Nuclear Sizes and the Weiszäcker Mass Formula*

ALEX E. S. GREEN

Los Alamos Scientific Laboratory, Los Alamos, New Mexico, and The Florida State University, Tallahassee, Florida†

I. EXPERIMENTAL DATA AND THEIR REPRESENTATION

THE painstaking measurement of atomic masses by experimentalists using mass spectrographic and nuclear techniques have brought forth an impressive array of precision data[1,2] whose interpretation continues to challenge the nuclear theorist. The strength of nuclear forces is immediately expressed in these data by the large deviations of observed atomic masses from the masses of the unassembled constituents. These deviations may be quantitatively expressed by the "nuclear energy" which for practical purposes (in view of smallness of atomic binding energies) may be defined by

$$E=M-Zm_h-Nm_n, \tag{1}$$

where M is the atomic mass of the neutral atom and m_h and m_n are the masses of the hydrogen atom and the neutron. The general trend of these nuclear energies for beta-stable nuclei is shown in Fig. 1. The smooth curve corresponds to an empirical formula defined later. The dots represent averages of experimental data for beta-stable nuclei in the neighborhoods of mass numbers 10, 20, $\cdots$ 250, i.e., the normal places. The most significant properties revealed in this figure are that nuclear energies build up in magnitude almost linearly with the number of particles, are large in magnitude, ranging up to 2000 mMU for very heavy nuclei and that the scatter of the "experimental" points are extremely small compared to the absolute values of the nuclear energies. It is convenient to define the quantity

$$\epsilon=E/A, \tag{2}$$

the nuclear energy per particle, represented by the curve labeled ϵ in Fig. 1. The extent to which ϵ is constant is an indication of the degree of saturation of nuclear energies. From the diagram it is clear that while the nucleus is approximately saturated, there are definite systematic departures from the constancy of the nuclear energy per particle. The two straight lines in Fig. 1 represent the general trends of the nuclear energies of the last neutrons and protons. The agreement and departure of the average of E_n and E_p from E are even better measures of the degree of saturation.

Two other ways of exhibiting nuclear mass data are

$$\Delta=M-A \tag{3}$$

and

$$f=\Delta/A=(M-A)/A. \tag{4}$$

The former has variously been called mass defect, mass excess, or mass decrement, the latter is called the packing fraction. Both quantities may conveniently be measured in millimass units (mMU). The experimental normal points and a smooth curve representing the empirical function[3]

$$\Delta_e=0.01(A-100)^2-64, \tag{5}$$

are shown in Fig. 2.

The corresponding empirical curve and normal points for the packing fraction are also shown. The fact that the mass decrement and packing fraction vary above and below the zero line indicates the appropriateness of the definition of the atomic unit as the mass per particle of O^{16}. If, as is frequently assumed in qualitative discussions, nuclear energies per particle all were equal to the value for O^{16} ($\approx$8.0 Mev) then mass decrements and packing fractions would lie along the horizontal line in Fig. 2. Departure from this horizontal line accordingly measures the stability relative to that of O^{16}.

To display the fine details and for making careful comparisons with theory it is helpful to present nuclear masses relative to a smooth reference function which

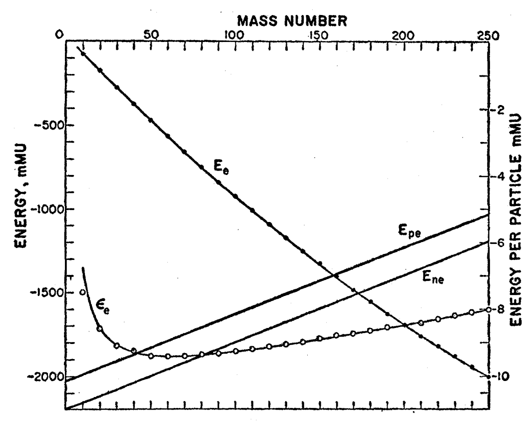

FIG. 1. Nuclear energies (E) nuclear energies per particle (ϵ) and nuclear energies of the last proton and neutron (E_p) and (E_n) for beta stable nuclei. The dots and circles represent averages of the experimental values in the neighborhoods of $A=10$, 20, $\cdots$ etc. The smooth curves are all based upon an empirical expression for mass decrements [see Eq. (5)].

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† On leave until September, 1958.

[1] A. H. Wapstra, Physica **21**, 367 (1956).

[2] J. R. Huizenger, Physica **21**, 410 (1956).

[3] A. Green, *Nuclear Physics* (McGraw-Hill Book Company, Inc., New York, 1955), Chaps. 8 and 9.

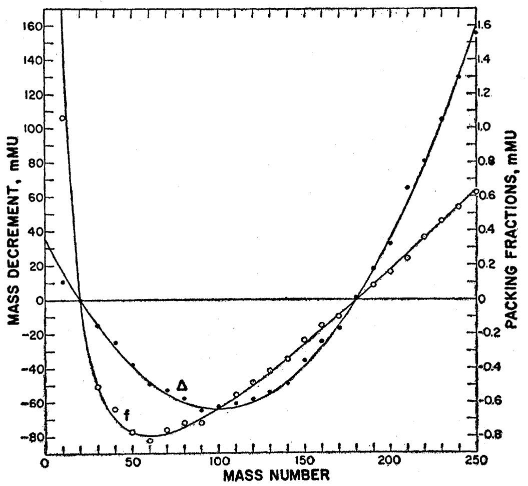

FIG. 2. Mass decrements (Δ) and packing fraction (f) for beta-stable nuclei. The dots and circles represent the experimental points, the smooth curves are based upon an empirical expression for mass decrements.

follows the general trends. The empirical function defined by Eq. (5) has proven helpful in these connections. We therefore define the mass residual by

$$R=M-M_e=\Delta_x-\Delta_e. \qquad (6)$$

The mass residuals of all beta-stable nuclei are presented in Fig. 3. Also shown are "the normal points" obtained from linear averages of residuals at equally spaced intervals centered about mass number 10, 20, $\cdots$ 250.

Perhaps the most conspicuous aspect of Fig. 3 is the short-range oscillations of the nuclear masses. These are associated with nuclear shell structure. From this figure it is apparent that it is possible to achieve local "fits" considerably better than the over-all fit attainable with a gradual curve. There is no obvious systematic separation between odd A and even A mass residuals and this is the basis for our use of all beta stable nuclide masses in arriving at the normal points.

The nuclear energy is related to the mass residual according to

$$E=R+\Delta^e-\tfrac{1}{2}(\Delta_n+\Delta_h)A-\tfrac{1}{2}(\Delta_n-\Delta_h)D, \qquad (7)$$

where $D=N-Z$ is the neutron excess. The curves in Fig. 1 are based upon Eqs. (5) and (7) and the empirical formula for the line of beta stability,[3]

$$D_m{}^e=40A^2/(A+200). \qquad (8)$$

Equation (8) is also useful for magnification purposes.

II. SEMIEMPIRICAL MASS EQUATIONS

Since the birth of modern nuclear physics in 1932, theorists have attempted to relate the nuclear energies and other properties of complex nuclei to the interactions of two nucleon systems. Certain difficulties were encountered and even today we are still struggling with them. Among the earliest efforts, the work of Weiszäcker[4] brought forth a number of concepts which still prevail. He divided the total nuclear energy into components called the volume energy, the surface energy, the Coulomb energy and the symmetry energy and allowed certain coefficients, which should be determinable from the two-body interaction, instead to be empirically adjusted in the light of stability and mass data of complex nuclei. In his original work and in subsequent theoretical efforts[5,6] the expression for total energy contained many small terms whose dependence upon the nuclear numbers A, Z, N, or D is quite complicated and rather untractable. Since the complications were unwarranted then, Bethe[5] chose a somewhat simplified form of the mass equation

$$E=-a_1A+a_2A^{\frac{2}{3}}+a_3Z^2/A^{\frac{1}{3}}+a_4D^2/4A. \qquad (9)$$

The coefficient of A in the first term, the volume term, is usually interpreted as the energy per particle in infinite nuclear matter at normal density. The second term is assumed to represent the extra energy of a finite nucleus by virtue of the unsaturated bonding of the surface particles. The third term is the added energy occasioned by Coulomb repulsion between protons. The fourth term represents an additional energy of rather

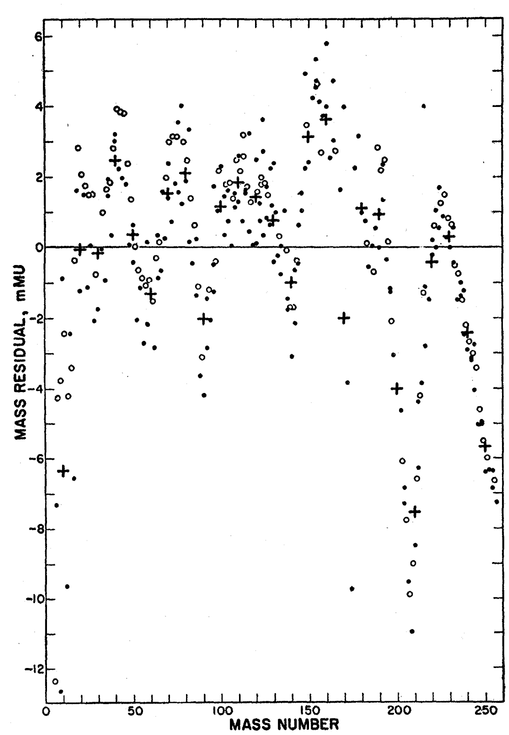

FIG. 3. Experimental mass residuals of beta stable nuclei. The circles correspond to even A nuclei, the dots to odd A nuclei. The crosses represent the "normal points."

[4] C. F. von Weiszäcker, Z. Physik **96**, 431 (1935).
[5] H. A. Bethe and R. F. Bacher, Revs. Modern Phys. **8**, 165 (1936).
[6] E. Feenberg, Revs. Modern Phys. **19**, 239 (1947).

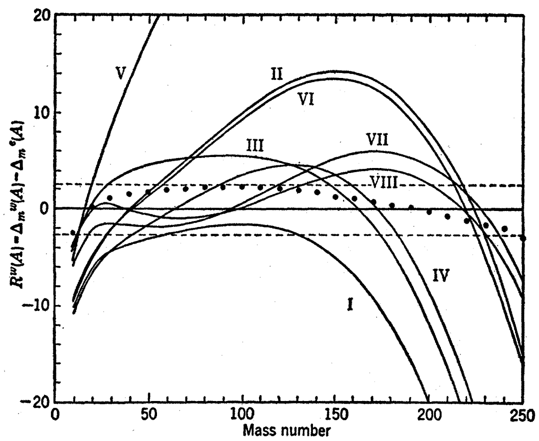

FIG. 4. Survey made in 1952[7] of fits to mass residuals attained using the semiempirical equation with various sets of constants (see reference 7). The dots represent a set of smoothed normal points inferred from the experimental data. Curve VIII and Curve VII represent the equations of Green and Engler, and Fowler.

complex origin required to express the tendency of nuclear matter toward equality of neutron and proton numbers.

Uncertainties as to the origin of nuclear forces makes the interpretation of a_1, a_2, and a_4 particularly obscure, but a simple classical interpretation is available for a_3. If one assumes uniformly distributed nuclear charge over a radius

$$R_c = r_c A^{\frac{1}{3}}, \tag{10}$$

then it follows from classical electrostatics that

$$a_3 = 3e^2/5r_c = U_c/r_c, \tag{11}$$

where $U_c = 0.8639$ Mev $= 0.9278$ mMU with r_c in units of 10^{-13} cm (all lengths quoted in this paper are in this unit).

Many attempts have been made to adjust the semiempirical equation to nuclear mass and stability information. Most of these attempts are quite successful in the light of the gross aspects of the data. When scrutinized more carefully using a presentation of nuclear data such as given in Fig. 2, and particularly the representation of mass residuals given in Fig. 3, it becomes apparent that there are considerable differences in the fits associated with the different sets of constants. Figure 4 shows a set of comparisons made in 1952.[7]

Here a smoothed set of normal points was used. The over-all fit of the curve labeled VIII (Green-Engler) is best, with curve VII (Fowler[8]) next best, but the over-all fit of the other curves is not good. All of the other adjustment procedures utilized the Coulomb energy constant from, or close to that obtained from mirror nuclei energy difference data ($\sim$0.63 mMU, $\sim$0.59 Mev) whereas Fowler and Green-Engler arrived at their much larger Coulomb constants ($\sim$0.75 mMU, $\sim$0.70 Mev) by procedures utilizing the atomic masses more directly.

The Green-Engler study, when interpreted[9,10] in the light of Eqs. (10) and (11), were indicative of a much smaller radius constant than had then been generally accepted. The coincidence of this result, together with the smaller radius determination made by μ-mesonic x-ray measurements[11] and electron scattering studies[12,13] played a role in the general acceptance of the nuclear collapse of 1953. A redetermination of the Coulomb radius constant from nuclear masses was made[14] using an objective criterion for best fit. Figure 5 illustrates the results. Essentially this established that within the framework of the Bethe-Weiszäcker equation, the fit attainable is quite sensitive to the Coulomb radius constant. Indeed the determination of r_c obtained was 1.216 with a calculated probable error of 1%. This value agreed quite precisely with the equivalent uniform

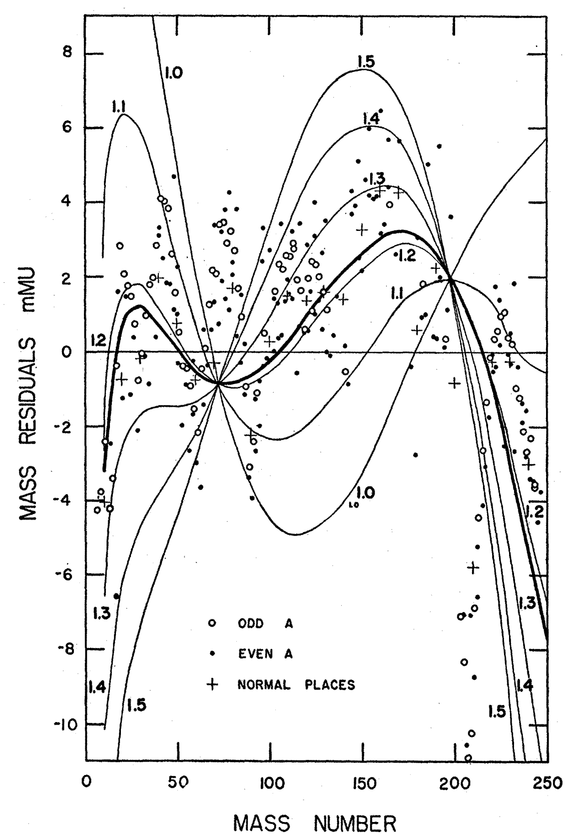

FIG. 5. Curves for the Bethe-Weisäcker equation determined by least square fitting to the 1954 normal points for radius constants 1.0, 1.1, 1.2, 1.3, 1.4, and 1.5. The dark curve represents the best fit; it corresponds to $r_c = 1.216$.

[7] A. Green and N. Engler, Phys. Rev. **91**, 40 (1953).
[8] W. Fowler (unpublished) quoted in W. E. Siri *Isotope Tracers and Nuclear Radiations* (McGraw-Hill Book Company, Inc., New York, 1949).
[9] A. Green, Gordon Conference on Nuclear Chemistry, June, 1953 (unpublished).
[10] F. Bitter and H. Feshbach, Phys. Rev. **92**, 837 (1953).
[11] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1953).
[12] Hofstadter, Fechter, and McIntyre, Phys. Rev. **92**, 978 (1953).
[13] L. M. Copper and E. M. Henley, Phys. Rev. **96**, 436 (1956).
[14] A. Green, Phys. Rev. **95**, 4, 1006 (1954).

radius defined by

$$R^2=5\langle r^2\rangle/3, \qquad (12)$$

obtained from μ-mesonic x-ray and electron scattering studies. If one fits the recently reported equivalent uniform radii for seven selected spherical nuclei by least squares to a function of the form given by Eq. (10) one obtains precisely $r_0=1.216$. A somewhat better fit to the Stanford equivalent uniform radii is given by

$$R=1.031A^{\frac{1}{3}}+0.897\approx 1.216A^{\frac{1}{3}}. \qquad (13)$$

Although the R defined by Eq. (12) furnishes a good basis for comparing almost uniform distributions, it actually measures a moment of the charge distribution which does not strictly correspond to the Coulomb energy radius. The most direct comparison of a Coulomb radius determination from nuclear masses may be made using the estimates of Coulomb energies obtained from the Stanford charge distribution measurements. Using the experimentally determined density function, the energies

$$E=\frac{1}{2}\int_0^\infty \rho(r)V(r)d\tau \qquad (14)$$

have been evaluated.[15] Equating these energies to $3Z^2e^2/5R_c$ one obtains a set of radii which are close but not quite equal to those from Eq. (12). These radii when fitted by least squares yield

$$R_c=1.052A^{\frac{1}{3}}+0.753\approx 1.204A^{\frac{1}{3}}, \qquad (15)$$

where the second expression is the least square fit to the restricted form.

The variation of the two term radius expression from the one term form helps to reconcile the larger Coulomb radius constant inferred from mirror nuclei mass differences with the smaller constant inferred from absolute masses. The two term form leads one to expect radii of the order of $1.30A^{\frac{1}{3}}$ for nuclei near mass number 25. This is still somewhat smaller than the radii inferred from mirror nuclei differences ($1.47A^{\frac{1}{3}}$) using the semi-empirical formula. The remainder of the discrepancy is largely accounted for when one considers the fact that the particle involved in mirror nuclei transitions tends to be in the outer regions. Detailed quantum-mechanical calculation based upon the shell model[13,16–19] bring out this aspect quite clearly. The Coulomb exchange energy, which also is relatively more important in mirror nuclei also contributes somewhat. Taking all of these considerations into account, the mirror nuclei proton radii are in accord with the other proton radii determinations.

The radius to the half fall-off point of the proton distribution conforms to

$$R_{\frac{1}{2}}=1.138A^{\frac{1}{3}}-0.280\approx 1.080A^{\frac{1}{3}}, \qquad (16)$$

which is significantlv smaller than either R_c or R.

III. INFLUENCE OF PERTURBATIONS UPON THE MASS FORMULA

While the precision of the mass formula method of determination of the Coulomb radius is good, in the absence of a rigorous basis for the complete mass formula, one might legitimately question the significance of the determination. It is difficult to put aside these concerns. From the theoretical viewpoint the number of corrective terms which have been proposed to the Bethe-Weizsäcker formula are almost innumerable. From the experimental viewpoint examination of Figs. 3 and 5 shows that the principal source of confusion is due to the fluctuations in nuclear masses related to shell structure. A number of efforts have been made to characterize these fluctuations. Wapstra[20] has suggested terms of the form

$$E_{sh}=A_i/(x_i^2+1), \qquad (18)$$

where

$$x_i=(Z-Z_i)/w_i, \qquad (19)$$

and A_i, Z_i, and w_i are empirical constants. This carries the mass surface smoothly through the closed shells regions, which is helpful in several applications. Green and Edwards,[21] proposed a discontinuous parabolic correction of the type

$$E_{sh}=-\alpha_i(N-N_i)^2, \qquad (20)$$

where α_i, N_i, are fixed constants within shell zones. These roughly conform to the prescriptions

$$\alpha_i=1/(N_u-N_l) \text{ in mMU} \quad\text{and}\quad N_i=\tfrac{1}{2}(N_u+N_l), \qquad (21)$$

where N_u and N_l represent the upper and lower magic numbers. Similar expressions apply to corrections for proton shells. One additional constant must be applied in each zone of the mass surface bounded by magic N and Z numbers. Its value depends upon the smooth mass surface used. While Green and Edwards did not pursue the parabolic shell correction to the limit of its capabilities, they did show that even using the rough prescriptions embodied in Eqs. (20) and (21) a very considerable improvement in the fit is accomplished. A recent study of Cameron[22] tends to confirm the separation of neutron and proton shell energies and the general nature of the above type of shell correction although Cameron presents empirical constants for every N and Z rather than a shell function.

The purely empirical equations for nuclear masses of Baker and Baker[23] and of Levy[24] might be viewed some-

[15] Hahn, Ravenhall, and Hofstadter, Phys. Rev. **101**, 1131 (1956).
[16] B. G. Jancovici, Phys. Rev. **95**, 389 (1954).
[17] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1956).
[18] O. Kofoed-Hansen, Nuclear Phys. **2**, 441 (1956).
[19] P. C. Sood and A. Green, Nuclear Phys. **5**, 274 (1958).
[20] A. H. Wapstra, Physica **18**, 2 (1952).
[21] A. Green and D. Edwards, Phys. Rev. **91**, 46 (1953).
[22] A. G. W. Cameron, Can. J. Phys. **35**, 1021 (1957).
[23] G. A. Baker, Jr., and G. A. Baker, Sr., Can. J. Phys. **34**, 423 (1956).
[24] H. B. Levy, Phys. Rev. **106**, 1265 (1957).

what in the context of local corrections to a Weiszäcker-type equation since the subtraction of a smooth major term should not necessitate a change in the form of their local functions although their parameters, of course, would be changed.

The most remarkable achievements in the way of local fitting of nuclear masses are those of Talmi, Thieberger,[25] and de-Shalit[26] based upon formulas inferred from the shell model. While these have not been carried out in the context of shell corrections to a smooth mass surface, the transformation of their results to such a view point should not be difficult. For heavy nuclei Thieberger and de-Shalit show that the total nuclear energies of a series of isotopes relative to one with a magic or submagic Z are accurately represented by the parabolic formula,

$$E=na_0+\tfrac{1}{2}n(n-1)a_1+y(n)a_2, \qquad (22)$$

where a_0, a_1, and a_2 are energy parameters, n represents the number of neutrons in the subshell, $y(n)=0$ for n even and 1 for n odd. A similar expression holds for a series of isotopes. For these series their formulas are equivalent to Green and Edwards. They determine by least squares the energy parameters which lead to fits with standard deviations which frequently lie within the experimental error.

One assumption underlying the derivation of Eq. (22) is that the single particle wave function entering the complete wave function is independent of the number of nucleons in the shell. It is quite likely that the formula which they propose has greater generality than implied by this restrictive assumption. One expects that by adding particles the radius of the nucleus expands. The principal energetic effect of this expansion is to increase the single nucleon energy (a_0). Studies with realistic potentials[27] indicate that in a small region this rate of increase is approximately linear with A (and hence n). Therefore in the empirical fitting of the local energy parameters, the correction term associated with the expansion would naturally be absorbed in the second term on the right of Eq. (22).

Another puzzling point in this recent study is that the experimentally inferred energy parameters do not change appreciably in going from one subshell to another, whereas the contrary might be expected from the shell model. The fact that subshell effects are difficult to detect in nuclear masses and nuclear decay energies had been noted earlier in a number of studies.[21,28] This absence of subshell effects suggests the importance of configuration mixing in ground-state masses and is probably closely related to spheroidal deformation of nuclei.

The pairing correction is perhaps the next most important perturbation. Since it has so short a "wavelength" it can be easily discounted in the study of statistical models as long as one is comparing the mass surfaces for comparable nuclear types. Following tradition we have been fitting the odd A mass surface. The fact that the experimental data for even A which are almost all of the EE type nuclei, agree on the average with the odd A beta-stable nuclei is due to the average compensation of the negative pairing correction with the positive parabolic correction.[29] The separation of the EE, odd A, and OO surfaces are fairly well represented by the function[21]

$$H(A)=12A^{-\frac{1}{2}}\,\mathrm{mMU}, \qquad (23)$$

which does somewhat better than the functions $36A^{-\frac{3}{4}}$ or $140A^{-1}$ which appeared earlier in the literature.[29,30] The shell model leads one to expect subshell effects in pairing energies and also differences in proton and neutron pairing energies. The former are not readily apparent[21,26] although the latter have been noted[31,21] particularly near closed major shell nuclei.

The fluctuating character of the shell and pairing corrections make it reasonable to expect that they would not seriously influence the determination of the Weiszäcker parameters. On the other hand, the influence of a slowly varying perturbation would be a matter for more serious concern since it might be expected to cause a systematic change in these parameters. An extension of the least squares procedure may be applied to the analysis of smooth perturbations of the mass surface. This method uses the close agreement attained to provide a surface of departure for studying perturbations which are smooth functions of the integral nuclear parameters. One assumes that any small perturbation may be placed in the form

$$\delta E=\delta_0(A)+\delta_1(A)\varphi+\delta_2(A)\varphi^2/2, \qquad (24)$$

where $\varphi=D-D_m{}^0$, δ_0, δ_1, and δ_2 are functions of A and $D_m{}^0$ is the unperturbed line of beta stability. If a perturbation is turned on the parameters $a_1, a_2, a_3,$ and a_4 will change in order to preserve the best fit of mass data. The net shift in the line of beta stability $\theta_s=D_m-D_m{}^0$ is given approximately by

$$\theta_s=\frac{\delta\rho 2A^{\frac{2}{3}}Z_m{}^0-2\delta_1 A a_4{}^{0-1}}{1+\rho^0A^{\frac{2}{3}}-2A\delta_2 a_4{}^{0-1}}, \qquad (25)$$

where $\rho=a_3/a_4$. The shift in the masses along the line of beta stability $R_s=\Delta_m-\Delta_m{}^0$ is given by

$$R_s=\delta_0+\delta_1Z_m{}^0+[(\Delta_n-\Delta_h)/4+a_4{}^0/4+\delta_1/2+\delta_2Z_m{}^0]\theta_s -\delta a_1A+\delta a_2A^{\frac{2}{3}}+\delta a_4(D_m{}^0+\theta_s)/4. \qquad (26)$$

In principle one might readjust the Weiszäcker constants for any perturbations to the Bethe-Weiszäcker

[25] I. Talmi and R. Thieberger, Phys. Rev. **103**, 718 (1956).
[26] R. Thieberger and A. de-Shalit, Phys. Rev. **108**, 378 (1957).
[27] A. Green, Phys. Rev. **102**, 1325 (1956).
[28] K. Way and M. Wood, Phys. Rev. **94**, 119 (1954).
[29] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).
[30] J. W. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).
[31] C. D. Coryell, Ann. Rev. Nuclear Sci. **2**, 305 (1953).

TABLE I. Changes in mass surface parameters (in mMU) induced by the addition of various perturbations. The case E_d is for $\gamma_d=-0.68$. The case E_{ss} is for $\gamma_{ss}=-25$ and a term $5D^2/A$ has been added to minimize the net perturbation.

		Δa_1	Δa_2	Δa_3	Δa_4	$\Delta r_c(\%)$
I	E_{cs}	−0.108	−0.364	0.0028	−2.47	−0.37
II	E_{cx}	0.062	0.133	0.0184	−1.91	−2.45
III	E_{cc}	0.123	0.138	0.0343	7.20	−4.57
IV	E_d	−0.234	−0.354	0.0452	−0.66	−6.03
V	E_{ss}	0.519	0.959	0.0443	5.44	−5.91
VI	E_{sh}	−0.038	−0.190	−0.0004	−0.06	−0.06

formula in the following way. Using Eq. (25) the $\delta\rho$ needed may be evaluated by least squares or more conveniently by requiring that θ_s vanish at an appropriately chosen mass number. This establishes a relation between δa_3 and δa_4, and the function θ_s. Using Eq. (26) the values of δa_1, δa_2, and δa_4 can be determined by least squares. The efficacy of the perturbation is then determined by how well the perturbed line of least masses and the perturbed line of beta stability fit the experimental trends of the mass residuals. In doing this one is confronted with the fact the nuclear masses predicted from the Bethe-Weizsäcker equation are in such good systematic agreement with experiment that it is difficult, in view of the scattering caused by shell effects, to infer any smooth correction which significantly improves the agreement. On the other hand, a small but probably significant systematic discrepancy appears to exist between the experimental line of beta stability and the predicted one (Fig. 3 and Fig. 4 of reference 7 or Figs. 8-8 and 9-2 of reference 3). In examining the influence of perturbations to the mass surface it therefore appears wise to look first at the changes in the line of beta stability. A number of perturbations have been so examined. These include the self-energy perturbation,

$$\delta E_{\mathrm{I}}=-a_3{}^0ZA^{-\frac{1}{3}}, \tag{27}$$

the Coulomb exchange perturbation,

$$\delta E_{\mathrm{II}}=-0.764a_3{}^0Z^{\frac{4}{3}}A^{-\frac{1}{3}}, \tag{28}$$

and the Coulomb radius compression correction,

$$\delta E_{\mathrm{III}}=a_3{}^0Z^2A^{-\frac{1}{3}}[(R^0/R_h)-1], \tag{29}$$

where R_h is given by a linear relationship as in Eq. (15) and R^0 does not have the constant, the direct symmetry term

$$\delta E_{\mathrm{IV}}=\gamma_d D, \tag{30}$$

and the surface symmetry energy of the form

$$\delta E_{\mathrm{V}}=\gamma_{ss}D^2/A^{\frac{4}{3}}. \tag{31}$$

Examination of these last two perturbations has been inspired by recent studies of Szamosi and Ziegler[32] and of Cameron.[22] A term linear in D arises from the Fermi gas model if the neutron distribution radius differs from the proton distribution radius. A similar term also arises in the Wigner model[3,30] and in the shell model equation of Talmi.[25] A surface symmetry energy is expected from refinements of the kinetic energy expression in the Fermi gas model.[33]

In treating a perturbation it is helpful to minimize its magnitude by subtracting away a constant times one of the terms in the Bethe-Weizsäcker equation which is close in functional form to the perturbation (e.g., $\gamma_{ss}D^2/5A$ for case V). After the analysis is completed, this subtracted term may be reintroduced by embodying it into the adjusted mass parameter. Thus relatively large changes in the mass parameters may be explored without exceeding the limits of the expansions used.

Changes in the mass parameters by these perturbations are listed in Table I. Because of the large shell structure "noise" the fits to the mass data when measured by standard deviations were not changed appreciably ($\sim\pm0.3$ out of 2.7). On the other hand, the fits to the line of beta stability occasioned by various perturbations seem to vary significantly. These effects are illustrated in Fig. 6 where the shifts corresponding to the five perturbations are shown along with a deviations curve which represents the trends of the departures of the experimental points from the Green-Engler line of beta stability.[7] The parameter change $\delta\rho$ was chosen to insure agreement of the line of beta stability with the Green-Engler line at $A=200$. If a least square adjustment were used instead the principal additional effect would be a slight rotation of these lines to minimize the standard deviations relative to the experimental line. It is apparent from Fig. 6 that all of the shifts lead to a deterioration of the fit. The size and directions of

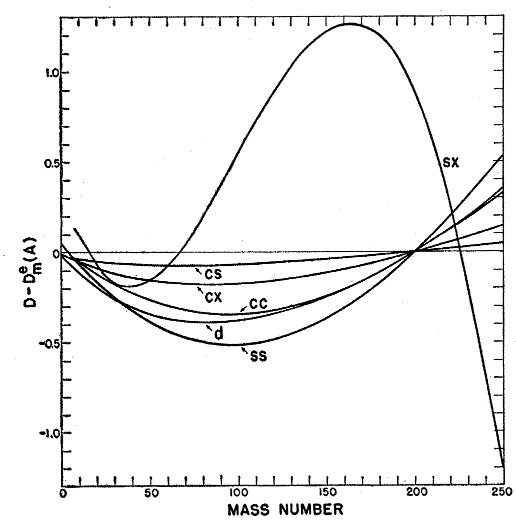

FIG. 6. Shifts relative to the G. E. line of beta stability induced by the perturbations given by Eqs. (27)–(31). The curve labeled d is for a direct symmetry coefficient $\gamma_d=0.68$. The curve labeled ss is for a surface symmetry with the coefficient $\gamma_{ss}=25$. The curve labeled sx corresponds to the trends of the experimental data.

[32] G. Szamosi and M. A. Ziegler, Acta Phys. Acad. Sci. Hung. 6, 346 (1956).

[33] D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953).

Coulomb self-energy, Coulomb exchange, and Coulomb radius compression are already determined. The sum of these three perturbations if added would considerably exaggerate the discrepancy The surface symmetry energy and the linear term in the neutron excess can be inverted by choosing their coefficients negative. Therefore a perturbation term of either of these types or another similar one is needed to improve the line of beta stability or to restore it after various Coulomb perturbations are added to the simple Bethe-Weiszäcker equation.

The recent work of Cameron[22] bears out the above conclusion relative to the influence of perturbations on the Bethe-Weiszäcker mass surface. He chooses a trapezoidal proton density distribution of constant surface thickness and with a radius constant to the half fall-off point given by

$$R_{\frac{1}{2}}=1.112A^{\frac{1}{3}}[1-0.62025A^{-\frac{2}{3}}]. \qquad (33)$$

On this model the Coulomb energy is computed using classical techniques but with corrections for the self-energy and Coulomb exchange terms. In addition to the usual volume and symmetry and surface energies with their free parameters, Cameron includes a surface symmetry energy term with a fourth free parameter. The two symmetry parameters are then adjusted by least squares to the line of beta stability. The free surface symmetry energy parameter assumes a rather large negative value with the volume symmetry energy going to a positive value which is considerably larger than the usual values. The combination $a_4-4\gamma/\bar{A}^{\frac{1}{3}}$ with the average value of $\bar{A}^{\frac{1}{3}}=5$ is however quite close to the earlier value.

The fit of Cameron's line of beta stability with the experimental normal points and a smooth curve based upon the experimental normal points is shown in Fig. 7. The curve labeled *F-G* represents the line of beta stability associated with Fermi's constants[7] and the

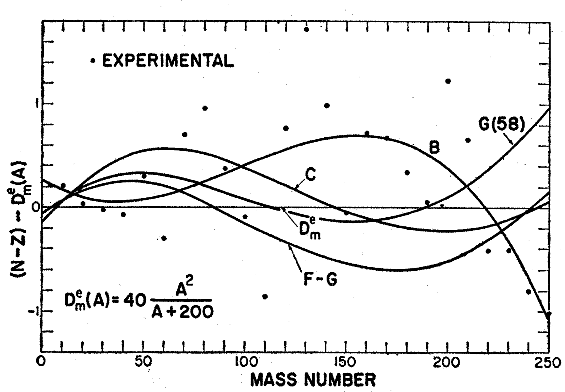

FIG. 7. Lines of beta stability. *F-G* corresponds to the line of beta stability associated with Fermi's constants and with earlier sets of constants used by Green. The curve labeled *C* represents Cameron's line. The curve *G*(58) represents a least square fit using the Bethe-Weiszäcker equation. The curve labeled *B* represents a "best" smooth fit to the experimental data. The dots represent the "normal place."

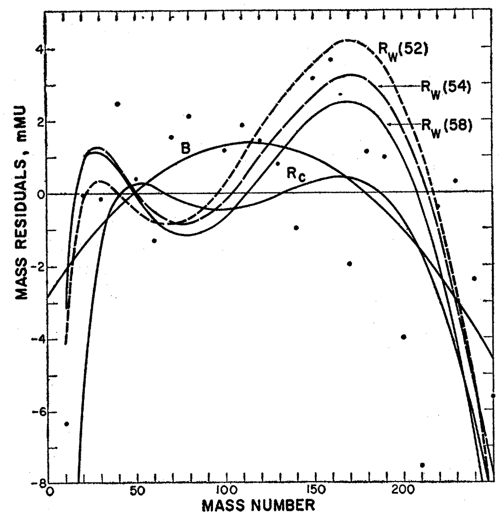

FIG. 8. Mass residuals *vs* mass number. $R_w(52)$, $R_w(54)$, and $R_w(58)$ correspond to the Bethe-Weiszäcker equation based on constants obtained in 52, 54, and in 58. R_c represents Cameron's equation. *B* represents the "best" smooth fit after corrections for the residual parabolic and pairing effects are applied.

various constants given previously by Green. That labeled *G*(58) represents a newly determined least square fit with the simple Bethe-Weiszäcker form. Line *C* makes somewhat smaller excursions from the experiment than line *F-G* but is not substantially different from *G*(58).

The comparative fit of the masses to the normal points are shown in Fig. 8. Since Cameron fitted his formula to odd-odd masses we have lowered the masses computed from his equation by $12/A^{\frac{1}{2}}$. Also shown in Fig. 8 are the curves of Green and Engler, the best fit based upon data available in 1954 and the best fit based upon smoothed normal places inferred from the recent Wapstra-Huizenger compilations. The standard deviations of the last three are $\sigma(C)=3.0$, $\sigma(54)=2.9$, $\sigma(58)=2.7$. These are somewhat larger than in our previous study, due to the extra point at $A=250$ and one or two normal points near closed shells which more recent experimental data have scattered somewhat. The fit to the masses obtained by Cameron leads to no overall improvement and indeed somewhat of a deterioration of fit particularly for light nuclei. This difficulty at the low mass numbers is also apparent from the asymmetrical nature of shell correction which Cameron introduces for each *Z* and *N* at low mass numbers. One might reasonably conclude that the three corrections, Coulomb self-energy, Coulomb exchange, and Coulomb radius compression, have overshot the requirements of experimental evidence in the light mass region. The fact that the large negative surface symmetry energy failed to accomplish a substantial improvement to the line of beta stability can be attributed to the fact that it was opposing the tendencies of the three Coulomb corrections.

It is quite likely that the difficulty at light nuclei associated with Cameron's formula is due to the overestimate of the Coulomb exchange which was calculated using free particle wave functions despite the fact that the requirement[5] $kR \gg 1$, does not prevail in light nuclei. This explanation is confirmed by the fact that calculations of the Coulomb exchange correction factor in mirror nuclei mass differences using the statistical formula leads to results about twice as large as those obtained with detailed calculations using Slater determinants and harmonic oscillator wave functions.[19] Despite this difficulty I believe that Cameron's study has been fruitful, particularly in bringing out the need for a negative surface symmetry term. Probably a more realistic calculation of the Coulomb exchange energy would lead to a smaller term in which case the readjusted mass surface would probably fit the experimental mass surface somewhat better than the Bethe-Weizsäcker surface. On the other hand, before finalizing upon particular perturbations it would appear wisest to see whether other perturbations should also be included. For example, a negative term proportional to D might also be helpful to the fit of the line of beta stability and may be of physically significance since such a term would arise if the neutron distribution is larger than the proton distribution. Whereas Szamosi and Ziegler derive a term with a negative energy coefficient, they arrive at a positive energy coefficient, a puzzling fact probably related to their manner of using mirror nuclei mass differences.

None of the perturbations investigated thus far take us back to the larger Coulomb radius constants characteristic of earlier mass surface determinations. An examination of the parameter shifts listed in Table I indicate that each of the perturbations studied here tends to decrease further the radius constant by a small amount. To test the sensitivity of these radius determinations to shell fluctuations we carried out a calculation in which the normal places for the masses were altered so that they would be close to the bottoms of the shells rather than the local centers of gravities of the masses. These shifts are listed in Table I. The Coulomb energy and symmetry energy parameters undergo negligible changes ($\sim 0.1\%$), but the surface and volume energy coefficients change slightly. This suggests that the Coulomb radius constant determination when carried out by a procedure involving fitting both the neutron excesses and the mass data along the line of beta stability is stable near $r_e = 1.20$. This stability may not persist in the face of additonal perturbation studies. There are indeed a large number of additional perturbations which might be investigated (e.g., compressibility terms, "length" energies[33] and various quantum-mechanical corrections) so that the present study is largely illustrative. Despite this my opinion is that because of recent progress in understanding nuclear structure, considerable value would be derived from the study of statistical perturbations to the Bethe-Weizsäcker formula, particularly if these are carried out with a close interplay of phenomenological and theoretical considerations.

One of the major obstacles which hampers the systematic study of perturbations is the large scatter associated mainly with shell structure. By using equally weighted normal points which fit local centers of gravity these fluctuations have been smoothed somewhat and the strong sampling bias of the actual mass data minimized. However, the fact that the shell structure "noise" dominates the standard deviations suggests that an initial data smoothing procedure would be of considerable value towards making standard deviations serve as sharp criteria of fit. While the best smoothing procedure would be one involving the use of accurate shell corrections, pending this the following averaging procedure has considerable merit.[34] Polynomials of various orders were fitted by least squares to the residuals normal points. For the beta-stability residuals the standard deviations obtained were $\sigma_0 = 0.669$, $\sigma_1 = 0.660$, $\sigma_2 = 0.559$, $\sigma_3 = 0.514$, $\sigma_4 = 0.513$, $\sigma_5 = 0.513$, $\sigma_6 = 0.509$, where the subscript denotes the order. For the mass residuals the standard deviations obtained were $\sigma_0 = 2.91$, $\sigma_1 = 2.83$, $\sigma_2 = 2.35$, $\sigma_3 = 2.32$, $\sigma_4 = 2.30$, $\sigma_5 = 2.22$, and $\sigma_6 = 2.01$. In the former case a third-order polynomial obviously provides the simplest good smooth representation, and has significance (using the F test) at the 95% confidence level. In the latter case a second-order polynomial suffices, and has significance at the 99% confidence level. One might in the latter case go to the sixth order but then the "wavelengths" introduced are comparable to what we wish to smooth out. The smooth curves arrived at by these least square procedures lie quite close to those intuitively inferred by Green and Engler.[7] In using smoothed residuals one should correct for the fact that actual beta stable nuclei do not lie precisely on the line of beta stability but fall at random between the stability limits. This randomness contributes considerably to the local scatter of points although on the average it should have a relatively small systematic effect upon the normal points. On the average the function

$$R_{pp} = (H-J)/3 - H^2/24J, \tag{34}$$

where J and H are parabolic and pairing functions,[3] approximately compensates for these differences. In Table II smoothed experimental normal points arrived at by the above procedures are presented. Experience with hand calculations suggests that this number of points is adequate for most investigations. Also given in Table II are corresponding values from the Bethe-Weizsäcker equation using a set of parameters adjusted to the smoothed data.

[34] Averaging procedures to eliminate shell effects without making detailed calculations of these effects have been used in statistical theories of nuclear reactions—C. F. Porter (private conversation).

IV. WEISZÄCKER FORMULA AND NUCLEAR POTENTIALS

Since our understanding of nuclear structure has increased considerably since the work of Weiszäcker and Bethe, it is appropriate to re-examine the derivation of the Weiszäcker formula, particularly as this bears upon determination of nuclear sizes and density distributions. Most derivations of the Weiszäcker equation have made use of Fermi-Thomas statistical approximations. Recent work of Brueckner,[35] Bethe,[36] and others relating infinite nuclear systems to the properties of two-body forces and the phemenological studies of Wilets[37] have also largely been within this framework.

The revival of the independent particle model (I.P.M.) of the nucleus constitutes one of the greatest successes of the past decade. The initial successes[38–40] (1948–1952) in the use of the shell model as a schematic guide to the understanding of nuclear moments and nuclear magic numbers were followed by successes in applying realistic potential wells to detailed interpretation of proton and neutron scattering data[41–43] and proton and neutron separation energies[27,44–48] (1953–1957). In view of these successes, the problem of explaining the origin of these potentials has supplanted the problem of inferring potentials from experiment as the main focus of current interest.

In probing the characteristics of nucleon-nucleus potentials it has been helpful to use probe particles with a variety of energies. The negative energy states are particularly useful since the combination $(E-V)$ in Schroedinger's equation, which determines the details of the nuclear wave function, undergoes a greater variation for negative E. While nuclear physicists can use negative energies corresponding to the last states of binding and low lying excited states immediately above them, no direct experimental method seems available to probe the inner energy states (the ultraviolet and x-ray levels of atomic spectra). Nuclear masses furnish an indirect way of doing so.

One of the clearest facts emerging from a host of recent studies is the need for velocity dependence of the effective nucleon-nucleus potential.[49–53] If one ignores this effective velocity dependence and attempts to compute nuclear energies using a realistic static potential of the type used in optical model analyses and last particle binding energy studies, one finds that the total nuclear nuclear energies computed using,

$$W=\sum_i \langle T_i\rangle+\tfrac{1}{2}\langle V_i\rangle, \qquad (35)$$

are considerably smaller than the experimental nuclear energies. Weisskopf[54] recently gave a qualitative argument based upon the approximate equality of last particle binding energies with average binding energies which demonstrates clearly the necessity for this velocity dependence.

The new self-consistent field methods of Brueckner and Bethe[55] and others in which correlations of nuclear wave functions are taken into account seem to provide the most promising path for solution of the nuclear structure problem. For finite nuclei this suggests that the Schroedinger equation in optical model and shell model calculations should be modified to the form

$$(\hbar^2/2m)\Delta^2\psi+E\psi=\int V(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}', \qquad (36)$$

where $V(\mathbf{r},\mathbf{r}')$ is a nonlocal potential generated from the two-body interaction. The complete self-consistent field problem involves generating the nonlocal potential from the two-body interactions and an initial set of wave functions with the use of the so-called K matrix, then using Eq. (36) to find an improved set of wave functions and continuing the cycle until convergence is achieved.

Brueckner *et al.*[53] have begun efforts toward programming this complete problem for individual nuclei. A more limited attempt toward relating total nuclear

TABLE II. Δ^{sx} (in mMU) and D^{sx} are values of the smoothed normal places. $\Delta^{w}(58)$ and $D^{w}(58)$ are values computed on the basis of the Bethe-Weiszäcker equation with the constants $a_1=16.996$, $a_2=19.298$, $a_3=0.7714$, $a_4=101.03$ (in mMU) or $a_1=15.826$, $a_2=17.970$, $a_3=0.7183$ and $a_4=94.068$ (in Mev). $\Delta^{sx}-\Delta^{w}$, and $D^{sx}-D^{w}$ are the residuals.

A	Δ^{sx}	D^{sx}	$\Delta^{w}(58)$	$D^{w}(58)$	$\Delta^{sx}-\Delta^{w}$	$D^{sx}-D^{w}$
20	−1.48	0.82	0.90	0.91	−2.38	0.09
40	−28.40	2.73	−27.36	2.97	−1.04	−0.25
60	−47.58	5.66	−48.74	5.84	1.17	−0.18
80	−59.00	9.39	−61.20	9.35	2.19	−0.04
100	−62.69	13.74	−64.69	13.41	2.00	0.33
120	−58.64	18.55	−59.60	17.96	0.96	0.59
140	−46.84	23.72	−46.41	22.94	−0.43	0.78
160	−27.31	29.13	−25.62	28.30	−1.68	0.83
180	−0.03	34.71	2.32	34.03	−2.35	0.68
200	34.97	40.37	37.00	40.09	−2.02	0.28
220	77.72	46.06	78.03	46.46	−3.07	−0.40
240	128.20	51.69	125.07	53.11	3.14	−1.42

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energies for all nuclei to such reasonable forms of nonlocal potentials is discussed below.

Frahn and Lemmer[56] have shown that in the case of almost local potentials of the form

$$V(\mathbf{r}',\mathbf{r})=V^*\xi\left(\left|\frac{\mathbf{r}+\mathbf{r}'}{2}\right|\right)\pi^{-\frac{3}{2}}b^{-3}\exp\left[-\left(\frac{\mathbf{r}-\mathbf{r}'}{b}\right)^2\right]. \quad (37)$$

It is possible to reduce Eq. (36) to

$$-\tfrac{1}{8}\hbar^2\left[\nabla^2\frac{1}{m(r)}+2\nabla\frac{1}{m(r)}\nabla+\frac{1}{m(r)}\nabla^2\right]\psi + V^*\xi(r)\psi(r)=E\psi, \quad (38)$$

where

$$m(r)=\frac{m_0}{1+(b^2m_0/2\hbar^2)V^*\xi(r)}=\frac{m_0}{1+\delta\xi(r)}. \quad (39)$$

The radial wave equation for this case may be placed in the form

$$G''+\left[\frac{\gamma^2\xi(r)}{1+\delta\xi}-\frac{\beta_e^2}{1+\delta\xi}-\frac{l(l+1)}{r^2}\right]G + \frac{\delta}{1+\delta\xi}\left[\xi''/4-\frac{\xi'}{2r}-\xi'\Gamma\right]G=0, \quad (40)$$

where

$$\gamma^2=2m_0V^*/\hbar^2, \quad (41)$$

$$\beta_e^2=2m_0|E|/\hbar^2, \quad (42)$$

and $\Gamma=-G'/G$.

The influence of the three small terms on the right of Eq. (40) can be estimated qualitatively. For a smooth monotonically decreasing form, the function ξ' is zero in the uniform region, goes negative to a peak in the surface region and then decays to zero. ξ'' is expected to go negative to a large peak, then cross over to a small positive peak and then decay to zero. For states of binding Γ starts out at long ranges at a positive value which can be estimated from the properties of spherical Bessel functions. It declines gradually towards zero at the first peak of the wave function on going radially inward. Since this first peak is practically never in the surface region for the case of the bound states of importance, Γ is on the average a positive quantity in the surface region. Therefore the term $\xi''/4$ acts to weaken the effective well and hence move the energy levels upward. (This term is not present if one uses[57] the form $\frac{1}{2}pm(r)^{-1}p$ for the kinetic energy) whereas the terms $-\xi'/2r$ and $-\xi'\Gamma$ tend to strengthen the effective well and move the energy levels downward.

These effects have been estimated quantitatively with the aid of a code constructed for the Oak Ridge Oracle.[58] The code accomplishes the numerical solution of Eq. (40) for form factors involving uniform interiors and arbitrarily shaped surface regions. Most attention has been given to the forms

$$\begin{aligned}\xi(r)&=1 &&\text{for}\quad r\geq a\\ \xi(r)&=\exp[-(r-a)/d] && r\geq a,\end{aligned} \quad (43)$$

and

$$\begin{aligned}\xi(r)&=1 &&\text{for}\quad r\leq a\\ \xi(r)&=2\exp[-(r-a)/d]-\exp[-2(r-a)/d].\end{aligned} \quad (44)$$

By turning on and off the perturbation terms we establish that the shifts associated with ξ'' are of the order of +0.8 Mev, the shifts associated with the other two terms are of the order of −1.6 Mev and that the net shifts are downward (stronger binding) of the order of −0.8 Mev. As might be expected the actual shifts fluctuate, tending to be smaller for s states particularly. Tentatively ignoring these corrections we need only to dispose of the terms $\delta\xi(r)$ in the denominators in order to place Eq. (40) in a familiar form. Let us equate

$$\langle\xi(r)\rangle=1-f_i. \quad (45)$$

The values of f_i are expected to vary from state to state tending to be smaller than the average in deeply bound states and larger than the average in outermost states. It is reasonable to replace f_i by an average $\bar{f}$. Defining now

$$\beta=\delta\langle\xi(r)\rangle=\delta(1-\bar{f}), \quad (46)$$

the radial equation becomes

$$G''+\left[\frac{\gamma^2}{1+\beta}\xi(r)-\frac{\beta_e^2}{1+\beta}-\frac{l(l+1)}{r^2}\right]G=0. \quad (47)$$

In many qualitative discussions of velocity dependence one assumes that

$$V=-V^*\xi(r)+\beta T. \quad (48)$$

Inserting this into Schroedinger's equation,

$$[T-V^*\xi(r)+\beta T-E]\psi=0, \quad (49)$$

one finds precisely Eq. (47) for the radial equation. Accordingly we have established contact between the frequently used approximation and the somewhat better one arising out of the nonlocal potential. This contact is established in such a way as to be able to estimate approximately the effects of the residual small surface terms.

Assuming the nucleon-nuclear potential arises strictly from two-body forces the total energy of a system of identical nucleons (i.e., same spin and isotopic spin) is given by

$$E_T(\Lambda)=\sum_{i=1}^{\Lambda}(E_i-\tfrac{1}{2}\langle V_i\rangle) = -\tfrac{1}{2}\Lambda V^*+\frac{1+\frac{1}{2}\beta}{1+\beta}E_0\sum\epsilon_i^2+\tfrac{1}{2}\frac{V^*}{1+\beta}\sum f_i, \quad (50)$$

[56] W. E. Frahn and R. H. Lemmer, Nuovo cimento 6, N3 (1957).
[57] Ross, Mark, and Lawson, Phys. Rev. **104**, 401 (1956).
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where

$$\epsilon_w^2 = -E_i/E_0, \quad \epsilon_0^2 = V^*/E_0, \quad \epsilon_i^2 = \epsilon_0^2 - \epsilon_w^2 \quad (51)$$

$$E_0 = \hbar^2/2m^*a^2 = U_0(1+\beta)/a^2, \quad (52)$$

and where a is a convenient scale factor and $U_0 = 20.734$ Mev. The problem is thus reduced to that of evaluating the sum of the dimensionless eigenvalues ϵ_i^2 and the leakage factors f_i.

The writer and a number of his associates[27,44–47] have investigated the eigenvalues and approximate analytic solutions of radial wave equations for the form function given by Eq. (43). Working backward from the systematics of last neutron binding energies and the nuclear size resonances at $A=55$ and 170, a family of static wells was inferred which is defined by Eq. (43) and the parameters

$$V_0 = 40 \text{ Mev} \quad a = 1.32A^{\frac{1}{3}} - 0.8 \quad d = 1. \quad (53)$$

Summarizing some of the main results of these studies, we may say:

(1) Static potentials arrived at in this way are quite similar to those obtained independently from low energy optical model analyses of neutron scattering.[59]

(2) With such a potential one can secure good neutron shell structure[46] throughout the entire mass range by the simple inclusion of a term about 45 times as large as the Thomas-Frenkel spin-orbit term.

(3) To bind protons at known binding energies after the Coulomb potential is turned on, it is necessary to increase the well depth of the nuclear potential affecting the protons.[27] This proton potential anomaly is about one-half the magnitude of the Coulomb potential. An even better characterization of the anomaly is[60]

$$v = -\lambda(1 \pm kD/A)V_0\xi(r), \quad (54)$$

where $\lambda \approx 0.125$, $k \approx 5.6$, with the upper sign for protons and the lower for neutrons. The neutron excess dependent term in Eq. (52) has the form appropriate for a Heisenberg force[61] although other explanations for such a term can be given.[60] The constant term which corresponds to a deepening of the nonexchange potential was chosen to preserve on the average the carefully adjusted features of the neutron well. Several investigations[62,19] suggest that these changes in the neutron well (deeper than 40 Mev in light nuclei, shallower than 40 in heavy nuclei) are demanded by experimental evidence. The problem of the origin of the proton potential anomaly is one of discriminating between many possible explanations rather than that of finding an explanation. We return to this question later.[60]

(4) The proton densities and radii ($R = 1.181A^{\frac{1}{3}} + 0.317 \approx 1.24A^{\frac{1}{3}}$) computed on the basis of these potentials, are in good agreement with the Stanford electron scattering results although they show a greater degree of density fluctuation.

(5) The neutron densities and radii based upon these potentials lead to a result ($R = 1.291A^{\frac{1}{3}} + 0.0561 \approx 1.30A^{\frac{1}{3}}$) which suggests that heavy nuclei have a thin film of neutrons ($\sim 0.06A^{\frac{1}{3}}$). The experimental evidence on this point is not conclusive. If this effect is not real it suggests that the form of the proton potential anomaly is such as to concentrate protons nearer the surface rather than a form comparable to the neutron potential as assumed.

(6) The equivalent uniform radius for the nuclear potential extends to about $R_v = 1.29A^{\frac{1}{3}} + 0.45 \approx 1.38A^{\frac{1}{3}}$. This on the average is about 0.61 fm beyond the neutron density and somewhat more beyond the proton density.

It would be extremely desirable if a derivation of the nuclear mass surface might be related to this effort. This is the concern of the balance of this work.

A previous attempt using the shell model was based upon a strict independent particle model. The total energies were computed simply by summing eigenvalues in a static nuclear potential.[63,64] The resulting mass surface gives too large a volume energy and too small a symmetry energy. Self-consistent field methods suggest that Eq. (35) must be used for total energies, in which case, a static well leads to too small a volume energy and again the symmetry energy is too small. Use of a velocity dependent potential has been looked to as the solution for both of these problems.[65,49]

It is possible to make use of the eigenvalues and eigenfunctions obtained in studies of last particle binding energies with static potentials by utilizing the dimensionless forms of the wave functions and eigenvalues and by embodying the reduced mass in the energy unit. In essence, what is available[27,46] are the eigenvalues and eigenfunctions for a dimensionless well with a boundary at $\rho = 1$ and a depth parameter ϵ_0^2 extending to about 140. The diffuseness parameter is related to the depth parameter in a particular way. Figure 9 shows the dimensionless wells. For more general applications, A may be regarded simply as a parameter serving as an index to the particular dimensionless well being used, the connection is $\epsilon_0 = 1.833A^{\frac{1}{3}} - 1.111$, which insures that the $3s$ and $4s$ low velocity neutron cross-section resonances occur at $A = 55$ and 170 in static potentials. With the velocity dependent cases, because of surface effects which are expected to act more drastically on zero energy s-wave neutrons, this built in feature of this family of potentials might be lost. However, we proceed in a way that holds last particle binding energies in approximate agreement with experimental values and holds radii of the various distributions close to those derived from the static potentials.

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[63] K. Lee and A. Green, Proc. Intern. Conf. Peaceful Uses Atomic Energy **2**, 113 (1956).
[64] K. Hammack, Ph.D. dissertation, Washington University (1951).
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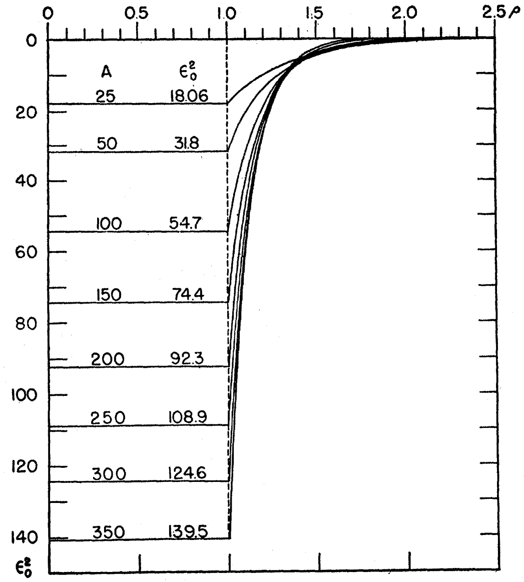

FIG. 9. Dimensionless potentials corresponding to the realistic potentials characterized by Eqs. (53) and (43) when scaled using the radius parameter a. Eigenvalues and eigenfunctions for these wells are given in references 27 and 46.

Returning to Eq. (47), the quantity ϵ_i^2 is a convenient one for forming eigenvalue sums since, being measured relative to the bottom of the well, it is rather insensitive to changes near the top. Furthermore, this quantity is expected to correspond closely to the dimensionless kinetic energy of particles and hence statistical expressions based upon the Fermi gas model might be used as a guide to fitting sums of ϵ_i^2. Using the eigenvalues shown in Fig. 7 of reference 27, sums were constructed by weighting each eigenvalue by $2l+1$, the degree of degeneracy of the state. An expression was found[63] which fitted eigenvalue sums in dimensionless wells with a variety of well depths and surface diffusenesses. In the dimensionless model now considered for which eigenfunctions have been determined, the well depth and diffuseness parameters are related. The modified expression for these eigenvalue sums are[66]

$$\sum_{i=1}^{\Lambda} \epsilon_i^2 = \Lambda^{5/3}(g_0+g_1\Lambda^{-\frac{1}{3}}-g_2\epsilon_0^{-1}), \tag{55}$$

where

$$g_0=3.50 \quad g_1=4.40 \quad \text{and} \quad g_2=12.60.$$

The fits of these expressions to the actual sums are shown in Fig. 10.

Since the wave functions have been determined for our realistic potential, it is possible now to evaluate f_i defined by Eq. (45). Using the dimensionless wave functions, one finds that

$$f_i = P_e[1-S_n(k)], \tag{56}$$

where P_e is the probability of a particle being found in the surface region and $S_n(k)$ is a function defined in reference 27. The only additional information one now needs to correct for the existence of two-body forces is the value of the $\sum_i f_i$. These sums have been evaluated for various occupation numbers and are fitted quite well by the expression[66]

$$\sum_i f_i = f_0\Lambda^{5/3}/\epsilon_0^2, \tag{57}$$

where

$$f_0=0.87. \tag{58}$$

Using the foregoing expressions in connection with Eq. (50), the total energy of Λ-identical particles in a velocity dependent well is[66]

$$E_T(\Lambda) = -\tfrac{1}{2}\Lambda V^* + U_0 a^{-2}(1+\tfrac{1}{2}\beta)\Lambda^{5/3} \times (g_0+g_1\Lambda^{-\frac{1}{3}}-g_2\epsilon_0^{-1}) - \tfrac{1}{2}f_0 U_0 \Lambda^{5/3} a^{-2}. \tag{59}$$

Previous studies[60] dealing with outermost neutron states indicated that to maintain last particle binding energies at approximately the experimental values at the fixed radius parameters one must relate V^* and β in a sharply restricted way. To a good approximation this prescription may be expressed by the relation

$$V^* = V_0 + V_1\beta = V_0(1+k_v\beta), \tag{60}$$

where $k_v = 0.7625$. Use may also be made of the corre-

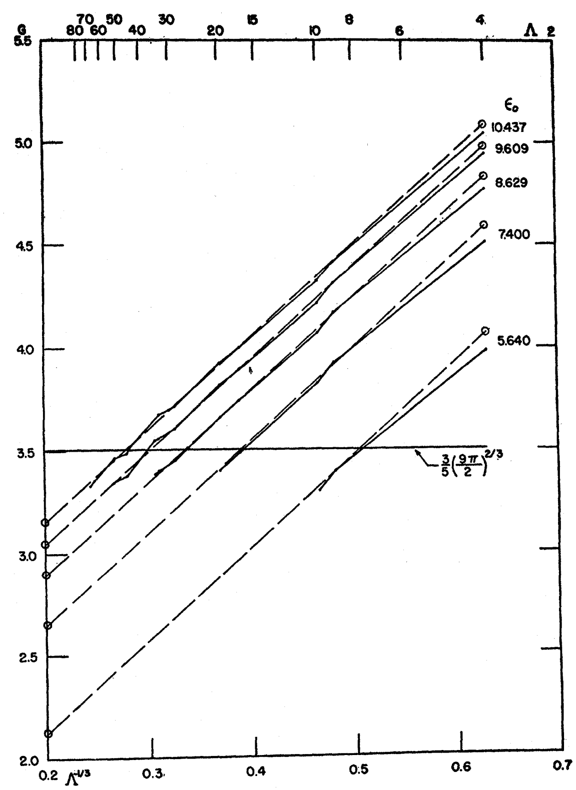

FIG. 10. $\Lambda^{-5/3}\sum \epsilon_i^2$ *vs* $\Lambda^{1/3}$ for various potential parameters ϵ_0. The straight lines correspond to Eq. (55). The horizontal line corresponds to the constant g_0 predicted for the kinetic energy on the Fermi gas model. (From Lee and Green, reference 66.)

[66] K. Lee and A. Green, Bull. Am. Phys. Soc. Ser. II, 1, 16 (1956).

sponding approximate relation

$$g_2(1+\tfrac{1}{2}\beta)\epsilon_0^{-1}=a^{-1}j_2(1+k_2\beta), \quad (61)$$

where

$$j_2=9.135 \quad \text{and} \quad k_2=0.5862.$$

The total energy now takes on the form

$$E_T(\Lambda)=-\tfrac{1}{2}\Lambda V_0+U_0a^{-2}\Lambda^{5/3}[(g_0-\tfrac{1}{2}f_0)+g_1\Lambda^{-\frac{1}{3}}-j_2a^{-1}] +\beta[-\tfrac{1}{2}\Lambda k_v V_0+U_0a^{-2}\Lambda^{5/3} \times(\tfrac{1}{2}g_0+\tfrac{1}{2}g_1\Lambda^{-\frac{1}{3}}-j_2k_2a^{-1})]. \quad (62)$$

The factors a^{-2} and a^{-3} require special consideration. To arrive at a result in a form comparable to a Weiszäcker-type equation we must handle the corrective term to the approximate representation, $a=r_0A^{\frac{1}{3}}$, with some care. Anticipating how these corrective terms will influence the mass equation, it is reasonable to choose them so as to "feed back" into the volume or surface terms rather than generate new terms in the mass equation. By least square fitting to the actual values of a^{-2} we find to a very good approximation that

$$a^{-2}=l_0A^{-\frac{2}{3}}+l_1A^{-1}$$
$$\text{with} \quad l_0=0.5220 \quad \text{and} \quad l_1=1.116, \quad (63)$$

and to fairly good approximations

$$a^{-2}=m_0A^{-\frac{2}{3}}+m_1A^{-\frac{1}{3}}$$
$$\text{with} \quad m_0=0.9838 \quad \text{and} \quad m_1=-0.0467, \quad (64)$$

and

$$a^{-3}=n_0A^{-1}+n_1A^{-\frac{2}{3}}$$
$$\text{with} \quad n_0=0.9563 \quad \text{and} \quad n_1=-0.0611. \quad (65)$$

To derive a mass surface (in the absence of the Coulomb potential) for an even-even nucleus, one considers now two independent systems with $\Lambda=\frac{1}{4}A(1\pm D/A)$ where the upper sign is for neutrons and the lower sign for protons. Assuming D/A is small, one expands $(1\pm D/A)^n$ and inserts the series into Eq. (62). The total energy is then

$$E=2E_T(\Lambda_n)+2E_T(\Lambda_p). \quad (66)$$

Neglecting terms of the order of D^4/A^4 the total energy may be placed in the form

$$E=-a_1A+a_2A^{\frac{2}{3}}+\alpha_4D^2/A-\alpha_5D^2/A^{\frac{4}{3}}, \quad (67)$$

where

$$a_1=\alpha_{10}+\alpha_{11}\beta, \quad a_2=\alpha_{20}+\alpha_{21}\beta, \quad \text{etc.}$$

For convenience, the divisor 4 in the usual symmetry term has been suppressed.

Table III shows the mass parameter coefficients obtained. We have introduced the symmetry coefficient $\alpha_s=\alpha_4-\alpha_5/\bar{A}^{\frac{1}{3}}$ with $\bar{A}^{\frac{1}{3}}=5$ to compare the results with the usual form of the Weiszäcker equation. The table shows that desired values of mass parameters would only be reached at very large values of β and that the symmetry energy particularly falls far short of the experimental value.

TABLE III. Mass surface parameters (in Mev) derived from an IPM model with a velocity dependent diffuse boundary potential [see Eq. (67)]. The numbers in the second column correspond to the parameters for the case $\beta=0$. The numbers in the fourth column give the values for $\beta=1$. The experimental values for α_4 and α_5 are rough estimates based upon this study of perturbations of the Weiszäcker equation and the work of Cameron.

	α_{10}	α_{11}	$\beta=1$	exp
a_1	4.92	6.38	11.30	15.826
a_2	12.80	2.20	15.01	17.970
α_s	6.92	3.73	10.66	23.52
α_4	9.27	5.37	14.65	~31.5
α_5	11.73	8.19	19.93	~40.0

Thus we are encountering again a difficulty closely related to the need for introducing a proton potential anomaly in static wells. According to the mass equation for nuclei near the line of beta stability

$$B_n-B_p\approx 2a_3ZA^{-\frac{1}{3}}-a_4DA^{-1}. \quad (68)$$

A weakness in the symmetry energy parameter thus would lead a large difference between neutron and proton binding energies. This situation arose in static well studies[60] and could be corrected by introducing perturbations of the type given by Eq. (54). In particular the terms $\pm\lambda kV_0\xi(r)D/A$ might be expected to help the symmetry energy.

In an unpublished note, Brueckner finds that if, as is done here, one identifies eigenvalues with the separation energies then in an infinite saturated system one must modify Eq. (35) to incorporate a "rearrangement energy." It is possible that the rather large deficiency in the volume energy found here at $\beta\sim 1$, which corresponds to a reduced mass of 0.5 might be related to the neglect of such a term. In any event the perturbation term $-\lambda V_0\xi(r)$ contained in Eq. (54) helps the volume energy perhaps by compensating for this effect, although this question is unsettled.

V. INFLUENCE OF PERTURBATIONS

Perturbing the potential by addition of a function of the form

$$v=-v_0\xi(r), \quad (69)$$

this perturbation enters the expression for the total energy directly through the main potential energy term and indirectly through the ϵ_0^{-1} surface term. To a good approximation one may expand

$$(V^*+v_0)^{-\frac{1}{2}}=(V^*)^{-\frac{1}{2}}(1-\tfrac{1}{2}v_0/V^*). \quad (70)$$

Thus the extra terms associated with an attractive perturbation of magnitude v_0 are

$$-\tfrac{1}{2}v_0\Lambda+\tfrac{1}{2}U_0a^{-3}j_2(1+k_2\beta)v_0V^{*-1}. \quad (71)$$

Using approximate techniques which have been developed in last particle binding energy studies let us now assume that a static perturbation is added of the form

$$v_n = -\lambda(1-kD/A)V_0\xi(r)$$
$$v_p = -\lambda(1+kD/A)V_0\xi(r) + 2U_c(Z-1)\alpha_e\xi(r)a^{-1} + 2U_c\beta_e(Z-1)a^{-1}, \quad (72)$$

where α_e and β_e are parameters depending upon the diffuseness and radii. They may be determined by the considerations given in reference 27 so as to approximate the influence of the classical Coulomb potential arising from a distribution of $(Z-1)$ protons when acting upon another proton in the nucleus. The extra terms which now enter the total energy expression are

$$E_{\text{pert}} = -\tfrac{1}{2}\lambda V_0 A + \tfrac{1}{2}\lambda k V_0 D^2/A + Z(Z-1)U_c(\alpha_e+\beta_e)a^{-1} - U_0 a^{-3} j_2 b Z^{5/3}(Z-1)U_c \alpha a^{-1} V_0^{-1} - (5/9)(4)^{-\frac{2}{3}}\lambda U_0 a^{-3} j_2 b D^2 A^{-\frac{1}{3}} + U_0 a^{-3}\lambda b A^{5/3}, \quad (73)$$

where

$$b = b_0 - b_1\beta \approx (1+k_2\beta)/(1+k_v\beta) \approx 0.95675 - 0.05675\beta. \quad (74)$$

The first three terms arise from the direct effect, the last three through the ϵ_0^{-1} term. The Coulomb term contains the classical result and an additional quantum-mechanical term. The latter might be described as the influence of the perturbation upon the wave functions causing a weakening of proton interactions. The combination $U_c A^{\frac{1}{3}}(\alpha_e+\beta_e)a^{-1}$ might be compared to the usual Coulomb energy constant. The calculated values for this number at various A values turn out to be practically constant at 0.697 Mev in rather good agreement with the desired value of about 0.71. Exactly what to do with the nonclassical Coulomb term is an open question. If one manipulates it into the form of the regular term and combines the two together for nuclides near the line of beta stability, one finds that now the Coulomb energy coefficient runs from about 0.64 to 0.67 from light to heavy nuclei which is still in fair agreement with the demands of the semiempirical equation.

To study the influence of the other perturbation terms we must fix the values of λ and k. We could of course choose these constants and β so as to arrive at desired values of a_1, a_2, and α_s. However, this *ad hoc* approach is rather repulsive and it would be more significant to use other information. Recent studies of the proton potential anomaly have suggested[67] that the absolute difference in the constant part of the proton and neutron wells needed to keep last particles at approximately the experimental values is rather insensitive to the degree of velocity dependence. Accordingly it is reasonable to examine the result of letting $\lambda \approx 0.125$ and $k \approx 5.6$, the values used in the static case. Table IV shows the mass parameters now obtained. Each of the mass parameters has been brought into the proper range by this perturbation at $\beta=1$, a value which corresponds to an average reduced mass of 0.5. This is about the same order of magnitude that has arisen in many recent studies of infinite nuclear matter. It is also satisfactory that the average slopes of isobaric mass differences[68] are approximately in accord with the requirements of experiment when $\beta \approx 1$. The calculated beta-decay energies, however, fluctuate markedly from their experimental values, an effect probably characteristic of an I.P.M. model which neglects interparticle couplings.

Using techniques familiar in uniform model calculations[3,30] one might derive a pairing energy function from our expression for the total energy of a system of particles. The result is

$$H(A) \approx 29A^{-1} - 40A^{-\frac{4}{3}}. \quad (75)$$

This is too small by a factor of about four and hence the pairing energy remains to be accounted for in another way.

VI. DISCUSSION

We have carried this effort to a point at which rather refined treatments and approximations are needed to go further. Before considering some of these fine details, let us return to the major consideration of the foregoing derivation of the Weiszäcker equation, namely, its bearing upon the question of nuclear sizes and density distributions. In this derivation the same scale factor used in the static well case was chosen in order to retain the rather satisfactory distributions obtained in a study based upon the static well. These are shown in Figs. 11 and 12.[47] In actuality a small decrease in the predicted matter radii (~3% in heavy nuclei) would appear to be desirable to improve agreement with the Stanford densities. Such a decrease would tend to increase the kinetic energy by about 6% and hence would require increasing the well depth by a comparable amount. It is likely that the net effect of all of these readjustments would be to increase the volume energy and improve somewhat the derived mass surface.

TABLE IV. Mass surface parameters (in Mev) derived after the inclusion of a perturbation inferred from last particle binding energies. The numbers in the second column give the parameters for the case $\beta=0$. The numbers in the fourth column give the values for $\beta=1$.

	α_{10}	α_{11}	$\beta=1$	exp
a_1	7.70	6.36	14.06	15.826
a_2	17.10	1.95	19.05	17.970
a_3	~0.66	~0	~0.66	0.7183
α_s	15.79	4.04	19.83	23.52
α_4	25.68	5.23	30.91	~31.5
α_5	49.46	5.96	55.42	~40.0

[67] A. Green and P. C. Sood (unpublished).

[68] R. J. Walker and A. Green, Bull. Am. Phys. Soc. Ser. II, 2, 288 (1957).

Quantitative estimates of this effect are rather difficult to make since it would also depend upon the slight reshaping of the well needed to restore the last particle binding energies.

The contributions of extra terms to the right of Eq. (40) are expected to increase the volume energy coefficient by about 0.5 Mev, a helpful increase. In addition the approximation of replacing of $\delta\xi$ in the denominators of Eq. (40) by the average β might be corrected by giving consideration to departures of f_i from $\bar{f}$ [see Eqs. (45) and (46)]. On the basis of Eqs. (57) and (58), f_i is estimated to be about 0.14. Individual state estimates using Eq. (56) range from about 0.05→0.3. This variation in the effective β corresponds to the type of variation of the reduced mass m^* found by Brueckner and Gammel.[69] Possibly these additional considerations related to the nonlocality of the effective potential will further improve the volume energy. It is unlikely that these fine features would change the over-all density distributions very greatly although they probably would act in a differential way upon the individual states.

The influence of the proton potential anomaly which is essential to the explanation of last particle separation energies, and to the attainment of a reasonable mass surface, also has a direct bearing upon nuclear sizes and density distributions. Using a representation of this anomaly involving a form factor similar to the neutron well, the anomaly has been found to cause a slight shrinkage of the charge radius.[46] This shrinkage is enough to upset the balancing effects of the Coulomb repulsion and the presence of extra neutrons which would otherwise lead to the approximate equality of proton and neutron radii. The net neutron skin thickness calculated is small, however, ($\sim 0.06A^{1/3}$) when compared to the earlier neutron skin estimates (~ 1) of Johnson and Teller.[70] This smaller result is close to that found recently by Wilets[71] using different theoretical techniques. The most recently reported experimental evidence by Cool[72] suggests the equality of the neutron and proton radii and appears to rule out a thick neutron skin. Further experimental clarification of this question should be of help in determining the exact origin of the proton potential anomaly. This anomaly unfortunately seems to have a large number of possible explanations (including the possibility that it doesn't exist[73]). On the basis of the reports of Fernbach[74] and Glassgold,[75] the considerations in the previous sections and other studies[19,76,77] it appears quite conclusively that there are indeed differences in the proton and neutron well depths. Derivation of the Weiszäcker formula and further detailed study of last particle binding energies[67] strongly suggests that the bulk of these differences are related to the neutron excess rather than the nuclear charge or to the velocity dependence of the potential. Since extra neutrons in heavy nuclei tend to lie in the outer regions of the nucleus, it is quite possible that the potential anomaly form factor should have a maximum near the surface and hence [see Eq. (72)] provide a

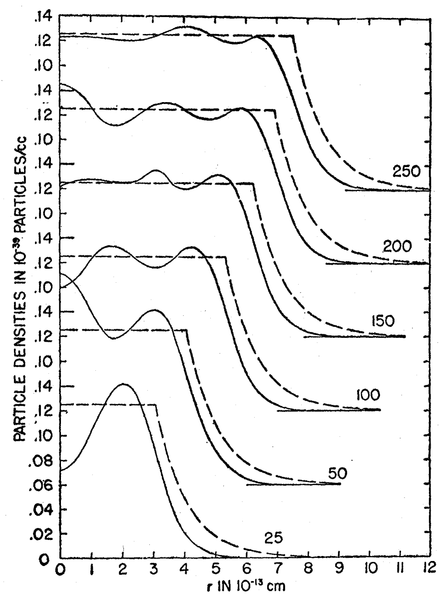

FIG. 12. Total particle densities and potentials *vs* radius.[47] The solid curves are calculated total particle densities. The dashed curves represent the form of the potential functions which underly this work. The heavy markers indicate ordinates at which the scale is broken by 0.06 scale unit.

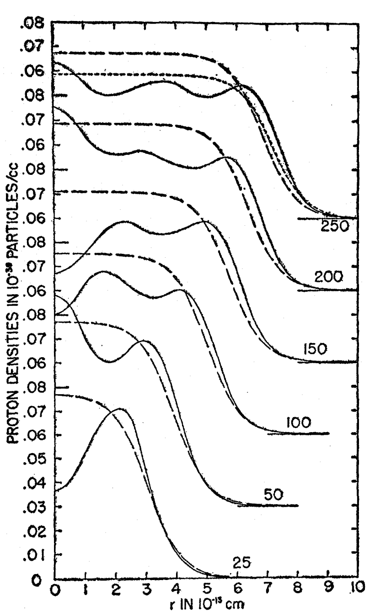

FIG. 11. Proton densities *vs* radius.[47] The solid curves are proton density distributions for mass numbers 25, 50, 100, 150, 200, and 250 calculated on the basis of the family of potentials arrived at in an earlier work. The dashed curves are density curves inferred from the Stanford electron scattering experiments. To avoid overlap, the curves for successive mass numbers are shifted upwards by 0.03 scale unit. The heavy markers denote the ordinates at which the scale is broken. The dotted line for mass number 250 represents a density curve with a radius parameter 5% larger than the radius parameter extrapolated from the Stanford experimental results.

[69] K. A. Brueckner and J. Gammel (to be published).

[70] M. H. Johnson and E. Teller, Phys. Rev. **93**, 357 (1954).
[71] L. Wilets, Phys. Rev. **101**, 1805 (1956).
[72] R. L. Cool, Revs. Modern Phys. (to be published).
[73] B. Margoles and V. F. Weisskopf, Phys. Rev. **107**, 641 (1957).
[74] S. Fernbach, Revs. Modern Phys. **30**, 414 (1958), this issue.
[75] A. E. Glassgold, Revs. Modern Phys. **30**, 419 (1958), this issue.
[76] J. P. Schiffer and L. L. Lee, Jr. (private communication).
[77] Johnson, Galonsky, and Ulrich (private communication).

local trough for protons and hill for neutrons near the surface. Whether the hill and trough would be sufficient to equalize the proton and neutron radii would depend critically upon the detailed relationship of the potential anomaly to the neutron excess density. Drell[61] points out the equality of the proton and neutron radii in conjunction with the larger potential radius would impose highly restrictive requirements upon the permissible two-body interactions. One might also expect an involvement of the sizes of the proton and neutron themselves[78] in the origin of the anomaly although the exact nature of this is unexplained. Finally one would also expect an involvement of the "rearrangement energy" question[54,79,80] since the final adjustment of the well parameters would depend upon the size of this effect in finite nuclei.

VII. CONCLUSIONS

It appears that a Bethe-Weiszäcker type mass surface provides its best account of nuclear masses at a Coulomb radius parameter in agreement with the equivalent parameter inferred from Stanford electron scattering. There is opportunity for further improvement, both in the form of fluctuating shell and pairing corrections, and in smooth statistical corrections.

The second portion of this work is devoted to an effort to relate the Weiszäcker equation directly to potentials obtained from scattering and shell models. Working with nonlocal or velocity dependent nuclear potentials such as might arise from Brueckner-Bethe self-consistent field methods, one can derive fairly directly a mass surface similar to the Weiszäcker formula by going to a reduced mass near the value determined in studies of infinite nuclei. The parameters, however, particularly the symmetry term, which includes a negative surface component, are not satisfactory. By adding an appropriate perturbing potential one gets a fairly satisfactory mass surface. This is accomplished at density distributions in good correspondence with experimentally inferred distributions. While many considerations have been left open, particularly questions of the self-consistency, stability, and "rearrangement energy," this type of analysis involving the use of deeply bound states may serve as a useful link in exploring characteristics of nucleon-nucleus potentials.

An interesting aspect of this derivation of the Weiszäcker equation from the IPM is the rather mixed origin of the terms which normally referred to as volume and surface terms. This mixing is occasioned largely by the fact that nuclear radii do not vary simply as $A^{\frac{1}{3}}$, and suggests that we must not be too literal in the interpretation of the separate terms as volume, surface, etc. energies. Perhaps some of the difficulties in the application of the mass surface to the interpretation of the theory of fission have been occasioned by this confusion.

In conclusion, atomic mass data constitute one of the most precise and extensive arrays of experimental information available for the interpretation of nuclear phenomena. When interpreted in the light of current theoretical ideas and other types of experimental information, these data can lead to rather detailed conclusions concerning nuclei. As our understanding of complex nuclei progresses, nuclear masses will play an increasingly important role in nuclear physics.

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[78] Hofstadter, Bumiller, and Yearian, Revs. Modern Phys. **30**, 482 (1958), this issue.
[79] K. A. Brueckner (unpublished note).
[80] D. J. Thouless, Bull. Am. Phys. Soc. Ser. II, **3**, 20 (1958).