Relation of Nuclear Size to Saturation and Nuclear Compressibility

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IN order to discuss the problem of nuclear size and structure and its relation to the problem of nuclear saturation it is convenient in first approximation to neglect the Coulomb interaction and to consider the idealized problem of nuclear matter, by which we mean a collection of nucleons of sufficient spatial extent so that surface effects can be neglected. Such an idealized situation does not exist, of course, in actual nuclei since the Coulomb repulsion actually determines the maximum possible nuclear size. Nevertheless it is possible to deduce properties of nuclear matter from those of actual nuclei by using the phenomenological approach of the semiempirical mass formulas.[1] These show that nuclear matter has a binding energy of 15 Mev per particle, with an uncertainty of about 1 Mev. The density is most accurately determined by electron scattering[2] and corresponds to a radius parameter of 1.07×10^{-13} cm, this being the radius of the mean volume per particle.

The energy and density parameters give the most important properties of nuclear matter and also of heavy nuclei. Their determination forms the basis of the study of nuclear size; this is the problem of nuclear saturation. The understanding of the saturation problem must underlie any theoretical attempt to predict other properties of the nucleus such as its surface structure. Thus we shall discuss the saturation of nuclear matter before we return to the surface problem.

The starting point of a saturation study is usually taken to be the properties of the two-body interactions as determined by studies of the two-body problem. Until the scattering studies at Berkeley starting in 1947 had been made, only the S-wave interaction at low energies had been experimentally determined. The first Berkeley studies showed that the two-body interaction did not have the exchange character required for saturation, this being a space exchange potential with four times the strength of the nonexchange potential. In its exchange characteristics, then, the two-body interaction was found to be nonsaturating. The way out of this difficulty was first pointed out by Jastrow[3] who showed that high-energy p-p scattering strongly pointed to a short-range repulsion in the two-body system. Such a repulsion would prevent collapse of the many-body system and hence lead to saturation. Such considerations left open the question of the actual density and also the binding energy of the saturated system.

More recent studies of the two-body problem, particularly those of Gammel, Christian, and Thaler,[4,5] have shown that a two-body interaction can be fixed with considerable precision if all available data on the two-body system are utilized. Their interactions were of Yukawa form outside a repulsive core, requiring specification of strength, range, and core radius to specify the potential. Including central, tensor, and spin-orbit interactions in states of both even and odd parity, their phenomenological interaction then has 20 parameters. Their best values are given in Table I.

The Gammel-Thaler potential is similar in its general exchange and range features to the prediction of pseudoscalar meson theory, except for the presence of very short-range spin-orbit forces. A somewhat similar result has also been obtained by Marshak and Signell[6] who investigated the effects of addition of a spin-orbit term to the meson-theoretic potential of Gartenhaus.[7]

As a result of these studies, the present knowledge of the interaction is sufficiently precise to provide a very well-defined starting point for further study of the many-body problem. One further problem remains, however, in the possible effects of many-body forces.

TABLE I.

State	Depth (Mev)	Inverse range (10^{13} cm^{-1})
(a) Parameter of 2-body interactions for Gammel-Christian-Thaler potentials. ($r_{core}=0.50\times 10^{-13}$ cm)		
Triplet central even	−6395	2.936
Triplet tensor even	−45	0.7342
Singlet even	−905.6	1.7
Triplet central odd	−150	1.5
Triplet tensor odd	57.5	1.12
Singlet odd	+113	1.0
(b) Parameters of 2-body interactions for Gammel-Thaler potential. ($r_c=0.40\times 10^{-13}$ cm)		
Triplet central even	−877.4	2.0909
Triplet tensor even	−159.4	1.0454
Singlet even	−434	1.45
LS even	−5000	3.70
Triplet central odd	−14.0	1.0
Triplet tensor odd	+22.0	0.8
Singlet odd	130	1.00
LS odd	−7315	3.70

[1] A. E. S. Green, Phys. Rev. **99**, 1410 (1955).
[2] R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).
[3] R. Jastrow, Phys. Rev. **79**, 389 (1950).
[4] Gammel, Christian, and Thaler, Phys. Rev. **105**, 311 (1957).
[5] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957); Phys. Rev. **107**, 1337 (1957).
[6] P. S. Signell and R. E. Marshak, Phys. Rev. **106**, 832 (1957).
[7] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).

These have been neglected in our studies since both meson theory and indirect evidence from nuclear structure show that many-body effects are small.

Let us now turn to the problem of determining the properties of nuclear matter, starting from the phenomenological two-body interactions. This is a typical many-body problem, complicated however by the complex nature of the interaction as well as by the presence of very strong repulsions. The treatment of this system was first considered by Levinson, Mahmoud, and the author in 1953.[8] Since then there has been much work done on the many-body problem by us[9] as well as others.[10] Most of this work has been concerned with the general theory problem of many-body systems to which the most important contributions to the specifically nuclear problem have come from Eden, Bethe, and Goldstone. Various attempts have also been made to predict nuclear properties, particularly by Martin and de Dominicis and by Weisskopf. In discussing actual results this paper will be confined to information obtained by John Gammel and the author at the computing center of Los Alamos Scientific Laboratory. These results are obtained with no important approximations and also are based on the accurate phenomenological interactions of Gammel and Thaler.[5]

Before going into the quantitative aspects of the saturation results, it is best to first describe briefly the solution of the nuclear many-body problem and also describe in physical terms the approximation method on which the solution is based. No attempt will be made to describe the formal theory which must be invoked to justify the plausibility arguments which are given; this is given in detail in various versions in the papers on the theory of the methods.

To solve the many-body problem, it is necessary to describe the correlated motion of N particles interacting through $N(N-1)/2$ potentials. The description must be accurate if an accurate result is to be obtained, particularly since the potentials are very strong and markedly perturb the wave function. The situation is further complicated by the nonexistence of a perturbation theory of the usual Rayleigh-Schrödinger form, the perturbation series diverging in all orders due to the repulsive cores in the potentials. Nevertheless, since we can hardly hope to solve the full problem exactly, some type of approximation must be used.

We consider first the treatment of a low-density nucleon gas. This problem can be solved exactly and consequently provides a useful starting point. At sufficiently low density, particle interactions can be considered to occur only pairwise with the colliding pair uninfluenced by the presence of other particles. In this limit it is therefore correct to determine the interaction energy of the many-body system by simply summing up the interaction energy of all pairs. The two-body problem is of course readily solved by the usual methods and one finds that the interaction energy is directly expressible in terms of the two-body scattering phase shifts. In this approximation the treatment of the hard-core repulsions leads to no difficulty and the correlation problem of the $N(N-1)/2$ pairs is solved exactly.

To go on to the case of finite density, we must take into account three different effects. First, since the nucleons are fermions, we must take this into account not only in the two-body collisions but also by properly antisymmetrizing the many-body wave function. Consequently the particles make up a fully degenerate Fermi gas, and transitions of particles to other states must be allowed only if the exclusion principle is not violated.

Another effect arises from the binding of the particles. In contrast to the situation at low densities where the kinetic energies are very large compared to the interaction energies, in the actual system at nuclear densities the interaction is sufficiently large to bind the system. Thus the single particle energies are very strongly shifted downward. The shift in energy will depend, in general, on the momentum of the particle so that the potential affecting a single particle will become velocity dependent. Alternatively we can say that the energy momentum relation for a particle is different from that for a free particle or that the medium becomes dispersive.

Taking these two effects into account, i.e., the exclusion effects and the altered dispersion law for particle motion, we then are still able to retain the low-density picture of particles propagating through the nuclear medium and interacting pairwise, but the medium affects particle motion through the exclusion principle and also through the altered dispersion law. Before showing how the alteration in the dispersion law is determined, let us consider the remaining corrections to this picture.

Clearly the inclusion of the two effects just described does not completely take into account the many-body complexity of the problem. To see the origin of the remaining corrections, we fix attention on a pair of interacting particles moving through the dispersive many-body medium. During their collision these two particles sense the remaining particles through the exclusion effects and the binding field; they may, however, while both excited make a hard collision with a

[8] Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954).

[9] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955); K. A. Brueckner, *ibid.* **100**, 36 (1955); *ibid.* **96**, 908 (1954); *ibid.* **97**, 1353 (1955); K. A. Brueckner, and W. Wada, *ibid.* **103**, 1008 (1956); K. A. Brueckner and J. L. Gammel, *ibid.* **105**, 1679 (1957); *ibid.* **109**, 1023, 1040 (1958).

[10] R. J. Eden and N. C. Francis, Phys. Rev. **97**, 1366 (1955); R. J. Eden, Proc. Roy. Soc. (London) **A235**, 408 (1956); J. Goldstone, *ibid.* **A293**, 267 (1957); K. M. Watson, Phys. Rev. **103**, 489 (1956); W. Riesenfeld and K. M. Watson, *ibid.* **104**, 492 (1956); J. Brenig, Nuclear Phys. **4**, 363 (1957); H. Kummel, Nuovo cimento **23**, 1 (1957); L. S. Rodberg, Ann. Phys. **1**, (1957); H. A. Bethe, Phys. Rev. **103**, 1353 (1956); P. Martin and C. de Dominicis, *ibid.* **105**, 1417 (1957); Gomes, Walecka, and Weisskopf (Ann. Phys., to be published); K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957); C. N. Yang and T. D. Lee, *ibid.* **105**, 1119 (1957).

TABLE II. Predicted nuclear parameters.

Potential	Energy (Mev)	$r_0(10^{-13}$ cm)	Compressibility parameter (Mev)
Gammel-Christian-Thaler	−18.5	0.95	167
Gammel-Thaler	−15.2	1.02	172
Experimental	−15.5	1.07	100 to 150

third particle of the medium and cause its excitation. This triple excitation may dissolve as the particles separate or may be followed by further excitation. This sequence of processes then provides a correction to the simple pair correlation picture; we have called it the series of linked cluster corrections. These effects have been evaluated and shown to be small in a medium of nuclear characteristics. It is interesting to note, however, that such sequences of multiple excitations are essential in the description of the electron or boson gas where they give rise to the collective properties. The small effect of these corrections in the nuclear problem is due to the relative diluteness of the nuclear medium and to the absence of important long range organized motion. In these results the action of the exclusion principle is very important since it tends to freeze particle motion and to minimize the effects of the excitation of multiple clusters.

Neglecting the cluster corrections, our problem is completely stated if we specify the dispersion law for the nuclear medium. This we can do by using the fact that the interaction energy of a particle is determined by the sum of its pairwise interaction energy with all other particles. In our approximation the energy of a particle is

$$E_i=\frac{p_i^2}{2m}+\sum_j \Delta E_{ij} \tag{1}$$

and the total energy

$$E=\sum_i \frac{p_i^2}{2m}+\tfrac{1}{2}\sum_{ij} \Delta E_{ij}. \tag{2}$$

The interaction energy $\Delta E(ij)$ is that determined for the pair (ij) in the velocity dependent binding potential with the exclusion effects included. In Eq. (2) the factor of $\frac{1}{2}$ takes account of the sum over (ij) running over all pairs twice.

This completes the statement of the approximation method. The basic equation to be solved is for the two-body interaction operator, the reaction matrix. It satisfies the integral equation

$$K_{kl,ij}=V_{kl,ij}+\sum_{\substack{mn\\ p_m\geqq p_F\\ p_n\geqq p_F}} V_{kl,mn}(E_i+E_j-E_m-E_n)^{-1}K_{mn,ij}, \tag{3}$$

where $V_{kl,ij}$ is a matrix element of the potential taken with respect to the plane wave states of the unperturbed medium. The interaction energy of the (ij) pair is given by the diagonal elements of K;

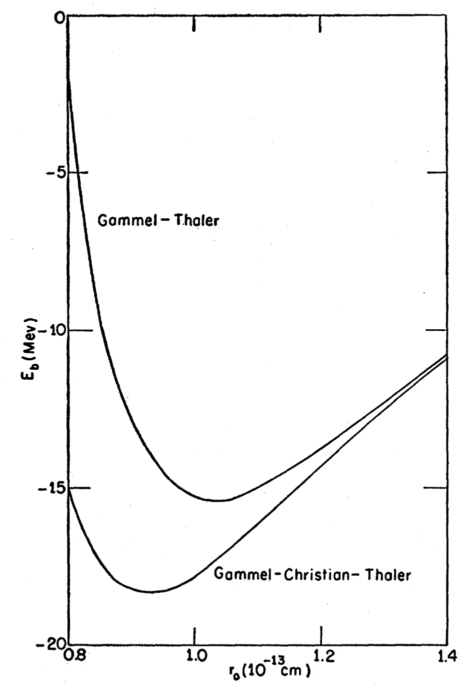

FIG. 1. Binding energy as a function of r_0 for a core radius of 0.20×10^{-13} cm.

$$\Delta E_{ij}=K_{ij,ij}-K_{ij,ji}, \tag{4}$$

which gives for a typical single particle energy, appearing in the energy denominator of Eq. (3)

$$E_i=\frac{p_i^2}{2m}+\sum_j (K_{ij,ij}-K_{ij,ji}). \tag{5}$$

Equations (3)–(5) form a very complicated nonlinear system of equations; they were solved only semiquantitatively until recently when Gammel and the author were able to obtain essentially exact solutions. These results will now be described.

The energy *versus* density curve obtained from the Gammel-Thaler[5] interactions is given in Fig. 1 together with an earlier result obtained from the best static potential (no spin-orbit term) determined by Gammel, Christian, and Thaler.[4] The equilibrium radius parameter and binding energy are given in Table II together with the nuclear compressibility parameter, defined by the equation

$$K=r_0^2(d^2E/dr_0^2). \tag{6}$$

Both sets of values are close to the observed values, particularly those from the Gammel-Thaler potential.

To understand these results in more detail, we have considered the effects of approximating to the solutions and also of changing the parameters of the two-body interactions. It is tempting, for example, to try to make

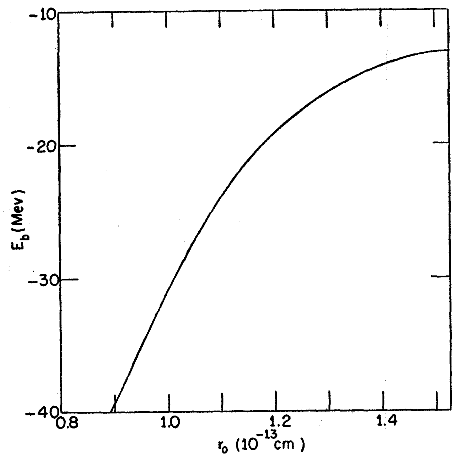

FIG. 2. Binding energy as a function of r_0.

use of the relative diluteness of nuclear matter and to use the low-density approximation for the interaction energy. This gives a binding energy of 55 Mev per particle at normal density, however, as well as failing to exhibit saturation at normal density. As another approximation, the exclusion effects can be included but the effects of the binding field neglected. The result was a binding energy of 34.4 Mev per particle and again no saturation occurred at normal density. We have also estimated the effects of the binding field with no many-body exclusion effects included; this approximation again overestimates the energy. Finally, inclusion of both the self-consistent binding field and the exclusion effects gives the computed result of 15.2 Mev. We therefore conclude that proper inclusion of the many-body effects in the K-matrix equation (3) is essential in describing the saturation process accurately.

We have also examined the effects of varying some of the parameters of the interaction. The repulsive core plays an essential role in saturation, of course, since the exchange character of the potential does not in itself lead to saturation. To investigate this in detail, the problem was solved with an interaction with a repulsive core of 0.2×10^{-13} cm instead of the 0.4×10^{-13} cm of the Gammel-Thaler potential. The interaction was chosen to agree with the low-energy scattering parameters. The result is given in Fig. 2 which shows that for this interaction the binding energy is over 30 Mev at normal density and saturation does not occur at this density. The effect of varying the exchange mixture was also estimated; if the exchange term in the potential is replaced by a Wigner potential (no exchange) then the equilibrium density increases by a factor of 5 or 6 with a similar rise in the binding energy.

The tensor force contribution to the binding energy was also determined. It is of particular interest since it vanishes in first order so that its contribution is a measure of higher order effects. The tensor force was found to give 6 Mev of binding per particle; its neglect also led to too low equilibrium density.

To summarize, it is useful to list the repulsive and attractive contributions to the energy which are balanced at the normal equilibrium density to give saturation. The repulsion of the cores, which prevents close packing, is the essential cause of saturation. To it must be added, however, the repulsion of the exchange potential and the repulsion of the normal kinetic energy. These three effects give roughly comparable contributions at normal density. The attractive contributions come largely from the S states of relative angular momentum but the D-state contribution is also important. The noncentral forces also cannot be neglected since although they give only about 10% of the interaction energy, this is 40% of the binding energy. Thus nuclear saturation and hence nuclear size is due to an interplay among many features of the interactions which individually cannot be appreciably altered and still maintain the observed properties.

To conclude, we now turn briefly to the problem of finite nuclei. A simple and easily evaluated effect arises from the Coulomb repulsion. This causes a "blowing up" of the nucleus, the increase in radius being determined by the nuclear compressibility. The effect computed for the compressibility given in Table II is an increase in the radius parameter of about 0.05×10^{-13} cm in lead. Somewhat similar effects also arise from the surface and symmetry energy repulsions, both of which tend to drop the density in a finite nucleus from that of nuclear matter.

A more difficult problem is the determination of the surface structure of a finite nucleus. Several important features of this problem must be emphasized. First, the surface depth or density falloff distance is obviously a sensitive function of the nuclear compressibility, since this determines the loss in energy as the density departs from its optimum value. Another important effect is the nonlocality of the potential seen by a nucleon, this being of the form

$$(\mathbf{r}|V|\mathbf{r}'). \qquad (7)$$

The nonlocality is due in part to polarization of the nuclear medium by the interacting particle, this effect having a range of 0.5 to 1.0×10^{-13} cm, and in part to exchange effects which lead to a nonlocality with range of the two-body interaction itself. Consequently, a local approximation to $(\mathbf{r}'|V|\mathbf{r})$ can be valid only for wave functions which are slowly varying over a range of roughly 2×10^{-13} cm.

An additional effect was mentioned by Wilets in his review of the theories of the nuclear surface.[11] As he emphasized, even in a local approximation to the single particle potential, the potential cannot be assumed to be a linear function of the density but must instead fall off less rapidly than linearly with density. This effect is closely associated with the saturation phenomenon and arises in both the effects of the

[11] Lawrence Wilets, Revs. Modern Phys. **30**, 542 (1958), this issue.

repulsive core and the exchange repulsion. A consequence is that the potential radius affecting an interacting nucleon is increased by about $\frac{1}{2}\times 10^{-13}$ cm relative to the density radius, this being accompanied by some change in the apparent surface depth. The finite range of interaction also alters the potential-density relationship, increasing the potential radius. Finally, since the repulsion in the two-body interaction has a range short compared to the attraction, the potential surface has a more complicated structure than the density surface.

The various effects just described must be incorporated into a realistic theory of the nuclear surface. This has recently been done by Gammel, Weitzner, and the author[12] and quantitative results will soon be available at the Los Alamos Scientific Laboratory.

[12] Brueckner, Gammel, and Weitzner, Phys. Rev. **110**, 431 (1958).

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Implications of the Direct-Interaction Model for Nuclear Structure*

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1. INTRODUCTION

THE physical basis for the direct-interaction model of nuclear reactions was given by Serber ten years ago.[1] In the intervening time we have seen its scope and usefulness greatly extended. At the same time several developments in technique of scattering theory have made it possible to put this model into a quantitative form. This quantitative form is the subject of this discussion.

By "direct-interaction model" one means (essentially) the attempt to describe the scattering of a particle by an atomic nucleus in terms of collisions (*one at a time*) of that particle with nuclear protons and neutrons. Furthermore, such binary collisions are considered as resulting from the same forces as cause scattering from a *free* proton or neutron. At high energies the *scattering cross section* from a bound nucleon is actually considered to be the same as that from a free nucleon.

The fundamental requirement for the correctness of the direct-interaction model is the condition that the interaction energy of the given particle with the nucleus be of the form

$$V=\sum_{i=1}^{A} V_i. \tag{1}$$

Here V_i is the interaction energy of the particle with the ith nucleon when that nucleon is removed from the nucleus. Aside from the condition of Eq. (1), the practical applicability of the direct-interaction (or Serber) model depends upon the complexity of nuclear structure and upon the energy of the scattered particle. Because of this dependence on nuclear structure we can hope to use it to learn something about nuclear properties. How one does this is the second point that I should like to describe. The third point is the possibility of using nuclear interaction to learn something about the forces between nucleons and "strange particles."

The Serber model has been sufficiently successful that one can feel some confidence in at least the approximate validity of Eq. (1). This makes it reasonable to assume that Eq. (1) is strictly correct and then to develop the model as completely as possible. As we shall argue, the model is susceptible of a much more quantitative development than has been made. Also, comparisons with experiment seem often to have been less precise than is justifiable. In other words, the limits on the accuracy of the direct-interaction model raise quantitative questions to which we are only beginning to find some answers.

The Serber model must be handled quite differently in different energy ranges. It is much simpler at high than at intermediate and low energies. The possible applicability of the model at low energies has been discussed by Brueckner and his collaborators.[2] Brueckner has just described this work,† which incorporates the physical basis of the direct-interaction model into a dynamical description of nuclear structure.

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[1] R. Serber, Phys. Rev. **72**, 114 (1957); Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).

[2] Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954); K. Brueckner, Phys. Rev. **100**, 36 (1955).

† K. A. Brueckner, Revs. Modern Phys. **30**, 561 (1958), preceding paper.