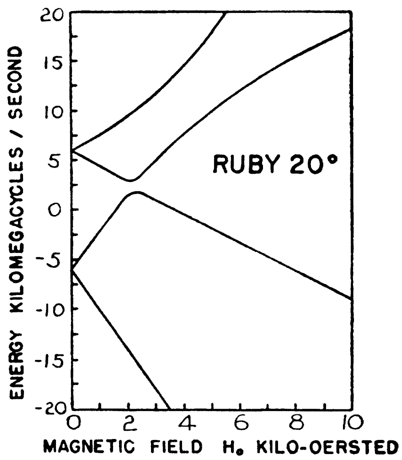

FIG. 10. Energy levels of Cr^{+++} in ruby *vs* applied magnetic field for H_o at 20° to c axis of the crystal.

Curves of energy levels as calculated with assistance of Dr. Schulz-DuBois are shown in Figs. 9–13. The four-level device using ruby has already been discussed. We are indebted to Professor Townes for the name "push-pull maser" for the four-level device of Makhov, Kikuchi, Lambe, and Terhune.‡

B. Characteristics of Some Ruby Masers

A ruby three-level maser has been developed by Alsop, Giordmaine, and Townes.[16] A large filling factor, approaching 0.9, is used. The "voltage gain" band-width product approaches 100 Mc, and the band width is approximately 5 Mc. The magnetic Q is about 400. The volume of the active material is about $\frac{1}{3}$ cm^3. A rectangular cavity is used, operating in the TE_{011} mode for the signal frequency and the TE_{012} mode for the saturation microwave field. The small cavity volume results from the high dielectric constant of ruby. About 30 mw of pumping power are required. The pump power is coupled to the wave guide by a remotely gear-controlled probe, and the signal power is coupled in by means of an iris. Liquid helium cooling with helium maintained at low pressure provided a bath temperature of 1.4°K.

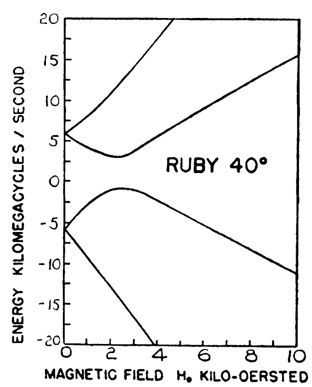

FIG. 11. Energy levels of Cr^{+++} in ruby *vs* applied magnetic field for H_o making an angle of 40° with the c axis of the crystal.

Morris, Kyhl, and Strandberg[17] have described the ruby maser illustrated in Fig. 14. All four levels are employed. The chromium concentration is about 0.01%. Levels 1–3 and 2–4 are saturated at 23 kMc/sec. Levels 2–3 are employed for amplification. The device is tunable over the range 8400–9700 Mc/sec, and has the new feature that cavity resonance is not needed at the saturation frequency if a saturation power of 100 mw is employed. The fringing fields near the coupling hole allow enough coupling to saturate the crystal at 23 kMc/sec.

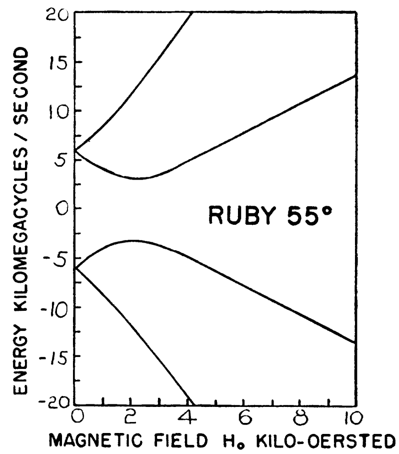

FIG. 12. Energy levels of Cr^{+++} in ruby *vs* applied magnetic field for H_o making an angle of 55° with the c axis of the crystal.

Design of masers is facilitated by the extensive tables of energy levels and transition probabilities now being prepared by W. S. Chang and A. E. Siegman of the Stanford Electronics Laboratories.[18]

IV. SUMMARY OF OTHER METHODS FOR OBTAINING MASER ACTION

Bloch[19] showed, in 1946, that inversion of level populations can be obtained by "adiabatic rapid pas-

‡ *Note added in proof.*—The paramagnetic resonance absorption for Cr^{+++} ions in emerald has been reported by Geusic, Peter, and Schulz-DuBois [Bull. Am. Phys. Soc. Ser. II, 4, 21 (1959)]. The spin Hamiltonian is

$$\mathcal{H}=\beta_m[g_{\parallel}H_zS_z+g_{\perp}(H_xS_x+H_yS_y)]+D[S_z^2-\tfrac{1}{3}S(S+1)],$$
$$2D=-53.6\text{ kmc},\quad g_{\parallel}=1.973\pm0.002,\quad g_{\perp}=1.97\pm0.01.$$

[16] Alsop, Giordmaine, Mayer, and Townes, Astron. J. **63**, 301 (1958).

[17] Morris, Kyhl, and Strandberg, Proc. Inst. Radio Engrs. **47**, 80 (1959).

[18] W. S. Chang and A. E. Siegman, Stanford Electronics Lab. Tech. Rept. 156–1 (May 16, 1958). Their machine calculations are presented in the Appendix to this paper. Note carefully that their calculations were done using positive D. To use their curves and data for the correct negative sign of D, change the sign of their energies and regard their level labeled number four as having the lowest energy. Note their comments at the end of the Appendix.

[19] F. Bloch, Phys. Rev. **70**, 460 (1946). See also R. K. Wangsness and F. Bloch, Phys. Rev. **89**, 728 (1953).

sage" through resonance. We have again a system of spins in a "constant" magnetic field H_o. If a microwave magnetic field is applied at right angles to the "constant" magnetic field the spin system will precess about H_o. Suppose that H_o is smaller than the value required for resonance at the microwave angular frequency ω. If H_o is steady for a long time, and is then suddenly increased through resonance and beyond, the spin system is turned over. The magnetization vector will be antiparallel to H_o until equilibrium is restored by the spin-lattice relaxation mechanism. The passage through resonance must be adiabatic, but rapid enough so that the sweep occurs in a time short compared with the spin-lattice relaxation time. While the magnetization is antiparallel to the field there are more moments antiparallel than parallel. This means more particles in excited states than in the ground state. Such a system will therefore amplify.

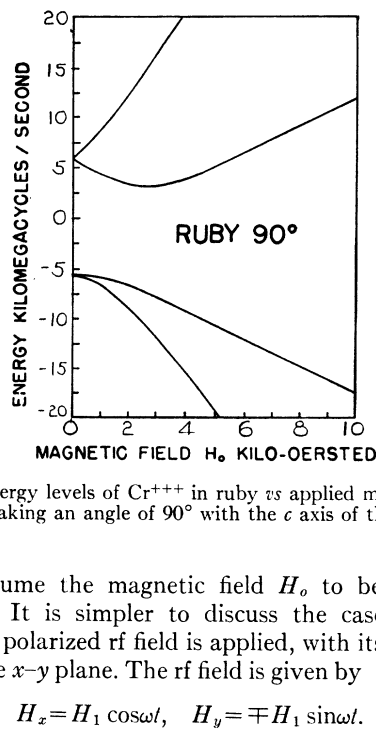

FIG. 13. Energy levels of Cr^{+++} in ruby *vs* applied magnetic field for H_o making an angle of 90° with the c axis of the crystal.

We assume the magnetic field H_o to be in the z direction. It is simpler to discuss the case where a circularly polarized rf field is applied, with its magnetic field in the x–y plane. The rf field is given by

$$H_x = H_1 \cos\omega t, \quad H_y = \mp H_1 \sin\omega t. \tag{36}$$

The − sign refers to a positive gyromagnetic ratio and the + sign to a negative one. The magnetization vector then satisfies the following equations.

$$\dot{M}_x - \gamma(M_y H_z - M_z H_y) + (M_x/T_2) = 0, \tag{37}$$

$$\dot{M}_y - \gamma(M_z H_x - M_x H_z) + (M_y/T_2) = 0, \tag{38}$$

$$\dot{M}_z - \gamma(M_x H_y - M_y H_x) + (M_z/T_1) = M_o/T_1. \tag{39}$$

γ is the gyromagnetic ratio, T_1 is the longitudinal (spin-lattice) relaxation time, T_2 is the transverse (spin-spin) relaxation time, and M_o is the value of the magnetization in the absence of a radio-frequency field.

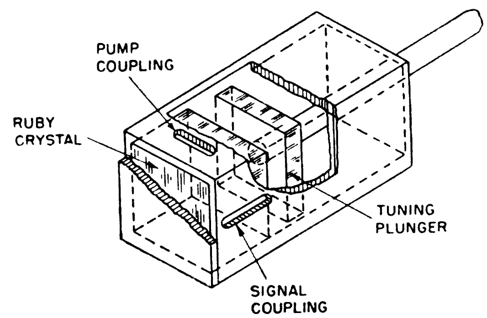

FIG. 14. Ruby maser of Morris, Kyhl, and Strandberg.

A quantity $\delta(t)$ is defined by the relation

$$\delta(t) = \frac{H_o(t) - (\omega/|\gamma|)}{H_1}. \tag{40}$$

$\delta(t)$ is zero at the resonant field $H_o = \omega/|\gamma|$. The magnetic field H_o is assumed to vary at a slow enough rate so that

$$|d\delta/dt| \ll |\gamma H_1|. \tag{41}$$

This is a statement of the adiabatic criterion. Subject to this condition Bloch gives the following solution of (37), (38), and (39):

$$M_x = \frac{M \cos\omega t}{(1+\delta^2)^{\frac{1}{2}}}, \tag{42}$$

$$M_y = \mp \frac{M \sin\omega t}{(1+\delta^2)^{\frac{1}{2}}}, \tag{43}$$

$$M_z = M\delta/(1+\delta^2)^{\frac{1}{2}}, \tag{44}$$

$$M(t) = \frac{1}{T_1}\int_{-\infty}^{t} \frac{\delta(t')e^{-[\theta(t)-\theta(t')]}M_o(t')}{[1+\delta^2(t')]^{\frac{1}{2}}}dt', \tag{45}$$

$$\theta(t) - \theta(t') = \frac{1}{T_1}\int_{t'}^{t}\left[\frac{\delta^2(t'') + (T_1/T_2)}{1+\delta^2(t'')}\right]dt''. \tag{46}$$

In order to discuss adiabatic rapid passage we assume that $\delta^2(t'')$ has been constant for a long time and then at time t_o is quickly increased through resonance, without violating (41). We have, for $\delta^2 \gg T_1/T_2$,

$$\theta(t) - \theta(t') = (t-t')/T_1. \tag{47}$$

The part of (45) resulting from the rapid change of δ is negligible. For the rest, since δ is large we have, from (45)

$$M(t) = \pm M(t_o). \tag{48}$$

Here the + sign refers to the situation where δ was positive up to t_o (H_o larger than the resonant value, initially), and the − sign refers to the situation where δ was negative up to t_o (H_o smaller than the resonant value, initially, then increased through resonance). Thus the sign of $M(t)$ does not change, but the sign of δ

changes as we go rapidly but adiabatically through resonance. According to (44) this means we have changed the sign of M_z which is now antiparallel to H_o.

An early unsuccessful attempt to develop an adiabatic fast passage solid-state maser was reported by Combrisson, Honig, and Townes.[20] Maser action due to adiabatic rapid passage using the paramagnetic electrons associated with the phosphorus donors in silicon was observed by Feher, Gordon, Buehler, Gere, and Thurmond.[21] Somewhat similar experiments using neutron irradiated quartz and magnesium oxide were done by Chester, Wagner, and Castle.[22] With a quartz sample containing $\sim 10^{18}$ spins the inverted state persisted for 2 msec at 4.2°K. A value of 5×10^6 sec^{-1} was obtained for the "voltage" gain-band-width product for gains between 8 and 21 db. This work was done at 9 kMc/sec with microwave powers for inversion of about 0.5 w, in 50–100 μsec pulses. A modulation structure of the pulsed power emitted from the cavity as the magnetic field is swept through resonance is observed. Senitzky[23] has suggested that this amplitude modulation of the power from the resonant cavity is due to the periodic transfer of energy between the cavity electromagnetic field and the spin system.

Purcell and Pound[24] were able to obtain inverted level populations in a magnetic resonance experiment using a single crystal of LiF which had a very long relaxation time. They removed their crystal from the strong field and inverted its spin system by means of other rapidly varying magnetic fields. When the crystal was reinserted in the strong field they were able to observe the return of the magnetization to its equilibrium value, from its "negative-temperature" state.

Weber studied the following method[1] in 1951. Consider a symmetric top molecule in an externally applied electrostatic field. The Stark effect has linear and quadratic terms. The linear effect is the dominant one if the field is not too strong. The energy levels are given by the formula

$$E_{JKM_j}^{(1)} = \frac{-\boldsymbol{\mu}\cdot\mathbf{E}KM_j}{J(J+1)}. \tag{49}$$

$\boldsymbol{\mu}$ is the electric dipole moment, $\mathbf{E}$ is the Stark field, J, K, and M_j are the quantum numbers associated with the symmetrical top. If the electrostatic (Stark) field is applied normal to the microwave electric field the $\Delta M = \pm 1$ transition will be allowed, and the frequency for such a transition will be

$$\nu = \boldsymbol{\mu}\cdot\mathbf{E}K/J(J+1)h. \tag{50}$$

If the gas is in equilibrium it will absorb microwaves at this frequency. There are more electric dipole moments parallel to the field than antiparallel to it. If the electrostatic field is suddenly reversed we have a negative temperature, and more dipole moments antiparallel than parallel to the field, and the device will amplify during roughly one relaxation time. A pulsed oscillator may be constructed as shown in Fig. 15. Here we have a resonant cavity with Stark electrode. If a square wave is applied there will be a microwave pulse emitted each time the field reverses and the TE_{10} mode (microwave electric field *parallel* to the Stark electrode) will be excited.

No experiments of this type were done because calculations showed that it would be difficult to achieve a useful gain-band width product in any gas-type maser amplifier, and the use of a solid was suggested.

A maser oscillator using a gas is, however, a very useful frequency standard, and compares favorably in many respects with a cesium beam clock. Work on an ammonia maser oscillator started at Columbia University in 1951 and the first maser oscillator and amplifier of this type was operated successfully by

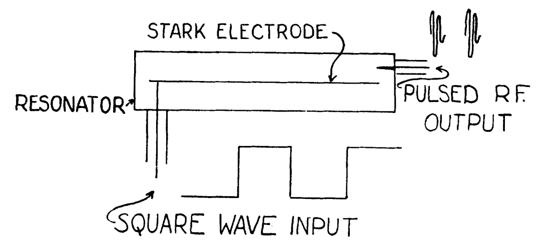

FIG. 15. Pulsed symmetric top maser oscillator.

Gordon and Zeiger[2] under the direction of Professor C. H. Townes in 1954. Similar work was done independently[3] by Basov and Prokhorov at the Lebedev Institute at about the same time. In addition to providing an exceptionally stable oscillator, the ammonia maser can also be employed as a very high resolution spectrometer. A sketch of the focuser and beam and cavity details is given in Fig. 16. The maser action is accomplished in the following way. The nitrogen atom in ammonia is in a potential which is symmetric on either side of the plane of the hydrogen atoms. In this kind of potential each harmonic oscillator level is split, and the ground state splitting provides microwave absorption lines of different frequencies for different rotational levels. In a Stark field the energy of the upper doublet level increases and that of the lower level decreases. The focuser of Fig. 16, provides an electric field with intensity approximately proportional to the displacement from the axis. If a molecular beam is now sent through the focuser the molecules in the lower doublet state drift outward to regions of lower energy and increasing field, and are removed. The center of the beam contains mostly higher energy particles. These

[20] Combrisson, Honig, and Townes, Compt. rend. **242**, 2451 (1956). Suggestions for a low-temperature adiabatic rapid passage maser were made independently by Strandberg in 1956.

[21] Feher, Gordon, Buehler, Gere, and Thurmond, Phys. Rev. **109**, 221 (1958).

[22] Chester, Wagner, and Castle, Phys. Rev. **110**, 281 (1958).

[23] I. R. Senitzky, Phys. Rev. Letters **1**, 167 (1958).

[24] E. M. Purcell and R. V. Pound, Phys. Rev. **81**, 279 (1951).

enter the resonant cavity and undergo stimulated emission. In this case, then, a higher population in excited states is achieved by the simple expedient of removing ground state particles from the beam.

The condition for sustained oscillation of a resonant cavity with volume V which contains n excited atoms or molecules may be obtained by setting the emitted power equal to the power lost in the cavity walls.

$$nW_{21}h\nu \geq P_w.$$

W_{21} is the transition probability. We write W_{21} in terms of the appropriate squared matrix elements $|\mu|^2$ and the line width $\Delta\nu$, and P_w in terms of the quality factor Q to obtain (Gordon, Zeiger, and Townes, see reference 2)

$$n \geqslant hV\Delta\nu/4\pi|\mu|^2Q.$$

The frequency range over which appreciable energy is distributed in a maser oscillator is given in terms of the

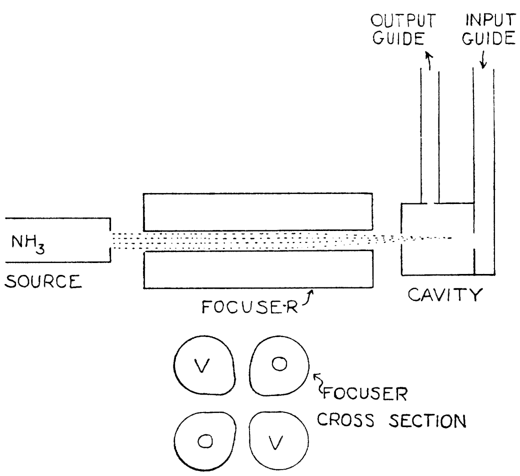

FIG. 16. Columbia ammonia maser.

total power output P, Boltzmann's constant, and the cavity wall temperature T as

$$\delta\nu = 4\pi kT(\Delta\nu)^2/P.$$

An upper state population ten times that of the lower state may be achieved in practice. The power output is approximately 10^{-9} w. An ammonia maser oscillator has been reported to have frequency stability over periods of one hour of the order of 1 part in 10^{10}. The stability is associated with the low noise property.

Another method of achieving inverted level populations in ammonia gas has been proposed by Dicke.[25] This uses a "hot grid" cell and again the quadratic Stark effect. A maser using this method of population inversion is being constructed by Dr. J. P. Wittke at RCA Laboratories.

Level population inversion can also be achieved by means of intense pulses of microwaves. For simplicity,

[25] J. P. Wittke, Proc. Inst. Radio Engrs. 45, 291 (1957).

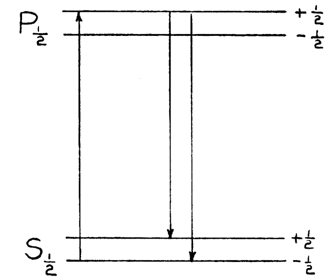

FIG. 17. Optical pumping.

consider, a two-level system with "permanent" electric dipole moment μ, driven at resonance[26] by a microwave field $E=E_o \sin(\omega t-\delta)$. The Hamiltonian is

$$\mathcal{H} = \mathcal{H}_o + \boldsymbol{\mu}\cdot\mathbf{E}_o \sin(\omega t-\delta). \tag{51}$$

Let the wave functions with $E_o=0$ be ψ_1 and ψ_2.

If we use this in the time-dependent Schroedinger equation and neglect sum frequency terms, an exact solution of the resulting equations gives for the wave function,

$$\Psi = \left[\cos\left(\frac{|\boldsymbol{\mu}_{12}\cdot\mathbf{E}_0|\,t}{2\hbar}\right)\right]\psi_1 e^{-i(E_1t/\hbar)} + \left[\sin\left(\frac{|\boldsymbol{\mu}_{12}\cdot\mathbf{E}_0|\,t}{2\hbar}\right)\right]\psi_2 e^{-iE_2t/\hbar}. \tag{52}$$

Here $\boldsymbol{\mu}_{12}$ is the dipole matrix element connecting the states 1 and 2. From this it follows that if the system is in the state ψ_1 at $t=0$, then at $t=(\pi\hbar)/(|\boldsymbol{\mu}_{12}\cdot\mathbf{E}_0|$, the microwave pulse has driven the system into state 2. This treatment is valid only for times short compared with the relaxation time since the effect of collisions is ignored.

Thus if we have thermal equilibrium at $t=0$ and apply a strong rf pulse of the correct length we can obtain inversion of level populations. This effect has been demonstrated by Dicke and Romer,[26] and by Kyhl, Standberg, Collins, and Park.[27]

The methods of optical pumping can also be employed[28] to achieve maser action. In Fig. 17 we have a system with a $^2S_{\frac{1}{2}}$ ground state and a $^2P_{\frac{1}{2}}$ excited state, in a magnetic field. In equilibrium almost all of the particles will be in the split S state. The $m=-\frac{1}{2}S$ state will have slightly more particles in it than the $m=+\frac{1}{2}$ state. Right circularly polarized optical resonance radiation will induce transitions with the selection rule $\Delta m=+1$. The excited particles can now spontaneously emit, either with $\Delta m=-1$, or 0. However this process only removes particles from the $m=-\frac{1}{2}S$ state and returns them to *both* the $m=-\frac{1}{2}$ and $m=+\frac{1}{2}$ states. Therefore the population of the $^2S_{+\frac{1}{2}}$ state will be in-

[26] R. H. Dicke and R. H. Romer, Rev. Sci. Instr. 26, 915 (1955)
[27] Kyhl, Strandberg, Collins, and Park, Signal Corps Symposium on Solid State Masers, Fort Monmouth, New Jersey (June 12–13, 1958).
[28] W. H. Culver, Science 126 810 (1957).

creased over that of the $^2S_{-\frac{1}{2}}$ state. A review of the work on optical pumping has been given by Kastler.[29]

All of the methods discussed so far use single quantum processes in which amplification results from more particles in excited states. A. Javan[30] has shown that two quantum processes can be used for amplification, with more particles in the lower state (Fig. 18).

Consider a two photon process in which a particle in a lower state absorbs a photon, then emits a photon, ending up in an excited state. Suppose the frequency of absorption corresponds to that of an intense (pump) oscillator and the emission is stimulated by a signal which it is desired to amplify. In this case we are obtaining emission in a particle transition from a state of lower energy to a state of higher energy. If all particles are in the ground state, amplification will result. Let the transition probability be W_{12} and let n_1 particles be in the ground state. The emitted power is $W_{12}n_1h\nu$. If the excited state has n_2 particles the reverse process may occur in which a photon of amplification frequency ν is absorbed and a pump frequency photon is emitted. The net power emitted will be $W_{12}h\nu(n_1-n_2)$, which is positive at positive temperatures.

For a *one* quantum process W_{12} may be written as

$$W_{12}=4\pi^2\tau|H_{12}'|^2/h^2. \tag{53}$$

Here τ is the relaxation time and H' is the interaction matrix element. For a two quantum process H' is replaced by

$$\sum_i H_{1i}'H_{i2}'/(E_1-E_i). \tag{54}$$

The summation is over all intermediate states of the particle and the quantized electromagnetic field. In Raman spectroscopy the intermediate states are usually quantum states which are different from the initial and final states, because the diagonal interaction matrix elements ordinarily vanish. If the diagonal matrix elements do not vanish in either the initial or final state, no intermediate state is necessary. For example we may have a diagonal magnetic dipole matrix element in a product of the following type

$$\langle -\tfrac{1}{2}, N_1, N_2|H'|-\tfrac{1}{2}, N_1+1, N_2\rangle \times\langle -\tfrac{1}{2}, N_1+1, N_2|H'|N_2-1, N_1+1, +\tfrac{1}{2}\rangle. \tag{55}$$

Here we use the nomenclature of quantum electrodynamics. The first bracket involves a matrix element between states in which the particle spin is $-\frac{1}{2}$, and the electromagnetic field oscillators of frequency ν_1 and ν_2 have N_1 and N_2 quanta, respectively, to a state with particle spin unchanged, N_1+1 quanta for the oscillator with frequency ν_1, and N_2 quanta for the oscillator with frequency ν_2. The second bracket then involves a final state in which the spin becomes $+\frac{1}{2}$ and the field oscillator of frequency ν_2 has lost a quantum. Three additional matrix elements, similar to (55) may be written. One of these corresponds to an intermediate state of the particle the same as the initial state but an absorption first of frequency ν_2 followed by emission of frequency ν_1. The other two matrix elements involve an intermediate state of the particle which is the same as the final state, with both possibilities for the order of absorption and emission. Some of these matrix elements may be small in comparison with the others, depending on circumstances of the experimental arrangement. The summation (54) may be written in terms of the (magnetic) dipole matrix elements M_{11} and M_{12} as

$$-\frac{[\mathbf{M}_{11}\cdot\mathbf{H}(\nu_1)][\mathbf{M}_{12}\cdot\mathbf{H}(\nu_2)]}{h\nu_1}+\frac{[\mathbf{M}_{12}\cdot\mathbf{H}(\nu_2)][\mathbf{M}_{22}\cdot\mathbf{H}(\nu_1)]}{h\nu_1}+\frac{[\mathbf{M}_{11}\cdot\mathbf{H}(\nu_2)][\mathbf{M}_{12}\cdot\mathbf{H}(\nu_1)]}{h\nu_2}-\frac{[\mathbf{M}_{12}\cdot\mathbf{H}(\nu_1)][\mathbf{M}_{22}\cdot\mathbf{H}(\nu_2)]}{h\nu_2}. \tag{56}$$

$\mathbf{H}(\nu_1)$ is the magnetic field at frequency ν_1 and $\mathbf{H}(\nu_2)$ is the magnetic field at frequency ν_2. A corresponding expression would contain the electric field vectors for electric dipole transitions. The transition probability can be written as

$$W=\frac{4\pi^2\tau}{h^2}\left|-\frac{[\mathbf{M}_{11}\cdot\mathbf{H}(\nu_1)][\mathbf{M}_{12}\cdot\mathbf{H}(\nu_2)]}{h\nu_1}+\frac{[\mathbf{M}_{12}\cdot\mathbf{H}(\nu_2)][\mathbf{M}_{22}\cdot\mathbf{H}(\nu_1)]}{h\nu_1}+\frac{[\mathbf{M}_{11}\cdot\mathbf{H}(\nu_2)][\mathbf{M}_{12}\cdot\mathbf{H}(\nu_1)]}{h\nu_2}-\frac{[\mathbf{M}_{12}\cdot\mathbf{H}(\nu_1)][\mathbf{M}_{22}\cdot\mathbf{H}(\nu_2)]}{h\nu_2}\right|^2. \tag{57}$$

The net power emitted by a two-level system which has n_1 particles in the lower state and n_2 particles in the upper state may then be written as

$$P=\frac{4\pi^2(n_1-n_2)\tau\nu_1}{h}\left|-\frac{[\mathbf{M}_{11}\cdot\mathbf{H}(\nu_1)][\mathbf{M}_{12}\cdot\mathbf{H}(\nu_2)]}{h\nu_1}+\frac{[\mathbf{M}_{12}\cdot\mathbf{H}(\nu_2)][\mathbf{M}_{22}\cdot\mathbf{H}(\nu_1)]}{h\nu_1}+\frac{[\mathbf{M}_{11}\cdot\mathbf{H}(\nu_2)][\mathbf{M}_{12}\cdot\mathbf{H}(\nu_1)]}{h\nu_2}-\frac{[\mathbf{M}_{12}\cdot\mathbf{H}(\nu_1)][\mathbf{M}_{22}\cdot\mathbf{H}(\nu_2)]}{h\nu_2}\right|^2. \tag{58}$$

[29] A. Kastler, J. Opt. Soc. Am. **47**, 460 (1957).
[30] A. Javan, Bull. Am. Phys. Soc. Ser. II, **3**, 213 (1958).

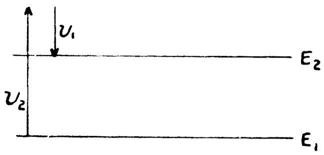

FIG. 18. Raman maser.

V. NOISE PERFORMANCE

Prior to 1950, methods had been developed for generation, amplification, and detection of signals in the electromagnetic spectrum from zero frequency to the mm wave region. At frequencies below 10 mc/sec the noise performance is already better than is ordinarily required for communications in the presence of atmospheric and other external noise. Indeed, a well designed vacuum tube amplifier in the vicinity of 1 mc/sec is capable of noise performance corresponding to an internal temperature of about 25°K. Microwave amplifiers, on the other hand, did not have good low-noise performance, mainly because of electron stream fluctuations originating at a hot cathode. Inasmuch as this type of noise must be absent in a maser, it was suggested that molecular beam maser type amplifiers would have very little noise.[31] It was very gratifying that the important sources of low-temperature solid state maser noise which were understood in 1956, are sufficiently small to make possible sensitivities which may approach detection of single microwave photons.

We consider first the inherent spontaneous emission noise of a maser, disregarding circuit noise, at spin temperature approaching minus 0°K. The earlier discussion did not consider the effect of spontaneous emission. This is a purely random process[32] and therefore contributes noise. In an ensemble of charged particles, each having two energy levels, those which are in the upper state have a transition probability for transitions to the lower state of the form

$$W_{21}=f(\nu,\mu^2\cdots)[N+1], \tag{59}$$

in consequence of the interaction with the electromagnetic field.§

Here f is a function of frequency and squared matrix elements and N is the number of quanta per radiation oscillator. The one which adds to N gives the effect of spontaneous emission. Let us calculate an equivalent temperature for spontaneous emission. In thermal equilibrium at temperature T, according to the Bose-Einstein statistics, N would be

$$\frac{1}{e^{h\nu/kT}-1}. \tag{60}$$

The positive temperature which black surroundings would need, to emit input noise equivalent to the spontaneous emission, will be denoted by T_E. From (59) we see that this (equivalent) T_E corresponds to an equivalent $N=1$.

Expressions (60) and (59) allow us to write

$$1=e^{h\nu/kT_E}-1, \tag{61}$$

giving

$$T_E=h\nu/k\ \ln_e 2. \tag{62}$$

Suppose we ask the question: how many microwave photons will double the randomly fluctuating output of a maser whose spin temperature approaches -0°K. Let the averaging time of the receiver be τ and let the equivalent input noise energy during this interval be U_n. This is given by the product of the Nyquist formula noise power for T_E and the averaging time τ.

$$U_n=\left[\frac{h\nu}{e^{h\nu/kT_E}-1}\right]\Delta\nu\tau. \tag{63}$$

Here $\Delta\nu$ is the band width. $\Delta\nu\tau\approx 1$. Employing expression (61) gives

$$U_n=h\nu. \tag{64}$$

The equivalent input (noise) energy over the averaging time is that of one photon. If one microwave photon is incident during this interval it will double the average noise output and therefore we are justified in saying that a maser with no circuit noise and spin temperature approaching -0°K may detect single microwave photons, and a flux of $\Delta\nu$ photons per sec, with reasonably large probability.

The foregoing analysis gives exactly what would be predicted by the Nyquist formula for *any* resistance at a temperature -0°K. The Nyquist formula noise energy (delivered to a "matched" load) over the averaging time τ is given by

$$U_n=\frac{h\nu\Delta\nu\tau}{|e^{h\nu/kT}-1|}. \tag{63a}$$

In the limit as $T\rightarrow -0°$ (63a) becomes

$$U_n=h\nu\Delta\nu\tau\approx h\nu, \tag{64a}$$

in agreement with (64). This makes it clear that the residual noise in a low-temperature maser is of the same type as discussed by Johnson and Nyquist[33] long ago.

If the spin temperature is not close to minus 0°K (64a) becomes

$$U_n=\frac{h\nu\Delta\nu\tau}{|e^{h\nu/kT_{\mathrm{spin}}}-1|}. \tag{64b}$$

[31] The Columbia work (Gordon, Zeiger, and Townes, reference 2) showed that good noise performance could be obtained with a molecular beam maser. Spontaneous emission from the molecules was not discussed. For a room temperature device this is small compared with input noise.

[32] It has been shown [J. Weber, Phys. Rev. **94**, 215 (1954)] that this type of noise is also present in conventional free electron amplifiers.

§ *Note added in proof.*—Expression (59) is given in Chapter V, *The Quantum Theory of Radiation*, by W. Heitler (Oxford University Press, London, 1954), third edition. It follows from the quantization of the Maxwell field. Each degree of freedom is like a harmonic oscillator and the $N+1$ arises from the squared harmonic oscillator matrix elements for downward transitions.

[33] J. B. Johnson, Phys. Rev. **32**, 47 (1928); H. Nyquist, Phys. Rev. **32**, 110 (1928).

For the case $(h\nu)/(kT)\ll 1$, the number of photons required to double the output is therefore

$$|kT_{\mathrm{spin}}/h\nu|. \quad (65)$$

Javan has remarked that the noise performance of a Raman type maser will be essentially the same as that of devices which employ greater populations of particles in excited states. This follows because the photon of frequency ν_1 (Fig. 18) can be spontaneously emitted in the absence of a signal, provided that intense "pump" radiation of frequency ν_2 is present.

A complete system will also have noise due to emission from circuit elements and the transmission system. A quantity which has been used to describe the performance of radio receivers is the noise figure, denoted by the symbol F. F is defined as the ratio of the total noise power output of the receiver to that part of the noise power output due to the source at the input of the receiver. An equivalent definition is that it is the quotient of the signal to noise power ratio at the input divided by the signal to noise power ratio at the output.

We now calculate the noise figure of a complete maser receiving system, considering first a traveling wave maser. Our approach is similar to that employed by Strandberg,[34] for two-level systems.[35] We consider three-level systems, taking effects of saturation[36] into account. Let $R_\nu d\nu$ be the number of modes per unit volume in a range $d\nu$ which can propagate, A be the cross-sectional area, and V_G the group velocity. The number of particles per unit volume in the states with energies E_1, E_2, and E_3 will be denoted by n_1, n_2, and n_3. Let β be the power gain per unit length. A quantity K is defined by the following relation

$$\beta=K(n_2-n_1). \quad (66)$$

Let N be the average number of quanta per mode in the vicinity of the operating frequency ν_{21} and let p_ν be the energy per mode. The change in power as a consequence of energy exchanged in a differential length leads to

$$\begin{aligned}&V_G A R_\nu d\nu(dp_\nu/dx)\\&\quad=V_G A R_\nu d\nu[Kn_2 h\nu(N+1)-Kn_1 h\nu N-\alpha_c N h\nu\\&\qquad+\alpha_c p_\nu(T_c)-\alpha_{13}N h\nu+\alpha_{13}p_\nu(T_{13})]. \quad (67)\end{aligned}$$

In expression (67) $p_\nu(T_c)$ is the average energy per mode in equilibrium at the temperature of the waveguide walls T_c, and from statistical mechanics $p_\nu(T)=h\nu[e^{h\nu/kT}-1]^{-1}$, $p_\nu(T_{13})$ is the average energy per mode at the temperature T_{13} associated with particles in the energy levels with energies E_1 and E_3, α_c is the power absorption coefficient of the transmission system when the maser material is absent and $\alpha_{13}(\nu_{21})$ is the absorption coefficient at frequency ν_{21} associated with the particles in the energy levels E_1 and E_3. The first term on the right gives the effect of both stimulated and spontaneous emission in inducing transitions out of the state with energy E_2, the second term gives the absorption by particles in the ground state, the third term gives the effect of absorption by conducting walls or loss other than the active solid, the fourth term gives the emission from the walls, the fifth and sixth terms represent the effects of absorption and emission from particles undergoing transitions between the first and third states. We define the temperatures T_{12} and T_{13} by

$$n_1/n_2=e^{h\nu_{21}/kT_{12}},\quad n_1/n_3=e^{h\nu_{31}/kT_{13}}. \quad (68)$$

Integrating expression (67) gives

$$(p_\nu)_{\mathrm{out}}=(p_\nu)_{\mathrm{in}}g^2+\left[\frac{\beta p_\nu(T_{12})-\alpha_c p_\nu(T_c)-\alpha_{13}p_\nu(T_{13})}{\beta-\alpha_c-\alpha_{13}}\right]\times(1-g^2). \quad (69)$$

Here g^2 is the power gain defined earlier. This tells us that the temperature of the environment of the maser enters in a subtle way, through the quantity $\alpha_c p_\nu(T_c)$. If the absorption coefficient of the environment, α_c, is small, then little noise is contributed even if T_c is not close to absolute zero. This is a consequence of Kirchhoff's law. Let the source temperature be T_s, the load temperature be T_L, and let the transmission line to the maser have a power loss factor t and a temperature T_t. Thermodynamic considerations enable us to express the output noise emission of the transmission line per cycle in terms of the input noise emission (per cycle) of the source, as

$$(p_\nu)_{\mathrm{line}}=tp_\nu(T_s)+(1-t)p_\nu(T_t). \quad (70)$$

Expression (70) is $(p_\nu)_{\mathrm{in}}$ for the maser.

We can now use (69), (70), and the previous definition of noise figure to write

$$\begin{aligned}F&=\frac{(p_\nu)_{\mathrm{out}}+p_\nu(T_L)}{g^2 t p_\nu(T_s)}\\&=1+\frac{1}{tp_\nu(T_s)}\Bigg\{(1-t)p_\nu(T_t)+(1-g^{-2})\\&\quad\times\left[\frac{\alpha_c p_\nu(T_c)+\alpha_{13}p_\nu(T_{13})-\beta p_\nu(T_{12})}{\beta-\alpha_c-\alpha_{13}}\right]+\frac{p_\nu(T_L)}{g^2}\Bigg\}. \quad (71)\end{aligned}$$

It was once thought that the saturation field would make a significant contribution to the noise because the temperature T_{13} associated with the (saturated) 1–3 system tends towards infinity. This is not the case; only a negligible contribution results, in the following way.[36]

[34] M. W. P. Strandberg, Phys. Rev. **106**, 617 (1957).

[35] Noise in maser type amplifiers has been discussed by Muller [Phys. Rev. **106**, 8 (1957)], Pound [Ann. Phys. **1**, 24 (1957)] and Shimoda, Takahasi, and Townes [J. Phys. Soc. Japan **12**, 686 (1957)]. The role of spontaneous emission was emphasized by R. H. Dicke at the Symposium on Amplification by Atomic and Molecular Resonance, Asbury Park, New Jersey (March 1, 1956).

[36] J. Weber, Phys. Rev. **108**, 537 (1957).

The quantity $\alpha_{13}(\nu_{21})$ may be written

$$\alpha_{13}(\nu_{21})=\frac{\chi n_1|\mu_{13}|^2}{kT_{13}}\left[\frac{\tau_{13}^{-1}}{4\pi^2(\nu_{21}-\nu_{31})^2+\tau_{13}^{-2}}\right], \quad (72)$$

where χ is a constant, and τ_{13} is the appropriate relaxation time.

β may be written in terms of χ and the 1–2 system relaxation time τ_{12}, as

$$\beta=\frac{\chi n_2|\mu_{12}|^2\tau_{12}}{|p_\nu(T_{12})|}. \quad (73)$$

The ratio of the terms involving α_{13} and β in the noise figure formula (71) is

$$\frac{\alpha_{13}p_\nu(T_{13})}{|\beta p_\nu(T_{12})|}=\frac{n_1|\mu_{13}|^2}{n_2|\mu_{12}|^2}\left[\frac{(\tau_{13}\tau_{12})^{-1}}{4\pi^2(\nu_{21}-\nu_{31})^2+\tau_{13}^{-2}}\right]. \quad (74)$$

For a typical amplifier in the microwave region, this gives a number of order 10^{-3}. Physically this is because a high-saturation temperature means a small absorption coefficient for the 1–3 system, consequently small noise emission.

In order to extend the previous results to a cavity type of maser we note that we have one mode with effective power width $(\pi/2)\Delta\nu$. $\Delta\nu$ is the band width over which the power exceeds one half of the maximum (exact resonance) power. The $\pi/2$ takes account of those parts of the response outside of the region where the power exceeds half the power at resonance. In order to calculate the noise we again employ detailed balancing on a power per cycle basis, making use of the definitions of the various Q's employed earlier. The noise power output per cycle is

$$P_{\text{noise}}=(g+1)^2\Bigg[tp_\nu(T_s)+(1-t)p_\nu(T_t)+\frac{Q_e}{Q_o}\left\{p_\nu(T_c)-\frac{1}{1+\gamma}(p_\nu(T_{12})+\gamma p_\nu(T_{13}))\right\}-\frac{g-1}{g+1}\left(\frac{p_\nu(T_{12})+\gamma p_\nu(T_{13})}{\gamma+1}\right)\Bigg]. \quad (75)$$

The noise figure is then given by

$$F=\frac{(g+1)^2}{g^2tp_\nu(T_s)}\Bigg[(1-t)p_\nu(T_t)+tp_\nu(T_s)+\frac{Q_e}{Q_o}\left\{p_\nu(T_c)-\frac{1}{1+\gamma}(p_\nu(T_{12})+\gamma p_\nu(T_{13}))\right\}-\frac{(g-1)}{(g+1)}\left(\frac{p_\nu(T_{12})+\gamma p_\nu(T_{13})}{1+\gamma}\right)\Bigg]+\frac{g^{-2}p_\nu(T_L)}{tp_\nu(T_s)}. \quad (76)$$

Here the quantity γ is defined by

$$\gamma=\frac{Q_m}{Q_{13}}=\frac{-\alpha_{13}(\nu_{21},T_{13})}{\beta}, \quad (77)$$

with Q_{13} the Q associated with energy absorption by the 1–3 system. The somewhat unusual appearance of (77) with the factor $(1+g)^2$ results from the fact that g^2 is the power gain at resonance. Different parts of the cavity response contribute noise, but with different effective gain.

The term noise temperature is often used, rather than noise figure. This may be defined in terms of the noise figure as

$$\left.\begin{aligned}[e^{h\nu/kT_n}-1]^{-1}&=(F-1)[e^{h\nu/kT_s}-1]^{-1}\\ T_n&\approx(F-1)T_s,\quad\text{for}\quad\frac{h\nu}{kT}\ll1\end{aligned}\right\}. \quad (78)$$

Here T_n is the noise temperature, and T_s is the source temperature.

McWhorter and Arams[37] have measured the noise temperature of a complete solid state maser system and found it to be 20±5°K.

The use of the concept of noise figure or noise temperature was very meaningful for the older type of microwave amplifiers. We propose a method of describing the noise performance which seems better for very quiet amplifiers. The number of photons received over the receiver averaging time which will double the noise output can be employed as a "noise number" to specify the performance of a low noise receiver. We showed earlier that the minimum noise number of a maser is one. The connection between noise number N_n and noise temperature T_n is

$$N_n=[e^{h\nu/kT_n}-1]^{-1}+[e^{h\nu/kT_s}-1]^{-1}. \quad (79)$$

VI. MICROWAVE PHOTON COUNTERS

The maser makes use of stimulated emission. If *all* of the particles were in the upper state of a two-level maser, it could amplify without absorbing any of the incident photons. However, under these conditions the noise number is one, so that at least one incident photon is necessary in order that its randomly fluctuating output be doubled over the receiver averaging time. The incidence of one photon could be interpreted as a spontaneous fluctuation. In an earlier paper[36] we remarked that a maser is a voltage amplifier, not a power amplifier, and that all voltage amplifiers have spontaneous emission noise. Detectors such as nuclear counters must absorb energy in order to operate. However, unlike a maser, the internal fluctuations of a counter or power amplifier can be made arbitrarily

[37] McWhorter and Arams, Lincoln Laboratory, MIT, Rept. M 37-22. See also Alsop, Giordmaine, Townes, and Wang, Phys. Rev. **107**, 1450 (1957); J. C. Helmer, Phys. Rev. **107**, 902 (1957).

small. Such a device has essentially zero output until it detects a photon. It was proposed[36] that power amplifiers and detectors be developed which employ particles initially in their ground states. Methods for doing this are now being studied here. Consider a three-level system, in which the frequency ν_{21} (Fig. 19) for transitions from the first to the second state, is in the optical region. A lamp illuminates with intense light of frequency ν_{21} which can be linearly polarized. Then transitions from state 1 to state 2 are allowed, since $\Delta m=0$. Excited particles in state 2 can spontaneously emit only linearly polarized light. We employ a detector which counts circularly polarized optical frequency photons, which can be emitted only in certain directions. In order to detect microwave photons we arrange to have them circularly polarized. Then a microwave photon can be absorbed in a transition to state 3 or 4, since $\Delta m=1$. Excited particles in states 3 or 4 can now spontaneously emit circularly polarized photons which will be counted. Thus if no microwave photons are present we have particles in states 1 and 2 and only linearly polarized photons. The device has no output. As soon as a microwave (or infrared) photon is absorbed we have a particle in state 3 or 4, and when this particle emits, we have a circularly polarized photon, which is counted. This method is similar to that which has been employed in optical pumping experiments. The transition probability for this type of third-order process can be calculated in the following way. Let the wave function for atom and electromagnetic field be $\Phi(t)$.

$$\Phi(t)=\sum a_i(t)\Psi_i.$$

The Ψ_i are unperturbed wave functions of particle and electromagnetic field. Let the ground state be denoted by the subscript M. Let $a_M=1$ at $t=0$, and $a_j=0$ for $j\neq M$, at $t=0$. Let H' be the interaction part of the Hamiltonian. We quantize the electromagnetic field, H' is then not time dependent. Our process is one involving three photons. One is absorbed from the lamp, one is absorbed from the microwave field, and one is then spontaneously emitted. The third-order probability amplitude coefficient is given by

$$a_L^{(3)}=\frac{h^{-3}H_{LK}'H_{KN}'H_{NM}'}{\nu_{NM}}\times\left[\frac{1}{\nu_{KM}}\left\{\frac{e^{i2\pi\nu_{LM}t}-1}{i\nu_{LM}}-\frac{e^{i2\pi\nu_{LK}t}-1}{i\nu_{LK}}\right\}-\frac{1}{\nu_{KN}}\left\{\frac{e^{i2\pi\nu_{LN}t}-1}{i\nu_{LN}}-\frac{e^{i2\pi\nu_{LK}t}-1}{i\nu_{LK}}\right\}\right].$$

The intermediate states are denoted by the subscripts N and K. We are interested in three photon transitions of the type in which energy is conserved or nearly conserved in all three steps. The subscript N refers to the state in which a particular oscillator excited by the lamp has lost one photon. The subscript K refers to the state in which a particular microwave field oscillator has lost a photon. In order to calculate the probability that the particle has returned to the ground state with emission of a circularly polarized photon we must square the probability amplitude and integrate over all lamp photons, microwave photons, and emitted circularly polarized photons. We denote by $\rho(N)$, $\rho(K)$, and $\rho(L)$ the density of intermediate and final states. The transition probability is then

$$W=\frac{1}{h^6t}\int\left|\frac{H_{LK}'H_{KN}'H_{NM}'}{\nu_{NM}}\right|^2\times\left[\frac{1}{\nu_{KM}}\left\{\frac{e^{i2\pi\nu_{LM}t}-1}{\nu_{LM}}-\frac{e^{i2\pi\nu_{LK}t}-1}{\nu_{LK}}\right\}-\frac{1}{\nu_{KN}}\left\{\frac{e^{i2\pi\nu_{LN}t}-1}{\nu_{KN}}-\frac{e^{i2\pi\nu_{LK}t}-1}{\nu_{LK}}\right\}\right]^2\times\rho(E_L)\rho(E_K)\rho(E_N)dE_LdE_KdE_N.$$

We need to evaluate this expression, under conditions such that energy is conserved or nearly conserved in all steps. All denominators may vanish. An approximate solution can be obtained in the following way. When integrating over dE_N, we are integrating over all lamp photons which can be absorbed. Let the lamp have intensity which is zero except for a range $\Delta\nu_1$ on each side of that required for resonance. Let the lamp intensity be assumed constant over the range $2\Delta\nu_1$. A study of the integrand shows that it is not singular in the range where all denominators vanish. The integral over E_N is along the real frequency axis from $-\Delta\nu_1$ to $+\Delta\nu_1$. We can equally well integrate along a semicircle in the lower half complex frequency plane from $-\Delta\nu_1$ on the real axis to $+\Delta\nu_1$ on the real axis. The second pair of terms which have the factor $1/(\nu_{KN})$ may then be neglected in comparison with the pair of terms which have the factor $1/(\nu_{KM})$. This integration along the semicircle is now readily performed. The integration over E_K can be performed in the same way if the relaxation time for interaction with other particles is much smaller than the spontaneous emission lifetime. This leaves one integration which may then be handled in the same way as for one photon processes. The result

FIG. 19. Energy levels for quantum mechanical amplifier without spontaneous emission noise.

then gives the transition probability

$$W=\frac{16\pi^2\rho(E_N)\rho(E_K)\rho(E_L)\,|H_{LK}'H_{KN}'H_{NM}'|^2}{h^3\Delta\nu_{NM}\Delta\nu_{KM}}.$$

All quantities are evaluated at resonance. This predicts modified absorption (or emission) at one frequency due to intense radiation at a different frequency, for single particles. $\rho(N)$ is calculated from the average number of field oscillators excited by the lamp. It is reasonable to suppose that each lamp excited field oscillator is unlikely to have more than one photon. The remaining quantities require, for calculation, a detailed knowledge of the transitions and the experimental arrangements. The excited states of the rare earth salts may be suitable for such counters and are now being investigated by Mr. U. E. Hochuli.

At a meeting held at the National Academy of Sciences in April, 1958, Bloembergen independently proposed a microwave (or infrared) photon counter which operates on similar principles (Fig. 20).

Consider a solid with at least three low-lying levels and one or more optical levels. The solid, such as a salt of the rare earth elements, is cooled to such a low temperature that only the ground state is occupied. $kT\ll E_2-E_1$. Strong optical illumination takes place at the frequency $h^{-1}(E_4-E_2)$. This light is normally not absorbed because the level E_2 is empty, unless it is excited by an incident quantum E_2-E_1. Then the system gets raised to level E_4. The spontaneous emission from level E_4 to E_3 can be detected by a photomultiplier tube. Again, discrimination from the strong incident background can be made by directional, frequency, or polarization filters.

VII. MASERS AT LOW FREQUENCIES AND IN THE INFRARED

There has not been a great deal of motivation for development of masers at low frequencies. The theory previously given would appear to be applicable. Nuclear spin energy levels and the levels associated with nuclear quadrupole spectra are available in this part of the spectrum. The small nuclear moment gives a weaker interaction with radiation, but this tends to be compensated by longer transverse relaxation times. As noted by Braunstein[38] a nucleus of spin $I>2$ which possesses an electric quadrupole moment will yield at least three unequally spaced levels in a crystal of lower than cubic symmetry. If the crystalline field has large deviations from axial symmetry the $\Delta m=2$ transitions (necessary for saturation) begin to have significantly large transition probabilities.

Allais[39] obtained maser action in nuclear resonance experiments in the megacycle region. This followed the work of Abragam, Combrisson, and Solomon[39] in the kilocycle region.

In the infrared and optical parts of the spectrum the sensitivity of a maser will be expected to be less than that of photon counting devices. At 8000 A the spontaneous emission equivalent temperature is 18 000°K. Achievement of coherent amplification would be desirable. This is a useful concept only when large numbers of photons are involved so that the phase can be well defined.[40] The availability of highly monochromatic sources would extend the resolution of spectroscopy in these regions. One problem is to provide a device at infrared and optical frequencies which has the energy storage capacity ordinarily provided by an electromagnetic cavity resonator, with wide mode separations. Dicke has proposed[41] a number of arrangements which employ a standing wave system between parallel planes. Similar proposals have been made by Schawlow and Townes.[42] They suggest that a multimode cavity could be employed. Single modes might be selected by reducing the cavity to partially transparent end plates, with open sides. The great directivity associated with diffraction from sources which are many wavelengths on a side may make it possible to select particular modes and suppress unwanted ones. They carry out calculations for a system using potassium vapor.

VIII. PARAMETRIC AMPLIFIERS

During the past few years another type of microwave amplifier has been developed, which has promise of low noise performance approaching that of a maser. We describe it in classical terms because nothing seems to be gained by a quantum mechanical description. In Fig. 21 two oscillators are coupled by a time-dependent capacitor. Let the natural frequencies of the oscillators be ν_1 and ν_2, and let the capacity C_3 have the time dependence

$$C_3=C_o+C_t\sin[2\pi(\nu_1+\nu_2)t].$$

It has been known since at least the time of Lord Rayleigh that two harmonic oscillators with natural frequencies ν_1 and ν_2, with this kind of time-dependent coupling may be unstable.

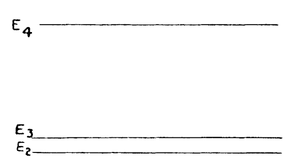

FIG. 20. Energy levels for quantum mechanical amplifier without spontaneous emission noise.

[38] R. Braunstein, Phys. Rev. **107**, 1195 (1958).

[39] Abragam, Combrisson, and Solomon, Compt. rend. **245**, 157 (1957); E. Allais, *ibid.* **246**, 2123 (1958).

[40] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London), third edition, p. 65.

[41] R. H. Dicke, U. S. Patent 2,851,652, issued September 9, 1958.

[42] A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1940 (1958); see also A. M. Prokhorov, J. Exptl. Theoret. Phys. **34**, 1658 (1958). Research activity is increasing along this line. Since low noise is not essential we suggest review of older methods in conjunction with parallel planes.

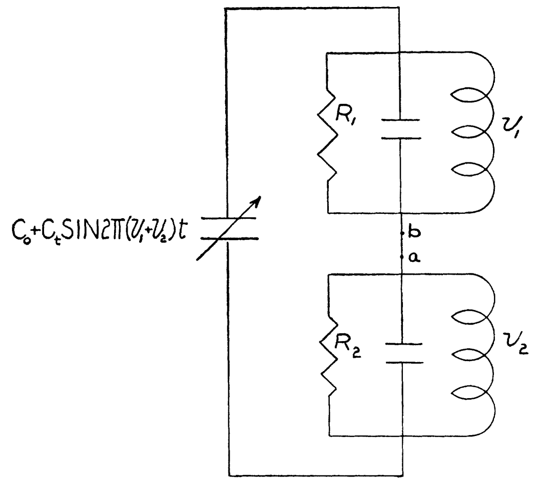

FIG. 21. Parametric amplifier.

Consider first a capacity alone with

$$C=C_t \sin[2\pi(\nu_1+\nu_2)t]$$

driven by a voltage $V=V_1\sin(2\pi\nu_1 t)+V_2\sin(2\pi\nu_2 t)$. The principal part[43] of the current is

$$i=-\pi\nu_1 C_t V_2\sin(2\pi\nu_1 t)-\pi\nu_2 C_t V_1\sin(2\pi\nu_2 t).$$

If, in the circuit of Fig. 21, a voltage $V_G\sin 2\pi\nu_1 t$ is applied between terminals a and b, the output voltage of frequency ν_1 can be shown to have increased amplitude

$$V_1=V_G\left[1-i2\pi C_oR_1\nu_1-\frac{\pi^2\nu_1\nu_2 C_t^2R_1R_2}{1-i2\pi\nu_2C_oR_2}\right]^{-1}.$$

In deducing this we assume that the oscillator of frequency ν_1 has impedance R_1 at ν_1 and zero impedance at all other frequencies. A corresponding assumption is made concerning the oscillator of frequency ν_2. The time-dependent capacitor or inductance need have very little random fluctuations. The low noise properties of this type of amplifier were pointed out unequivocally by van der Ziel[44] in 1948. This pioneering theoretical work appears to represent the earliest solution to the problem of low noise microwave amplification. Experimental verification of the low noise properties was accomplished by Salzberg and Sard.[45] The noise figure of a parametric amplifier has been given as[43]

$$F=1+\frac{G_1}{G_g}+\frac{\nu_1}{\nu_2}\frac{G_{T1}}{G_g}+F_S+F_G,$$

where G_1= effective conductance of unloaded amplifying resonant circuit, G_g= effective conductance coupled into the amplifying resonant circuit by the generator, G_{T1} = effective conductance of the loaded amplifying resonant circuit, ν_1= amplifying frequency, ν_2= idling frequency, F_S= shot noise, and F_G= gain fluctuation noise.

The revival of interest in this type of amplifier appears to have started with the work of Suhl[46] and Weiss[47] on the ferromagnetic amplifier. Low noise "up frequency conversion" is also possible with parametric devices.[48] Room temperature operation is possible, with moderately good noise performance.

IX. APPLICATIONS OF MASERS TO EXPERIMENTAL PHYSICS AND RADIO ASTRONOMY

The very low noise, coupled with adequate gain-bandwith product, make the low-temperature solid state maser an obvious tool for radio astronomy and radar.|| A modified Dicke-type radiometer in which a maser is used for pre-amplification has been described.[49] The ruby maser of Alsop, Giordmaine, and Townes, which was described earlier, was mounted near the focus of the Naval Research Laboratory 50-ft reflector in order to minimize transmission line losses. Liquid helium cooling was provided by a stainless steel Dewar with 3 l capacity which was maintained under partial vacuum. This allowed about 15 hr of observation before recharging was needed. The complete radiometer installation made it possible to observe with an rms output fluctuation of 0.03°K for an output time constant of 5 sec. The intermediate frequency amplifiers had a band width of 5 Mc.

The great stability of oscillators such as the cesium beam clock and the molecular beam masers make possible certain experimental tests of both special and general relativity. These possibilities have been discussed by Moller.[50] A molecular beam maser has been employed to repeat the Michelson-Morley experiment.[51] If there is a fixed ether a difference in frequency would

[43] We are following the treatment of H. Heffner and G. Wade, Meeting of Institute of Radio Engineers Professional Group on Electron Devices, Washington, D. C. (October, 1957).

[44] A. van der Ziel, J. Appl. Phys. **19**, 999 (1948).

[45] B. Salzberg and E. W. Sard, Proc. Inst. Radio Engrs. **46**, 1303 (1958).

[46] H. Suhl Phys. Rev. **106**, 384 (1957); J. Appl. Phys. **28**, 1225 (1957).

[47] M. T. Weiss, Phys. Rev. **107**, 317 (1957).

[48] Herrmann, Uenohara, and Uhlir, Proc. Inst. Radio Engrs. **46**, 1301(L) (1958).

|| *Note added in proof.*—Operation of a ruby maser at 60°K was reported by C. R. Ditchfield and P. A. Forrester [Phys. Rev. Letters **1**, 448 (1958)], and independently by T. H. Maiman (Hughes Aircraft Company Report). A maser employing a slow wave structure with band width of 67 mc per sec and gain of 13 decibels was described by R. W. DeGrasse (Bell Telephone Laboratories Report entitled "Slow Wave Structures for Unilateral Solid State Maser Amplifiers"). Amplifiers employing ruby which have wide tunability were reported by G. K. Wessel [380–450 mc, Proc. Inst. Radio Engrs. **47**, 590 (1959)], and by F. R. Arams and S. Okwit [850–2000 mc, Proc. Inst. Radio Engrs. **47**, 992 (1959)]. A proposal for generating higher frequencies than the pump frequency was made by A. E. Siegman and R. J. Morris [Phys. Rev. Letters **2**, 302 (1959)]. They suggest a staircase scheme in which say the spin populations of the lowest pair of states are reversed by adiabatic rapid passage, followed by adiabatic rapid passage reversal of the populations of levels 2 and 3. Then levels 3 and 1 can be employed for power generation.

[49] Mayer, McCullough, and Sloanmaker, Astrophys. J. **127**, 1 (1958).

[50] C. Moller, Nuovo cimento **6**, Suppl., 381 (1957).

[51] Cedarholm, Bland, Havens, and Townes, Phys. Rev. Letters **1**, 342 (1958).

be expected between two molecular beam type masers with beam velocities parallel and antiparallel to the earth's orbital motion. Let u be the velocity of the molecules relative to the cavity which is assumed to be moving with velocity v with respect to the fixed ether (u is parallel to v). It is assumed that the photons radiated by the molecules move with a velocity **c** relative to the fixed ether. The photon velocity relative to the cavity must be normal to u. This requires that **c** be tilted forward of the normal to u by an angle $\phi \approx (v/c)$. The expected doppler shift would be $\nu u\phi/c = uv\nu/c^2$. For a thermal velocity of 0.6 kMs/sec and for the earth's orbital velocity the difference in frequency between two *oppositely* directed beams is $2uv\nu/c^2$, which is ~ 10 cps if $\nu = 23\,870$ Mc/sec. The experiment was done by mounting two maser oscillators on a rack which could be rotated about a vertical axis. The oscillators were adjusted so that their frequencies differed by a small amount. This difference was recorded and the apparatus was then rotated through 180°. A slight change of about 1/50 cps was observed under the best conditions of operation. This is smaller by a factor of 1000 than what would be expected on the basis of a fixed ether. Inasmuch as the special theory of relativity is one of the most securely established of all physical theories, such experiments may be regarded as a search for other effects such as perhaps an anisotropy of space in this part of our galaxy. The use of stable oscillators to test the gravitational red shift has also been considered.[50,52] This is perhaps more a test of whether or not these devices are natural clocks (whose intervals are invariants) than a test of the fundamental postulates of the general theory of relativity.

A series of experiments is being planned by the author to search for gravitational radiation. The theory of an antenna for such radiation has been discussed.[53] Two masses which are separated will have forces exerted upon them by a gravitational wave. The phase difference at the two masses results in one being driven relative to the other. Also strains can be set up in a material by a gravitational wave. Under some conditions it is desirable to make use of acoustical resonance, under other conditions where acoustic phase reversal is troublesome, it is better not to employ acoustical resonance. The gravitational wave interacts both with the mass of the piezoelectric crystal and the conducting masses shown. Very low-frequency search is planned. The output voltages are amplified as shown in Fig. 22. Liquid helium temperatures and low noise receivers may make it possible to observe correlation in the outputs, in the presence of electrical noise.

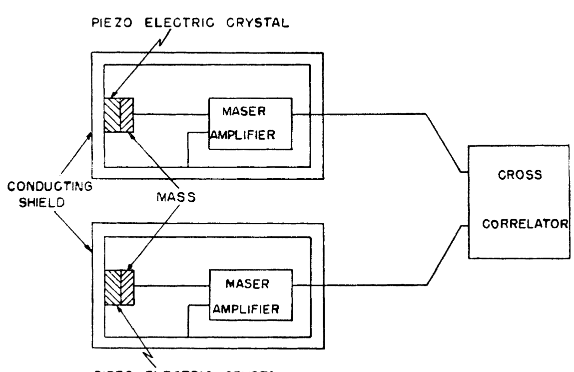

FIG. 22. Masers for detection of gravitational radiation at radio-frequencies.

[52] S. F. Singer, Phys. Rev. **104**, 11 (1956). J. R. Zacharias has considered terrestrial tests employing a cesium beam clock.

[53] J. Weber, Gravity Research Foundation Essay, New Boston, New Hampshire (April, 1958) and April, 1959.

X. CONCLUSION

The new microwave amplifiers are the result of electronics research to develop millimeter wave techniques, magnetic resonance research, and research in microwave spectroscopy. Maser amplifiers bring to the microwave region a detection sensitivity of the order of a few microwave photons. Most of these amplifiers employ the magnetic moment of the electron, and therefore fill in the gap in modern electronics pointed out by Sommerfeld. We may look forward to important advances in radio astronomy, spectroscopy and solid state physics, in consequence of improved ability to distinguish weak signals.

ACKNOWLEDGMENTS

We thank Professor Bloembergen, Professor Strandberg, and Professor Townes for helpful criticism and a profitable correspondence. Suggestions for improving the manuscript were also made by Dr. L. S. Nergaard and Dr. E. O. Schulz-DuBois. We acknowledge stimulating discussions with Dr. R. K. Wangsness and Professor R. A. Ferrell.

APPENDIX. MACHINE CALCULATIONS OF MATRIX ELEMENTS AND ENERGY LEVELS FOR RUBY¶

The following data are entirely the work of W. S. Chang and A. E. Siegman of the Stanford University Electronics Laboratories. We thank them for their kindness and cooperation in allowing this work to be published here.

The spin Hamiltonian used was

$$\mathcal{H} = g\beta_M \mathbf{H}_{dc}\cdot\mathbf{S} + D(S_z^2 - 5/4),$$

with $g = 1.99$ and $2D = +11.46$ kmc/s. The small anisotropy in the g tensor was ignored.

¶ This material originally appeared as Tech. Rept. No. 156–2 under Air Force Contract AF33(600)-27784 of the Stanford Electronics Laboratories, Stanford University, California. A similar set of calculations have also been performed for the material potassium chromicyanide, $K_3Cr(CN)_6$, and reported earlier in Tech. Rept. No. 156–1 under the same contract. Because of the lower symmetry of the chromicyanide, the results are too lengthy to present here. Copies of the report can be obtained directly from the Stanford Electronics Laboratories. The support of this work by the Wright Air Development Center is gratefully acknowledged.

Table I(a). Ruby $\theta = 10°$.

17453 49			50000 49	
64626 50N	50051 50N	36736 50	77941 50	
24804 47N	16853 49	98548 49 •	20405 48	for E_1
16158 48	98556 49N	16850 49 •	40671 47	for E_2
29417 45N	57731 46	20799 48	99978 49	for E_3
99986 49N	16344 48N	25300 46 •	17341 47	for E_4
18817 50	20007 50N	33252 49 •	1-2	
33277 49N	33277 49N	92644 47 •	2-3	
20548 47N	21808 47	51172 47 •	3-4	
16999 50N	16991 50N	40608 48 •	1-3	
17118 50	17127 50N	32422 48 •	2-4	
26263 49N	25668 49	43304 47	1-4	

17453 49			10000 50	
72064 50N	42862 50N	16288 50	98638 50	
41318 47N	16798 49	98466 49 •	46866 48	for E_1
89242 48	98534 49N	16764 49 •	11935 48	for E_2
19840 46N	38882 47	48205 48	99882 49	for E_3
99956 49N	29520 48N	82463 46 •	73835 46	for E_4
18657 50	20049 50N	33263 49 •	1-2	
38480 49N	38487 49N	31531 48 •	2-3	
10939 48N	11069 48	16925 47 •	3-4	
16911 50N	16835 50N	92312 48 •	1-3	
17157 50	17158 50N	58765 48 •	2-4	
23407 49N	23142 49	65152 47	1-4	

17453 49			15000 50	
79668 50N	35733 50N	39737 49 •	11937 51	
52844 47N	16691 49	98251 49 •	82286 48	for E_1
40030 48	98496 49N	16476 49 •	33220 48	for E_2
97454 46N	19060 48	86662 48	99605 49	for E_3
99918 49	40362 48	14941 47	23014 46	for E_4
18210 50	20260 50N	33512 49 •	1-2	
45328 49N	45401 49N	10387 49 •	2-3	
42584 48	42537 48N	29690 47 •	3-4	
16916 50N	16409 50N	15754 49 •	1-3	
17180 50N	17176 50	80461 48	2-4	
20950 49	21003 49N	76920 47 •	1-4	

17453 49			20000 50	
87539 50N	28974 50N	23629 50 •	14014 51	
60918 47N	16498 49	97748 49 •	13142 49	for E_1
47405 48	95305 49N	12460 49 •	27186 49	for E_2
12793 48N	24904 49	17029 49	95331 49	for E_3
99877 49	49432 48	21514 47	58699 46	for E_4
13570 50	23397 50N	38710 49 •	1-2	
47065 49N	54913 49N	99548 49 •	2-3	
45137 49	45107 49N	28066 48 •	3-4	
20122 50N	11086 50N	16809 49 •	1-3	
16601 50N	16598 50	94719 48	2-4	
18846 49	19007 49N	82257 47 •	1-4	

17453 49			25000 50	
95864 50N	43740 50N	21331 50 •	16093 51	
66087 47N	16141 49	96594 49 •	20212 49	for E_1
57094 47	11108 49N	18568 49	97629 49	for E_2
56523 48N	97894 49	18018 49	77442 48	for E_3
99836 49	57128 48	27512 47	28468 46	for E_4
12909 50N	18533 50N	43067 49 •	1-2	
67367 49	68455 49N	36998 49 •	2-3	
17136 50	17138 50N	11392 49 •	3-4	
19265 50N	17613 50	28622 49	1-3	
16617 49N	16626 49	94096 47	2-4	
16947 49	16996 49N	80871 47 •	1-4	

17453 49			30000 50	
10500 51N	62448 50N	14294 50 •	18174 51	
68019 47N	15452 49	93942 49 •	30586 49	for E_1
48383 47	93557 48N	29424 49	95112 49	for E_2
63276 48N	98148 49	17573 49	42498 48	for E_3
99796 49	63739 48	33155 47	22405 46	for E_4
11226 50N	17885 50N	61087 49 •	1-2	
87435 49	88034 49N	26571 49 •	2-3	
17187 50	17195 50N	12749 49 •	3-4	
17943 50N	17414 50	27904 49	1-3	
11871 49N	11877 49	69073 47	2-4	
14900 49	14918 49N	76059 47 •	1-4	

17453 49			35000 50	
11562 51N	79711 50N	72369 49 •	20257 51	
65428 47N	14131 49	88111 49 •	45125 49	for E_1
55808 47	10689 49N	43954 49	89181 49	for E_2
69049 48N	98172 49	17441 49	32132 48	for E_3
99757 49	69477 48	38421 47	34837 46	for E_4
70231 49N	17688 50N	83522 49 •	1-2	
11419 50	11485 50N	26872 49 •	2-3	
17200 50	17213 50N	13914 49 •	3-4	
16214 50N	15997 50	25027 49	1-3	
11756 49N	11779 49	66547 47	2-4	
12450 49	12497 49N	68489 47 •	1-4	

17453 49			40000 50	
12861 51N	94608 50N	18315 48 •	22341 51	
57376 47N	11992 49	77641 49 •	61867 49	for E_1
68069 47	12847 49N	60576 49	78517 49	for E_2
74094 48N	98161 49	17381 49	27167 48	for E_3
99721 49	74502 48	43117 47	20662 46	for E_4
59294 48N	17593 50N	10025 50 •	1-2	
14375 50	14466 50N	29690 49 •	2-3	
17205 50	17225 50N	14929 49 •	3-4	
13580 50N	13488 50	20351 49	1-3	
12579 49N	12576 49	76933 47	2-4	
96076 48	96045 48N	52662 47 •	1-4	

17453 49			45000 50	
14435 51N	10676 51N	68636 49	24425 51	
46566 47N	95477 48	64794 49 •	75566 49	for E_1
79205 47	14663 49N	74165 49	65451 49	for E_2
78548 48N	98140 49	17346 49	24255 48	for E_3
99686 49	78940 48	47503 47	27320 46	for E_4
57030 49	17537 50N	10173 50 •	1-2	
16660 50	16785 50N	32204 49 •	2-3	
17207 50	17234 50N	15827 49 •	3-4	
10597 50N	10557 50	15220 49	1-3	
13065 49N	13073 49	80524 47	2-4	
68736 48	68808 48N	39305 47 •	1-4	

17453 49			50000 50	
16218 51N	11683 51N	13903 50	26511 51	
37184 47N	75430 48	53728 49 •	84000 49	for E_1
86825 47	15735 49N	82539 49	54211 49	for E_2
82509 48N	98116 49	17324 49	22341 48	for E_3
99654 49	82887 48	51465 47	23836 46	for E_4
99216 49	17500 50N	93462 49 •	1-2	
17970 50	18126 50N	33587 49 •	2-3	
17207 50	17241 50N	16625 49 •	3-4	
81402 49N	81207 49	11171 49	1-3	
12997 49N	12991 49	82791 47	2-4	
48638 48	48596 48N	27000 47 •	1-4	

17453 49			55000 50	
18125 51N	12565 51N	20937 50	28597 51	
30440 47N	61370 48	45688 49 •	88739 49	for E_1
91833 47	16301 49N	87250 49	46052 49	for E_2
86054 48N	98091 49	17309 49	20989 48	for E_3
99624 49	86420 48	55099 47	21835 46	for E_4
12345 50	17474 50N	83745 49 •	1-2	
18649 50	18831 50N	34230 49 •	2-3	
17207 50	17247 50N	17339 49 •	3-4	
63995 49N	63888 49	84195 48	1-3	
12618 49N	12600 49	82495 47	2-4	
35445 48	35350 48N	18580 47 •	1-4	

17453 49			60000 50	
20100 51N	13381 51N	27965 50	30685 51	
25772 47N	51759 48	40041 49 •	91486 49	for E_1
95370 47	16608 49N	89983 49	40326 49	for E_2
89247 48N	98066 49	17299 49	19981 48	for E_3
99596 49	89601 48	58497 47	30924 46	for E_4
13739 50	17455 50N	75515 49 •	1-2	
19012 50	19215 50N	34526 49 •	2-3	
17206 50	17251 50N	17982 49 •	3-4	
51955 49N	51890 49	65877 48	1-3	
12114 49N	12118 49	79763 47	2-4	
26793 48	26810 48N	15690 47 •	1-4	

17453 49			65000 50	
22112 51N	14158 51N	34988 50	32772 51	
22478 47N	45030 48	36000 49 •	93185 49	for E_1
98073 47	16785 49N	91676 49	36231 49	for E_2
92136 48N	98043 49	17291 49	19201 48	for E_3
99569 49	92479 48	61619 47	38302 46	for E_4
14586 50	17440 50N	69045 49 •	1-2	
19220 50	19438 50N	34667 49 •	2-3	
17204 50	17255 50N	18563 49 •	3-4	
43424 49N	43380 49	53340 48	1-3	
11594 49N	11616 49	77066 47	2-4	
20974 48	21061 48N	14027 47 •	1-4	

17453 49			70000 50	
24148 51N	14913 51N	42007 50	34860 51	
20079 47N	40150 48	33016 49 •	94306 49	for E_1
10026 48	16896 49N	92794 49	33208 49	for E_2
94763 48N	98020 49	17285 49	18582 48	for E_3
99544 49	95095 48	64457 47	38247 46	for E_4
15131 50	17428 50N	63999 49 •	1-2	
19347 50	19578 50N	34737 49 •	2-3	
17202 50	17259 50N	19091 49 •	3-4	
37165 49N	37134 49	44421 48	1-3	
11103 49N	11121 49	75124 47	2-4	
16927 48	16991 48N	11321 47 •	1-4	

17453 49			75000 50	
26198 51N	15653 51N	49028 50	36948 51	
18272 47N	36487 48	30742 49 •	95087 49	for E_1
10210 48	16968 49N	93572 49	30905 49	for E_2
97163 48N	97998 49	17280 49	18075 48	for E_3
99521 49	97484 48	67097 47	40659 46	for E_4
15503 50	17418 50N	60020 49 •	1-2	
32419 49N	32396 49	37851 48	2-3	
13974 48	14044 48N	99553 46 •	3-4	
19430 50	19670 50N	34772 49 •	1-3	
10636 49N	10657 49	72886 47	2-4	
17200 50	17262 50N	19574 49 •	1-4	

TABLE I(b). Ruby $\theta=20°$.

34906 49			50000 49	
65568 50N	49333 50N	37805 50	77096 50	
89413 47N	30842 49	95044 49 •	38052 48	for E_1
31103 48	95069 49N	30821 49 •	14643 48	for E_2
23034 46N	21829 47	40713 48	99916 49	for E_3
99947 49	32345 48	11070 47	41471 46	for E_4
16039 50	20029 50N	58867 49 •	1-2	
61063 49N	61076 49N	33442 48 •	2-3	
83498 47	83075 47N	19082 47 •	3-4	
16213 50N	16181 50N	74688 48 •	1-3	
16662 50N	16651 50	62833 48	2-4	
47202 49	47302 49N	15828 48 •	1-4	

34906 49			10000 50	
74222 50N	41641 50N	18752 50	97111 50	
14771 48N	30538 49	94831 49 •	85000 48	for E_1
56798 48	95031 49N	30315 49 •	42024 48	for E_2
14878 47N	14020 48	93768 48	99549 49	for E_3
99827 49	58609 48	33527 47	28799 46	for E_4
15579 50	20176 50N	59093 49 •	1-2	
69877 49N	70032 49N	11065 49 •	2-3	
41003 48	41024 48N	48084 47 •	3-4	
15917 50N	15646 50N	16058 49 •	1-3	
16792 50N	16776 50	11538 49	2-4	
41744 49	41729 49N	23233 48 •	1-4	

34906 49			15000 50	
83397 50N	34329 50N	43895 48	11728 51	
18622 48N	29999 49	94296 49 •	14305 49	for E_1
78067 48	94857 49N	28679 49 •	10891 49	for E_2
65644 47N	61350 48	16888 49	98370 49	for E_3
99675 49	80302 48	59583 47	64534 46	for E_4
14390 50	20778 50N	60611 49 •	1-2	
79330 49N	80821 49N	33272 49 •	2-3	
14319 49	14318 49N	17613 48 •	3-4	
15845 50N	14300 50N	24415 49 •	1-3	
16833 50N	16818 50	15876 49	2-4	
36788 49	36802 49N	26254 48 •	1-4	

34906 49			20000 50	
93284 50N	28764 50N	15536 50 •	13758 51	
21020 48N	29155 49	93203 49 •	21415 49	for E_1
89064 48	89761 49N	18492 49 •	41232 49	for E_2
37388 48N	34269 49	31150 49	88550 49	for E_3
99510 49	98478 48	85679 47	90722 46	for E_4
88982 49	23721 50N	70165 49 •	1-2	
63055 49N	93695 49N	13519 50 •	2-3	
66513 49	66516 49N	82947 48 •	3-4	
18024 50N	71820 49N	17630 49 •	1-3	
15604 50N	15598 50	17893 49	2-4	
32205 49	32216 49N	26638 48 •	1-4	

34906 49			25000 50	
10413 51N	37067 50N	16782 50 •	15798 51	
22204 48N	27917 49	91235 49 •	29963 49	for E_1
42619 48	39749 49N	17202 49	90032 49	for E_2
10385 49N	86665 49	37134 49	31658 49	for E_3
99343 49	11389 49	11052 48	11805 47	for E_4
47650 49N	21462 50N	76345 49 •	1-2	
81185 49	10656 50N	12767 50 •	2-3	
15813 50	15824 50N	21604 49 •	3-4	
17658 50N	11132 50	30403 49	1-3	
62755 49N	62750 49	76655 48	2-4	
27817 49	27832 49N	25353 48 •	1-4	

34906 49			30000 50	
11621 51N	51996 50N	10241 50 •	17845 51	
22310 48N	26232 49	88093 49 •	39323 49	for E_1
31530 48	28905 49N	31622 49	90302 49	for E_2
12165 49N	91185 49	35181 49	17293 49	for E_3
99179 49	12712 49	13352 48	13702 47	for E_4
46180 49N	19397 50N	86467 49 •	1-2	
11291 50	12468 50N	85483 49 •	2-3	
16582 50	16615 50N	25205 49 •	3-4	
15368 50N	13077 50	35327 49	1-3	
39528 49N	39523 49	49135 48	2-4	
23575 49	23575 49N	22889 48 •	1-4	

34906 49			35000 50	
12974 51N	66101 50N	31387 49 •	19898 51	
21519 48N	24156 49	83723 49 •	49012 49	for E_1
30510 48	27536 49N	42441 49	86204 49	for E_2
13435 49N	92011 49	34448 49	12906 49	for E_3
99022 49	13657 49	15470 48	16114 47	for E_4
20062 49N	18689 50N	98067 49 •	1-2	
13132 50	14056 50N	74563 49 •	2-3	
16742 50	15798 50N	27757 49 •	3-4	
13594 50N	12494 50	32959 49	1-3	
33156 49N	33153 49	41743 48	2-4	
19538 49	19543 49N	19867 48 •	1-4	

34906 49			40000 50	
14478 51N	78841 50N	40509 49	21957 51	
20130 48N	21886 49	78484 49 •	57939 49	for E_1
31652 48	28099 49N	51718 49	80781 49	for E_2
14483 49N	92253 49	34092 49	10830 49	for E_3
98874 49	14858 49	17406 48	18325 47	for E_4
11013 49	18340 50N	10616 50 •	1-2	
14577 50	15457 50N	71176 49 •	2-3	
16803 50	16885 50N	29906 49 •	3-4	
11910 50N	11307 50	28997 49	1-3	
30316 49N	30317 49	38695 48	2-4	
15899 49	15910 49N	16716 48 •	1-4	

34906 49			45000 50	
16117 51N	90270 50N	11251 50	24019 51	
18500 48N	19672 49	73012 49 •	65412 49	for E_1
33273 48	29038 49N	59305 49	75023 49	for E_2
15393 49N	92314 49	33887 49	96257 48	for E_3
98734 49	15739 49	19165 48	19750 47	for E_4
40117 49	18133 50N	10982 50 •	1-2	
15680 50	16574 50N	70105 49 •	2-3	
16831 50	16940 50N	31772 49 •	3-4	
10345 50N	99856 49	24867 49	1-3	
28502 49N	28496 49	37041 48	2-4	
12830 49	12830 49N	13717 48 •	1-4	

34906 49			50000 50	
17867 51N	10061 51N	18442 50	26085 51	
16901 48N	17695 49	67861 49 •	71265 49	for E_1
34843 48	29897 49N	65199 49	69591 49	for E_2
16195 49N	92304 49	33756 49	88405 48	for E_3
98603 49	16521 49	20773 48	21343 47	for E_4
64277 49	17997 50N	11000 50 •	1-2	
16480 50	17406 50N	69755 49 •	2-3	
16844 50	16978 50N	33417 49 •	3-4	
89721 49N	87428 49	21161 49	1-3	
27018 49N	27010 49	35675 48	2-4	
10363 49	10362 49N	11227 48 •	1-4	

34906 49			55000 50	
19701 51N	11014 51N	25618 50	28154 51	
15474 48N	16022 49	63324 49 •	75702 49	for E_1
36215 48	30575 49N	69653 49	64810 49	for E_2
16910 49N	92260 49	33666 49	82886 48	for E_3
98481 49	17218 49	22245 48	23033 47	for E_4
83115 49	17900 50N	10814 50 •	1-2	
17047 50	18007 50N	69611 49 •	2-3	
16848 50	17008 50N	34881 49 •	3-4	
78171 49N	76620 49	18062 49	1-3	
25673 49N	25668 49	34375 48	2-4	
84392 48	84418 48N	92751 47 •	1-4	

34906 49			60000 50	
21595 51N	11907 51N	32779 50	30225 51	
14259 48N	14644 49	59464 49 •	79041 49	for E_1
37380 48	31084 49N	73002 49	60748 49	for E_2
17550 49N	92200 49	33600 49	78797 48	for E_3
98366 49	17844 49	23591 48	24486 47	for E_4
97441 49	17827 50N	10534 50 •	1-2	
17446 50	18437 50N	69518 49 •	2-3	
16846 50	17033 50N	36190 49 •	3-4	
68654 49N	67551 49	15548 49	1-3	
24427 49N	24423 49	33155 48	2-4	
69563 48	69602 48N	77257 47 •	1-4	

34906 49			65000 50	
23534 51N	12756 51N	39925 50	32298 51	
13244 48N	13518 49	56227 49 •	81571 49	for E_1
38368 48	31463 49N	75541 49	57347 49	for E_2
18128 49N	92132 49	33551 49	75647 48	for E_3
98259 49	18408 49	24825 48	25673 47	for E_4
10830 50	17771 50N	10226 50 •	1-2	
17731 50	18749 50N	69435 49 •	2-3	
16842 50	17053 50N	37369 49 •	3-4	
60857 49N	60038 49	13528 49	1-3	
23272 49N	23265 49	32011 48	2-4	
58111 48	58134 48N	64793 47 •	1-4	

34906 49			70000 50	
25505 51N	13573 51N	47056 50	34373 51	
12399 48N	12595 49	53520 49 •	83518 49	for E_1
39214 48	31747 49N	77496 49	54507 49	for E_2
18651 49N	92062 49	33513 49	73145 48	for E_3
98159 49	18920 49	25960 48	26774 47	for E_4
11661 50	17726 50N	99239 49 •	1-2	
17937 50	18979 50N	69354 49 •	2-3	
16835 50	17070 50N	38435 49 •	3-4	
54448 49N	53818 49	11899 49	1-3	
22202 49N	22194 49	30912 48	2-4	
49172 48	49183 48N	54957 47 •	1-4	

34906 49			75000 50	
27499 51N	14367 51N	54176 50	36449 51	
11695 48N	11835 49	51247 49 •	85042 49	for E_1
39946 48	31962 49N	79027 49	52125 49	for E_2
19128 49N	91991 49	33482 49	71113 48	for E_3
98065 49	19386 49	27006 48	27751 47	for E_4
12307 50	17689 50N	96411 49 •	1-2	
18091 50	19152 50N	69276 49 •	2-3	
16827 50	17085 50N	39404 49 •	3-4	
49138 49N	48638 49	10574 49	1-3	
21215 49N	21204 49	29872 48	2-4	
42109 48	42104 48N	47006 47 •	1-4	

TABLE I(c). Ruby $\theta=30°$.

52359 49			50000 49	
66856 50N	48388 50N	39534 50	75709 50	
17546 48N	41329 49	90895 49 •	61782 48	for E_1
44365 48	90933 49N	41270 49 •	28467 48	for E_2
75341 46N	44760 47	58917 48	99826 49	for E_3
99886 49	47653 48	24248 47	79319 46	for E_4
12895 50	20061 50N	75770 49 •	1-2	
81998 49N	82077 49N	65108 48 •	2-3	
17592 48	17511 48N	45626 47 •	3-4	
15264 50N	15196 50N	99631 48 •	1-3	
16137 50N	16111 50	90544 48	2-4	
62792 49	62930 49N	30804 48 •	1-4	

52359 49			10000 50	
77133 50N	40154 50N	22691 50	94596 50	
28808 48N	40699 49	90614 49 •	11148 49	for E_1
82129 48	90890 49N	40110 49 •	79164 48	for E_2
45562 47N	26721 48	13402 49	99060 49	for E_3
99619 49	86824 48	74774 47	12364 47	for E_4
12222 50	20330 50N	76694 49 •	1-2	
92392 49N	93215 49N	20691 49 •	2-3	
82211 48	82208 48N	15973 48 •	3-4	
14787 50N	14212 50N	19860 49 •	1-3	
16390 50N	16348 50	16924 49	2-4	
54615 49	54652 49N	44396 48 •	1-4	

52359 49			15000 50	
88263 50N	32964 50N	73738 49	11384 51	
35972 48N	39685 49	89958 49 •	17877 49	for E_1
11334 49	90461 49N	36680 49 •	18516 49	for E_2
17314 48N	99647 48	23668 49	96630 49	for E_3
99275 49	11940 49	13415 48	22440 47	for E_4
10686 50	21151 50N	80050 49 •	1-2	
99263 49N	10525 50N	54597 49 •	2-3	
25032 49	25030 49N	49350 48 •	3-4	
14418 50N	12060 50N	26391 49 •	1-3	
16410 50N	16365 50	23323 49	2-4	
47143 49	47200 49N	48886 48 •	1-4	

52359 49			20000 50	
10035 51N	28707 50N	43124 49 •	13337 51	
40209 48N	38298 49	88783 49 •	25188 49	for E_1
12987 49	85189 49N	24775 49 •	44274 49	for E_2
58677 48N	32565 49	38727 49	86053 49	for E_3
98897 49	14676 49	19414 48	31536 47	for E_4
67946 49	23020 50N	89646 49 •	1-2	
73366 49N	11780 50N	13473 50 •	2-3	
70029 49	70060 49N	14195 49 •	3-4	
14975 50N	60112 49N	17463 49 •	1-3	
15185 50N	15156 50	26091 49	2-4	
40370 49	40410 49N	48236 48 •	1-4	

52359 49			25000 50	
11353 51N	32817 50N	67653 49 •	15311 51	
42261 48N	36582 49	87010 49 •	32756 49	for E_1
10690 49	63971 49N	12347 48	76103 49	for E_2
12775 49N	65425 49	49206 49	55991 49	for E_3
98512 49	16996 49	25146 48	40838 47	for E_4
96914 48	22790 50N	98469 49 •	1-2	
30988 49	13027 50N	17439 50 •	2-3	
13067 50	13092 50N	28559 49 •	3-4	
14720 50N	40860 49	13346 49	1-3	
10548 50N	10535 50	19759 49	2-4	
34249 49	34289 49N	44852 48 •	1-4	

52359 49			30000 50	
12784 51N	43240 50N	19503 49 •	17303 51	
42710 48N	34631 49	84674 49 •	40158 49	for E_1
86546 48	49972 49N	18994 49	84065 49	for E_2
16633 49N	77091 49	49600 49	36332 49	for E_3
98133 49	18981 49	30467 48	49224 47	for E_4
34071 48N	20946 50N	10361 50 •	1-2	
89839 49	14196 50N	14389 50 •	2-3	
15143 50	15212 50N	36383 49 •	3-4	
13066 50N	81397 49	27057 49	1-3	
73330 49N	73255 49	14173 49	2-4	
28823 49	28855 49N	40137 48 •	1-4	

52359 49			35000 50	
14325 51N	54571 50N	47278 49	19310 51	
42074 48N	32579 49	81926 49 •	46999 49	for E_1
80022 48	45178 49N	29670 49	83752 49	for E_2
18954 49N	80431 49	48941 49	27859 49	for E_3
97769 49	20693 49	35343 48	57004 47	for E_4
64746 48	19891 50N	10951 50 •	1-2	
11480 50	15231 50N	12540 50 •	2-3	
15754 50	15879 50N	41158 49 •	3-4	
11615 50N	89139 49	29409 49	1-3	
59131 49N	59066 49	11641 49	2-4	
24126 49	24154 49N	35074 48 •	1-4	

52359 49			40000 50	
15965 51N	65491 50N	11870 50	21327 51	
40799 48N	30557 49	78981 49 •	53023 49	for E_1
78777 48	43650 49N	37388 49	81453 49	for E_3
20736 49N	81662 49	48458 49	23524 49	for E_2
97425 49	22182 49	39788 48	63919 47	for E_4
21963 49	19295 50N	11434 50 •	1-2	
12917 50	16102 50N	11614 50 •	2-3	
15999 50	16185 50N	44841 49 •	3-4	
10380 50N	87126 49	28343 49	1-3	
51491 49N	51426 49	10292 49	2-4	
20167 49	20188 49N	30212 48 •	1-4	

52359 49			45000 50	
17689 51N	75798 50N	19151 50	23354 51	
39226 48N	28666 49	76042 49 •	58141 49	for E_1
79475 48	43300 49N	43376 49	78615 49	for E_2
22197 49N	82168 49	48134 49	20943 49	for E_3
97101 49	23486 49	43814 48	70251 47	for E_4
37747 49	18921 50N	11748 50 •	1-2	
13883 50	16811 50N	11114 50 •	2-3	
16115 50	16363 50N	47914 49 •	3-4	
93084 49N	81943 49	26838 49	1-3	
46517 49N	46450 49	94247 48	2-4	
16896 49	16914 49N	25878 48 •	1-4	

52359 49			50000 50	
19483 51N	85529 50N	26469 50	25388 51	
37579 48N	26959 49	73251 49 •	62396 49	for E_1
80876 48	43386 49N	48128 49	75735 49	for E_2
23462 49N	82364 49	47909 49	19243 49	for E_3
96798 49	24635 49	47476 48	75870 47	for E_4
52038 49	18666 50N	11912 50 •	1-2	
14576 50	17375 50N	10820 50 •	2-3	
16171 50	16484 50N	50556 49 •	3-4	
83789 49N	75900 49	23922 49	1-3	
42836 49N	42763 49	87888 48	2-4	
14233 49	14245 49N	22137 48 •	1-4	

52359 49			55000 50	
21331 51N	94772 50N	33787 50	27430 51	
35989 48N	25454 49	70688 49 •	65895 49	for E_1
82480 48	43621 49N	51938 49	73017 49	for E_2
24570 49N	82407 49	47746 49	18043 49	for E_3
96515 49	25656 49	50807 48	81083 47	for E_4
64349 49	18482 50N	11962 50 •	1-2	
15090 50	17820 50N	10635 50 •	2-3	
16197 50	16572 50N	52866 49 •	3-4	
75764 49N	69921 49	21710 49	1-3	
39882 49N	39807 49	82754 48	2-4	
12074 49	12085 49N	19015 48 •	1-4	

52359 49			60000 50	
23223 51N	10362 51N	41091 50	29476 51	
34518 48N	24143 49	68381 49 •	68769 49	for E_1
84080 48	43890 49N	55018 49	70539 49	for E_2
25552 49N	82370 49	47683 49	17152 49	for E_3
96252 49	26567 49	53841 48	85800 47	for E_4
74719 49	18343 50N	11937 50 •	1-2	
15478 50	18171 50N	10509 50 •	2-3	
16204 50	16640 50N	54906 49 •	3-4	
68857 49N	64368 49	19711 49	1-3	
37398 49N	37319 49	78404 48	2-4	
10325 49	10335 49N	16416 48 •	1-4	

52359 49			65000 50	
25149 51N	11215 51N	48376 50	31527 51	
33188 48N	23007 49	66327 49 •	71135 49	for E_1
85594 48	44149 49N	57530 49	68321 49	for E_2
26428 49N	82289 49	47526 49	16466 49	for E_3
96006 49	27384 49	56612 48	90092 47	for E_4
83383 49	18233 50N	11865 50 •	1-2	
15777 50	18449 50N	10420 50 •	2-3	
16200 50	16695 50N	56721 49 •	3-4	
62912 49N	59358 49	17948 49	1-3	
35248 49N	35165 49	74594 48	2-4	
89038 48	89121 48N	4261 48 •	1-4	

52359 49			70000 50	
27103 51N	12043 51N	55640 50	33583 51	
32000 48N	22024 49	64509 49 •	73097 49	for E_1
86994 48	44382 49N	59599 49	66350 49	for E_2
27216 49N	82184 49	47449 49	15920 49	for E_3
95777 49	28121 49	59148 48	93993 47	for E_4
90616 49	18146 50N	11766 50 •	1-2	
16011 50	18671 50N	10354 50 •	2-3	
16189 50	16741 50N	58346 49 •	3-4	
57782 49N	54897 49	16409 49	1-3	
33351 49N	33266 49	71193 48	2-4	
77405 48	77474 48N	12468 48 •	1-4	

52359 49			75000 50	
29078 51N	12852 51N	62883 50	35641 51	
30944 48N	21171 49	62900 49 •	74737 49	for E_1
88277 48	44587 49N	61320 49	64604 49	for E_2
27929 49N	82066 49	47386 49	15477 49	for E_3
95563 49	28788 49	61476 48	97559 47	for E_4
96677 49	18073 50N	11654 50 •	1-2	
16197 50	18851 50N	10304 50 •	2-3	
16173 50	16780 50N	59810 49 •	3-4	
53338 49N	50945 49	15071 49	1-3	
31659 49N	31371 49	68114 48	2-4	
67813 48	67870 49N	10971 48 •	1-4	

TABLE I(d). Ruby $\theta=40°$.

69813 49			50000 49	
68258 50N	47405 50N	41852 50	73811 50	
26912 48N	49068 49	86874 49 •	61592 48	for E_1
55900 48	86910 49N	48958 49 •	42875 48	for E_2
17264 47N	69979 47	74705 48	99717 49	for E_3
99807 49	61920 48	41262 47	10057 47	for E_4
10008 50	20098 50N	86421 49 •	1-2	
97346 49N	97605 49N	98061 48 •	2-3	
28478 48	28472 48N	80529 47 •	3-4	
14331 50N	14218 50N	11578 49 •	1-3	
15647 50N	15607 50	11545 49	2-4	
74075 49	74145 49N	46798 48 •	1-4	

69813 49			10000 50	
80255 50N	38751 50N	27869 50	91136 50	
44126 48N	48190 49	86577 49 •	12745 49	for E_1
10522 49	86797 49N	47164 49 •	11446 49	for E_2
96345 47N	38143 48	16676 49	98521 49	for E_3
99342 49	11371 49	13116 48	30786 47	for E_4
92676 49	20465 50N	88431 49 •	1-2	
10758 50N	10989 50N	29582 49 •	2-3	
12593 49	12594 49N	35662 48 •	3-4	
13568 50N	12823 50N	21268 49 •	1-3	
16027 50N	15935 50	22003 49	2-4	
63452 49	63585 49N	66531 48 •	1-4	

69813 49			15000 50	
93302 50N	31957 50N	16161 50	10909 51	
54992 48N	46901 49	85955 49 •	19533 49	for E_1
14599 49	86073 49N	42442 49 •	24017 49	for E_2
31366 48N	12013 49	28364 49	95085 49	for E_3
98725 49	15723 49	23895 48	55651 47	for E_4
78100 49	21284 50N	93334 49 •	1-2	
10941 50N	12196 50N	68186 49 •	2-3	
33391 49	33404 49N	96663 48 •	3-4	
12970 50N	10249 50N	25188 49 •	1-3	
16008 50N	15898 50	30321 49	2-4	
53895 49	54064 49N	71854 48 •	1-4	

69813 49			20000 50	
10739 51N	28685 50N	85182 49	12756 51	
61520 48N	45317 49	84969 49 •	26241 49	for E_1
17003 49	82167 49N	31315 49 •	44483 49	for E_2
78727 48N	28614 49	42276 49	85626 49	for E_3
98035 49	19392 49	34944 48	80800 47	for E_4
53652 49	22384 50N	10200 50 •	1-2	
81636 49N	13334 50N	12855 50 •	2-3	
70115 49	70221 49N	21158 49 •	3-4	
12634 50N	55159 49N	17068 49 •	1-3	
14944 50N	14850 50	34090 49	2-4	
45494 49	45671 49N	69720 48 •	1-4	

69813 49			25000 50	
12249 51N	31045 50N	71112 49	14642 51	
65062 48N	43557 49	83653 49 •	32597 49	for E_1
16750 49	72689 49N	13758 49 •	65164 49	for E_2
14326 49N	48090 49	52840 49	68483 49	for E_3
97323 49	22501 49	45540 48	10449 48	for E_4
28271 49	22469 50N	11016 50 •	1-2	
14742 49N	14362 50N	16876 50 •	2-3	
11000 50	11049 50N	35560 49 •	3-4	
12004 50N	28188 48	97516 47	1-3	
12392 50N	12324 50	31134 49	2-4	
38265 49	38430 49N	64075 48 •	1-4	

69813 49			30000 50	
13850 51N	37907 50N	10804 50	16560 51	
66602 48N	41736 49	82094 49 •	38394 49	for E_1
15532 49	63801 49N	15689 48	75403 49	for E_2
19471 49N	59623 49	56809 49	53278 49	for E_3
96619 49	25149 49	55375 48	12619 48	for E_4
18477 49	21557 50N	11529 50 •	1-2	
42581 49	15258 50N	16852 50 •	2-3	
13282 50	13410 50N	46607 49 •	3-4	
10940 50N	38977 49	14045 49	1-3	
99014 49N	98483 49	26035 49	2-4	
32161 49	32308 49N	57159 48 •	1-4	

69813 49			35000 50	
15531 51N	46523 50N	16800 50	18504 51	
66897 48N	39950 49	80402 49 •	43526 49	for E_1
14789 49	58774 49N	12123 49	78612 49	for E_2
23063 49N	64789 49	57853 49	43856 49	for E_3
95940 49	27418 49	64354 48	14578 48	for E_4
20902 49	20687 50N	11935 50 •	1-2	
75760 49	16016 50N	15875 50 •	2-3	
14312 50	14543 50N	54257 49 •	3-4	
98881 49N	54175 49	19756 49	1-3	
82492 49N	82022 49	22233 49	2-4	
27085 49	27212 49N	50183 48 •	1-4	

69813 49			40000 50	
17281 51N	55534 50N	23665 50	20468 51	
66338 48N	38265 49	78677 49 •	47975 49	for E_1
14499 49	56262 49N	19584 49	78998 49	for E_2
25766 49N	67136 49	58084 49	38143 49	for E_3
95298 49	29375 49	72482 48	16336 48	for E_4
28215 49	20061 50N	12270 50 •	1-2	
95321 49	16644 50N	15075 50 •	2-3	
14799 50	15146 50N	60022 49 •	3-4	
89694 49N	59213 49	21618 49	1-3	
71669 49N	71219 49	19633 49	2-4	
22907 49	23015 49N	43734 48 •	1-4	

69813 49			45000 50	
19088 51N	64488 50N	30869 50	22449 51	
65388 48N	36717 49	76991 49 •	51781 49	for E_1
14457 49	55035 49N	25176 49	78282 49	for E_2
27942 49N	68244 49	58092 49	34455 49	for E_3
94696 49	31074 49	79812 48	17906 48	for E_4
36848 49	19619 50N	12526 50 •	1-2	
10793 50	17159 50N	14522 50 •	2-3	
15045 50	15514 50N	64661 49 •	3-4	
81769 49N	59735 49	21739 49	1-3	
64023 49N	63577 49	17755 49	2-4	
19485 49	19578 49N	38035 48 •	1-4	

69813 49			50000 50	
20940 51N	73252 50N	38205 50	24445 51	
64230 48N	35318 49	75393 49 •	55019 49	for E_1
14539 49	54455 49N	29544 49	77138 49	for E_2
29763 49N	68754 49	58036 49	31918 49	for E_3
94135 49	32558 49	86420 48	19317 48	for E_4
45355 49	19296 50N	12708 50 •	1-2	
11668 50	17579 50N	14143 50 •	2-3	
15171 50	15763 50N	68532 49 •	3-4	
74908 49N	58239 49	21090 49	1-3	
58226 49N	57778 49	16306 49	2-4	
16686 49	16766 49N	33128 48 •	1-4	

69813 49			55000 50	
22830 51N	81802 50N	45586 50	26452 51	
62997 48N	34067 49	73908 49 •	57769 49	for E_1
14679 49	54206 49N	33052 49	75853 49	for E_2
31323 49N	68951 49	57963 49	30078 49	for E_3
93614 49	33863 49	92378 48	20574 48	for E_4
53193 49	19052 50N	12827 50 •	1-2	
12305 50	17923 50N	13877 50 •	2-3	
15230 50	15943 50N	71830 49 •	3-4	
68941 49N	55871 49	20120 49	1-3	
53596 49N	53143 49	15135 49	2-4	
14391 49	14460 49N	28942 48 •	1-4	

69813 49			60000 50	
24751 51N	90151 50N	52974 50	28469 51	
61768 48N	32954 49	72543 49 •	60110 49	for E_1
14844 49	54130 49N	35925 49	74558 49	for E_2
32680 49N	68975 49	57889 49	28690 49	for E_3
93131 49	35017 49	97763 48	21704 48	for E_4
60201 49	18862 50N	12897 50 •	1-2	
12787 50	18204 50N	13685 50 •	2-3	
15250 50	16082 50N	74681 49 •	3-4	
63730 49N	53183 49	19044 49	1-3	
49760 49N	49301 49	14152 49	2-4	
12500 49	12561 49N	25396 48 •	1-4	

69813 49			65000 50	
26696 51N	98324 50N	60350 50	30494 51	
60586 48N	31967 49	71299 49 •	62111 49	for E_1
15016 49	54144 49N	38314 49	73313 49	for E_2
33874 49N	68900 49	57819 49	27606 49	for E_3
92683 49	36044 49	10264 49	22722 48	for E_4
66382 49	18710 50N	12932 50 •	1-2	
13161 50	18435 50N	13543 50 •	2-3	
15246 50	16192 50N	77171 49 •	3-4	
59161 49N	50447 49	17966 49	1-3	
46498 49N	46033 49	13306 49	2-4	
10934 49	10987 49N	22396 48 •	1-4	

69813 49			70000 50	
28662 51N	10634 51N	67705 50	32526 51	
59471 48N	31089 49	70168 49 •	63831 49	for E_1
15186 49	54205 49N	40325 49	72145 49	for E_2
34934 49N	68767 49	57754 49	26739 49	for E_3
92269 49	36962 49	10707 49	23644 48	for E_4
71806 49	18585 50N	12942 50 •	1-2	
13458 50	18627 50N	13435 50 •	2-3	
15228 50	16282 50N	79364 49 •	3-4	
55136 49N	47795 49	16933 49	1-3	
43672 49N	43204 49	12565 49	2-4	
96288 48	96755 48N	19854 48 •	1-4	

69813 49			75000 50	
30644 51N	11423 51N	75038 50	34564 51	
58433 48N	30309 49	69143 49 •	65318 49	for E_1
15349 49	54288 49N	42037 49	71063 49	for E_2
35881 49N	68602 49	57694 49	26029 49	for E_3
91884 49	37786 49	11110 49	24476 48	for E_4
76561 49	18481 50N	12933 50 •	1-2	
13697 50	18788 50N	13351 50 •	2-3	
15199 50	16358 50N	81308 49 •	3-4	
51575 49N	45289 49	15967 49	1-3	
41190 49N	40719 49	11909 49	2-4	
85324 48	85734 48N	17685 48 •	1-4	

TABLE I(e). Ruby $\theta=50°$.

87266 49			50000 49	
69599 50N	46507 50N	44662 50	71444 50	
36243 48N	64965 49	83184 49 •	67857 48	for E_1
65664 48	83198 49N	84802 49 •	56130 48	for E_2
32933 47N	92835 47	87535 48	99611 49	for E_3
99717 49	74794 48	61314 47	19011 47	for E_4
75117 49	20133 50N	93174 49 •	1-2	
10870 50N	10932 50N	12813 49 •	2-3	
39682 48	39731 48N	15375 48 •	3-4	
13478 50N	13312 50N	12450 49 •	1-3	
15218 50N	15148 50	13725 49	2-4	
82493 49	82631 49N	62600 48 •	1-4	

87265 49			10000 50	
83200 50N	37595 50N	34000 50	86795 50	
59674 48N	53971 49	82880 49 •	13503 49	for E_1
12599 49	82906 49N	52564 49 •	14303 49	for E_2
16858 48N	45722 48	19073 49	98043 49	for E_3
99009 49	13880 49	20181 48	66049 47	for E_4
68199 49	20562 50N	96363 49 •	1-2	
11754 50N	12223 50N	36488 49 •	2-3	
16560 49	16565 49N	67002 48 •	3-4	
12532 50N	11593 50N	21141 49 •	1-3	
15718 50N	15535 50	26692 49	2-4	
69812 49	70221 49N	88283 48 •	1-4	

87266 49			15000 50	
97949 50N	31289 50N	26111 50	10312 51	
74695 48N	52624 49	82335 49 •	19895 49	for E_1
17630 49	81881 49N	47348 49 •	27251 49	for E_2
47869 48N	12334 49	31062 49	94127 49	for E_3
98032 49	19338 49	37587 48	12101 48	for E_4
56484 49	21269 50N	10229 50 •	1-2	
11452 50N	13406 50N	74878 49 •	2-3	
39060 49	39110 49N	16277 49 •	3-4	
11710 50N	89166 49N	23041 49 •	1-3	
15681 50N	15432 50	36808 49	2-4	
58534 49	59077 49N	94188 48 •	1-4	

87266 49			20000 50	
11370 51N	28675 50N	22120 50	12025 51	
84143 48N	51097 49	81574 49 •	25765 49	for E_1
20908 49	78949 49N	37853 49 •	43553 49	for E_2
10032 49N	24116 49	43378 49	86232 49	for E_3
96909 49	23967 49	55753 48	17675 48	for E_4
41700 49	21313 50N	11016 50 •	1-2	
88112 49N	14448 50N	12158 50 •	2-3	
69451 49	69747 49N	30379 49 •	3-4	
10988 50N	52156 49N	16414 49 •	1-3	
14773 50N	14517 50	41666 49	2-4	
48861 49	49446 49N	90341 48 •	1-4	

87266 49			25000 50	
13031 51N	30272 50N	22554 50	13802 51	
89908 48N	49519 49	80656 49 •	31008 49	for E_1
22117 49	74000 49N	25539 49 •	58158 49	for E_2
16321 49N	35975 49	52807 49	75171 49	for E_3
95726 49	27882 49	73288 48	22915 48	for E_4
29607 49	21964 50N	11731 50 •	1-2	
41889 49N	15336 50N	15512 50 •	2-3	
98125 49	99151 49N	45878 49 •	3-4	
10198 50N	14648 49N	51200 48 •	1-3	
13073 50N	12837 50	40756 49	2-4	
40779 49	41349 49N	82332 48 •	1-4	

87266 49			30000 50	
14764 51N	35082 50N	26401 50	15632 51	
93264 48N	47976 49	79649 49 •	35598 49	for E_1
22176 49	69127 49N	14156 49 •	67298 49	for E_2
21953 49N	44121 49	58098 49	64775 49	for E_3
94547 49	31191 49	89536 48	27661 48	for E_4
24558 49	21507 50N	12251 50 •	1-2	
22558 48	16077 50N	16766 50 •	2-3	
11736 50	11971 50N	59083 49 •	3-4	
93446 49N	11962 49	44069 48	1-3	
11267 50N	11052 50	37024 49	2-4	
34136 49	34663 49N	73119 48 •	1-4	

87266 49			35000 50	
14555 51N	41615 50N	32149 50	17502 51	
75067 48N	46521 49	78611 49 •	39567 49	for E_1
22026 49	65645 49N	52947 48 •	71954 49	for E_2
26439 49N	48690 49	60691 49	56980 49	for E_3
93410 49	13945 49	10426 49	31876 48	for E_4
25240 49	20941 50N	12639 50 •	1-2	
34212 49	16687 50N	16922 50 •	2-3	
12826 50	13239 50N	69314 49 •	3-4	
85395 49N	27179 49	10264 49	1-3	
97926 49N	95803 49	33079 49	2-4	
26726 49	25200 49N	64132 48 •	1-4	

87266 49			40000 50	
18396 51N	48924 50N	38811 50	19407 51	
95880 48N	45179 49	77587 49 •	42978 49	for E_1
21969 49	63450 49N	13786 48	74090 49	for E_2
29993 49N	51080 49	61970 49	51485 49	for E_3
92335 49	36380 49	11745 49	35583 48	for E_4
28894 49	20446 50N	12944 50 •	1-2	
55922 49	17187 50N	16747 50 •	2-3	
13417 50	14033 50N	77273 49 •	3-4	
78281 49N	34926 49	13403 49	1-3	
86377 49N	84425 49	29702 49	2-4	
24332 49	24754 49N	55981 48 •	1-4	

87266 49			45000 50	
20275 51N	56536 50N	45897 50	21339 51	
96074 48N	43959 49	76604 49 •	45903 49	for E_1
22026 49	62115 49N	64921 48	74929 49	for E_2
32874 49N	52279 49	62631 49	47575 49	for E_3
91333 49	38421 49	12919 49	38828 48	for E_4
33724 49	20047 50N	13186 50 •	1-2	
70927 49	17595 50N	16520 50 •	2-3	
13732 50	14563 50N	83623 49 •	3-4	
72089 49N	38570 49	14919 49	1-3	
77462 49N	75550 49	26919 49	2-4	
20762 49	21135 49N	48846 48 •	1-4	

87266 49			50000 50	
22186 51N	64244 50N	53176 50	23293 51	
95885 48N	42859 49	75677 49 •	48415 49	for E_1
22159 49	61308 49N	10515 49	75097 49	for E_2
35264 49N	52822 49	62986 49	44709 49	for E_3
90407 49	40176 49	13964 49	41671 48	for E_4
38820 49	19730 50N	13376 50 •	1-2	
81669 49	17929 50N	16317 50 •	2-3	
13890 50	14938 50N	88806 49 •	3-4	
66694 49N	39923 49	15511 49	1-3	
70332 49N	68445 49	24612 49	2-4	
17849 49	18178 49N	42714 48 •	1-4	

87266 49			55000 50	
24123 51N	71957 50N	60535 50	25265 51	
95468 48N	41871 49	74815 49 •	50580 49	for E_1
22335 49	60822 49N	13758 49	74916 49	for E_2
37281 49N	52998 49	63178 49	42540 49	for E_3
89554 49	41696 49	14894 49	44167 48	for E_4
43761 49	19476 50N	13524 50 •	1-2	
89626 49	18205 50N	16151 50 •	2-3	
13956 50	15217 50N	93110 49 •	3-4	
61971 49N	40008 49	15584 49	1-3	
64477 49N	62609 49	22672 49	2-4	
15460 49	15750 49N	37489 48 •	1-4	

87266 49			60000 50	
26081 51N	79635 50N	67921 50	27252 51	
94920 48N	40984 49	74017 49 •	52455 49	for E_1
22533 49	60533 49N	16425 49	74552 49	for E_2
39008 49N	52962 49	63280 49	40852 49	for E_3
88771 49	43021 49	15722 49	46361 48	for E_4
48369 49	19269 50N	13638 50 •	1-2	
95693 49	18432 50N	16019 50 •	2-3	
13967 50	15433 50N	96732 49 •	3-4	
57812 49N	39391 49	15366 49	1-3	
59564 49N	57713 49	21015 49	2-4	
13486 49	13744 49N	33046 48 •	1-4	

87266 49			65000 50	
28056 51N	87263 50N	75306 50	29252 51	
94304 48N	40189 49	73283 49 •	54086 49	for E_1
22737 49	60368 49N	18656 49	74098 49	for E_2
40504 49N	52805 49	63327 49	39504 49	for E_3
88053 49	44182 49	16464 49	48303 48	for E_4
52586 49	19098 50N	13725 50 •	1-2	
10043 50	18622 50N	15913 50 •	2-3	
13945 50	15607 50N	99816 49 •	3-4	
54131 49N	38396 49	14988 49	1-3	
55369 49N	53536 49	19583 49	2-4	
11845 49	12074 49N	29270 48 •	1-4	

87266 49			70000 50	
30046 51N	94836 50N	82675 50	31262 51	
93659 48N	39474 49	72609 49 •	55515 49	for E_1
22941 49	60279 49N	20546 49	73606 49	for E_2
41810 49N	52577 49	63342 49	38407 49	for E_3
87394 49	45207 49	17129 49	50026 48	for E_4
56410 49	18954 50N	13790 50 •	1-2	
10421 50	18782 50N	15829 50 •	2-3	
13901 50	15749 50N	10246 50 •	3-4	
50856 49N	37207 49	14528 49	1-3	
51738 49N	49924 49	19330 49	2-4	
10471 49	10675 49N	26050 48 •	1-4	

87266 49			75000 50	
32048 51N	10235 51N	90024 50	33281 51	
93010 48N	38829 49	71990 49 •	56772 49	for E_1
23138 49	60240 49N	22168 49	73104 49	for E_2
42961 49N	52311 49	63337 49	37497 49	for E_3
86789 49	46116 49	17727 49	51563 48	for E_4
59862 49	18832 50N	13839 50 •	1-2	
10727 50	18916 50N	15761 50 •	2-3	
13845 50	15868 50N	10476 50 •	3-4	
47927 49N	35935 49	14032 49	1-3	
48558 49N	46765 49	17225 49	2-4	
93113 48	94945 48N	23297 48 •	1-4	

TABLE I(f). Ruby $\theta=60°$.

10471 50			50000 49	
70754 50N	45768 50N	47857 50	68664 50	
45080 48N	59683 49	79794 49 •	70995 48	for E_1
73510 48	79767 49N	59480 49 •	67200 48	for E_2
57507 47N	10940 48	97054 48	99520 49	for E_3
99625 49	85927 48	83456 47	40402 47	for E_4
53524 49	20164 50N	97495 49 •	1-2	
11725 50N	11849 50N	15290 49 •	2-3	
49949 48	49938 48N	29120 48 •	3-4	
12719 50N	12489 50N	12721 49 •	1-3	
14842 50N	14711 50	15531 49	2-4	
89144 49	89610 49N	77651 48 •	1-4	

10471 50			10000 50	
85708 50N	36726 50N	40781 50	81653 50	
74914 48N	58690 49	79459 49 •	13619 49	for E_1
14414 49	79188 49N	57050 49 •	16330 49	for E_2
27022 48N	48354 48	20572 49	97704 49	for E_3
98634 49	16163 49	28660 48	13241 48	for E_4
47874 49	20620 50N	10171 50 •	1-2	
12391 50N	13192 50N	41127 49 •	2-3	
19753 49	19768 49N	11685 49 •	3-4	
11652 50N	10536 50N	20127 49 •	1-3	
15472 50N	15110 50	30841 49	2-4	
74701 49	75812 49N	10945 49 •	1-4	

10471 50			15000 50	
10183 51N	30864 50N	36666 50	96034 50	
94620 48N	57435 49	78963 49 •	19401 49	for E_1
20428 49	77810 49N	51983 49 •	28737 49	for E_2
68578 48N	11366 49	32126 49	93763 49	for E_3
97191 49	22748 49	55058 48	24644 48	for E_4
39471 49	21200 50N	10826 50 •	1-2	
11698 50N	14381 50N	77183 49 •	2-3	
42612 49	42771 49N	26111 49 •	3-4	
10717 50N	7 221 49N	20730 49 •	1-3	
15463 50N	14915 50	42600 49	2-4	
61 36 49	63384 49N	11609 49 •	1-4	

10471 50			20000 50	
11888 51N	28669 50N	35960 50	11159 51	
10768 49N	56100 49	78363 49 •	24413 49	for E_1
24661 49	75323 49N	44240 49 •	41964 49	for E_2
12801 49N	19328 49	42809 49	87349 49	for E_3
95456 49	28381 49	83310 48	36265 48	for E_4
30761 49	21585 50N	11562 50 •	1-2	
92452 49N	15388 50N	11505 50 •	2-3	
68478 49	69244 49N	44343 49 •	3-4	
98761 49N	49499 49N	15447 49 •	1-3	
14679 50N	14038 50	48366 49	2-4	
50928 49	52664 49N	11042 49 •	1-4	

10471 50			25000 50	
13664 51N	29889 50N	38408 50	12812 51	
11638 49N	54790 49	77709 49 •	28701 49	for E_1
27059 49	72160 49N	35348 49 •	53019 49	for E_2
19348 49N	26336 49	50882 49	79642 49	for E_3
93583 49	33124 49	11076 49	47058 48	for E_4
24708 49	21580 50N	12210 50 •	1-2	
57561 49N	16214 50N	14340 50 •	2-3	
90558 49	92797 49N	62478 49 •	3-4	
90747 49N	22750 49N	78502 48 •	1-3	
13355 50N	12687 50	48498 49	2-4	
41960 49	43708 49N	99715 48 •	1-4	

10471 50			30000 50	
15497 51N	33588 50N	43103 50	14545 51	
12223 49N	53557 49	77042 49 •	32350 49	for E_1
28273 49	69208 49N	27145 49 •	60613 49	for E_2
25360 49N	31095 49	56057 49	72439 49	for E_3
91695 49	37079 49	13603 49	56576 48	for E_4
22349 49	21296 50N	12717 50 •	1-2	
24054 49N	16879 50N	15953 50 •	2-3	
10569 50	11035 50N	78153 49 •	3-4	
83230 49N	31739 48N	11614 48 •	1-3	
11909 50N	11243 50	45518 49	2-4	
34721 49	36378 49N	87769 48 •	1-4	

10471 50			35000 50	
17374 51N	38796 50N	49110 50	16342 51	
12618 49N	52425 49	76389 49 •	35453 49	for E_1
28944 49	66923 49N	20390 49 •	65327 49	for E_2
30419 49N	33821 49	59138 49	66583 49	for E_3
89875 49	40360 49	15857 49	64749 48	for E_4
22817 49	20919 50N	13110 50 •	1-2	
26698 48	17409 50N	16748 50 •	2-3	
11476 50	12258 50N	90681 49 •	3-4	
76462 49N	94968 48	36039 48	1-3	
10597 50N	99416 49	41575 49	2-4	
28924 49	30446 49N	76363 48 •	1-4	

10471 50			40000 50	
19285 51N	44853 50N	55806 50	18189 51	
12886 49N	51398 49	75766 49 •	38098 49	for E_1
29407 49	65299 49N	15026 49 •	68157 49	for E_2
34567 49N	35183 49	60954 49	62062 49	for E_3
88171 49	43084 49	17835 49	71668 48	for E_4
24897 49	20553 50N	13421 50 •	1-2	
22642 49	17831 50N	17121 50 •	2-3	
11975 50	13110 50N	10049 50 •	3-4	
70502 49N	17227 49	66960 48	1-3	
94829 49N	88403 49	37679 49	2-4	
24290 49	25664 49N	66187 48 •	1-4	

10471 50			45000 50	
21224 51N	51380 50N	62847 50	20077 51	
13067 49N	50474 49	75181 49 •	40365 49	for E_1
29792 49	64178 49N	10768 49 •	69840 49	for E_2
37963 49N	35727 49	62043 49	58591 49	for E_3
86604 49	45354 49	19557 49	77495 48	for E_4
27728 49	20230 50N	13674 50 •	1-2	
37429 49	18168 50N	17293 50 •	2-3	
12219 50	13723 50N	10818 50 •	3-4	
65283 49N	21782 49	86093 48	1-3	
85524 49N	79234 49	34165 49	2-4	
20576 49	21804 49N	57402 48 •	1-4	

10471 50			50000 50	
23185 51N	58170 50N	70051 50	21997 51	
13190 49N	49644 49	74636 49 •	42318 49	for E_1
30141 49	63407 49N	73423 48 •	70830 49	for E_2
40766 49N	35798 49	62707 49	55895 49	for E_3
85179 49	47258 49	21052 49	82403 48	for E_4
30817 49	19955 50N	13880 50 •	1-2	
48504 49	18438 50N	17373 50 •	2-3	
12311 50	14181 50N	11427 50 •	3-4	
60703 49N	24353 49	97453 48	1-3	
77734 49N	71587 49	31093 49	2-4	
17583 49	18676 49N	49938 48 •	1-4	

10471 50			55000 50	
25164 51N	65108 50N	77326 50	23942 51	
13271 49N	48899 49	74133 49 •	44011 49	for E_1
30469 49	62876 49N	45416 48 •	71397 49	for E_2
43103 49N	35603 49	63118 49	53763 49	for E_3
83890 49	48865 49	22352 49	86553 48	for E_4
33904 49	19723 50N	14051 50 •	1-2	
56949 49	18657 50N	17407 50 •	2-3	
12313 50	14533 50N	11914 50 •	3-4	
56668 49N	25690 49	10377 49	1-3	
71158 49N	65156 49	28432 49	2-4	
15152 49	16125 49N	43637 48 •	1-4	

10471 50			60000 50	
27158 51N	72130 50N	84624 50	25908 51	
13324 49N	48229 49	73669 49 •	45488 49	for E_1
30777 49	62506 49N	22159 48 •	71699 49	for E_2
45071 49N	35262 49	63374 49	52046 49	for E_3
82727 49	50234 49	23485 49	90080 48	for E_4
36861 49	19526 50N	14192 50 •	1-2	
63509 49	18836 50N	17420 50 •	2-3	
12260 50	14813 50N	12309 50 •	3-4	
53094 49N	26253 49	10685 49	1-3	
65550 49N	59696 49	26128 49	2-4	
13161 49	14029 49N	38326 48 •	1-4	

10471 50			65000 50	
29164 51N	79199 50N	91920 50	27891 51	
13357 49N	47625 49	73242 49 •	46785 49	for E_1
31068 49	62247 49N	25694 47 •	71832 49	for E_2
46744 49N	34846 49	63531 49	50640 49	for E_3
81677 49	51408 49	24478 49	93097 48	for E_4
39632 49	19357 50N	14310 50 •	1-2	
68696 49	18983 50N	17482 50 •	2-3	
12177 50	15040 50N	12632 50 •	3-4	
49912 49N	26330 49	10781 49	1-3	
60722 49N	55016 49	24127 49	2-4	
11519 49	12294 49N	33840 48 •	1-4	

10471 50			70000 50	
31180 51N	86293 50N	99204 50	29888 51	
13375 49N	47080 49	72850 49 •	47931 49	for E_1
31341 49	62065 49N	14138 48	71858 49	for E_2
48180 49N	34396 49	63624 49	49471 49	for E_3
80730 49	52423 49	25352 49	95694 48	for E_4
42199 49	19212 50N	14409 50 •	1-2	
72861 49	19104 50N	17419 50 •	2-3	
12076 50	15229 50N	12899 50 •	3-4	
47067 49N	26102 49	10741 49	1-3	
56527 49N	50970 49	22381 49	2-4	
10151 49	10846 49N	30038 48 •	1-4	

10471 50			75000 50	
33204 51N	93399 50N	10647 51	31897 51	
13383 49N	46586 49	72488 49 •	48948 49	for E_1
31596 49	61937 49N	28545 48	71814 49	for E_2
49423 49N	33938 49	63676 49	48485 49	for E_3
79873 49	53307 49	26125 49	97944 48	for E_4
44562 49	19085 50N	14492 50 •	1-2	
76254 49	19207 50N	17414 50 •	2-3	
11968 50	15387 50N	13122 50 •	3-4	
44510 49N	25683 49	10613 49	1-3	
52853 49N	47442 49	20848 49	2-4	
90039 48	96286 48N	26801 48 •	1-4	

TABLE I(g). Ruby $\theta=70°$.

12217 50			50000 49	
71636 50N	45223 50N	51319 50	65539 50	
53100 48N	63657 49	76606 49 •	71413 48	for E_1
79208 48	76525 49N	63435 49 •	75513 48	for E_2
10063 48N	11645 48	10308 49	99455 49	for E_3
99539 49	94972 48	10656 48	78833 47	for E_4
34391 49	20187 50N	10018 50 •	1-2	
12368 50N	12609 50N	17121 49 •	2-3	
58015 48	58014 48N	53564 48 •	3-4	
12056 50N	11732 50N	12508 49 •	1-3	
14503 50N	14265 50	16882 49	2-4	
94767 49	95892 49N	91636 48 •	1-4	

12217 50			10000 50	
87606 50N	36133 50N	47925 50	75814 50	
89479 48N	62747 49	76204 49 •	13248 49	for E_1
15925 49	75576 49N	61054 49 •	17520 49	for E_2
43619 48N	45507 48	21214 49	97520 49	for E_3
98220 49	18172 49	38993 48	26968 48	for E_4
30449 49	20651 50N	10518 50 •	1-2	
12734 50N	14052 50N	43829 49 •	2-3	
22011 49	22065 49N	20743 49 •	3-4	
10960 50N	95991 49N	18568 49 •	1-3	
15331 50N	14584 50	34156 49	2-4	
78464 49	81429 49N	13063 49 •	1-4	

12217 50			15000 50	
10474 51N	30606 50N	47386 50	67959 50	
11443 49N	61662 49	75699 49 •	18340 49	for E_1
22962 49	73745 49N	56522 49 •	28975 49	for E_2
10102 49N	93084 48	31834 49	93796 49	for E_3
96123 49	25936 49	78359 48	51193 48	for E_4
25139 49	21129 50N	11208 50 •	1-2	
11671 50N	15343 50N	77363 49 •	2-3	
44484 49	44998 49N	43617 49 •	3-4	
10033 50N	70797 49N	18301 49 •	1-3	
15441 50N	14191 50	47069 49	2-4	
63694 49	68073 49N	13892 49 •	1-4	

12217 50			20000 50	
12271 51N	28666 50N	49588 50	10179 51	
13181 49N	60565 49	75164 49 •	22546 49	for E_1
28225 49	71229 49N	50377 49 •	39896 49	for E_2
17345 49N	13880 49	40790 49	88558 49	for E_3
93427 49	32641 49	12187 49	75691 48	for E_4
20313 49	21371 50N	11915 50 •	1-2	
92995 49N	16419 50N	11002 50 •	2-3	
66709 49	68860 49N	69165 49 •	3-4	
92184 49N	45919 49N	13974 49 •	1-3	
14773 50N	13172 50	52942 49	2-4	
51154 49	56241 49N	13135 49 •	1-4	

12217 50			25000 50	
14129 51N	29688 50N	53907 50	11707 51	
14417 49N	59531 49	74638 49 •	26022 49	for E_1
31710 49	68586 49N	43873 49 •	48635 49	for E_2
24731 49N	17068 49	47272 49	82839 49	for E_3
90415 49	38216 49	16415 49	97509 48	for E_4
17321 49	21329 50N	12526 50 •	1-2	
64196 49N	17264 50N	13536 50 •	2-3	
83633 49	89198 49N	92552 49 •	3-4	
84745 49N	25396 49N	85275 48 •	1-3	
13605 50N	11817 50	52776 49	2-4	
40975 49	46183 49N	11715 49 •	1-4	

12217 50			30000 50	
16032 51N	32786 50N	59568 50	13354 51	
15315 49N	58588 49	74142 49 •	26910 49	for E_1
33938 49	66281 49N	37986 49 •	54881 49	for E_2
31309 49N	18695 49	51491 49	77580 49	for E_3
87368 49	42716 49	20211 49	11551 49	for E_4
16296 49	21110 50N	13018 50 •	1-2	
37625 49N	17900 50N	15244 50 •	2-3	
94135 49	10474 50N	11137 50 •	3-4	
77961 49N	10540 49N	37699 48 •	1-3	
12282 50N	10425 50	49300 49	2-4	
32956 49	37920 49N	10149 49 •	1-4	

12217 50			35000 50	
17969 51N	37216 50N	65969 50	15094 51	
15982 49N	57739 49	73682 49 •	31329 49	for E_1
35423 49	64463 49N	33054 49 •	59135 49	for E_2
36769 49N	19168 49	54106 49	73164 49	for E_3
84485 49	46294 49	23468 49	12974 49	for E_4
16688 49	20828 50N	13414 50 •	1-2	
16083 49N	18368 50N	16334 50 •	2-3	
99560 49	11621 50N	12549 50 •	3-4	
71860 49N	53806 47N	20101 47 •	1-3	
11017 50N	91615 49	44678 49	2-4	
26733 49	31278 49N	86806 48 •	1-4	

12217 50			40000 50	
19932 51N	42478 50N	72746 50	16905 51	
16487 49N	56982 49	73262 49 •	33373 49	for E_1
36488 49	63089 49N	29034 49 •	62010 49	for E_2
41169 49N	18951 49	55706 49	69589 49	for E_3
81864 49	49127 49	26200 49	14079 49	for E_4
17911 49	20543 50N	13735 50 •	1-2	
50454 47	18712 50N	17030 50 •	2-3	
10167 50	12470 50N	13578 50 •	3-4	
66431 49N	59698 48	23010 48	1-3	
98937 49N	80753 49	40034 49	2-4	
21924 49	25996 49N	74005 48 •	1-4	

12217 50			45000 50	
21915 51N	48264 50N	79705 50	18771 51	
16877 49N	56307 49	72879 49 •	[illegible] 49	for E_1
37306 49	62063 49N	25766 49 •	63371 49	for E_2
44687 49N	18371 49	56692 49	66713 49	for E_3
79538 49	51380 49	28472 49	14931 49	for E_4
19549 49	20280 50N	14000 50 •	1-2	
13110 49	18966 50N	17485 50 •	2-3	
10182 50	13111 50N	14322 50 •	3-4	
61625 49N	10132 49	40000 48	1-3	
69237 49N	71626 49	35810 49	2-4	
18194 49	21803 49N	63222 48 •	1-4	

12217 50			50000 50	
23914 51N	54389 50N	86748 50	20678 51	
17184 49N	55705 49	72531 49 •	36616 49	for E_1
37970 49	61293 49N	23089 49 •	65331 49	for E_2
47510 49N	17627 49	57307 49	64404 49	for E_3
77496 49	53190 49	30363 49	15593 49	for E_4
21350 49	20046 50N	14223 50 •	1-2	
22731 49	19156 50N	17792 50 •	2-3	
10090 50	13608 50N	14860 50 •	3-4	
57375 49N	12757 49	51316 48	1-3	
80953 49N	64001 49	32123 49	2-4	
15276 49	18462 49N	54274 48 •	1-4	

12217 50			55000 50	
25926 51N	60740 50N	93823 50	22618 51	
17428 49N	55167 49	72215 49 •	37916 49	for E_1
38529 49	60708 49N	20870 49 •	66289 49	for E_2
49794 49N	16827 49	57696 49	62517 49	for E_3
75711 49	54660 49	31943 49	16110 49	for E_4
23174 49	19840 50N	14411 50 •	1-2	
30157 49	19300 50N	18005 50 •	2-3	
99438 49	14002 50N	15254 50 •	3-4	
53609 49N	14375 49	58709 48	1-3	
73882 49N	57618 49	28957 49	2-4	
12970 49	15783 49N	46881 48 •	1-4	

12217 50			60000 50	
27948 51N	67245 50N	10090 51	24582 51	
17625 49N	54685 49	71928 49 •	39053 49	for E_1
39011 49	60259 49N	19008 49 •	66974 49	for E_2
51661 49N	16030 49	57946 49	60461 49	for E_3
74151 49	55869 49	33274 49	16519 49	for E_4
24946 49	19661 50N	14572 50 •	1-2	
35970 49	19412 50N	18159 50 •	2-3	
97727 49	14321 50N	15546 50 •	3-4	
50263 49N	15331 49	63398 48	1-3	
67823 49N	52243 49	26250 49	2-4	
11126 49	13614 49N	40767 48 •	1-4	

12217 50			65000 50	
29979 51N	73860 50N	10799 51	26566 51	
17786 49N	54250 49	71666 49 •	40054 49	for E_1
39435 49	59908 49N	17428 49 •	67469 49	for E_2
53205 49N	15265 49	58106 49	59663 49	for E_3
72784 49	56875 49	34405 49	16846 49	for E_4
26629 49	19503 50N	14710 50 •	1-2	
40586 49	19499 50N	18272 50 •	2-3	
95942 49	14585 50N	15764 50 •	3-4	
47280 49N	15846 49	66220 48	1-3	
62602 49N	47682 49	23932 49	2-4	
96352 48	11842 49N	35690 48 •	1-4	

12217 50			70000 50	
32017 51N	80553 50N	11507 51	28565 51	
17920 49N	53857 49	71428 49 •	40941 49	for E_1
39812 49	59630 49N	16071 49 •	67829 49	for E_2
54496 49N	14546 49	58210 49	58567 49	for E_3
71581 49	57722 49	35373 49	17111 49	for E_4
28208 49	19364 50N	14831 50 •	1-2	
44302 49	19569 50N	18358 50 •	2-3	
94178 49	14806 50N	15928 50 •	3-4	
44608 49N	16067 49	67747 48	1-3	
58073 49N	43781 49	21940 49	2-4	
84153 48	10382 49N	31450 48 •	1-4	

12217 50			75000 50	
34061 51N	87303 50N	12214 51	30577 51	
18031 49N	53501 49	71209 49 •	41732 49	for E_1
40151 49	59408 49N	14896 49 •	68092 49	for E_2
55585 49N	13876 49	58276 49	57632 49	for E_3
70519 49	58443 49	36209 49	17328 49	for E_4
29678 49	19241 50N	14936 50 •	1-2	
47333 49	19625 50N	18424 50 •	2-3	
92484 49	14994 50N	16054 50 •	3-4	
42206 49N	16090 49	68372 48	1-3	
54117 49N	40419 49	20218 49	2-4	
74072 48	91669 48N	27865 48 •	1-4	

TABLE I(h). Ruby $\theta = 80°$.

13962 50			50000 49	
72188 50N	44891 50N	54929 50	62149 50	
60029 48N	67167 49	73513 49 •	69495 48	for E_1
82494 48	73376 49N	66951 49 •	80821 48	for E_2
21352 48N	10775 48	10556 49	99412 49	for E_3
99455 49	10163 49	13428 48	18832 48	for E_4
16819 49	22202 50N	10167 50 •	1-2	
12787 50N	13336 50N	18355 49 •	2-3	
63116 48	63168 48N	12019 49 •	3-4	
11530 50N	10971 50N	11857 49 •	1-3	
14252 50N	13711 50	17598 49	2-4	
99271 49	10257 50N	10489 49 •	1-4	

13962 50			10000 50	
88785 50N	35790 50N	55159 50	69417 50	
10300 49N	66387 49	73007 49 •	12514 49	for E_1
17073 49	72009 49N	64809 49 •	17968 49	for E_2
85619 48N	33851 48	20971 49	97341 49	for E_3
97616 49	19897 49	54707 48	67156 48	for E_4
14805 49	20665 50N	10714 50 •	1-2	
12565 50N	15128 50N	45704 49 •	2-3	
23151 49	23433 49N	45158 49 •	3-4	
10609 50N	85607 49N	16340 49 •	1-3	
15537 50N	13630 50	35597 49	2-4	
79600 49	89157 49N	15484 49 •	1-4	

13962 50			15000 50	
10653 51N	30467 50N	57824 50	79176 50	
13378 49N	65495 49	72426 49 •	16903 49	for E_1
25179 49	69605 49N	60985 49 •	28318 49	for E_2
18117 49N	52670 48	29938 49	93529 49	for E_3
94121 49	28944 49	11782 49	12830 49	for E_4
12254 49	21081 50N	11422 50 •	1-2	
10807 50N	16827 50N	78550 49 •	2-3	
43816 49	46192 49N	89160 49 •	3-4	
98751 49N	59917 49N	15025 49 •	1-3	
16032 50N	12566 50	47235 49	2-4	
60926 49	75663 49N	16791 49 •	1-4	

13962 50			20000 50	
12506 51N	28665 50N	62308 50	91417 50	
15520 49N	64627 49	71861 49 •	20377 49	for E_1
31566 49	66661 49N	56196 49 •	37440 49	for E_2
28323 49N	56691 48	36368 49	88560 49	for E_3
89204 49	36706 49	18849 49	18438 49	for E_4
10106 49	21250 50N	12114 50 •	1-2	
81673 49N	18150 50N	10963 50 •	2-3	
60116 49	68627 49N	12955 50 •	3-4	
91771 49N	37738 49N	11068 49 •	1-3	
15503 50N	10901 50	49889 49	2-4	
45234 49	62267 49N	15806 49 •	1-4	

13962 50			25000 50	
14412 51N	29588 50N	67866 50	10584 51	
17292 49N	63834 49	71342 49 •	23158 49	for E_1
36131 49	63809 49N	51407 49 •	44499 49	for E_2
37091 49N	47393 48	40340 49	83513 49	for E_3
83783 49	42790 49	25302 49	22560 49	for E_4
88671 48	21188 50N	12706 50 •	1-2	
55661 49N	19008 50N	13398 50 •	2-3	
68673 49	87330 49N	15793 50 •	3-4	
84705 49N	21522 49N	69736 48 •	1-3	
14269 50N	91490 49	46401 49	2-4	
33338 49	50119 49N	13772 49 •	1-4	

13962 50			30000 50	
16357 51N	32386 50N	73970 50	12199 51	
18569 49N	63129 49	70879 49 •	25415 49	for E_1
39303 49	61384 49N	47179 49 •	49611 49	for E_2
43762 49N	31456 48	42631 49	79103 49	for E_3
78709 49	47295 49	30540 49	25204 49	for E_4
84931 48	20995 50N	13191 50 •	1-2	
34643 49N	19503 50N	15115 50 •	2-3	
71141 49	10177 50N	17468 50 •	3-4	
77842 49N	10755 49N	37321 48 •	1-3	
12796 50N	76174 49	40671 49	2-4	
24867 49	40057 49N	11562 49 •	1-4	

13962 50			35000 50	
18331 51N	36412 50N	80344 50	13937 51	
19566 49N	62506 49	70470 49 •	27275 49	for E_1
41548 49	59441 49N	43651 49 •	53244 49	for E_2
48598 49N	13951 48	43961 49	75521 49	for E_3
74357 49	50574 49	34579 49	26786 49	for E_4
87215 48	20755 50N	13587 50 •	1-2	
18998 49N	19771 50N	16284 50 •	2-3	
70286 49	11270 50N	18359 50 •	3-4	
71509 49N	38821 48N	14159 48 •	1-3	
11378 50N	63850 49	34929 49	2-4	
18954 49	32164 49N	95855 48 •	1-4	

13962 50			40000 50	
20325 51N	41243 50N	86861 50	15763 51	
20362 49N	61959 49	70110 49 •	28827 49	for E_1
43194 49	57912 49N	40762 49 •	55845 49	for E_2
52080 49N	26987 47N	44772 49	72683 49	for E_3
70761 49	52981 49	37661 49	27701 49	for E_4
93038 48	20511 50N	13914 50 •	1-2	
76090 48N	19914 50N	17085 50 •	2-3	
68010 49	12103 50N	18796 50 •	3-4	
65843 49N	47170 47	17871 47	1-3	
10128 50N	54204 49	29934 49	2-4	
14796 49	26107 49N	79507 48 •	1-4	

13962 50			45000 50	
22335 51N	46609 50N	93465 50	17649 51	
21007 49N	61476 49	69791 49 •	30139 49	for E_1
44449 49	56709 49N	38397 49 •	57740 49	for E_2
54618 49N	17664 48N	45302 49	70437 49	for E_3
67621 49	54785 49	40030 49	28218 49	for E_4
10067 49	20281 50N	14188 50 •	1-2	
72370 47	19989 50N	17644 50 •	2-3	
65300 49	12750 50N	18985 50 •	3-4	
60346 49N	32399 48	12643 48	1-3	
90646 49N	46666 49	25861 49	2-4	
11816 49	21468 49N	66386 48 •	1-4	

13962 50			50000 50	
24358 51N	52345 50N	10012 51	19579 51	
21539 49N	61050 49	69509 49 •	31261 49	for E_1
45438 49	55752 49N	36446 49 •	59149 49	for E_2
56503 49N	30825 48N	45672 49	68642 49	for E_3
65412 49	56169 49	41881 49	28500 49	for E_4
10906 49	20073 50N	14420 50 •	1-2	
69162 48	20028 50N	18046 50 •	2-3	
62602 49	13261 50N	19042 50 •	3-4	
56458 49N	50078 48	20016 48	1-3	
81686 49N	40717 49	22542 49	2-4	
96284 48	17889 49N	55935 48 •	1-4	

13962 50			55000 50	
26390 51N	58341 50N	10682 51	21541 51	
21984 49N	60671 49	69259 49 •	32229 49	for E_1
46240 49	54981 49N	34819 49 •	60219 49	for E_2
57933 49N	42321 48N	45948 49	67189 49	for E_3
63420 49	57254 49	43352 49	28642 49	for E_4
11760 49	19887 50N	14619 50 •	1-2	
11606 49	20048 50N	18342 50 •	2-3	
60093 49	13674 50N	19030 50 •	3-4	
52602 49N	61348 48	25006 48	1-3	
74138 49N	35958 49	19849 49	2-4	
79848 48	15094 49N	47590 48 •	1-4	

13962 50			60000 50	
28431 51N	64526 50N	11356 51	23527 51	
22361 49N	60334 49	69035 49 •	33073 49	for E_1
46906 49	54351 49N	33448 49 •	61047 49	for E_2
59040 49N	52366 48N	46165 49	65996 49	for E_3
61756 49	58122 49	44543 49	28701 49	for E_4
12594 49	19720 50N	14792 50 •	1-2	
15226 49	20057 50N	18566 50 •	2-3	
57827 49	14013 50N	18981 50 •	3-4	
49201 49N	68442 48	28358 48	1-3	
67746 49N	32093 49	17648 49	2-4	
67229 48	12883 49N	40877 48 •	1-4	

13962 50			65000 50	
30478 51N	70853 50N	12032 51	25530 51	
22683 49N	60031 49	68834 49 •	33814 49	for E_1
47470 49	53830 49N	32279 49 •	61700 49	for E_2
59913 49N	61174 48N	46341 49	65002 49	for E_3
60352 49	58830 49	45522 49	28709 49	for E_4
13392 49	19572 50N	14943 50 •	1-2	
18070 49	20059 50N	18739 50 •	2-3	
55808 49	14296 50N	18915 50 •	3-4	
46189 49N	72765 48	30569 48	1-3	
62291 49N	28912 49	15831 49	2-4	
57350 48	11110 49N	35429 48 •	1-4	

13962 50			70000 50	
32530 51N	77287 50N	12711 51	27547 51	
22962 49N	59759 49	68653 49 •	34471 49	for E_1
47953 49	53393 49N	31273 49 •	62222 49	for E_2
60613 49N	68935 48N	46490 49	64165 49	for E_3
59153 49	59417 49	46339 49	28687 49	for E_4
14145 49	19440 50N	15075 50 •	1-2	
20342 49	20059 50N	18875 50 •	2-3	
54016 49	14535 50N	18842 50 •	3-4	
43508 49N	75223 48	31979 48	1-3	
57600 49N	26257 49	14315 49	2-4	
49484 48	96715 48N	30961 48 •	1-4	

13962 50			75000 50	
34587 51N	83805 50N	13391 51	29576 51	
23205 49N	59512 49	68489 49 •	35055 49	for E_1
48374 49	53022 49N	30399 49 •	62645 49	for E_2
61181 49N	75811 48N	46618 49	63450 49	for E_3
58121 49	59911 49	47029 49	28648 49	for E_4
14850 49	19322 50N	15193 50 •	1-2	
22183 49	20057 50N	18984 50 •	2-3	
52426 49	14739 50N	18767 50 •	3-4	
41109 49N	76411 48	32819 48	1-3	
53532 49N	24015 49	13038 49	2-4	
43122 48	84891 48N	27263 48 •	1-4	

TABLE I(i). Ruby $\theta=90°$.

15707 50			50000 49	
72375 50N	44779 50N	58425 50	58729 50	
65592 48N	70405 49	70405 49 •	65592 48	for E_1
83105 48	70220 49N	70220 49 •	83105 48	for E_2
70412 49N	65600 48N	65584 48	70398 49	for E_3
70228 49	83112 48	83098 48	70212 49	for E_4
20000 42N	20207 50N	10214 50 •	1-2	
18873 46	19157 50N	25894 49 •	2-3	
75774 45N	64937 48N	29772 50 •	3-4	
15175 50N	14703 46	14979 45	1-3	
19176 50N	21057 46	28597 45	2-4	
16656 46	14974 50N	15933 49 •	1-4	

15707 50			10000 50	
89185 50N	35678 50N	61285 50	63578 50	
11507 49N	69767 49	69767 49 •	11507 49	for E_1
17784 49	68437 49N	68437 49 •	17784 49	for E_2
69768 49N	11508 49N	11507 49	69767 49	for E_3
68438 49	17784 49	17784 49	68436 49	for E_4
43000 43	20669 50N	10777 50 •	1-2	
17040 45	20399 50N	58696 49 •	2-3	
15520 45N	23886 49N	29057 50 •	3-4	
13191 50N	10520 45	29580 44	1-3	
19997 50N	29810 45	79800 44	2-4	
19460 45	12286 50N	22438 49 •	1-4	

15707 50			15000 50	
10713 51N	30423 50N	65288 50	72273 50	
15219 49N	69053 49	69053 49 •	15219 49	for E_1
27000 49	65352 49N	65352 49 •	27000 49	for E_2
69053 49N	15219 49N	15219 49	69053 49	for E_3
65352 49	27000 49	27000 49	65352 49	for E_4
90000 42	21064 50N	11491 50 •	1-2	
56000 44N	21035 50N	91975 49 •	2-3	
53800 44	46569 49N	27898 50 •	3-4	
11511 50N	26500 44N	60000 43 •	1-3	
19327 50N	27200 44N	12500 44 •	2-4	
17200 44N	95985 49N	22389 49 •	1-4	

15707 50			20000 50	
12585 51N	28664 50N	70050 50	84464 50	
18039 49N	68370 49	68370 49 •	18039 49	for E_1
34649 49	61639 49N	61639 49 •	34649 49	for E_2
68370 49N	18039 49N	18039 49	68370 49	for E_3
61639 49	34649 49	34649 49	61639 49	for E_4
23000 43	21211 50N	12179 50 •	1-2	
33000 43N	21211 50N	11990 50 •	2-3	
43000 43N	68549 49N	26536 50 •	3-4	
10132 50N	40000 42N	30000 43	1-3	
17545 50N	27600 44	16100 44	2-4	
14700 44	72880 49N	19336 49 •	1-4	

15707 50			25000 50	
14507 51N	29558 50N	75329 50	99308 50	
20223 49N	67756 49	67756 49 •	20223 49	for E_1
40372 49	58052 49N	58052 49 •	40372 49	for E_2
67756 49N	20223 49N	20223 49	67756 49	for E_3
58052 49	40372 49	40372 49	58052 49	for E_4
31000 43N	21142 50N	12765 50 •	1-2	
84000 43N	21150 50N	14064 50 •	2-3	
16900 44	86749 49N	25233 50 •	3-4	
90056 49N	49000 43N	82000 43 •	1-3	
15402 50N	38000 43N	42000 43 •	2-4	
22000 43N	55122 49N	15732 49 •	1-4	

15707 50			30000 50	
16467 51N	32264 50N	80970 50	11596 51	
21950 49N	67217 49	67217 49 •	21950 49	for E_1
44484 49	54965 49N	54965 49 •	44484 49	for E_2
67217 49N	21950 49N	21950 49	67217 49	for E_3
54965 49	44484 49	44484 49	54964 49	for E_4
22000 43N	20957 50N	13247 50 •	1-2	
26000 43N	21007 50N	15527 50 •	2-3	
58000 43	10084 50N	24120 50 •	3-4	
80805 49N	62000 43N	82000 43 •	1-3	
13390 50N	73000 43	35000 43	2-4	
22000 43	42205 49N	12587 49 •	1-4	

15707 50			35000 50	
18452 51N	36165 50N	86873 50	13381 51	
23341 49N	66746 49	66746 49 •	23341 49	for E_1
47441 49	52433 49N	52433 49 •	47441 49	for E_2
66746 49N	23341 49N	23341 49	66746 49	for E_3
52433 49	47441 49	47441 49	52433 49	for E_4
25000 43N	20728 50N	13643 50 •	1-2	
24000 43N	20855 50N	16551 50 •	2-3	
10700 44	11159 50N	23213 50 •	3-4	
73137 49N	44000 43N	65000 43 •	1-3	
11677 50N	16000 43N	35000 43 •	2-4	
28000 43N	32933 49N	10102 49 •	1-4	

15707 50			40000 50	
20457 51N	40860 50N	92970 50	15246 51	
24482 49N	66336 49	66336 49 •	24482 49	for E_1
49610 49	50386 49N	50386 49 •	49610 49	for E_2
66336 49N	24482 49N	24482 49	66336 49	for E_3
50386 49N	49610 49	49610 49	50386 49	for E_4
25000 43	20496 50N	13972 50 •	1-2	
80000 42N	20720 50N	17278 50 •	2-3	
73000 43	11985 50N	22484 50 •	3-4	
66711 49N	70000 42	42000 43	1-3	
10267 50N	14000 43	20000 42 •	2-4	
12000 43N	26222 49N	81964 48 •	1-4	

15707 50			45000 50	
22476 51N	46092 50N	99212 50	17164 51	
25433 49N	65978 49	65978 49 •	25433 49	for E_1
51241 49	48726 49N	48726 49 •	51241 49	for E_2
65978 49N	25433 49N	25433 49	65978 49	for E_3
48726 49	51241 49	51241 49	48726 49	for E_4
34000 43	20278 50N	14249 50 •	1-2	
10000 42	20608 50N	17806 50 •	2-3	
38000 43	12631 50N	21895 50 •	3-4	
61267 49N	26000 43	56000 43	1-3	
91162 49N	58000 43	29000 43	2-4	
50000 42	21279 49N	67394 48 •	1-4	

15707 50			50000 50	
24506 51N	51700 50N	10556 51	19120 51	
26235 49N	65663 49	65663 49 •	26235 49	for E_1
52500 49	47367 49N	47367 49 •	52500 49	for E_2
65663 49N	26235 49N	26235 49	65663 49	for E_3
47367 49	52500 49	52500 49	47367 49	for E_4
20000 42	20078 50N	14484 50 •	1-2	
60000 42	20516 50N	18198 50 •	2-3	
	13146 50N	21416 50 •	3-4	
56608 49N	80000 42	60000 42 •	1-3	
81717 49N	11300 44	88000 43	2-4	
35000 43	17565 49N	56165 48 •	1-4	

15707 50			55000 50	
26546 51N	57578 50N	11201 51	21102 51	
26921 49N	65385 49	65385 49 •	26921 49	for E_1
53493 49	46242 49N	46242 49 •	53493 49	for E_2
65385 49N	26921 49N	26921 49	65385 49	for E_3
46242 49	53493 49	53493 49	46242 49	for E_4
21000 43	19898 50N	14687 50 •	1-2	
60000 42	20442 50N	18496 50 •	2-3	
55000 43	13564 50N	21021 50 •	3-4	
52583 49N	20000 42	11000 43	1-3	
73897 49N	21000 43	40000 42	2-4	
50000 42N	14719 49N	47402 48 •	1-4	

15707 50			60000 50	
28593 51N	63656 50N	11853 51	23105 51	
27512 49N	65138 49	65138 49 •	27512 49	for E_1
54294 49	45300 49N	45300 49 •	54294 49	for E_2
65138 49N	27512 49N	27512 49	65138 49	for E_3
45300 49	54294 49	54294 49	45300 49	for E_4
40000 42N	19737 50N	14864 50 •	1-2	
90000 42	20381 50N	18727 50 •	2-3	
25000 43	13909 50N	20692 50 •	3-4	
49075 49N	19000 43N	24000 43 •	1-3	
67351 49N	38000 43	27000 43	2-4	
12000 43	12498 49N	40469 48 •	1-4	

15707 50			65000 50	
30645 51N	69885 50N	12510 51	25123 51	
28028 49N	64918 49	64918 49 •	28028 49	for E_1
54951 49	44500 49N	44500 49 •	54951 49	for E_2
64918 49N	28028 49N	28028 49	64918 49	for E_3
44500 49	54951 49	54951 49	44500 49	for E_4
18000 43N	19592 50N	15019 50 •	1-2	
13000 43	20332 50N	18909 50 •	2-3	
62000 43	14197 50N	20414 50 •	3-4	
45993 49N	17000 43N	41000 43 •	1-3	
61812 49N	11000 43N	10000 43 •	2-4	
11000 43	10735 49N	34910 48 •	1-4	

15707 50			70000 50	
32703 51N	76231 50N	13173 51	27153 51	
28481 49N	64720 49	64720 49 •	28481 49	for E_1
55498 49	43816 49N	43816 49 •	55498 49	for E_2
64720 49N	28481 49N	28481 49	64720 49	for E_3
43816 49	55498 49	55498 49	43816 49	for E_4
15000 43N	19463 50N	15155 50 •	1-2	
12000 43	20291 50N	19055 50 •	2-3	
24000 43	14442 50N	20176 50 •	3-4	
43266 49N	26000 43N	44000 43 •	1-3	
57077 49N	20000 43	15000 43	2-4	
10000 43	93160 48N	30396 48 •	1-4	

15707 50			75000 50	
34764 51N	82668 50N	13839 51	29191 51	
28883 49N	64542 49	64542 49 •	28883 49	for E_1
55961 49	43223 49N	43223 49 •	55961 49	for E_2
64542 49N	28883 49N	28883 49	64542 49	for E_3
43223 49	55961 49	55961 49	43223 49	for E_4
32000 43	19346 50N	15277 50 •	1-2	
14000 43	20257 50N	19174 50 •	2-3	
12000 43	14652 50N	19971 50 •	3-4	
40838 49N	18000 43	29000 43	1-3	
52989 49N	56000 43	57000 43	2-4	
80000 42	81567 48N	26686 48 •	1-4	

θ in radians			H_{dc} in kilogauss	
E_1	E_2	E_3	E_4	E_j in Kmc/s
a_1	b_1	c_1	d_1	for E_1
a_2	b_2	c_2	d_2	for E_2
a_3	b_3	c_3	d_3	for E_3
a_4	b_4	c_4	d_4	for E_4
α	β	γ	1-2 transition	
α	β	γ	2-3 transition	
α	β	γ	3-4 transition	
α	β	γ	1-3 transition	
α	β	γ	2-4 transition	
α	β	γ	1-4 transition	

FIG. 23. Format of each sub-block in tables for ruby.

The energy eigenvectors $|j\rangle$ and the energy eigenvalues E_j are defined by the equations

$$\mathcal{H}|j\rangle = E_j|j\rangle$$

and

$$|j\rangle = a_j|\tfrac{3}{2}\rangle + b_j|\tfrac{1}{2}\rangle + c_j|-\tfrac{1}{2}\rangle + d_j|-\tfrac{3}{2}\rangle,$$

where $j=1$ through 4 denotes the four energy levels.

The axes were chosen so that the c axis of the ruby crystal is the z axis, the dc magnetic field H_{dc} lies in the x–z plane at an angle θ from the z axis, and the y axis is perpendicular to the plane containing H_{dc} and the c axis.

The transition probability matrix elements between levels k and l are given by

$$\frac{(\bar{H}')_{kl}}{\beta_M H_{rf}} = \frac{1}{\beta_M H_{rf}}\langle k|2\beta_M \mathbf{H}_{rf}\cdot\mathbf{S}|l\rangle$$
$$= (\alpha\phi_1 + \gamma\phi_3) + j\beta\phi_2,$$

where H_{rf} is the peak amplitude of the linearly-polarized rf magnetic field, and ϕ_1, ϕ_2, ϕ_3 are the direction cosines of $\mathbf{H}_{rf}$ with respect to the x, y, z axes.

For each operating point, characterized by H_{dc} and θ, the four energy eigenvalues E_j in units of kmc/s, the eigenvector expansion coefficients $a_j\cdots d_j$ for each level j, and the values of α, β, γ for each possible transition are presented in tabular form in Tables I(a) through I(i). The tables cover $\theta=10°$ to $\theta=90°$ in 10° increments, with H_{dc} increasing from 500 gauss to 7500 gauss by 500 gauss increments in each table.

The format of each subblock in the tables is shown in Fig. 23. The numbers in the blocks are expressed in

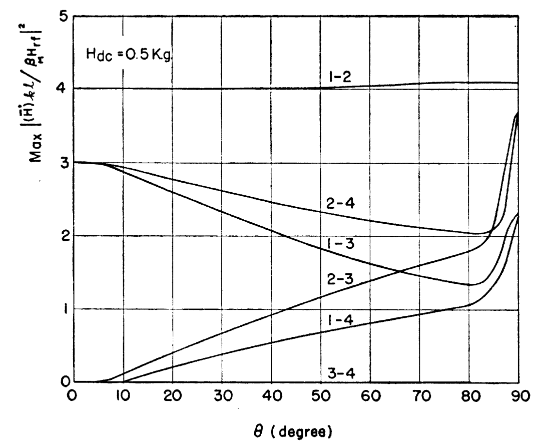

FIG. 24.

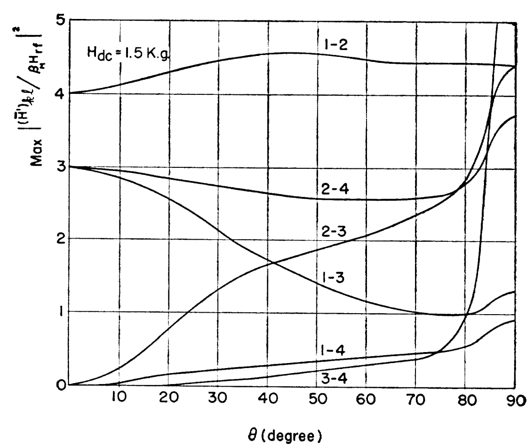

FIG. 25.

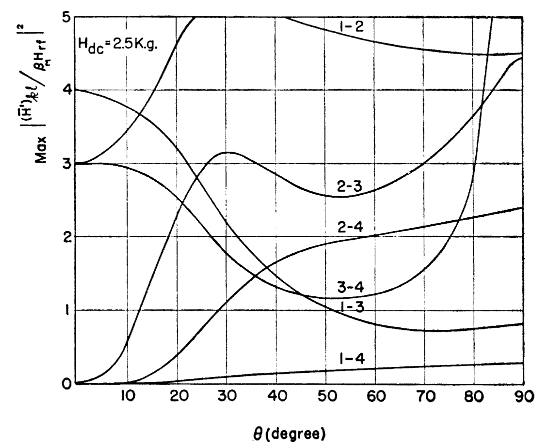

FIG. 26.

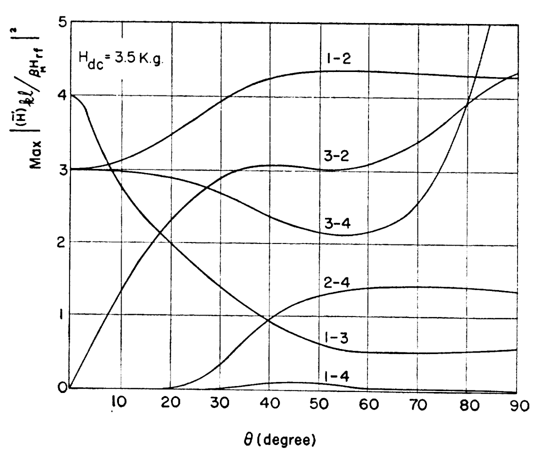

FIG. 27.

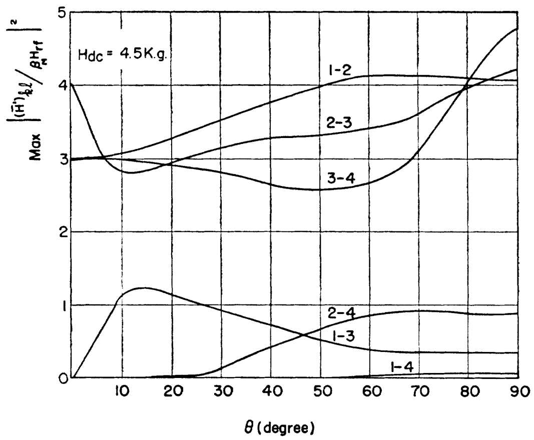

FIG. 28.

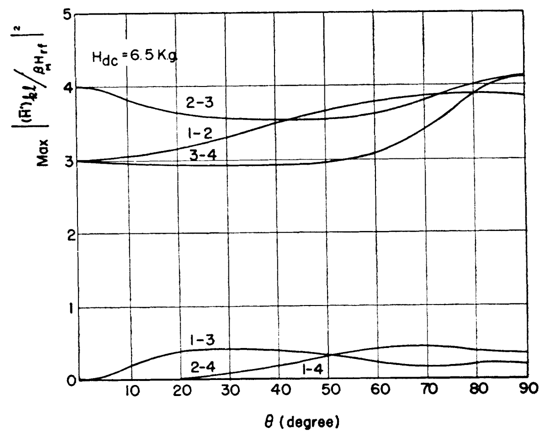

FIG. 30.

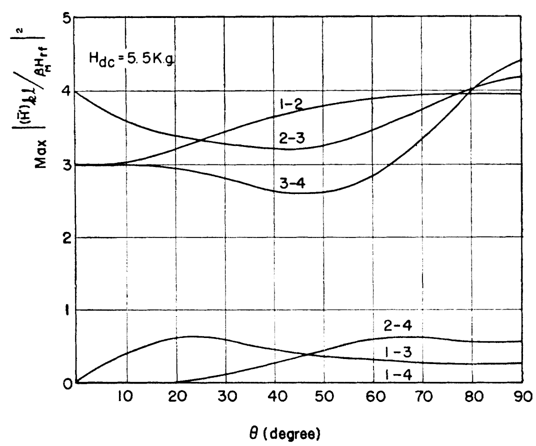

FIG. 29.

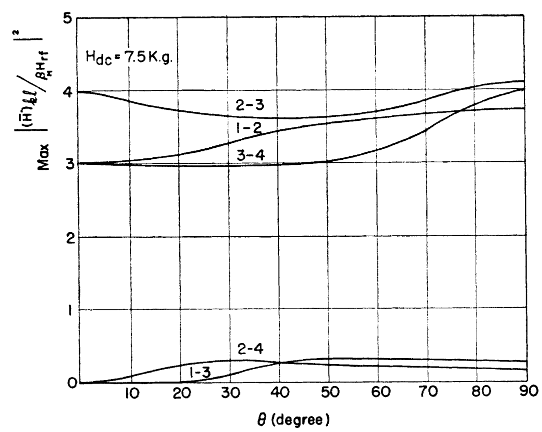

FIG. 31.

floating-point decimal form, since the blocks were prepared directly from the printed output of the computer to avoid transcription errors. The numbers can be converted to ordinary form by following the rule

$$abcde\ gh \equiv a \cdot bcde \times 10^{(gh\ 50)}.$$

The symbols *, *N*, or CR following a number indicate that the number is negative. For example, 56 423 51CR in the tables equals −56.423 in ordinary decimal form.

The maximum possible transition probability matrix element for optimum orientation of $\mathbf{H}_{rf}$ is either $\beta_M{}^2$ or $(\alpha^2+\gamma^2)$, whichever is larger. The maximum values for all the transitions have been plotted *vs* θ for several different values of H_{dc} in Figs. 24–31. The operation of the selection rules and the rapid variations in regions where levels curve strongly are very evident.

Since the calculations were performed, it has been found that D is negative, not positive as used in these calculations. This does not invalidate the calculations, but does require some simple modifications. To obtain correct answers for negative D, the signs of the E_j's should be reversed, and the signs of α and γ should be reversed. This of course does not change the squares of the matrix elements. When these corrections are made, the subscript J equals one in the tables and curves of this Appendix refers to the highest energy level, while J equals four refers to the lowest energy level. For example the subscript one dash two in the figures or tables refers to transitions between the two top-most levels of the ruby energy level spectrum.